Probability Curves Showing Poisson's Exponential Summation

By GEORGE A. CAMPBELL

In many important practical operations the constant probability of an event happening in a single trial is extremely small, but the number of trials is so large that the event may actually occur a sufficient number of times to become a matter of importance. The curves of Figs. 1 and 2 show the probability P of such an event happening at least c times in a number of trials for which the average number of occurrences is a. The probability range shown is from 0.000001 to 0.999999 and the average extends from 0 to 15 in Fig. 1 and to 200 in Fig. 2. An open scale is obtained at both ends, even when the probability approaches to within one part in a million of the limits 0 and 1, by employing an ordinate scale corresponding to the normal probability integral.

In the practical use of these curves the first question which arises is—What number of trials is necessary to make the curves applicable? In practice an infinite number of trials, which is the case for which the curves are drawn, can never be attained; and if we had absolutely no knowledge of the relation between the probabilities for an infinite number and a finite number of trials, the curves would have a theoretical interest only. We do, however, know in a general way when a finite number of trials approximates to the limiting case; the more complete and precise our knowledge on this point, the more generally useful the curves will become. Without attempting to go into the question exhaustively, which would require most careful analysis, a general answer will be found to the question as to the number of trials required by plotting the simple functions $(a/c)^c$, $\frac{1}{2}(c-a-1)$, and $\frac{1}{2}c(c-a-1)$.

The characteristic of all probability curves when n is either finite or infinite, is shown by Fig. 3, where $P(c,n,a)$ denotes the probability of an event happening at least c times in n trials when the average number of occurrences is a. Any curve $P(c,n,a)$ is contained between the ordinates at $a=0$ and $a=n$ and is asymptotic to these ordinates; it cuts $P=\frac{1}{2}$ between $a=c-1$ and $c-0.3$. Thus as n decreases from infinity to c, the central portion of the c curve changes but little, but the curve is confined to the narrowing band to the left of $a=n$ and becomes steeper. On reducing n to $c-1$ the c curve disappears entirely, since c cases cannot occur; the number of trials

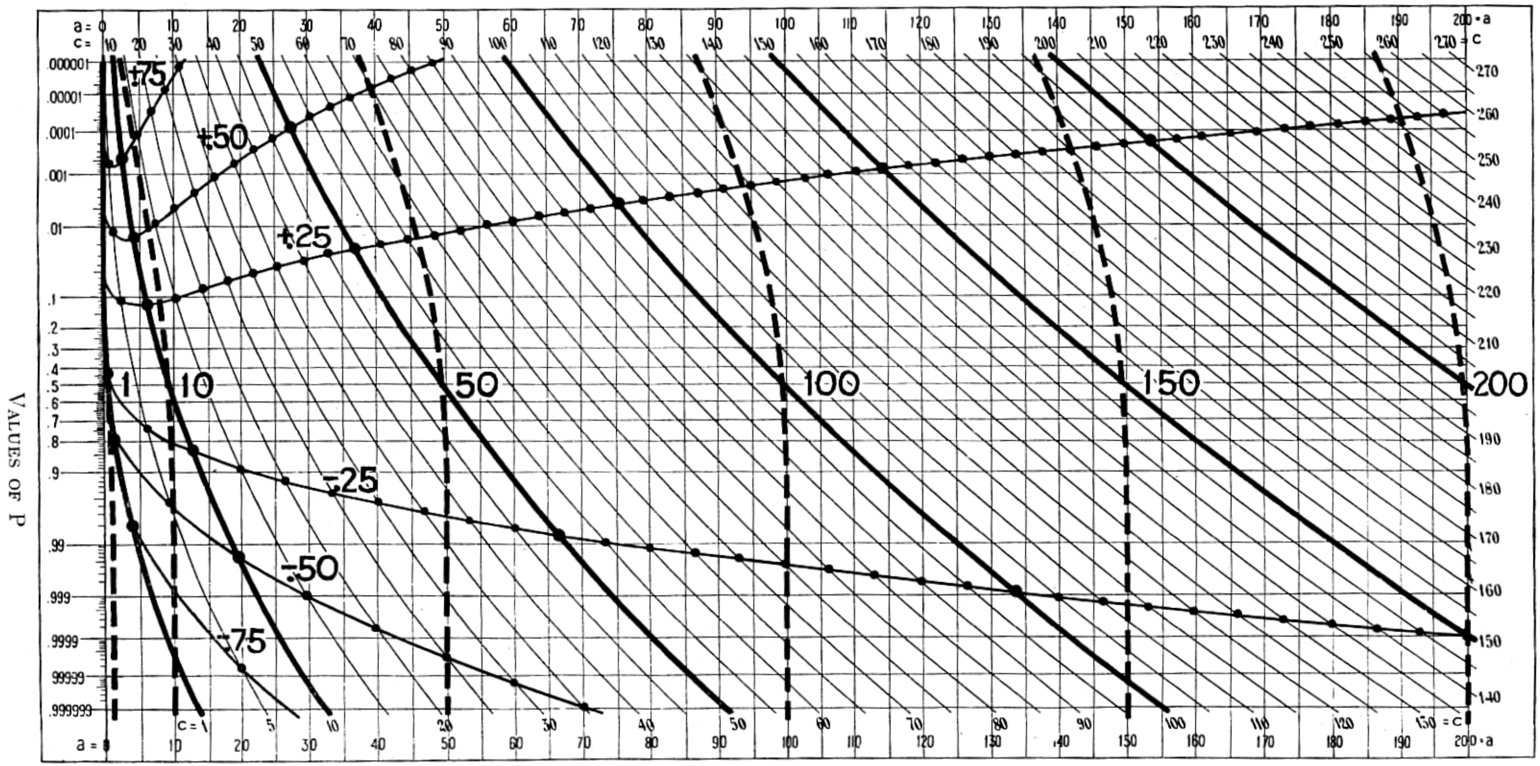

Fig. 3—Extreme curves of the point binomial; the heavy line curves are for the Poisson exponential $P(c,\infty,a)$ and the dashed curves are for $P(c,c,a)=(a/c)^c$ which are the special cases $n=\infty$ and $n=c$ of the general point binomial,

$$P(c,n,a)=\frac{\Gamma(n+1)}{\Gamma(c)\Gamma(n-c+1)}\int_0^{a/n} x^{c-1}(1-x)^{n-c}dx.$$

The beaded curves show the corresponding relative increment in the average, $\Delta a/a=[a(c,\,c,\,P)-a(c,\infty,P)]/a(c,\infty,P)$.

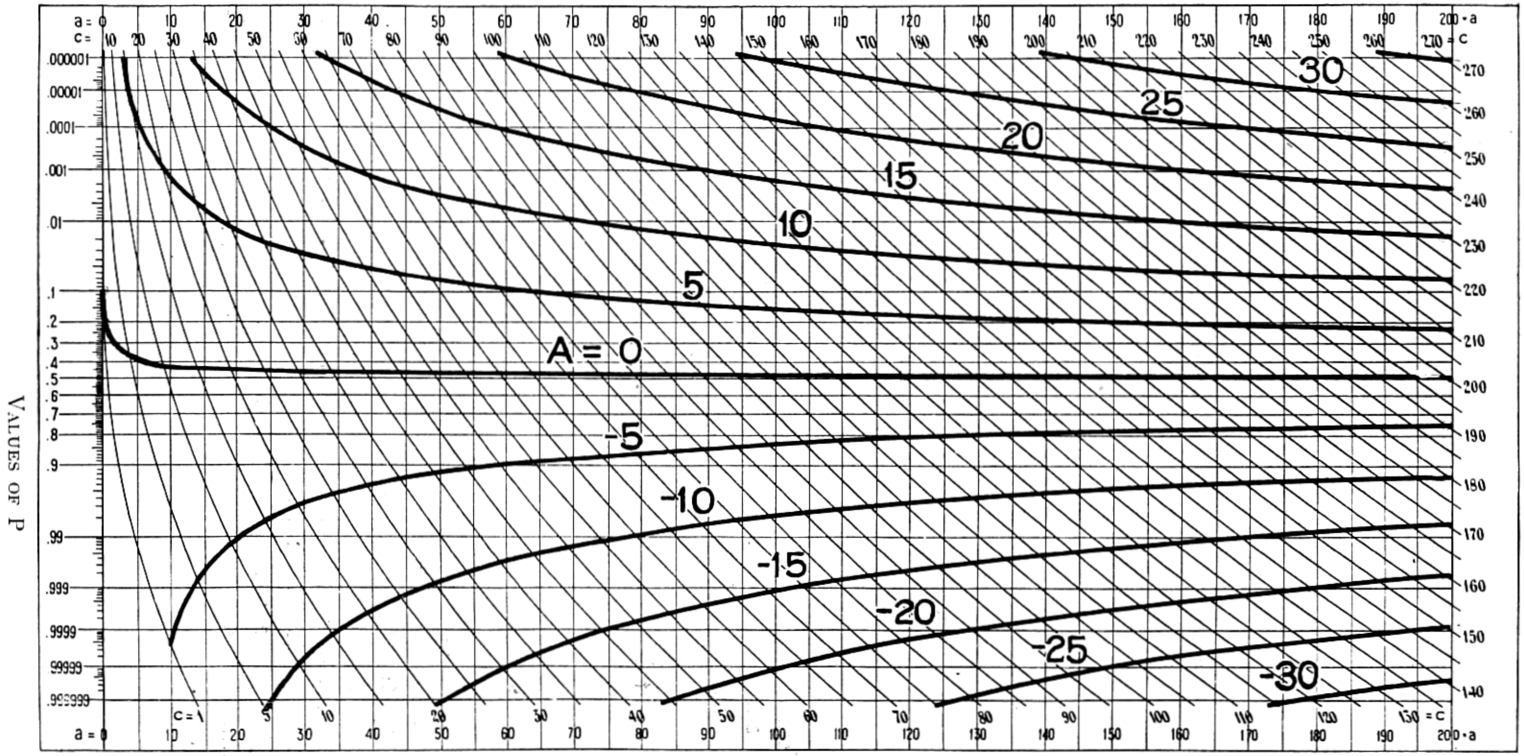

Fig. 4—Curves showing the initial rate of change, $[da/d(1/n)]/a$, as n decreases from infinity, in the point binomial probability curve. The values A attached to the curves are the value of the first coefficient in the following expansion for the relative change in the average,

$$\frac{a(c,n,P)-a(c,\infty,P)}{a(c,\infty,P)}=An^{-1}+\frac{1}{12}[14A^2+(3a+2)A+a]n^{-2}+\ \ldots,\ \text{where } A=\tfrac{1}{2}(c-a-1) \text{ and } a=a(c,\infty,P).$$

n is an integer which cannot be less than the average a or the number of occurrences c.

Fig. 3, making use of Fig. 2 as a background, shows for $c=1$, 10, 50, 100, 150 and 200, the curves of the point binomial with $n=\infty$ and $n=c$, as heavy full and dashed lines, respectively. Each pair of curves, with the exception of the first, crosses in the neighborhood of $P=\frac{1}{2}$, and, except near this crossing, all of the intermediate curves of each family of c curves lie between these extreme curves. The relative change in the probability P or $1-P$, when these probabilities are small, due to reducing n to this lower limit c, for the c curve, is great, but the relative increase in the average a is only moderate over the greater part of the range covered by Fig. 3. The extreme relative change in the average a is shown by dots placed on each of the Poisson exponential curves, each dot being located at the point where the extreme relative increase in the average is $\pm.25$, $\pm.50$, or $\pm.75$. The relative increment in the average ranges, for Fig. 3, from a decrease of 93 per cent at $P=.999999$ on $c=1$ to an increase of 97 per cent at $P=.000001$ on $c=9$ and 10, but the greater part of the field is included between the beaded curves for ± 50 per cent. Having thus obtained, by examining Fig. 3, a general idea of the relative and absolute numerical magnitudes of the extreme changes to which the probability curves are subject, we are in a better position to make practical use of the curves of Figs. 4 and 5 for the small initial shift in the curves occurring when the number of trials is finite but still large compared with c.

The rate at which the probability curves start to shift, when the number of trials is decreased from infinity, is shown by Fig. 4, which gives the value of the first coefficient A in the expansion, in descending powers of n, for the relative increment in the average. In the upper part of the curves the shift is to the right and in the lower part of the curves it is to the left. The point at which the curve remains initially at rest is shown by the intersection of the c curve with the curve for $A=0$. Since $A=35$ is the largest arithmetical value occurring on Fig. 4 and $n=700$ will make the first term of the series equal 1/20, and the next term is then still smaller, it follows that Fig. 2 redrawn for 700 trials would not show a difference of more than about 5 per cent in any value of the average. For Fig. 1 the corresponding number of trials is 220; it may be shown by direct computation that n may even be reduced to the lower limit 1 with only a small percentage change in the abscissas of the upper portion of the curve $c=1$.

Curves similar to Fig. 4 showing the exact number of trials producing a given relative or absolute shift in the average would be useful. Still another variation is shown by Fig. 5 where the curves give the first coefficient in the expansion, in descending powers of n, of the ratio of the increments in probability, due to a decrease in n and to unit increase in c. These curves therefore show the initial rate at which any c curve approaches the $c+1$ curve above it, if the scale of ordinates were made linear; below the curve $A=0$ the initial shift is downward as indicated by the negative sign for the A's. If sets of curves corresponding to Figs. 1 and 2 were drawn for the number of trials $n=400$ and 2000, respectively, no curve would be shifted by as much as the original distance between the curves shown, since the maximum values on Fig. 5 up to $a=15$ and 200 are 400 and 10,000, respectively; Fig. 2 shows only every fifth curve; the second term of the series indicates that the initial maximum rate of shift is not maintained as n decreases at these points.

The second question arising in connection with the use of the curves is their accuracy. Fig. 1 was drawn with the greatest care on a scale somewhat larger than that of the reproduction, and errors are believed to be only of the order of uncertainty of reading such curves with the unaided eye. Fig. 2 was drawn with less skill and shows larger deviations but it has proved accurate enough for ordinary applications.[2]

The third question which may arise is that of going beyond the curves either in range or in accuracy.[3] The exact calculated values employed in plotting the curves up to $c=101$ are contained in Table II, every entry having been independently checked by two persons. The greater part of the table was calculated by means of a new formula which so expresses the average in terms of P and c as to readily give accurate results for the central range of P with large values of c, which

[1] Cf. Soper, H.E., The Numerical Evaluation of the Incomplete B-Function, 1921, p. 41, and Fisher, A., Mathematical Theory of Probabilities, 2nd Edition, 1922, p. 276.

[2] These claims for the accuracy of the curves of Figs. 1 and 2 have been confirmed by comparison with Pearson's Tables of the Incomplete Γ-Function, 1922, which has been received during the proof-reading of this paper. His tabulated function $I(u, p)$ is, in the notation of the present paper, the probability P corresponding to the average $a=u\sqrt{p+1}$ and the number of occurrences $c=p+1$.

[3] When c is not greater than 51, Pearson's tables may be employed. If the probability is assigned, as in many practical engineering problems, finding the corresponding average from the tables requires interpolation. Formula (1) of the present paper gives the average directly, that is, it gives the inverse incomplete gamma function. The following formula gives c in terms of a:

$$c=a\left[1-ta^{-\frac{1}{2}}+\tfrac{1}{6}(t^2+2)a^{-1}+\frac{1}{72}(t^3+2t)a^{-\frac{3}{2}}+\ldots\right]$$

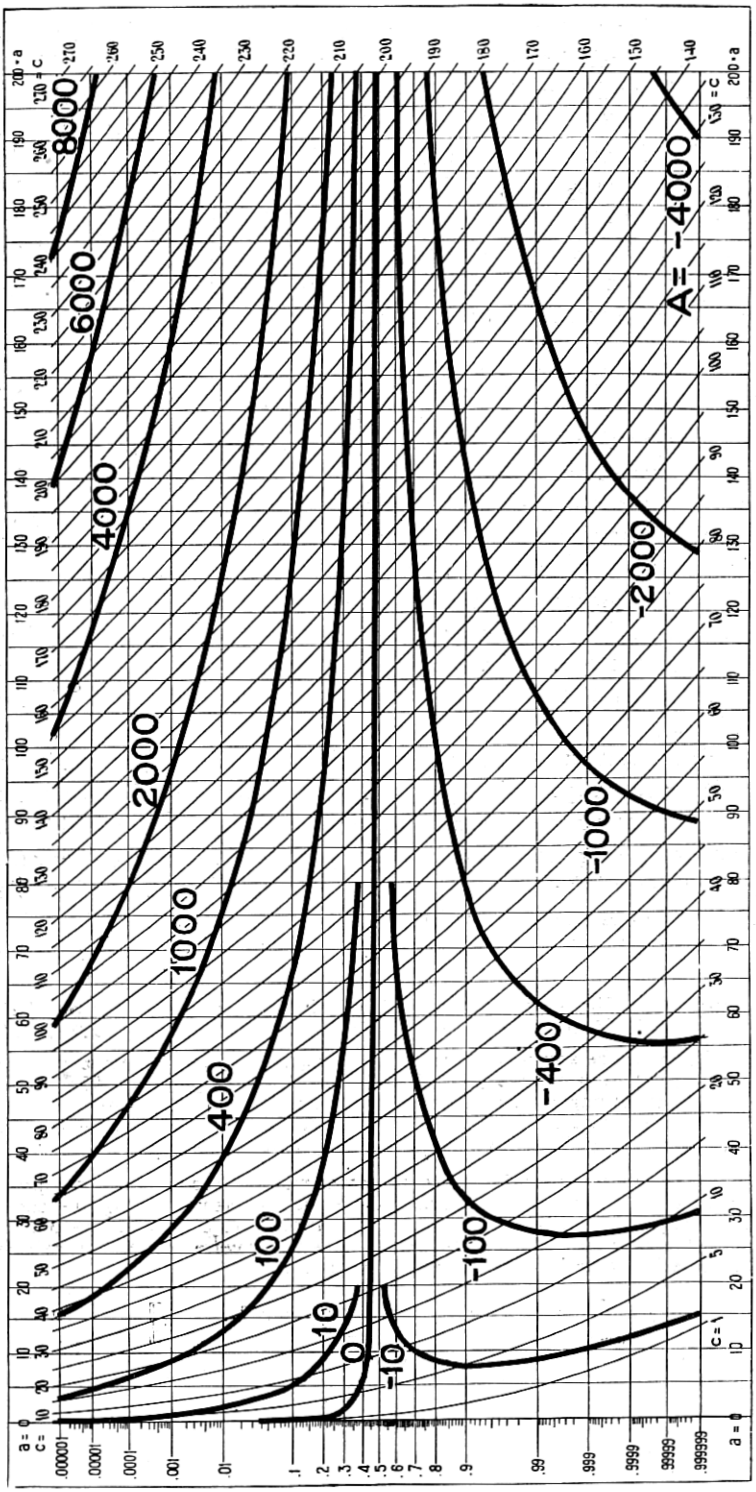

Fig. 5—Curves showing the initial rate of change in the probability, $dP/d(1/n)$, as n decreases from infinity, divided by the increment in P due to unit increase in c. The values of A attached to the curves are the value of the first coefficient in the expansion,

$$\frac{P(c,n,a)-P(c,\infty,a)}{P(c+1,\infty,a)-P(c,\infty,a)} = An^{-1} - \frac{c}{24}(3D^3-7aD+a^2-c^2+1)n^{-2}+\dots,$$

where $A = \frac{1}{2}c(c-a-1)$, and $D=(c-a-1)$.

is the domain in which the ordinary formulas are not convenient for calculation. This is formula (1) below which involved transforming the normal probability integral to fit the skew probability summation of Poisson's exponential binomial limit. The reason for thinking that this transformation would prove useful is made clear by noting that in Figs. 1 and 2 the curves become more and more uniformly spaced with increasing values of the average a and thus the probability approximates more and more closely to the normal probability integral, since this is the scale employed for the ordinates. The results of the mathematical work are summed up in the following formula:

For Poisson's exponential binomial limit the average a is expressed as a function of the probability P of at least c occurrences by the infinite series

$$a = c\sum_{n=0}^{\infty} Q_n c^{-\frac{1}{2}n}, \qquad (1)$$

where the coefficients Q_n are functions of the argument t corresponding to the probability P expressed in the form of the normal probability integral,

$$P = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{1}{2}t^2}dt; \qquad (2)$$

twelve of these coefficients are given in the following table:

TABLE I COEFFICIENTS IN FORMULA (1) FOR THE AVERAGE

n	Q_n
0	1
1	t
2	$(t^2-1)/3$
3	$(t^3-7t)/2^2 3^2$
4	$(-3t^4-7t^2+16)/2^1 3^4 5$
5	$(9t^5+256t^3-433t)/2^5 3^5 5$
6	$(12t^6-243t^4-923t^2+1{,}472)/2^3 3^6 5^1 7$
7	$(-3{,}753t^7-4{,}353t^5+289{,}517t^3+289{,}717t)/2^7 3^8 5^2 7$
8	$(270t^8+4{,}614t^6-9{,}513t^4-104{,}989t^2+35{,}968)/2^4 3^9 5^2 7$
9	$(-5{,}139t^9-547{,}848t^7-2{,}742{,}210t^5+7{,}016{,}224t^3+37{,}501{,}325t)/2^{11}3^{10}5^2 7$

10 $(-364{,}176t^{10}+6{,}208{,}146t^{8}+125{,}735{,}778t^{6}+303{,}753{,}831t^{4}-672{,}186{,}949t^{2}-2{,}432{,}820{,}224)/2^{7}3^{13}5^{3}7^{1}11$

11 $(199{,}112{,}985t^{11}+1{,}885{,}396{,}761t^{9}-31{,}857{,}434{,}154t^{7}-287{,}542{,}736{,}226t^{5}-556{,}030{,}221{,}167t^{3}+487{,}855{,}454{,}729t)/2^{13}3^{14}5^{3}7^{2}11$

For any given value of P the corresponding value of t in (2) can be found from tables of the probability integral. The value of a for this value of P and for any value of c can then be determined by (1). In this way values of a were calculated for every integral value of c from 1 to 101 and for eleven particular values of P: 0.000001, 0.0001, 0.01, 0.1, 0.25, 0.5, 0.75, 0.9, 0.99, 0.9999, 0.999999. These results are presented in Table II. The numerical values of the coefficients Q_1 to Q_7, corresponding to the particular values of P used in Table II, are given in Table VII.

From the information given in Table II, two sets of curves were drawn, Figs. 1 and 2, the first for each integral value of c in the range $a=0$ to $a=15$ and $P=0.000001$ to $P=0.999999$, and the second for every fifth integral value of c in the range $a=0$ to $a=200$ and the same range of P. From these curves any one of the variables (P, c, a) may be found corresponding to assigned values of the other two, subject to the practical condition that c is to be an integer.

Proof

The well-known expressions for the summation of Poisson's exponential binomial limit are:

$$P=\frac{a^c e^{-a}}{c!}+\frac{a^{c+1}e^{-a}}{(c+1)!}+\frac{a^{c+2}e^{-a}}{(c+2)!}+\dots$$

$$=\sum_{s=c}^{\infty}\frac{a^s e^{-a}}{s!}$$

$$=1-\left[1+\frac{a}{1!}+\frac{a^2}{2!}+\dots+\frac{a^{c-1}}{(c-1)!}\right]e^{-a}$$

$$=1-\sum_{s=0}^{c-1}\frac{a^s e^{-a}}{s!}$$

$$=\frac{1}{\Gamma(c)}\int_0^a a^{c-1}e^{-a}da. \qquad (3)$$

The series expansion (1) is determined by equating the integrands of (2) and (3),

$$\frac{1}{\Gamma(c)}a^{c-1}e^{-a}da=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt, \tag{4}$$

and solving for positive values of a with the condition that $t=-\infty$ when $a=0$.

$$\left.\begin{aligned}&\text{Let}\quad c=\frac{1}{b^2},\qquad a=\frac{1}{b^2}Q,\qquad Q=e^L,\\ &\frac{\Gamma(c)}{\sqrt{2\pi}}=b(b^2e)^{-1/b^2}e^M,\\ &R=\frac{Q-L-1}{b^2}-\tfrac{1}{2}t^2+M.\end{aligned}\right\} \tag{5}$$

Substituting these values (5) in equation (4),

$$L'=be^R, \tag{6}$$

where L' is written for dL/dt.

$$\text{Let}\ L=\sum_{s=0}^{\infty}L_sb^s,\quad M=\sum_{s=0}^{\infty}M_sb^s,\quad R=\sum_{s=0}^{\infty}R_sb\ ,\quad Q=\sum_{s=0}^{\infty}Q_sb^s, \tag{7}$$

where the coefficients are polynomials in t (constants in the case of the series for M). Upon substituting these series expansions for the functions in the last equality of (5) and equating coefficients of like powers of b, we obtain

$$\begin{aligned}0&=Q_0-L_0-1,\\ 0&=Q_1-L_1,\\ R_0&=Q_2-L_2-\tfrac{1}{2}t^2+M_0,\\ R_1&=Q_3-L_3+M_1,\\ R_2&=Q_4-L_4+M_2,\\ &\dots\dots\dots\\ R_n&=Q_{n+2}-L_{n+2}+M_n,\ (n=1,\ 2,\ 3\ \dots).\end{aligned} \tag{8}$$

From (5) we obtain $Q_o=e^{L_o}$, and then $L_o+1=e^{L_o}$, the only real solution of which is $L_o=0$, and therefore, $Q_o=1$.

Utilizing these initial values we obtain

$$Q_1 = L_1,$$

$$Q_2 = L_2 + \tfrac{1}{2} L_1 Q_1,$$

$$Q_3 = L_3 + \tfrac{2}{3} L_2 Q_1 + \tfrac{1}{3} L_1 Q_2,$$

$$\cdots\cdots\cdots\cdots$$

$$Q_n = \sum_{s=o}^{n-1} \frac{n-s}{n} L_{n-s} Q_s, \; (n=1,\, 2,\, 3 \ldots). \tag{9}$$

From (6) we obtain $L_1' = e^{R_o}$, and from that $L_1' = e^{\frac{1}{2}L_1^2 - \frac{1}{2}t^2 + M_0}$.

Since L_1 is a polynomial in t, and M_0 a constant, we must have $L_1' = 1$, $L_1 = \pm t$, $M_0 = 0$, that is, $L_1 = t$, and hence $Q_1 = t$.

Then $L_2' = R_1$,

$$L_3' = R_2 + \tfrac{1}{2} R_1 L_2',$$

$$L_4' = R_3 + \tfrac{2}{3} R_2 L_2' + \tfrac{1}{3} R_1 L_3',$$

$$\cdots\cdots\cdots\cdots$$

$$L_{n+1}' = \sum_{s=o}^{n-1} \frac{n-s}{n} R_{n-s} L_{s+1}', \; (n=1,\, 2,\, 3 \ldots). \tag{10}$$

The next set of coefficients can now be deduced, as follows:

$$Q_3 = L_3 + \tfrac{2}{3} L_2 Q_1 + \tfrac{1}{3} L_1 Q_2,$$

$$R_1 = Q_3 - L_3 + M_1,$$

$$R_1 = \tfrac{2}{3} L_2 Q_1 + \tfrac{1}{3} L_1 Q_2 + M_1,$$

$$L_2' = R_1,$$

$$Q_2 = L_2 + \tfrac{1}{2} L_1 Q_1,$$

$$L_2' = \tfrac{2}{3} L_2 Q_1 + \tfrac{1}{3} L_1 L_2 + \tfrac{1}{6} L_1^2 Q_1 + M_1,$$

$$L_1 = t,$$

$$Q_1 = t,$$

$$L_2' = L_2 t + \tfrac{1}{6} t^3 + M_1.$$

But M_1 is a constant, and L_2 is a polynomial in t. Let

$$L_2 = c_2 t^2 + c_1 t + c_o,$$

it being evident that L_2 is of the second degree. Then

$$L_2' = 2c_2 t + c_1.$$

Substituting and equating coefficients of like powers of t

$$c_2+\tfrac{1}{6}=0,\ c_1=0,\ c_o=2c_2,\ M_1=c_1.$$

Hence

$$\left.\begin{aligned} L_2&=(-t^2-2)/6, & R_1&=-t/3,\\ M_1&=0, & Q_2&=(t^2-1)/3. \end{aligned}\right\} \tag{11}$$

Starting with these initial values equations (8)–(10) are sufficient to determine as many coefficients in the expansions (7) as are required. In order to demonstrate this, assume that all the coefficients up to and including L_k, M_{k-1}, R_{k-1}, Q_k have been determined. It can then be shown that the next coefficient in each expansion can be obtained from these data, as follows:

For $n=k+2$, equation (9) can be written

$$Q_{k+2}=L_{k+2}+\frac{k+1}{k+2}L_{k+1}t+\sum_{s=2}^{k}\frac{k+2-s}{k+2}L_{k+2-s}Q_s+\frac{1}{k+2}Q_{k+1}t, \tag{12}$$

where Q_{k+1}, Q_{k+2}, L_{k+1}, L_{k+2} are the unknown quantities. For $n=k$ and $n=k-1$, equation (8) assumes the forms

$$R_k=Q_{k+2}-L_{k+2}+M_k, \tag{13}$$

and

$$R_{k-1}=Q_{k+1}-L_{k+1}+M_{k-1}, \tag{14}$$

respectively, where all the quantities are unknown except R_{k-1} and M_{k-1}. For $n=k$, equation (10) can be written in the form

$$L_{k+1}=R_k+\sum_{s=1}^{k-1}\frac{k-s}{k}R_{k-s}L'_{s+1}, \tag{15}$$

where L_{k+1} and R_k are the unknown quantities. Substituting in (12) the value of $(Q_{k+2}-L_{k+2})$ found from (13), and then substituting the value of R_k found from (15) and the value of Q_{k+1} from (14),

$$\begin{aligned} L'_{k+1}= M_k&+\frac{k+1}{k+2}L_{k+1}t+\frac{1}{k+2}(R_{k-1}+L_{k+1}-M_{k-1})t\\ &+\sum_{s=2}^{k}\frac{k+2-s}{k+2}L_{k+2-s}Q_s+\sum_{s=1}^{k-1}\frac{k-s}{k}R_{k-s}L'_{s+1}. \end{aligned} \tag{16}$$

This is a linear differential equation in L_{k+1} as a function of t, all the coefficients being known functions of t with the exception of M_k which is an undetermined numerical constant. By a suitable choice of the constant M_k, (16) may be solved for L_{k+1} as a polynomial in

t of the $(k+1)$st degree. R_k may then be determined by (15) and Q_{k+1} by (14). From these results, the next set of coefficients may be found, and so on. The values of the coefficients for $k=2$ (L_2, M_1, R_1, Q_2) have been found, and equations (8)–(10) are valid for the particular values of n utilized in the above method. Hence the next set of coefficients (L_3, M_2, R_2, Q_3) may be found, and in the same way, as many more as are desired. The detailed work of the first step is indicated below:

Substituting $k=2$ in (16),

$$L_3' = M_2 + \tfrac{3}{4}L_3 t + \tfrac{1}{4}(R_1 + L_3 - M_1)t + \tfrac{1}{2}L_2 Q_2 + \tfrac{1}{2}R_1 L_2'. \qquad (17)$$

Substituting in (17) the values known from (11),

$$L_3' = L_3 t + (-t^4 - 2t^2 + 2)/36 + M_2. \qquad (18)$$

Let L_3 be a polynomial of the form $(A_3 t^3 + A_2 t^2 + A_1 t + A_0)$ and substitute in (18). Upon equating coefficients of like powers of t, we find that $A_3 = 1/36$, $A_2 = 0$, $A_1 = 5/36$, $A_0 = 0$, and $M_2 = 1/12$. R_2 is then obtained by substituting these values in (15) and Q_3 from (14). The results are as follows:

$$\left.\begin{array}{ll} L_3 = (t^3 - 5t)/36, & R_2 = (t^2 - 5)/36, \\ M_2 = 1/12, & Q_3 = (t^3 - 7t)/36. \end{array}\right\} \qquad (19)$$

The actual work of computing these coefficients has been performed up to and including $k=11$ (L_{11}, M_{10}, R_{10}, Q_{11}). These results are presented in the attached tables: Q_n in I, L_n in III, L_n' in IV, R_n in V, and M_n in VI. From this information the next coefficient in the series (1), Q_{12}, can be computed by the method outlined above.

It may be pointed out in conclusion that the expansion of M presented in Table VI is the asymptotic series obtained in Stirling's expansion of $\Gamma(c)$, as is to be expected from equations (5). This in itself constitutes a partial check upon the determination of the coefficients.

Additional Properties of the Curves

At the probability $P=0.5$, the difference $(c-a)=1/3$, approximately.[4] Discrepancies are so small as not to be positively dis-

[4] This recalls the approximate rule that the median lines one-third of the distance from the mean towards the mode. (Yule, Theory of Statistics, 1911, p. 121.) But in the Poisson exponential the median never lies between the mean and the mode; the median occurs at the first integer above or below the mean, whichever integer corresponds to the c curve cutting $P=0.5$ next below the mean, while the mode is always at the first integer less than the mean. For the range of cases having a given mode, however, the mean and the median are, on the average, greater than the mode by $\frac{1}{2}$ and approximately $\frac{1}{3}$, respectively; thus the median must line one-third of the distance from the mean towards the mode in the case of the corresponding heterogeneous samplings.

cernible on Fig. 1, but Table II gives for $c=1, 2, 3 \ldots 100$, the differences $(c-a)=0.3069$, 0.3217, 0.3259, . . . 0.3331, which differ but little from 0.3333 . . . , which is approached more and more closely for large values of c.

At $P=0.5$ and large values of c the derivative along the c curve is $dP/da=1/\sqrt{2\pi c}$, as found by differentiating (3), substituting $a=c-1/3$ and Stirling's expression for the gamma function. Thus, for large values of c the slope of the curve at $P=0.5$ decreases numerically as the square root of c increases. For large values of c the curves are approximately straight over the wide range of probability shown in the figures. This, in connection with the additional fact that the standard deviation $\sqrt{npq}$ is always equal to $\sqrt{a}$ for the Poisson exponential, is an alternative way of arriving at the expression for the derivative given above.

I am indebted to Miss Edith Clarke for extending the series of formula (1) to seven terms, for making all of the original computations and for drawing Fig. 1, and to Miss Sallie E. Pero for extending the formula to its present eleven terms, and for checking all of the preceding work; the single error which she found occurred in the seventh term of the expansion where it was without effect on the final numerical results. Finally, the work was entirely rechecked, without discovering additional errors, by Mr. Ronald M. Foster, who also put the mathematical work into its present form, pointed out the asymptotic nature of the expansion and compared the overlapping numerical results with those obtained by direct summation by Miss Lucy Whitaker[5] and more recently by Mr. E. C. Molina, as well as with his earlier table.[6]

[5] Tables for Statisticians and Biometricians, 1914, Table LII.

[6] Computation Formula for the Probability of an Event Happening at Least C Times in N Trials, *American Mathematical Monthly*, XX, June, 1913, p. 193.

TABLE II. VALUES OF THE AVERAGE a CORRESPONDING TO GIVEN P AND c IN POISSON'S EXPONENTIAL SUMMATION

$P=$	0.000001	0.0001	0.01	0.1	0.25	0.5	0.75	0.9	0.99	0.9999	0.999999
$c=$ 1	0.00*	0.000*	0.010	0.1054	0.2877	0.6931	1.3863	2.3026	4.605	9.210	13.82
2	0.00*	0.014	0.149	0.5318	0.9613	1.6783	2.6926	3.8897	6.638	11.756	16.69
3	0.02	0.086	0.436	1.1021	1.7273	2.6741	3.9204	5.3223	8.406	13.928	19.13
4	0.07	0.232	0.823	1.7448	2.5353	3.6721	5.1094	6.6808	10.045	15.914	21.35
5	0.17	0.444	1.279	2.4326	3.3686	4.6709	6.2744	7.9936	11.605	17.782	23.43
6	0.31	0.714	1.785	3.1519	4.2192	5.6702	7.4227	9.2747	13.108	19.567	25.41
7	0.50	1.030	2.330	3.8948	5.0827	6.6696	8.5585	10.5321	14.571	21.290	27.32
8	0.73	1.387	2.906	4.6561	5.9561	7.6692	9.6844	11.7709	16.000	22.962	29.16
9	0.99	1.778	3.507	5.4325	6.8376	8.6690	10.8024	12.9947	17.403	24.595	30.96
10	1.28	2.198	4.130	6.2213	7.7259	9.6687	11.9138	14.2060	18.783	26.193	32.71
11	1.60	2.643	4.771	7.0208	8.6198	10.6685	13.0196	15.4066	20.145	27.762	34.43
12	1.94	3.111	5.428	7.8293	9.5186	11.6684	14.1206	16.5981	21.490	29.306	36.11
13	2.31	3.600	6.099	8.6459	10.4217	12.6682	15.2173	17.7816	22.821	30.829	37.77
14	2.69	4.106	6.782	9.4696	11.3286	13.6681	16.3102	18.9580	24.139	32.331	39.41
15	3.10	4.629	7.477	10.2996	12.2388	14.6680	17.3999	20.1280	25.446	33.816	41.02
16	3.52	5.167	8.181	11.1353	13.1521	15.6679	18.4865	21.2924	26.743	35.286	42.62
17	3.96	5.718	8.895	11.9761	14.0680	16.6679	19.5704	22.4516	28.031	36.741	44.19
18	4.42	6.281	9.616	12.8217	14.9865	17.6678	20.6518	23.6061	29.310	38.182	45.75
19	4.88	6.856	10.346	13.6715	15.9073	18.6677	21.7310	24.7563	30.581	39.612	47.30
20	5.36	7.442	11.082	14.5253	16.8301	19.6677	22.8080	25.9025	31.845	41.031	48.83
21	5.86	8.037	11.825	15.3827	17.7550	20.6676	23.8831	27.0451	33.103	42.440	50.34
22	6.36	8.641	12.574	16.2436	18.6816	21.6676	24.9564	28.1843	34.355	43.839	51.85
23	6.87	9.255	13.329	17.1076	19.6099	22.6675	26.0281	29.3203	35.601	45.229	53.35
24	7.40	9.876	14.088	17.9746	20.5397	23.6675	27.0982	30.4533	36.841	46.610	54.83
25	7.93	10.505	14.853	18.8443	21.4710	24.6675	28.1668	31.5836	38.077	47.984	56.30
26	8.47	11.141	15.623	19.7167	22.4038	25.6674	29.2340	32.7112	39.308	49.351	57.77
27	9.02	11.783	16.397	20.5915	23.3378	26.6674	30.3000	33.8364	40.535	50.711	59.23
28	9.57	12.432	17.175	21.4687	24.2730	27.6674	31.3647	34.9593	41.757	52.064	60.67
29	10.14	13.088	17.957	22.3480	25.2094	28.6674	32.4283	36.0799	42.975	53.411	62.11
30	10.71	13.748	18.742	23.2294	26.1469	29.6673	33.4907	37.1985	44.190	54.752	63.55
31	11.29	14.415	19.532	24.1128	27.0855	30.6673	34.5521	38.3151	45.401	56.087	64.97
32	11.87	15.086	20.324	24.9981	28.0250	31.6673	35.6126	39.4298	46.609	57.417	66.39
33	12.46	15.763	21.120	25.8852	28.9655	32.6673	36.6720	40.5427	47.813	58.742	67.81
34	13.06	16.444	21.919	26.7740	29.9069	33.6673	37.7306	41.6540	49.015	60.062	69.21

*These values which require more decimals are $a=0.0000010$ and 0.0001000 for $c=1$ and 0.0014149 for $c=2$.

TABLE II. (*Continued*)

$P =$	0.000001	0.0001	0.01	0.1	0.25	0.5	0.75	0.9	0.99	0.9999	0.999999
$c = 35$	13.66	17.130	22.721	27.6645	30.8492	34.6672	38.7883	42.7635	50.213	61.377	70.61
36	14.27	17.821	23.525	28.5565	31.7923	35.6672	39.8452	43.8715	51.409	62.688	72.01
37	14.88	18.515	24.333	29.4500	32.7361	36.6672	40.9013	44.9780	52.601	63.995	73.40
38	15.50	19.214	25.143	30.3449	33.6808	37.6672	41.9566	46.0831	53.791	65.298	74.79
39	16.13	19.916	25.955	31.2413	34.6262	38.6672	43.0112	47.1868	54.979	66.596	76.16
40	16.75	20.622	26.770	32.1389	35.5723	39.6672	44.0651	48.2891	56.164	67.891	77.54
41	17.39	21.332	27.587	33.0379	36.5190	40.6672	45.1184	49.3902	57.347	69.183	78.91
42	18.02	22.045	28.406	33.9380	37.4664	41.6671	46.1709	50.4900	58.528	70.470	80.28
43	18.66	22.762	29.228	34.8394	38.4145	42.6671	47.2229	51.5886	59.707	71.755	81.64
44	19.31	23.481	30.051	35.7419	39.3631	43.6671	48.2742	52.6861	60.884	73.036	83.00
45	19.95	24.204	30.877	36.6455	40.3123	44.6671	49.3250	53.7825	62.059	74.314	84.35
46	20.61	24.930	31.704	37.5502	41.2621	45.6671	50.3752	54.8778	63.231	75.589	85.70
47	21.26	25.659	32.534	38.4560	42.2125	46.6671	51.4248	55.9721	64.402	76.860	87.05
48	21.92	26.391	33.365	39.3627	43.1633	47.6671	52.4739	57.0654	65.571	78.129	88.39
49	22.58	27.125	34.198	40.2704	44.1147	48.6671	53.5225	58.1576	66.738	79.396	89.73
50	23.25	27.862	35.032	41.1791	45.0666	49.6671	54.5706	59.2490	67.903	80.659	91.06
51	23.92	28.602	35.869	42.0886	46.0190	50.6671	55.6182	60.3394	69.067	81.920	92.40
52	24.59	29.344	36.707	42.9991	46.9718	51.6670	56.6654	61.4290	70.230	83.179	93.72
53	25.27	30.089	37.546	43.9104	47.9251	52.6670	57.7121	62.5177	71.390	84.435	95.05
54	25.95	30.836	38.387	44.8226	48.8788	53.6670	58.7584	63.6055	72.549	85.688	96.37
55	26.63	31.585	39.229	45.7355	49.8330	54.6670	59.8042	64.6926	73.707	86.940	97.69
56	27.31	32.337	40.073	46.6493	50.7876	55.6670	60.8496	65.7788	74.863	88.189	99.01
57	28.00	33.090	40.918	47.5638	51.7425	56.6670	61.8946	66.8643	76.018	89.435	100.33
58	28.69	33.846	41.765	48.4791	52.6979	57.6670	62.9392	67.9490	77.172	90.680	101.64
59	29.38	34.604	42.612	49.3951	53.6537	58.6670	63.9835	69.0330	78.324	91.922	102.95
60	30.08	35.364	43.462	50.3118	54.6098	59.6670	65.0273	70.1163	79.475	93.163	104.25
61	30.77	36.126	44.312	51.2292	55.5663	60.6670	66.0708	71.1989	80.625	94.401	105.56
62	31.47	36.890	45.164	52.1473	56.5232	61.6670	67.1139	72.2808	81.773	95.638	106.86
63	32.17	37.656	46.016	53.0661	57.4804	62.6670	68.1567	73.3620	82.921	96.873	108.16
64	32.88	38.423	46.870	53.9855	58.4380	63.6670	69.1991	74.4426	84.067	98.106	109.45
65	33.59	39.193	47.726	54.9055	59.3959	64.6670	70.2412	75.5226	85.212	99.337	110.75
66	34.29	39.964	48.582	55.8262	60.3541	65.6670	71.2830	76.6020	86.355	100.566	112.04
67	35.01	40.736	49.439	56.7474	61.3126	66.6670	72.3245	77.6807	87.498	101.793	113.33
68	35.72	41.511	50.298	57.6692	62.2714	67.6670	73.3656	78.7589	88.640	103.019	114.62

TABLE II. (*Continued*)

$P=$	0.000001	0.0001	0.01	0.1	0.25	0.5	0.75	0.9	0.99	0.9999	0.999999
$c=69$	36.43	42.287	51.157	58.5917	63.2306	68.6670	74.4065	79.8365	89.781	104.244	115.90
70	37.15	43.065	52.017	59.5146	64.1900	69.6670	75.4470	80.9135	90.920	105.466	117.19
71	37.87	43.844	52.879	60.4382	65.1497	70.6669	76.4873	81.9900	92.059	106.687	118.47
72	38.59	44.625	53.741	61.3622	66.1097	71.6669	77.5273	83.0659	93.197	107.907	119.75
73	39.31	45.407	54.604	62.2868	67.0700	72.6669	78.5670	84.1413	94.333	109.125	121.03
74	40.04	46.191	55.469	63.2119	68.0306	73.6669	79.6064	85.2162	95.469	110.341	122.30
75	40.77	46.976	56.334	64.1375	68.9914	74.6669	80.6456	86.2906	96.604	111.556	123.58
76	41.49	47.763	57.200	65.0636	69.9525	75.6669	81.6845	87.3645	97.738	112.770	124.85
77	42.22	48.551	58.067	65.9902	70.9139	76.6669	82.7231	88.4379	98.871	113.982	126.12
78	42.96	49.341	58.935	66.9173	71.8755	77.6669	83.7615	89.5108	100.003	115.193	127.39
79	43.69	50.131	59.803	67.8448	72.8373	78.6669	84.7997	90.5833	101.135	116.402	128.66
80	44.42	50.923	60.673	68.7728	73.7994	79.6669	85.8376	91.6553	102.265	117.610	129.92
81	45.16	51.717	61.543	69.7013	74.7617	80.6669	86.8753	92.7268	103.395	118.817	131.18
82	45.90	52.511	62.414	70.6302	75.7243	81.6669	87.9127	93.7980	104.524	120.023	132.45
83	46.64	53.307	63.286	71.5595	76.6871	82.6669	88.9499	94.8686	105.652	121.227	133.71
84	47.38	54.104	64.159	72.4893	77.6501	83.6669	89.9869	95.9389	106.779	122.430	134.97
85	48.12	54.903	65.032	73.4194	78.6133	84.6669	91.0237	97.0087	107.906	123.632	136.22
86	48.87	55.702	65.906	74.3500	79.5768	85.6669	92.0602	98.0781	109.032	124.833	137.48
87	49.62	56.503	66.781	75.2810	80.5404	86.6669	93.0966	99.1471	110.157	126.033	138.73
88	50.36	57.305	67.657	76.2124	81.5043	87.6669	94.1327	100.2158	111.281	127.231	139.99
89	51.11	58.107	68.533	77.1442	82.4684	88.6669	95.1686	101.2840	112.405	128.428	141.24
90	51.86	58.911	69.410	78.0763	83.4326	89.6669	96.2043	102.3518	113.528	129.625	142.49
91	52.61	59.716	70.288	79.0088	84.3971	90.6669	97.2398	103.4193	114.650	130.820	143.74
92	53.37	60.523	71.166	79.9418	85.3618	91.6669	98.2752	104.4864	115.772	132.014	144.99
93	54.12	61.330	72.045	80.8750	86.3266	92.6669	99.3103	105.5531	116.893	133.207	146.23
94	54.88	62.138	72.925	81.8086	87.2917	93.6669	100.3452	106.6195	118.014	134.399	147.48
95	55.63	62.947	73.805	82.7426	88.2569	94.6669	101.3800	107.6855	119.133	135.590	148.72
96	56.39	63.757	74.686	83.6770	89.2224	95.6669	102.4146	108.7512	120.252	136.780	149.96
97	57.15	64.568	75.568	84.6116	90.1880	96.6669	103.4490	109.8165	121.371	137.969	151.20
98	57.91	65.381	76.450	85.5466	91.1537	97.6669	104.4832	110.8815	122.489	139.157	152.44
99	58.67	66.194	77.333	86.4820	92.1197	98.6669	105.5172	111.9462	123.606	140.344	153.68
100	59.44	67.008	78.216	87.4176	93.0858	99.6669	106.5511	113.0105	124.723	141.530	154.92
101	60.20	67.823	79.100	88.3536	94.0521	100.6669	107.5848	114.0745	125.839	142.715	156.16

TABLE III

n	L_n
0	0
1	t
2	$(-t^2-2)/2^1 3$
3	$(t^3+5t)/2^2 3^2$
4	$(-6t^4-59t^2-58)/2^2 3^4 5$
5	$(9t^5+232t^3+599t)/2^5 3^5 5$
6	$(24t^6-45t^4-817t^2+592)/2^4 3^6 5^1 7$
7	$(-3{,}753t^7-44{,}853t^5-149{,}683t^3-418{,}583t)/2^7 3^8 5^2 7$
8	$(540t^8+12{,}396t^6+77{,}283t^4+226{,}939t^2+217{,}112)/2^5 3^9 5^2 7$
9	$(-5{,}139t^9-416{,}952t^7-4{,}411{,}314t^5-17{,}022{,}320t^3-24{,}039{,}619t)/2^{11} 3^{10} 5^2 7$
10	$(-728{,}352t^{10}-2{,}418{,}858t^8+84{,}239{,}766t^6+514{,}580{,}817t^4+428{,}031{,}517t^2-2{,}293{,}097{,}728)/2^8 3^{13} 5^3 7^1 11$
11	$(199{,}112{,}985t^{11}+4{,}293{,}113{,}877t^9+28{,}888{,}236{,}342t^7+124{,}692{,}719{,}238t^5+654{,}335{,}303{,}761t^3+2{,}373{,}932{,}511{,}173t)/2^{13} 3^{14} 5^3 7^2 11$

TABLE IV

n	L'_n
0	0
1	1
2	$-t/3$
3	$(3t^2+5)/2^2 3^2$
4	$(-12t^3-59t)/2^1 3^4 5$
5	$(45t^4+696t^2+599)/2^5 3^5 5$
6	$(72t^5-90t^3-817t)/2^3 3^6 5^1 7$
7	$(-26{,}271t^6-224{,}265t^4-449{,}049t^2-418{,}583)/2^7 3^8 5^2 7$
8	$(2{,}160t^7+37{,}188t^5+154{,}566t^3+226{,}939t)/2^4 3^9 5^2 7$
9	$(-46{,}251t^8-2{,}918{,}664t^6-22{,}056{,}570t^4-51{,}066{,}960t^2-24{,}039{,}619)/2^{11} 3^{10} 5^2 7$
10	$(-3{,}641{,}760t^9-9{,}675{,}432t^7+252{,}719{,}298t^5+1{,}029{,}161{,}634t^3+428{,}031{,}517t)/2^7 3^{13} 5^3 7^1 11$
11	$(2{,}190{,}242{,}835t^{10}+38{,}638{,}024{,}893t^8+202{,}217{,}654{,}394t^6+623{,}463{,}596{,}190t^4+1{,}963{,}005{,}911{,}283t^2+2{,}373{,}932{,}511{,}173)/2^{13} 3^{14} 5^3 7^2 11$

TABLE V

n	R_n
0	0
1	$-t/3$
2	$(t^2+5)/2^2 3^2$
3	$(t^3-43t)/2^2 3^4 5$
4	$(-21t^4-49t^2+112)/2^4 3^5 5$
5	$(45t^5+488t^3+787t)/2^5 3^6 7$
6	$(-1{,}056t^6-32{,}103t^4-145{,}639t^2-150{,}452)/2^5 3^8 5^2 7$
7	$(-2{,}727t^7+34{,}773t^5+500{,}803t^3+1{,}282{,}103t)/2^7 3^9 5^2 7$
8	$(9{,}990t^8+112{,}614t^6+62{,}577t^4-1{,}193{,}539t^2-1{,}732{,}352)/2^8 3^{10} 5^2 7$
9	$(-28{,}663{,}299t^9-723{,}162{,}744t^7-4{,}907{,}564{,}946t^5$ $-14{,}409{,}113{,}392t^3-22{,}453{,}298{,}291t)/2^{11} 3^{13} 5^3 7^1 11$
10	$(12{,}763{,}008t^{10}+897{,}127{,}182t^8+11{,}273{,}606{,}766t^6$ $+58{,}618{,}777{,}197t^4+161{,}552{,}157{,}577t^2+172{,}910{,}387{,}072)/$ $2^9 3^{14} 5^3 7^2 11$

TABLE VI

n	M_n
0	0
1	0
2	1/12
3	0
4	0
5	0
6	−1/360
7	0
8	0
9	0
10	1/1260

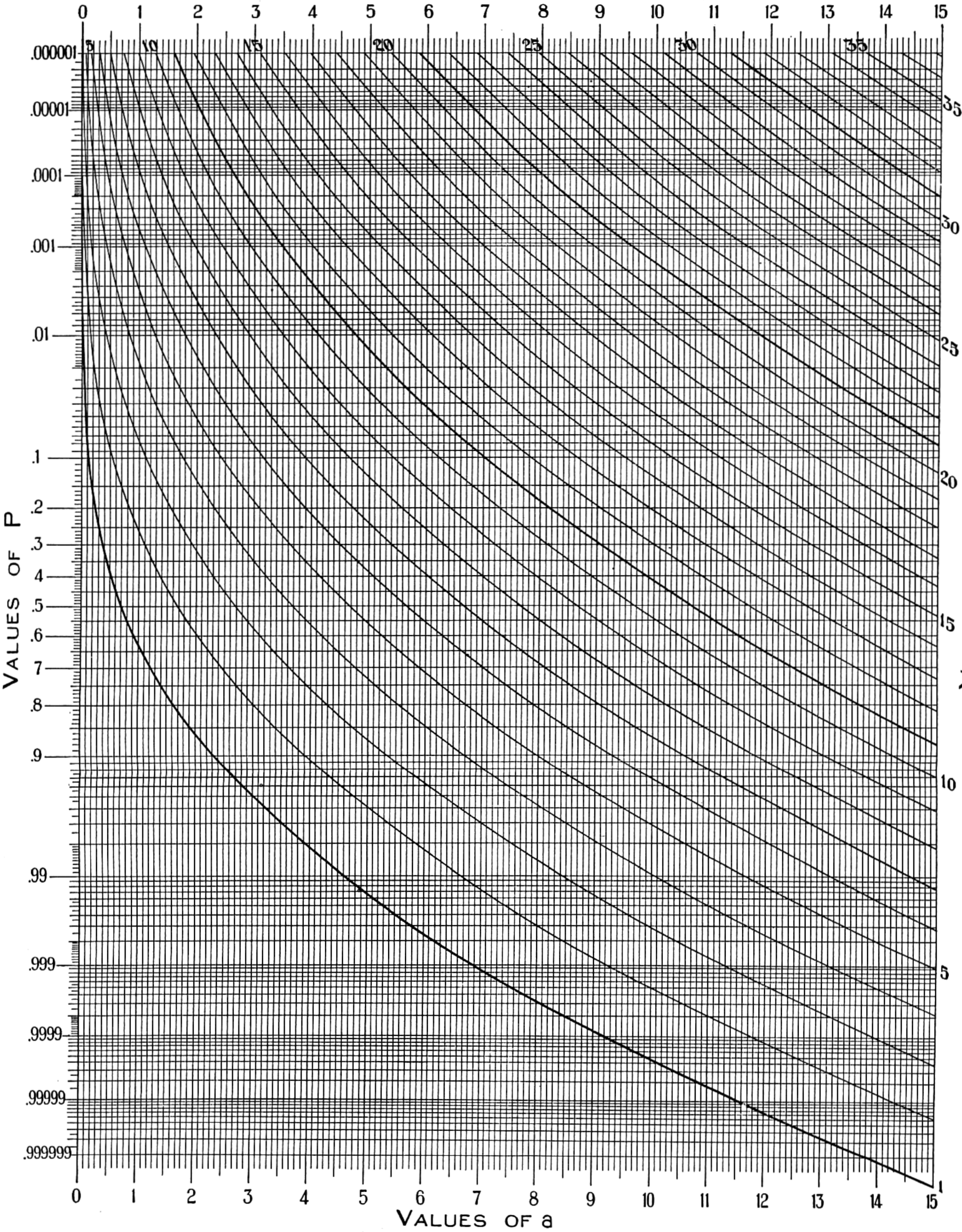

Fig. 1.—Probability Curves Showing Poisson's Exponential Summation $P=1-\left(1+\frac{a}{1!}+\frac{a^2}{2!}+\ldots+\frac{a^{c-1}}{(c-1)!}\right)e^{-a}$

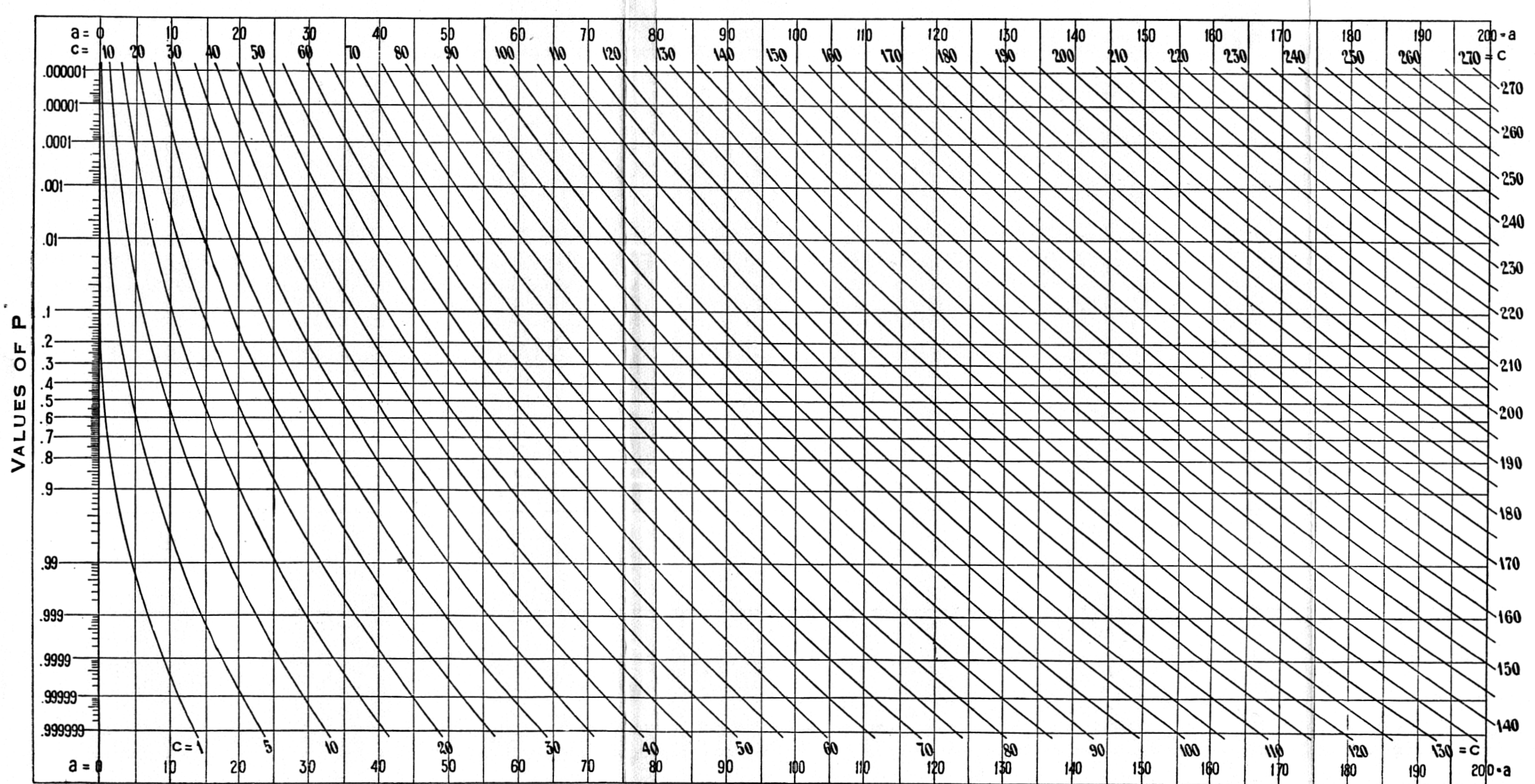

Fig. 2.—Probability Curves Showing Poisson's Exponential Summation $P=1-\left(1+\frac{a}{1!}+\frac{a^2}{2!}+\ldots+\frac{a^{c-1}}{(c-1)!}\right)e^{-a}$

TABLE VII.

P	θ	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7
0.5	0	0	−0.333333	0	0.019753	0	0.007211	0
0.75	0.47693628	0.67448975	−0.181688	−0.122627	0.015055	−0.005459	0.004913	0.001928
0.9	0.90619380	1.2815516	0.214125	−0.190724	−0.004431	0.000386	−0.003166	0.006425
0.99	1.6449764	2.3263479	1.470632	−0.102625	−0.135493	0.072761	−0.042809	0.017953
0.999	2.1851242	3.0902323	2.849845	0.218852	−0.400528	0.225125	−0.093337	−0.012844
0.9999	2.6297418	3.7190165	4.277028	0.705692	−0.808289	0.461955	−0.127519	−0.163690
0.99999	3.0157332	4.2648908	5.729764	1.325587	−1.362810	0.789917	−0.115119	−0.535981
0.999999	3.3611785	4.7534242	7.198347	2.059162	−2.066386	1.216005	−0.024576	−1.251181

NOTE—By substituting $t=\theta\sqrt{2}$, equation (2) may be written $2P-1=\frac{2}{\sqrt{\pi}}\int_0^{\theta} e^{-\theta^2}d\theta$. Values of θ were found directly from tables of the probability integral. Upon changing P to $1-P$, θ, Q_1, Q_3, Q_5, and Q_7 change sign, while Q_2, Q_4, and Q_6 remain unchanged.