A Reactance Theorem

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SYNOPSIS: The theorem gives the most general form of the driving-point impedance of any network composed of a finite number of self-inductances, mutual inductances, and capacities. This impedance is a pure reactance with a number of resonant and anti-resonant frequencies which alternate with each other. Any such impedance may be physically realized (provided resistances can be made negligibly small) by a network consisting of a number of simple resonant circuits (inductance and capacity in series) in parallel or a number of simple anti-resonant circuits (inductance and capacity in parallel) in series. Formulas are given for the design of such networks. The variation of the reactance with frequency for several simple circuits is shown by curves. The proof of the theorem is based upon the solution of the analogous dynamical problem of the small oscillations of a system about a position of equilibrium with no frictional forces acting.

AN important theorem[1] gives the driving-point impedance[2] of any network composed of a finite number of self-inductances, mutual inductances, and capacities; showing that it is a pure reactance with a number of resonant and anti-resonant frequencies which alternate with each other; and also showing how any such impedance may be physically realized by either a simple parallel-series or a simple series-parallel network of inductances and capacities, provided resistances can be made negligibly small. The object of this note is to give a full statement of the theorem, a brief discussion of its physical significance and its applications, and a mathematical proof.

THE THEOREM

The most general driving-point impedance S obtainable by means of a finite resistanceless network is a pure reactance which is an odd rational function of the frequency $p/2\pi$ and which is completely determined, except for a constant factor H, by assigning the resonant and anti-resonant frequencies, subject to the condition that they alternate and include both zero and infinity. Any such impedance may be physically

[1] The theorem was first stated, in an equivalent form and without his proof, by George A. Campbell, *Bell System Technical Journal*, November, 1922, pages 23, 26, and 30. By an oversight the theorem on page 26 was made to include unrestricted dissipation. Certain limitations, which are now being investigated, are necessary in the general case of dissipation. The theorem is correct as it stands when there is no dissipation, that is, when all the R's and G's vanish; this is the only case which is considered in the present paper.

A corollary of the theorem is the mutual equivalence of simple resonant components in parallel and simple anti-resonant components in series. This corollary had been previously and independently discovered by Otto J. Zobel as early as 1919, and was subsequently published by him, together with other reactance theorems, *Bell System Technical Journal*, January, 1923, pages 5–9.

[2] The driving-point impedance of a network is the ratio of an impressed electromotive force at a point in a branch of the network to the resulting current at the same point.

constructed either by combining, in parallel, resonant circuits having impedances of the form $iLp+(iCp)^{-1}$, *or by combining, in series, anti-resonant circuits having impedances of the form* $[iCp+(iLp)^{-1}]^{-1}$. *In more precise form,*

$$S=-iH\frac{(p_1^2-p^2)\,(p_3^2-p^2)\,\ldots\,(p_{2n-1}^2-p^2)}{p(p_2^2-p^2)\,\ldots\,(p_{2n-2}^2-p^2)}, \tag{1}$$

where $H\geq 0$ *and* $0=p_0\leq p_1\leq p_2\leq\,\ldots\,\leq p_{2n-1}\leq p_{2n}=\infty$.[3] *The inductances and capacities for the n resonant circuits are given by the formula,*

$$L_j=\frac{1}{C_jp_j^2}=\left(\frac{ipS}{p_j^2-p^2}\right)_{p=p_j}\quad (j=1,\,3,\,\ldots,\,2n-1), \tag{2}$$

and the inductances and capacities of the $n+1$ *anti-resonant circuits are given by the formula,*

$$C_j=\frac{1}{L_jp_j^2}=\left(\frac{ip}{S(p_j^2-p^2)}\right)_{p=p}\quad (j=0,\,2,\,4,\,\ldots\,,2n-2,\,2n), \tag{3}$$

which includes the limiting values,

$$C_0=\frac{p_2^2\,\ldots\,p_{2n-2}^2}{Hp_1^2p_3^2\,\ldots\,p_{2n-1}^2},\quad L_0=\infty,\ C_{2n}=0,\ L_{2n}=H.$$

Formula (1) may be stated in several mutually equivalent forms.[4] This particular form is the driving-point impedance of the most general symmetrical network in which every branch contains an inductance and a capacity in series, with mutual inductance between each pair of branches. This includes as special cases the driving-point impedances of every other finite resistanceless network.

[3] Since the impedance S is an odd function of the frequency, resonance or anti-resonance for $p=P$ implies resonance or anti-resonance for $p=-P$. In enumerating the resonant and anti-resonant frequencies it is customary, however, to exclude negative values of the frequency. Thus, in the present case, we say that there are n resonant points (p_1, p_3, . . . , p_{2n-1}) and $n+1$ anti-resonant points ($p_0=0$, p_2, p_4, . . . , p_{2n-2}, $p_{2n}=\infty$).

[4] The expression for S given by formula (1) may be written in the mutually equivalent forms,

$$\left[-iH\frac{(p_1^2-p^2)\,(p_3^2-p^2)\,\ldots\,(p_{2n-1}^2-p^2)}{p\,(p_2^2-p^2)\,\ldots\,(p_{2n-2}^2-p^2)}\right]^{\pm 1}\quad\text{and}\quad\left[iHp\frac{(p_2^2-p^2)\,\ldots\,(p_{2n-2}^2-p^2)}{(p_1^2-p^2)\,\ldots\,(p_{2n-3}^2-p^2)}\right]^{\pm 1}$$

If the constant H and all the p_j's of these formulas are restricted to finite values greater than zero, the four cases, obtained by separating the plus and minus exponents, are mutually exclusive, but together they cover the entire field. If p_1 is allowed to be zero, either the first or the second pair covers the entire field. Finally, if in addition p_{2n-1} or p_{2n-2} is allowed to become infinite, while Hp_{2n-1}^2 or Hp_{2n-2}^2 is maintained finite, any one of the four expressions covers the entire field. Sometimes one, sometimes another way of covering the field is the more convenient. Formulas (2) and (3) apply to all of these expressions for S provided the p_j's include all the resonant points and all the anti-resonant points, respectively.

PHYSICAL DISCUSSION

The variation of the reactance $X=S/i$ with frequency is illustrated by the curves of Fig. 1 in all the typical cases of formula (1) for $n=1$ and for $n=2$. For every curve the reactance increases with the frequency,[5] except for the discontinuities which carry it back from a positive infinite value to a negative infinite value at the anti-resonant points. Thus between every two resonant frequencies there is an anti-resonant frequency, no matter how close together the two resonant frequencies may be. The effect of increasing n by one unit is to add one resonant point, and thus to introduce one additional branch to the reactance curve, this branch increasing from a negative infinite value through zero to a positive infinite value.

That formula (1) includes several familiar circuits is seen by considering the most general network with one mesh, that is, an inductance and a capacity in series, with the impedance $iLp+(iCp)^{-1}$. This expression is given immediately by (1) upon setting $n=1$, $H=L$, and $p_1=1/\sqrt{LC}$. Since L and C are both positive these constants satisfy the conditions stipulated under (1), thus verifying the theorem for circuits of one mesh. This general one-mesh circuit includes as special cases a single inductance L by setting $H=L$ and $p_1=0$, and a single capacity C by setting $H=0$ and $p_1=\infty$ such that $Hp_1^2=1/C$.

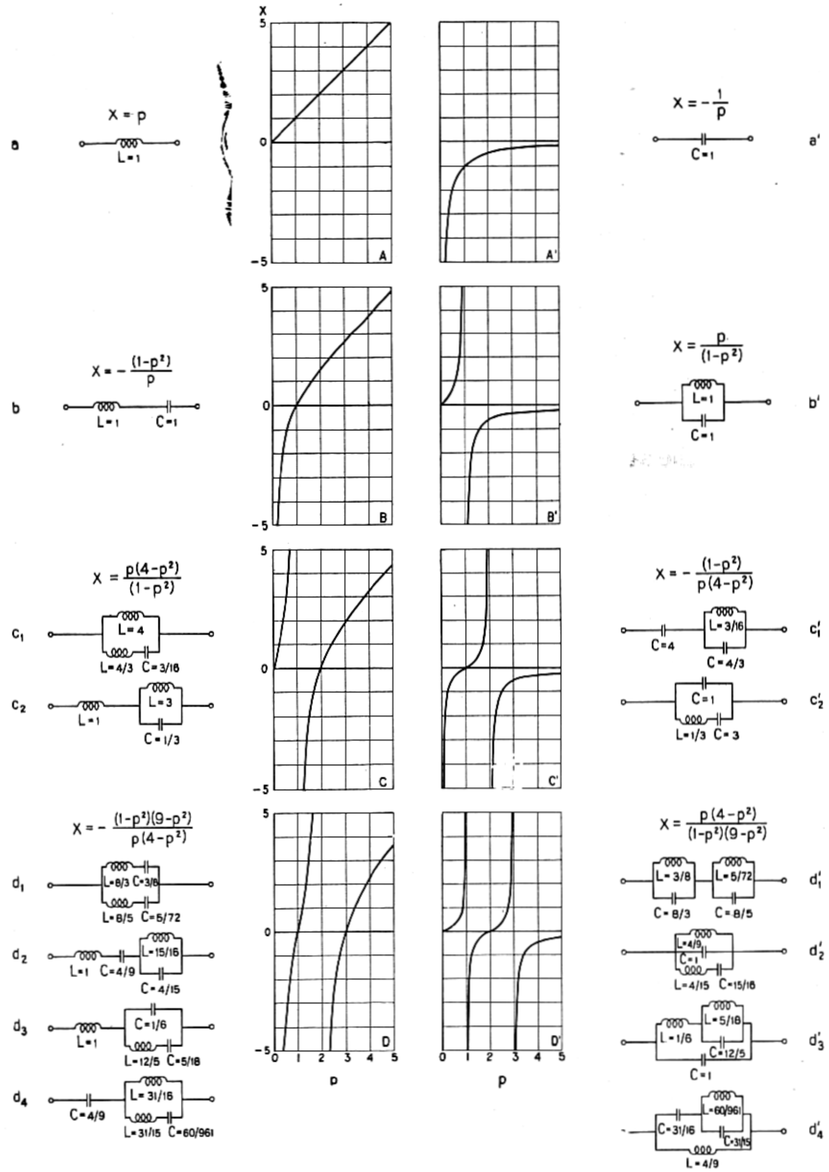
In Fig. 1 the reactances shown by the curves on the right are the negative reciprocals of those on the left. Fig. 1 also shows networks which give the several reactance curves, the networks being computed by means of formulas (2) and (3). The networks are arranged in pairs with reciprocal driving-point impedances and with the networks themselves reciprocally related, that is, the geometrical forms of the networks are conjugate,[6] and inductances correspond to capacities of the same numerical value and vice versa. This relation is a natural consequence of the reciprocal relation between an inductance and a capacity of the same numerical value, these being the elements from which the networks are constructed.

For $n=1$, formulas (2) and (3) give identical networks, as illustrated by the reactances A, B, A', and B' of Fig. 1, each of which is realized by a single network. For the reactances C and C' the two formulas give distinct networks, c_1 and c_2, c_1' and c_2', respectively, these

[5] This has been proved by Otto J. Zobel (loc. cit., pp. 5, 36), using the formula for the most general driving-point impedance given by George A. Campbell (loc. cit., p. 30).

[6] For a further treatment of conjugate or inverse networks, see P. A. MacMahon, *Electrician*, April 8, 1892, pages 601, 602, and Otto J. Zobel, loc. cit., pages 5, 36, and 37.

two being the only networks with the minimum number of elements which give the specified impedance. In general, however, there are four ways of realizing a given impedance when $n=2$, as illustrated by D and D' of Fig. 1; formulas (2) and (3) give only the first two



1—Reactance curves and networks for simple cases of formula (1).

networks, d_1 and d_2, d_1' and d_2', respectively. The total number of possible ways of realizing a given impedance increases very rapidly for values of n greater than 2; for $n=3$, there are, in general, 32 distinct networks giving a specified impedance.

Formulas (2) and (3) are to be used for determining the constants of the circuits which have certain specified characteristics, whereas most network formulas are for the determination of the characteristics of the circuit from the given constants of the circuit. The application of these formulas is illustrated by the following numerical problem:

To design a reactance network which shall be resonant at frequencies of 1000, 3000, 5000, and 7000 cycles; anti-resonant at 2000, 4000, and 6000 cycles, as well as at zero and infinite frequencies; and have a reactance of 2500 ohms at a frequency of 10,000 cycles.

By formula (1) the reactance of such a network must be

$$X=-H\frac{(p_1^2-p^2)\,(p_3^2-p^2)\,(p_5^2-p^2)\,(p_7^2-p^2)}{p\ (p_2^2-p^2)\,(p_4^2-p^2)\,(p_6^2-p^2)}, \tag{4}$$

where p_1, p_3, p_5, and p_7 are determined by the resonant frequencies to be $1000\times 2\pi$, $3000\times 2\pi$, $5000\times 2\pi$, and $7000\times 2\pi$, respectively; p_2, p_4, and p_6 are determined by the anti-resonant frequencies to be $2000\times 2\pi$, $4000\times 2\pi$, and $6000\times 2\pi$, respectively; and H must be made equal to 0.0596 in order that the reactance at $p=10{,}000\times 2\pi$ may be 2500. The variation of the reactance with the frequency is shown by the curve of Fig. 2.

A network having this reactance may be constructed by combining $n=4$ simple resonant circuits in parallel, or $n+1=5$ simple anti-resonant circuits in series. These two networks are shown by Fig. 2. The numerical values of the elements are determined as follows: Applying formula (2) we have

$$L_1=\frac{1}{C_1p_1^2}=H\frac{(p_3^2-p_1^2)\,(p_5^2-p_1^2)\,(p_7^2-p_1^2)}{(p_2^2-p_1^2)\,(p_4^2-p_1^2)\,(p_6^2-p_1^2)}=0.349,$$

$$L_3=\frac{1}{C_3p_3^2}=H\frac{(p_1^2-p_3^2)\,(p_5^2-p_3^2)\,(p_7^2-p_3^2)}{(p_2^2-p_3^2)\,(p_4^2-p_3^2)\,(p_6^2-p_3^2)}=0.323,$$

$$L_5=\frac{1}{C_5p_5^2}=H\frac{(p_1^2-p_5^2)\,(p_3^2-p_5^2)\,(p_7^2-p_5^2)}{(p_2^2-p_5^2)\,(p_4^2-p_5^2)\,(p_6^2-p_5^2)}=0.264,$$

$$L_7=\frac{1}{C_7p_7^2}=H\frac{(p_1^2-p_7^2)\,(p_3^2-p_7^2)\,(p_5^2-p_7^2)}{(p_2^2-p_7^2)\,(p_4^2-p_7^2)\,(p_6^2-p_7^2)}=0.142;$$

and applying formula (3) we have

$$C_0=\frac{p_2^2p_4^2p_6^2}{Hp_1^2p_3^2p_5^2p_7^2}=0.0888\times10^{-6}, L_0=\infty,$$

$$C_2=\frac{1}{L_2p_2^2}=\frac{-p_2^2(p_4^2-p_2^2)(p_6^2-p_2^2)}{H(p_1^2-p_2^2)(p_3^2-p_2^2)(p_5^2-p_2^2)(p_7^2-p_2^2)}=0.0461\times10^{-6},$$

$$C_4=\frac{1}{L_4p_4^2}=\frac{-p_4^2(p_2^2-p_4^2)(p_6^2-p_4^2)}{H(p_1^2-p_4^2)(p_3^2-p_4^2)(p_5^2-p_4^2)(p_7^2-p_4^2)}=0.0523\times10^{-6},$$

$$C_6=\frac{1}{L_6p_6^2}=\frac{-p_6^2(p_2^2-p_6^2)(p_4^2-p_6^2)}{H(p_1^2-p_6^2)(p_3^2-p_6^2)(p_5^2-p_6^2)(p_7^2-p_6^2)}=0.0725\times10^{-6},$$

$$C_8=0, \qquad L_8=H=0.0596.$$

These formulas give the numerical values of the inductances in henries and the capacities in farads. The entire set of numerical values is shown in Fig. 2. It is to be noted that the anti-resonant circuit corresponding to $p_0=0$ consists of a simple capacity since the inductance is infinite and thus does not appear in the network, whereas for $p_8=\infty$ the anti-resonant circuit consists of a simple inductance, the capacity being zero and thus not appearing in the network.

Mathematical Proof

We shall first prove that the driving-point impedance S, as given by (1), may be physically realized by either a simple parallel-series or a simple series-parallel network of inductances and capacities, provided resistances can be made negligibly small.

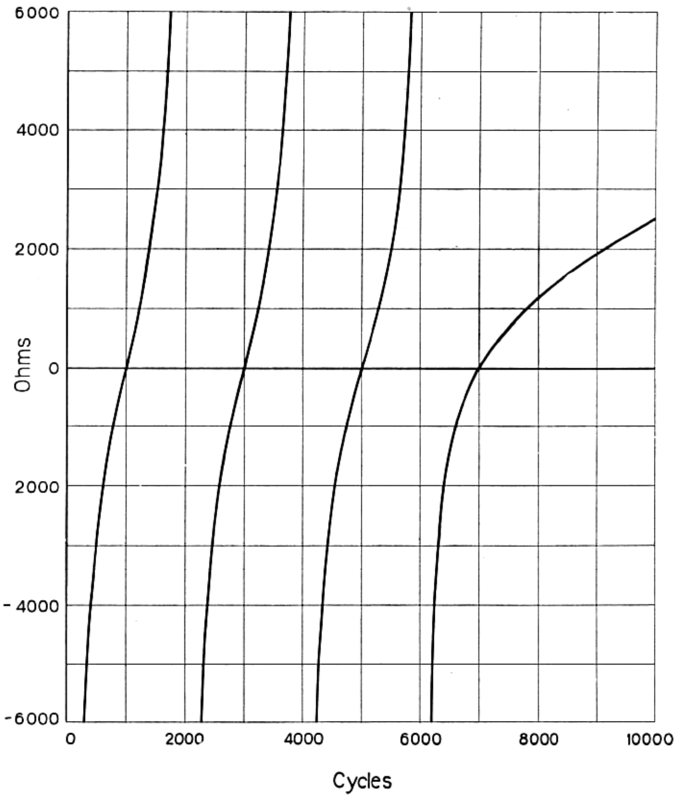
The rational function $1/S$ can be expanded in partial fractions,

$$\frac{1}{S}=\frac{iH_1p}{p_1^2-p^2}+\frac{iH_3p}{p_3^2-p^2}+\dots+\frac{iH_{2n-1}p}{p_{2n-1}^2-p^2},$$

where $$H_j=\left(\frac{p_j^2-p^2}{ipS}\right)_{p=p_j} \quad (j=1, 3, \dots, 2n-1).$$

Hence S is equal to the impedance of the parallel combination of the n circuits having the impedances $(p_j^2-p^2)/(iH_jp)=iH_j^{-1}p+[i(H_jp_j^{-2})p]^{-1}$, that is, n simple resonant circuits in parallel, each circuit consisting of an inductance and a capacity in series, with the numerical values given by (2). Furthermore, these numerical values of the inductances and capacities given by (2) are all positive, an even number of negative factors being obtained upon substituting $p=p_j$, since in every case $p_j\leq p_{j+1}$. Hence the network defined by (2) has the impedance S as given by (1) and is physically realizable.

Likewise, by expanding S in partial fractions, it can be shown that the network defined by (3) has the impedance S as given by (1) and is physically realizable.



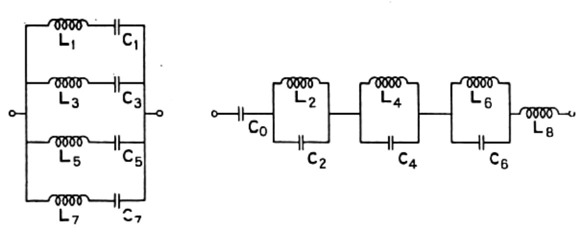


Fig. 2—Reactance curve and networks for formula (4).

The values of the inductances and capacities are (in henries and microfarads):

$L_1 = 0.349$	$C_1 = 0.0726$	$L_2 = 0.137$	$C_0 = 0.0888$
$L_3 = 0.323$	$C_3 = 0.00872$	$L_4 = 0.0302$	$C_2 = 0.0461$
$L_5 = 0.264$	$C_5 = 0.00384$	$L_6 = 0.00971$	$C_4 = 0.0523$
$L_7 = 0.142$	$C_7 = 0.00363$	$L_8 = 0.0596$	$C_6 = 0.0725$

The electrical problem of the free oscillations of a resistanceless network is formally the same as the dynamical problem of the small oscillations of a system about a position of equilibrium with no frictional forces acting. The proof of formula (1) may be derived from the treatment of this dynamical problem as given, for example, by Routh.[7]

In any network the driving-point impedance in the qth mesh, S_q, is equal to the ratio A/A_q, where A is the determinant[8] of the network and A_q the principal minor of this determinant obtained by striking out the qth row and the qth column. The determinant of a network has the element Z_{jk} in the jth row and kth column, Z_{jk} being the mutual impedance between meshes j and k (self-impedance when $j=k$), the determinant including n independent meshes of the network.

Hence the determinant A has the element $Z_{jk}=iL_{jk}p+(iC_{jk}p)^{-1}$, where L_{jk} is the total inductance and C_{jk} the total capacity common to the meshes j and k. Upon taking the factor $(ip)^{-1}$ from each row and substituting $-p^2=x$, the expression for A may be put in the form $A=(ip)^{-n}D$, where D is a determinant with $L_{jk}x+1/C_{jk}$ as the element in the jth row and the kth column. This is of exactly the same form as the determinant given by Routh [9] for the solution of the dynamical problem; it is proved there that this determinant, regarded as a polynomial, has n negative real roots which are separated by the $n-1$ negative real roots of every first principal minor of the determinant.

Hence, we may write $D=E(x_1+x)(x_3+x)\ldots(x_{2n-1}+x)$, where $x_1, x_3, \ldots, x_{2n-1}$ are all positive and arranged in increasing order of magnitude, and where E is also positive since D must be positive for $x=0$. The determinant D_q may be expressed in similar manner since it is of the same form as D but of lower order.

[7] E. J. Routh, "Advanced Rigid Dynamics," sixth edition, 1905, pages 44–55. In the notation of the dynamical problem as presented here, the coefficients A_{jk} correspond to the inductances, $1/C_{jk}$ to the capacities, $p/(i2\pi)$ to the frequency, and θ', ϕ', etc., to the branch currents in the electrical problem.

A complete proof of formula (1) has been worked out for the electrical problem, without depending in any way upon the solution of the corresponding dynamical problem. This proof has not been published here in view of the great simplification made by using the results already worked out for the dynamical problem.

[8] A complete discussion of the solution of networks by means of determinants has been given by G. A. Campbell, Transactions of the A. I. E. E., 30, 1911, pages 873–909.

[9] The determinant given by Routh (loc. cit., p. 49) has the element $A_{jk}p^2+C_{jk}$.

The driving-point impedance is given by

$$S_q = \frac{A}{A_q} = (ip)^{-1}\frac{D}{D_q} = (ip)^{-1}\frac{E(x_1+x)(x_3+x)\ldots(x_{2n-1}+x)}{E_q(x_2+x)\ldots(x_{2n-2}+x)},$$

where $0 \leq x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_{2n-2} \leq x_{2n-1}$, since the roots of D are separated by the roots of D_q. Upon substituting $x = -p^2$ and introducing the notation $H = E/E_q$ and $p_1^2, p_2^2, \ldots, p_{2n-1}^2 = x_1, x_2, \ldots, x_{2n-1}$, respectively, we see that formula (1) is completely verified as the most general driving-point impedance obtainable by means of a finite resistanceless network.