A Generalization of the Reciprocal Theorem

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THE Reciprocal Theorem, an interesting and extremely important relation of wide applicability, which was discovered by Lord Rayleigh, is stated by him in the language of electric circuit theory as follows:

"Let there be two circuits of insulated wire A and B, and in their neighborhood any combination of wire circuits or solid conductors in communication with condensers. A periodic electromotive force in the circuit A will give rise to the same current in B as would be excited in A if the electromotive force operated in B."[1]

Before proceeding with the generalization which is the subject of this paper, Rayleigh's theorem, in the following modified form, will first be stated and proved:

I. *Let a set of electromotive forces* $V_1' \ldots . V_n'$, *all of the same frequency, acting in the* n *branches of an invariable network, produce a current distribution* $I_1' \ldots . I_n'$, *and let a second set of electromotive forces* $V_1'' \ldots . V_n''$ *of the same frequency produce a second current distribution* $I_1'' \ldots . I_n''$. *Then*

$$\sum_{j=1}^{n} V_j' I_j'' = \sum_{j=1}^{n} V_j'' I_j'. \tag{1}$$

To prove this theorem we start with the equations of the network

$$\sum_{k=1}^{n} Z_{jk} I_k = V_j, \qquad j=1, 2, \ldots n, \tag{2}$$

and observe that, provided the network is invariable, contains no internal source of energy or unilateral device, and provided that the applied electromotive forces $V_1 \ldots V_n$ are all of the same frequency, say $\omega/2\pi$, the mutual impedances satisfy the reciprocal relations $Z_{jk}=Z_{kj}$. Consequently if (2) is solved for the currents, we get

$$I_j = \sum_{k=1}^{n} A_{jk} V_k, \qquad j=1, 2 \ldots n, \tag{3}$$

and the coefficients also obey the reciprocal relations $A_{jk}=A_{kj}$.

Now consider two independent and arbitrary sets of equi-periodic applied electromotive forces, $V_1' \ldots . V_n'$ and $V_1'' \ldots . V_n''$: then

[1] Rayleigh, Theory of Sound, Vol. I, p. 155.

in accordance with (3), the corresponding distributions of network currents $I_1' \ldots . I_n'$ and $I_1'' \ldots . I_n''$ are given by

$$I_j' = \sum_{k=1}^{n} A_{jk} V_k', \qquad j = 1, 2 \ldots n, \tag{4}$$

$$I_j'' = \sum_{k=1}^{n} A_{jk} V_k''. \tag{5}$$

Now form the product sum $\sum V_j'' I_j'$; by means of (4) it is easy to show that, since $A_{jk} = A_{kj}$,

$$\sum_{j=1}^{n} V_j'' I_j' = \sum_{j=1}^{n} \sum_{k=1}^{n} A_{jk} (V_j' V_k'' + V_j'' V_j') - \sum A_{jj} V_j' V_j''.$$

Since this is symmetrical in the two sets of applied forces $V_1' \ldots . V_n'$ and $V_1'' \ldots . V_n''$, it follows at once that

$$\sum V_j'' I_j' = \sum V_j' I_j'',$$

which proves the theorem.

Now if we analyze the foregoing proof it is seen to depend on the assumption, first that the network can be described in terms of a set of simultaneous equations with constant coefficients, and secondly on the reciprocal relation in the coefficients, $Z_{jk} = Z_{kj}$. In other words, it is assumed that the currents flow in linear, invariable circuits, and that the system is what is called quasi-stationary.[2] What this means is that the network may be treated as a dynamical system defined by n coordinates, the n currents $I_1 \ldots . I_n$ being the velocities of the n coordinates. More precisely stated, the underlying assumption is that the magnetic energy, the electric energy, and the dissipation function can be expressed as homogeneous quadratic functions of the following form

$$T = \tfrac{1}{2} \sum \sum L_{jk} I_j I_k,$$

$$W = \tfrac{1}{2} \sum \sum S_{jk} Q_j Q_k, \qquad I_j = d/dt\ Q_j,$$

and

$$D = \tfrac{1}{2} \sum \sum R_{jk} I_j I_k,$$

where the coefficients L_{jk}, S_{jk}, R_{jk} are constants. Subject to these assumptions, which, it may be remarked, underlie the whole of electric circuit theory, the direct application of Lagrange's equations to the quadratic functions T, W, D leads at once to the circuit equations (1) and the reciprocal relation $Z_{jk} = Z_{kj}$. This is merely a very brief outline of Maxwell's dynamical theory of quasi-stationary systems or networks.

[2] See Theorie der Electrizitat, Abraham u. Foppl, Vol. I, p. 254.

Now in view of the foregoing assumptions and restrictions which underlie all the proofs of the Reciprocal Theorem, known to the writer, it is by no means obvious that the theorem is valid when we have to do with currents in continuous media as well as in linear circuits, and when, furthermore we have to take account of radiation phenomena.[3] The proof or disproof of the theorem in the electromagnetic case is, however, extremely important. The writer therefore, offers the following generalized Reciprocal Theorem, subject to the restriction noted below.

II. *Let a distribution of impressed periodic electric intensity* $\boldsymbol{F}'=\boldsymbol{F}'(x, y, z)$ *produce a corresponding distribution of current intensity* $\boldsymbol{u}'=\boldsymbol{u}'(x, y, z)$, *and let a second distribution of equi-periodic impressed electric intensity* $\boldsymbol{F}''=\boldsymbol{F}''(x, y, z)$ *produce a second distribution of current intensity* $\boldsymbol{u}''=\boldsymbol{u}''(x, y, z)$, *then*

$$\int(\boldsymbol{F}'\cdot\boldsymbol{u}'')\mathrm{dv}=\int(\boldsymbol{F}''\cdot\boldsymbol{u}')\mathrm{dv}, \tag{6}$$

the volume integration being extended over all conducting and dielectric media. $\boldsymbol{F}$ and $\boldsymbol{u}$ are vectors and the expression $(\boldsymbol{F}\cdot\boldsymbol{u})$ denotes the scalar product of the two vectors.

The only serious restriction on the generality of this theorem, as proved below, is that magnetic matter is excluded: in other words it is assumed that all conducting and dielectric media in the field have unit permeability. This restriction is theoretically to be regretted, but is not of serious consequence in important practical applications.

Proof of Generalized Reciprocal Theorem[4]

In order to prove the generalized theorem stated above it is necessary to discard the special assumption of quasi-stationary systems underlying Rayleigh's theorem, and start with the fundamental equations of electromagnetic theory. These may be formulated as follows:

$$\mathrm{div}\ \boldsymbol{B}=0,$$

$$\mathrm{div}\ \boldsymbol{E}=4\pi\rho,$$

$$\mathrm{curl}\ \boldsymbol{E}=-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B},$$

$$\mathrm{curl}\ \boldsymbol{B}=4\pi\boldsymbol{u}+\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{E},$$

where c is the velocity of light.

[3] The theory of quasi-stationary systems expressly excludes radiation.

[4] In the following proof it is necessary to assume a knowledge on the part of the reader of the elements of vector analysis; the notation is that employed by Abraham.

It will be noted that there are only two field vectors, $\boldsymbol{E}$ and $\boldsymbol{B}$, instead of the usual four vectors $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$, where $\boldsymbol{D}=k\boldsymbol{E}$ and $\boldsymbol{B}=\mu\boldsymbol{H}$, and that the constants of the medium k and μ do not explicitly appear. This formal simplification is effected by taking as the current density

$$\boldsymbol{u}=\overline{\boldsymbol{u}}+\frac{1}{c}\frac{\partial \boldsymbol{P}}{\partial t}+\text{curl } \boldsymbol{M}$$

where $\overline{\boldsymbol{u}}$ is the conduction current density, $\boldsymbol{P}$ is the polarization, defined as

$$\boldsymbol{P}=\frac{k-1}{4\pi}\boldsymbol{E},$$

and $\boldsymbol{M}$ is defined as

$$\boldsymbol{M}=\frac{1}{4\pi}\frac{\mu-1}{\mu}\boldsymbol{B}.$$

The equation of continuity

$$\text{div } \boldsymbol{u}=-\frac{1}{c}\frac{\partial}{\partial t}\rho$$

then determines the charge density ρ.

The advantage of this formulation is that $\boldsymbol{E}$ and $\boldsymbol{B}$ can now be expressed in terms of the retarded scalar and vector potentials Φ and $\boldsymbol{A}$, as follows:

$$\boldsymbol{E}=-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}-\nabla\Phi,$$

$$\boldsymbol{B}=\text{curl } \boldsymbol{A},$$

where

$$\Phi=\int\frac{\rho(t-r/c)}{r}\text{dv},$$

$$\boldsymbol{A}=\int\frac{\boldsymbol{u}(t-r/c)}{r}\text{dv}.$$

The notation $\rho(t-r/c)$ and $\boldsymbol{u}(t-r/c)$ indicates that ρ and $\boldsymbol{u}$ are taken not at time t but at time $t-r/c$ in evaluating the integrals. It will be observed that with ρ and $\boldsymbol{u}$ defined as above, all effects are transmitted with the velocity of light, independently of the characteristics of the medium, a point of view in accordance with the modern development of electromagnetic theory.

In the application of the preceding equations to our problem, it will be assumed that $\boldsymbol{M}$ is everywhere zero, so that

$$\boldsymbol{u}=\overline{\boldsymbol{u}}+\frac{1}{c}\frac{\partial \boldsymbol{P}}{\partial t}.$$

It will be assumed further that $\overline{\boldsymbol{u}}=\sigma\boldsymbol{E}$ and, since $\boldsymbol{P}=\dfrac{k-1}{4\pi}\boldsymbol{E}$,

$$\boldsymbol{u}=\left(\sigma+\frac{k-1}{4\pi}\frac{1}{c}\frac{\partial}{\partial t}\right)\boldsymbol{E}$$

and is therefore a linear function of $\boldsymbol{E}$. σ and k are in general point functions of the medium. The reason for setting $\boldsymbol{M}=0$, is that it appears essential to the following proof that $\boldsymbol{u}$ shall be linear in $\boldsymbol{E}$; that is, that the current density at any point be proportional to the electric intensity.[5]

With the foregoing very brief review of the fundamental equations, we are now prepared to prove the generalized reciprocal therorem. Assuming a periodic steady state, so that $\partial/\partial t=i\omega$, we start with the vector equation

$$\boldsymbol{E}=\boldsymbol{F}-\frac{i\omega}{c}\boldsymbol{A}-\nabla\Phi \tag{7}$$

where

$$\boldsymbol{A}=\int\frac{1}{r}\exp\left(-\frac{i\omega}{c}r\right)\boldsymbol{u}\,\mathrm{dv},$$

$$\Phi=\int\frac{1}{r}\exp\left(-\frac{i\omega}{c}r\right)\rho\,\mathrm{dv}.$$

Here F is the *impressed intensity:* that is, *the electric intensity which is not due to the currents and charges of the system itself.* Also by virtue of the assumption $\boldsymbol{M}=0$,

$$\boldsymbol{u}=\left(\sigma+\frac{k-1}{4\pi}\frac{i\omega}{c}\right)\boldsymbol{E}=\lambda\boldsymbol{E},$$

whence (7) can be written as

$$\frac{1}{\lambda}\boldsymbol{u}+\frac{i\omega}{c}\int\frac{1}{r}\exp\left(-\frac{i\omega}{c}r\right)\boldsymbol{u}\,\mathrm{dv}=\boldsymbol{G}, \tag{8}$$

where $\boldsymbol{G}=\boldsymbol{F}-\nabla\Phi$.

[5] The question as to whether the generalized theorem itself, and not merely the foregoing proof, is restricted in general to the case where $\boldsymbol{M}$ is everywhere zero has not as yet received a conclusive answer. There are reasons, however, which cannot be fully entered into here, which make it appear probable that the theorem itself is in general restricted to the case where the current density contributing to the retarded vector potential is linear in the electric intensity and the two vectors are parallel. Subject to the hypothesis and assumptions of quasi-stationary systems, however, the restriction $\boldsymbol{M}=0$ is not necessary. The writer hopes to deal with these questions in a future paper.

Equation (8) is a vector integral equation [6] in $\boldsymbol{u}$. The nucleus or kernel of the equation, $\exp\left(\frac{i\omega}{c}r\right)/r$, is symmetrical with respect to any two points $(x_1y_1z_1)$ and $(x_2y_2z_2)$, the distance between which is r. By virtue of this symmetry the following reciprocal relation is easily established.[7]

If $\boldsymbol{u}'=\boldsymbol{u}'(x,y,z)$ *is a function satisfying equation* (8) *when* $\boldsymbol{G}=\boldsymbol{G}'=\boldsymbol{G}'(x, y, z)$ *and* $\boldsymbol{u}''=\boldsymbol{u}''(x, y, z)$ *a second function satisfying* (8) *when* $\boldsymbol{G}=\boldsymbol{G}''=\boldsymbol{G}''(x, y, z)$, *then*

$$\int(\boldsymbol{u}'\cdot\boldsymbol{G}'')dv=\int(\boldsymbol{u}''\cdot\boldsymbol{G}')dv. \tag{9}$$

Consequently since $\boldsymbol{G}=\boldsymbol{F}-\nabla\Phi$

$$\int(\boldsymbol{u}'\cdot\boldsymbol{F}'')dv-\int(\boldsymbol{u}''\cdot\boldsymbol{F}')dv=\int\Big\{(\boldsymbol{u}'\cdot\nabla\Phi'')-(\boldsymbol{u}''\cdot\nabla\Phi')dv. \tag{10}$$

The proof of the theorem is now reduced to showing that

$$\int\Big\{(\boldsymbol{u}'\cdot\nabla\Phi'')-(\boldsymbol{u}''\cdot\nabla\Phi')\Big\}dv=0.$$

Now integrating by parts

$$\int(\boldsymbol{u}'\cdot\nabla\Phi'')dv=-\int\Phi''\operatorname{div}\boldsymbol{u}'\,dv,$$
$$=\frac{i\omega}{c}\int\Phi''\rho'\,dv,$$

since, from the equations of continuity, div $\boldsymbol{u}=-\frac{i\omega}{c}\rho$. But from the fundamental field equations:

$$4\pi\rho'=-\nabla^2\Phi'+\left(\frac{i\omega}{c}\right)^2\Phi'$$

whence

$$\int\Big\{(\boldsymbol{u}'\cdot\nabla\Phi'')-(\boldsymbol{u}''\cdot\nabla\Phi')\Big\}dv=\frac{1}{4\pi}\left(\frac{i\omega}{c}\right)\int\Big\{\Phi'\nabla^2\Phi''-\Phi''\nabla^2\Phi'\Big\}dv,$$

and by Greens Theorem, the right hand volume integral is equal to the surface integral

$$\frac{1}{4\pi}\left(\frac{i\omega}{c}\right)\int\left\{\Phi'\frac{\partial}{\partial n}\Phi''-\Phi''\frac{\partial}{\partial n}\Phi'\right\}dS,$$

the surface being any surface which totally encloses the volume, and $\partial/\partial n$ denoting differentiation along the normal to the surface.

[6] The formulation of the electromagnetic field equations in this form is of considerable importance. The integral equation furnishes a basis for developing electric circuit theory from the fundamental field equations. In addition it leads to the solution of problems in wave propagation which can not be directly solved from the wave equation itself.

[7] Perhaps the easiest way to prove this proposition is to regard the integral equation as the limit of a set of simultaneous equations, a point of view which forms the basis of Fredholm's researches on integral equations.

Now if the surface be taken as a sphere of radius R, centered at or near the system, it is easily shown that if R is taken sufficiently large

$$\frac{\partial}{\partial n}\Phi' = \frac{\partial}{\partial R}\Phi' = -\frac{i\omega}{c}\Phi',$$

$$\frac{\partial}{\partial n}\Phi'' = -\frac{i\omega}{c}\Phi'',$$

and the surface integral vanishes. Consequently we have established the *generalized reciprocal theorem*

$$\int(\boldsymbol{u}' \cdot \boldsymbol{F}'')\mathrm{dv} = \int(\boldsymbol{u}'' \cdot \boldsymbol{F}')\mathrm{dv}.$$

The Reciprocal Theorem I has long been employed in electric circuit theory, and has proved extremely useful. As an example of the practical utility of the generalized theorem II it may be remarked that it enables us to deduce the transmitting properties of an antenna system from its receiving properties. The latter may sometimes be approximately deduced quite simply, as in the case of the wave antenna, whereas a direct threoretical determination of the former presents enormous difficulties.