Impedance of Loaded Lines, and Design of Simulating and Compensating Networks

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SYNOPSIS: A knowledge of the impedance characteristics of loaded lines is of considerable importance in telephone engineering, and particularly in the engineering of telephone repeaters. The first half of the present paper deals with the impedance of non-dissipative loaded lines as a function of the frequency and the line constants, by means of description accompanied by equations transformed to the most suitable forms and by graphs of those equations; and it outlines qualitatively the nature of the modifications produced by dissipation. The characteristics are correlated with those of the corresponding smooth line.

The somewhat complicated effects produced by the presence of distributed inductance are investigated rather fully. In the absence of distributed inductance a loaded line would have only one transmitting band, extending from zero frequency to the critical frequency. Actually, however, every line—even a cable—has some distributed inductance; and the effect of distributed inductance, besides altering the nominal impedance and the critical frequency, is to introduce into the attenuating range above the critical frequency a series of relatively narrow transmitting bands—here termed the "minor transmitting bands"—spaced at relatively wide intervals. The paper is concerned primarily with the impedance in the first or major transmitting band; but it investigates the minor transmitting bands sufficiently to determine how they depend on the distributed inductance, and to derive general formulas and graphical methods for finding their locations and widths—an investigation involving rather extensive analysis.

The latter half of the paper describes various networks devised for simulating and for compensating the impedance of loaded lines; it furnishes design-formulas and supplementary design-methods for all of the networks depicted; and outlines a considerable number of applications pertaining to lines and to repeaters.

INTRODUCTION

THE present paper on periodically loaded lines (of the series type) is to some extent a sequel to a previous paper on smooth lines.[1]

The reader may be reminded that the transmission of alternating currents over any transmission line between specified terminal impedances depends only on the propagation constant and the characteristic impedance of the line. In this sense, then, the characteristics of transmission lines may be classed broadly as propagation characteristics and impedance characteristics. In telephony we are concerned primarily with the dependence of these characteristics on the frequency, over the telephonic frequency range.

Prior to the application of telephone repeaters to telephone lines the propagation characteristics of such lines were more important than

[1] "Impedance of Smooth Lines, and Design of Simulating Networks," this *Journal*, April, 1923. Two typographical errors in that article may here be noted: p. 37, formula for C_7/C_3, affix an exponent ^{2} to the last parenthesis; p. 39, value for C_1, replace comma by decimal point.

their impedance characteristics, because the received energy depended much more on the former than on the latter. Indeed, the object of loading[2] was to improve the propagation characteristics of transmission lines; the effects on the impedance characteristics were incidental, and of quite secondary importance.

The application of the two-way telephone repeater greatly altered the relative importance of these two characteristics, decreasing the need for high transmitting efficiency of a line but greatly increasing the dependence of the results on the impedance of the line. As well known, this is because the amplification to which a two-way repeater can be set without singing, or even without serious injury to the intelligibility of the transmission, depends strictly on the degree of impedance-balance between the lines or between the lines and their balancing networks. In the case of the 21-type repeater the two lines must closely balance each other throughout the telephonic frequency range. In the case of the 22-type repeater, which for long lines requiring more than one repeater is superior to the 21-type, impedance-networks are required for closely balancing the impedances of the two lines throughout the telephonic frequency range. Such balancing networks are necessary also in connection with the so-called four-wire repeater circuit.[3]

In Parts I, II, and III of this paper there is presented in a simple yet fairly comprehensive manner the dependence of the characteristic impedance of periodically loaded lines (of the series type) on the frequency and on the line constants, by means of description accompanied by equations transformed to the most suitable forms and by graphs of those equations. Also, the dependence of the attenuation constant on the frequency is presented to the extent necessary for exhibiting the disposition of the transmitting and the attenuating bands and thus enabling the characteristic impedance to be described with reference to those bands, and the important correlation between the characteristic impedance and the attenuation constant thereby exhibited; for the characteristic impedance by itself is not fully significant.

Parts IV to VIII, inclusive, relate to the simulation and the compensation of the impedance of periodically loaded lines by means of

[2] For the fundamental theory of loaded lines, reference may be made to the original papers of Pupin and of Campbell (Pupin: Trans. A. I. E. E., March 22, 1899 and May 19, 1900; *Electrical World*, October 12, 1901 and March 1, 1902. Campbell: *Phil. Mag.*, March, 1903).

[3] Regarding the broad subject of repeaters and repeater circuits, reference may be made to the paper by Gherardi and Jewett: "Telephone Repeaters," Trans. A. I. E. E., 1919, pp. 1287–1345.

the simulating and the compensating[4] networks for loaded lines devised by the writer at various times within about the last twelve years. Of course, the impedance of any loaded line could be simulated, as closely as desired, by means of an artificial model constructed of many short sections each having lumped constants; but such structures would be very expensive and very cumbersome. Compared with them the networks described in this paper are very simple non-periodic structures that are relatively inexpensive and are quite compact; yet the most precise of them have proved to be adequate for simulating with high precision the characteristic impedance of any periodically loaded line, while even the least precise (which are the simplest) suffice for a good many applications. The compensating networks also are of simple form. Design-formulas are included for all of the networks depicted; and certain supplementary design-methods are indicated. Finally, a considerable number of practical applications are outlined (Part VIII).

PART I

Impedance of Loaded Lines—General Considerations

Before proceeding to the more precise and detailed treatment of the impedance of periodically loaded lines in Parts II and III, it seems desirable to furnish a background by outlining broadly the salient facts. For this purpose the loaded line will be compared with its "corresponding smooth line," that is, the smooth line having the same total constants (inductance, capacity, resistance, leakance).

Comparison with the Corresponding Smooth Line

At sufficiently low frequencies the impedance of a periodically loaded line approximates to that of the corresponding smooth line;[1] but at higher frequencies departs widely. Moreover, the impedance of the loaded line depends very much on its relative termination—fractional end-section or end-load ("load" is here used with the same meaning as "load coil" or "loading coil").

To bring out simply and sharply the contrast between a periodically loaded line and the corresponding smooth line, the effects of dissipation will at first be ignored, although the contrast is somewhat heightened thereby.

It will be recalled that the attenuation constant, the phase velocity, and the characteristic impedance of a non-dissipative smooth line are

[4] Defined in the second paragraph of Part IV.

independent [5] of frequency; such a line having a transmitting band (that is, a non-attenuating band) extending from zero frequency to infinite frequencies, and a characteristic impedance which is a pure and constant resistance.

In contrast, the corresponding characteristics of a non-dissipative periodically loaded line depend very greatly on the frequency; such a line has an infinite sequence of alternate transmitting and attenuating bands* wherein the impedance varies enormously with frequency, while at the transition frequencies its nature undergoes a sudden change. In this connection it may be remarked that, because of its special practical importance in being the upper boundary frequency of the first or principal transmitting band, the lowest transition frequency is termed the "critical frequency" to distinguish it from the other transition frequencies; though in its essential nature each transition frequency is a "critical" frequency. In the ordinary case, where the distributed inductance is small compared with the load inductance, each transmitting band is very narrow compared with the succeeding attenuating band. In the limiting case of no distributed inductance there is only one transmitting band and one attenuating band, the former extending from zero frequency to the critical frequency and the latter from the critical frequency to infinite frequencies.

The characteristic impedance of any non-dissipative transmission line is or is not pure reactance according as the contemplated frequency is in an attenuating band or in a transmitting band. For in an attenuating band the line cannot receive energy, since it cannot dissipate any energy and cannot transmit any energy to an infinite distance; while in a transmitting band the line must receive energy, because it does transmit. Thus, at the transition frequency between an attenuating band and a transmitting band the characteristic impedance undergoes a sudden change in its nature; the frequency-derivative of the impedance (namely, the derivative of the impedance with respect to the frequency) is discontinuous, so that the graph of the impedance has a corner (salient point) at a transition frequency. Moreover, at certain of the transition frequencies of a non-dissipative periodically loaded line the impedance is zero, and at others is infinite. The mid-point impedances are pure resistances throughout every transmitting band. (The "mid-point" terminations are "mid-load" and "mid-section," that is, "half-load" and "half-section" respectively.)

[5] Except for slight change of the inductance, and even of the capacity, with frequency.

* For distinction, the first (lowest) or principal transmitting band may be termed the "major" transmitting band; the others, the "minor" transmitting bands.

Clearly the characteristic impedance of any dissipative line cannot be pure reactance at any frequency; for the line receives at its sending end the energy dissipated within itself. Also, the presence of dissipation renders the frequency-derivative of the impedance continuous at all frequencies; that is, it rounds off the corners on the graph of the impedance. Dissipation prevents the impedance from becoming either zero or infinite at any frequency; and in general it prevents the mid-point impedances from being pure resistances in the transmitting bands.

In the neighborhood of the transition frequencies of the loaded line, the effects of even ordinary amounts of dissipation may be very large, thus preventing the impedance from attaining the very extreme values of the non-dissipative line; but with that exception it may be said that the contrast between a loaded line and the corresponding smooth line is merely softened or dulled by the presence of ordinary amounts of dissipation: The impedance of the smooth line is no longer pure resistance, and it varies somewhat or even considerably with the frequency.[1] The impedance of the loaded line no longer varies quite so rapidly with the frequency nor attains such extreme values; but, except at low frequencies, it continues to depart widely from the impedance of the corresponding smooth line, and to vary much more rapidly than the smooth line with frequency, besides varying greatly with its relative termination (fractional end-section or end-load).

Non-Dissipative Loaded Lines

Except in the neighborhood of zero frequency and of the transition frequencies, the characteristic impedance of an efficient loaded line is dependent mainly on the inductance and capacity, only relatively little on the wire resistance and load resistance, and very much less still on the leakance. The present paper is confined mainly to non-dissipative loaded lines; it deals first with the limiting case of no distributed inductance, and then with the case where distributed inductance is present. By the neglect of all dissipation the number of independent variables is sufficiently reduced to enable a comprehensive, though only approximate, view to be obtained of the characteristic impedance of loaded lines. Such a view is a valuable guide in engineering work even though in most cases it may be necessary, for final calculations or verifications, to resort to exact formulas (Appendix D) or graphs thereof.

Notation and Terminology

The meanings of the fundamental symbols employed in this paper can be readily seen from inspection of Fig. 1. Thus, C and L denote the capacity and the inductance of each whole section between loads, and L' the inductance of each whole load; the ratio L/L' is denoted by λ. Figs. 1a and 1b represent infinitely long loaded lines terminating

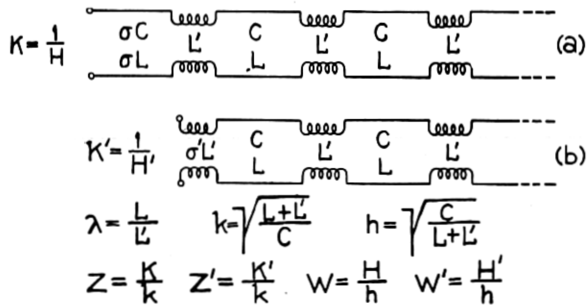

Fig. 1—A Non-Dissipative Infinitely Long Loaded Line Terminating at: (a) σ-Section, (b) σ'-Load

at σ-section and σ'-load respectively; the ratios σ and σ' will be termed the "relative terminations." K and K' denote the corresponding characteristic impedances, and H and H' the characteristic admittances. Stated more fully, K denotes the σ-section characteristic impedance, and K' the σ'-load characteristic impedance; similarly for the admittances H and H'. The "nominal impedance" and the "nominal admittance" are denoted by k and h, respectively; that is,

$$k=1/h=\sqrt{(L+L')/C}=\sqrt{(1+\lambda)L'/C}, \tag{1}$$

the nominal impedance of a periodically loaded line being defined as equal to the nominal impedance of the corresponding smooth line.[1] $Z=X+iY$ and $Z'=X'+iY'$ denote relative impedances and $W=U+iV$ and $W'=U'+iV'$ the corresponding relative admittances, as defined by the equations

$$Z=K/k, \qquad Z'=K'/k, \qquad W=H/h, \qquad W'=H'/h; \tag{2}$$

the real components being X, X', U, U', and the imaginary components Y, Y', V, V', respectively. By (2),

$$ZW=Z'W'=KH=K'H'=1. \tag{2.1}$$

r denotes the relative frequency, namely, the ratio of any frequency $f=\omega/2\pi$ to the critical frequency f_c; that is, $r=f/f_c=\omega/\omega_c$. i denotes the imaginary operator $\sqrt{-1}$.

Besides depending on the frequency f, the quantities K, H, Z, W and K', H', Z', W' depend on the relative terminations σ and σ' respectively (Fig. 1). This dependence will not usually need to be indicated explicitly, but in case of such need the subscript notation will be found convenient. Thus, K_σ will denote the σ-section characteristic impedance (Fig. 1a); and $K_{1-\sigma}$ the "complementary characteristic impedance," that is, the characteristic impedance of the same loaded line if beginning at the "complementary termination"—namely, $(1-\sigma)$-section. As an application of this notation we may note here the relations

$$K_0=K_1', \qquad H_0=H_1', \qquad K_1=K_0', \qquad H_1=H_0'; \tag{2.2}$$

the first two relations subsisting because of the coincidence of the points σ-section and σ'-load for $\sigma=0$ and $\sigma'=1$, and the second two because of the coincidence for $\sigma=1$ and $\sigma'=0$.

PART II

Impedance of Non-Dissipative Loaded Lines without Distributed Inductance

Transmitting Band and Attenuating Band

As already stated, a periodically loaded line without distributed inductance (Fig. 1, with $L=0$) has only one transmitting band and only one attenuating band; the former extending from zero frequency to the critical frequency f_c, and the latter from the critical frequency to infinite frequencies. The formula for f_c is

$$f_c=1/\pi\sqrt{L'C}, \tag{3}$$

L' denoting the inductance of each load and C the capacity of each line-section between loads.

From the energy considerations already adduced, it is known that the characteristic impedance must be pure reactance throughout the attenuating band, but cannot be pure reactance anywhere in the transmitting band.

Formulas for the Relative Impedances

The impedance of even a loaded line without distributed inductance (Fig. 1, with $L=0$) depends on no less than four independent variables—namely, the frequency f, load inductance L', section-capacity C, and one or the other of the relative terminations σ and σ'. But it is found that these quantities enter in such a way that the relative impedances $Z=K/k$ and $Z'=K'/k$ and the relative admittances $W=H/h$ and $W'=H'/h$ depend on only two ratios,—namely, the relative frequency $r=f/f_c$, and the appropriate relative termination σ or σ',—as expressed by the equations [6]

$$Z=\frac{1}{W}=\frac{1}{\sqrt{1-r^2}+i(2\sigma-1)r}=\frac{\sqrt{1-r^2}+i(1-2\sigma)r}{1-4\sigma(1-\sigma)r^2}, \tag{4}$$

$$Z'=\frac{1}{W'}=\sqrt{1-r^2}+i(2\sigma'-1)r=\frac{1-4\sigma'(1-\sigma')r^2}{\sqrt{1-r^2}+i(1-2\sigma')r}. \tag{5}$$

In particular, for $\sigma=0.5$ amd $\sigma'=0.5$, respectively,

$$Z_{.5}=1/W_{.5}=1/\sqrt{1-r^2}, \tag{6}$$

$$Z'_{.5}=1/W'_{.5}=\sqrt{1-r^2}. \tag{7}$$

Equations (4) and (5) are not restricted to values of σ and σ' less than unity. On the contrary they are valid for any (real) values of these quantities—though values much exceeding unity are of infrequent occurrence in practice.

Miscellaneous Properties and Relations

Some of the most useful and interesting simple facts deducible from equations (4) and (5) are noted in the next five paragraphs:

In agreement with the general conclusion already reached from energy considerations, equations (4) and (5) show that each of the relative impedances and relative admittances is pure imaginary in the attenuating band ($r>1$). In the transmitting band ($0<r<1$), each is seen to be complex for all values of the relative terminations (σ and σ'), except that each degenerates to a real value when the relative termination becomes 0.5.

Throughout the transmitting band ($0<r<1$), a certain conjugate property is possessed by each of the quantities Z, W, Z', W'—namely, each changes merely to its conjugate when σ is changed to $1-\sigma$, as is readily seen from (4) and (5); that is,

$$Z_\sigma=\overline{Z}_{1-\sigma}, \qquad W_\sigma=\overline{W}_{1-\sigma}, \qquad Z'_\sigma=\overline{Z}'_{1-\sigma}, \qquad W'_\sigma=\overline{W}'_{1-\sigma}, \tag{8}$$

the bar over a symbol denoting the conjugate of the same symbol without the bar. Thus, complementary characteristic impedances are mutually conjugate throughout the transmitting band.

At all values of r,

$$W_\sigma+W_{1-\sigma}=2W_{.5}, \qquad Z'_\sigma+Z'_{1-\sigma}=2Z'_{.5}; \tag{9}$$

[6] The equations were written in this sequence because, in practice, section-termination occurs much more frequently than load-termination.

although relations of this form do not hold for Z and for W'. Each of the relations (8) and (9) can be inferred also from simple physical considerations.

Equations (4) and (5) show that W and Z' are alike in form, and also W' and Z, when σ and σ' are regarded as corresponding to each other; in fact, when $\sigma=\sigma'$,

$$ZZ'=WW'=W/Z'=W'/Z=KK'/k^2=HH'/h^2=1. \qquad (10)$$

Besides, there is the set of perfectly general relations (2.1), which, of course, continue to hold when $\sigma=\sigma'$.

Equations (4) and (5) show also the existence of the following more special relations, holding when the relative terminations (σ and σ') have the values 0 and 1, as indicated by the subscripts:

$$Z_0Z_1=Z_0'Z_1'=W_0W_1=W_0'W_1'=1, \qquad (11)$$

$$|Z_0|=|Z_1|=|Z_0'|=|Z_1'|=|W_0|=|W_1|=|W_0'|=|W_1'|=1. \qquad (12)$$

Graphical Representations

Graphical representations of the relative impedances $Z=X+iY$ and $Z'=X'+iY'$, based on equations (4) and (5), will be taken up in the following paragraphs. Evidently it will not be necessary to consider also the relative admittances $W=U+iV$ and $W'=U'+iV'$ explicitly, since these are of the same functional forms as Z' and Z respectively—as noted in connection with equation (10).

One graphical method of representing the dependence of Z on r and σ is by means of a network of equi-r and equi-σ curves of Z in the Z-plane; likewise the dependence of Z' on r and σ', by means of the equi-r and equi-σ' curves of Z'. The analytic-geometric properties of these curves, as deduced from equations (4) and (5), may be formulated as follows, for any (real) values of σ and σ' but for r restricted to the range 0 to 1:

(*a*) r fixed, σ varied: Z moves on the circle

$$(X-1/2\sqrt{1-r^2})^2+Y^2=1/4(1-r^2),$$

of radius $1/2\sqrt{1-r^2}$ with center at $Z=1/2\sqrt{1-r^2}$.

(*b*) σ fixed, r varied: Z moves on the curve

$$(X^2+Y^2)^2-X^2-Y^2/(2\sigma-1)^2=0.$$

(*c*) r fixed, σ' varied: Z' moves on the straight line

$$X'=\sqrt{1-r^2},$$

which is parallel to the X'-axis at a distance $\sqrt{1-r^2}$ therefrom.

(*d*) σ' fixed, r varied: Z' moves on the ellipse

$$(X'/1)^2+(Y'/[2\sigma'-1])^2=1,$$

whose center is at $Z'=0$ and whose semi-axes along the X' and Y' axes have the lengths 1 and $2\sigma'-1$ respectively.

For values of r, σ, σ' each between 0 and 1, these facts are exhibited graphically in Fig. 2. This is a complex-plane chart of the equi-r

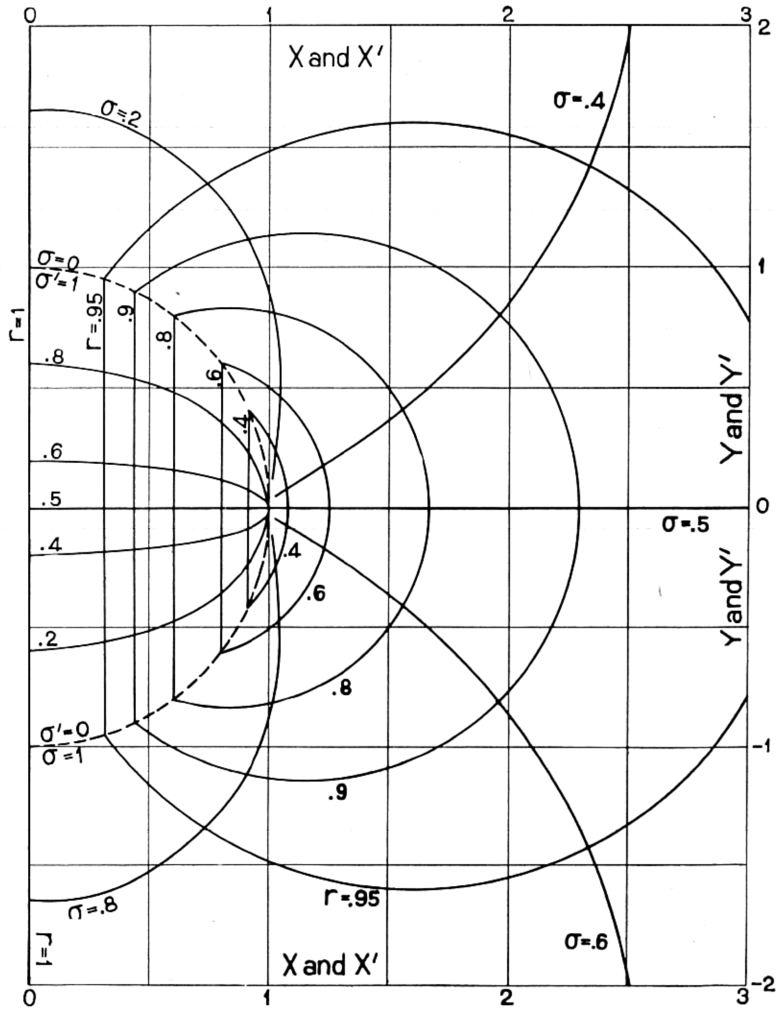

Fig. 2—Complex-Plane Chart of the σ-Section Relative Impedance $Z=X+iY$ and the σ'-Load Relative Impedance $Z'=X'+iY'$

and the equi-σ curves of Z, and the equi-r and the equi-σ' curves of Z'. The equi-r and the equi-σ curves constitute a curvilinear network superposed on the rectangular background of $Z=X+iY$; for any assigned pair of values of r and σ the value of Z can be obtained by finding the intersection of those particular curves of r and σ, and at that point reading off the value of Z on the rectangular background. Similarly for the evaluation of Z' by means of the network of equi-r and equi-σ' curves.

For the σ'-range and the σ-range contemplated in Fig. 2—namely, $0<\sigma'<1$ and $0<\sigma<1$—the Z'-realm and the Z-realm are distinct; their mutual boundary (drawn dashed) is the unit semi-circle, that is, the semi-circle of unit radius having its center at the origin. The Z'-realm is the region inside; the Z-realm is all the region outside, extending to infinity in all directions through the positive real half of the complex-plane.

If the ranges of σ' and σ are extended to include values exceeding unity, the Z'-realm and the Z-realm will cease to be distinct but will overlap. The Z'-realm will expand upward, beyond the unit semi-circle, and ultimately will fill the region of unit width extending upward to infinity; the Z-realm will expand into and ultimately will fill the lower half of the unit semi-circle. Hence for values of σ' and σ exceeding unity it is preferable to employ individual charts in representing Z' and Z.

In the language of function-theory it may be said that, when $\sigma'=\sigma$, the Z'-realm and the Z-realm are inverse realms with respect to the unit semi-circle. The straight lines and the circles are inverse curves; the ellipses, and the curves characterized by the equation $(X^2+Y^2)^2-X^2-Y^2/(2\sigma-1)^2=0$ are also inverse curves.

For $r=0$ it is seen that $Z'=Z=1$ for all values of σ' and σ.

For values of r equal to or greater than unity, Z' and Z are pure imaginary, for all values of σ' and σ. For $r=1$, Z' lies somewhere on that part of the imaginary axis constituting the vertical diameter of the unit semi-circle, its position thereon depending on the particular value of σ' contemplated; while Z lies somewhere on the remainder of the imaginary axis. When r approaches infinity, Z' approaches infinity and Z approaches zero, along the imaginary axis.

Another graphical method of representing the relative impedances $Z=X+iY$ and $Z'=X'+iY'$, based on equations (4) and (5), is by means of the Cartesian curves of the components X, Y and X', Y', with the relative frequency r taken as the independent variable and the relative termination (σ or σ') as the parameter.

In this way, Fig. 3 represents X' and Y', and Fig. 4 represents X and Y, all to the same scale. In each of these figures the r-range is 0 to 1.5, thus including the entire transmitting band and a portion of the attenuating band half as wide as the transmitting band. In the

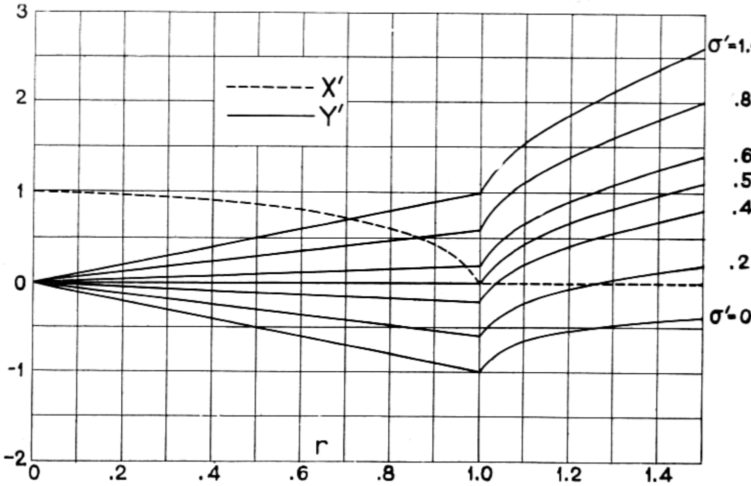

Fig. 3—Components of the σ'-Load Relative Impedance $Z'=X'+iY'$

attenuating band, Z' and Z are pure imaginary; in the transmitting band they are complex in general, though real for $\sigma'=0.5$ and $\sigma=0.5$.

Because in practical applications the transmitting band is much more important than the attenuating band, Fig. 5 has been supplied in order to represent X and Y in the transmitting band only, but to a considerably larger scale and for more values of σ.

If σ is read for σ', Fig. 3 will represent U and V instead of X' and Y' respectively. If σ' is read for σ, Fig. 4 will represent U' and V' instead of X and Y; so also will Fig. 5.

From Fig. 5 it will be observed that, in a certain range of σ, each curve of X has a maximum at some point within the transmitting band $(0<r<1)$. For any fixed value of σ (in the range found below) the corresponding maximum of X and the particular value of r (critical value) at which the maximum occurs are expressed by the formulas

$$\text{Max. } X=|1/4(1-2\sigma)\sqrt{\sigma(1-\sigma)}|,$$

$$\text{Crit. } r=\sqrt{\frac{8\sigma(1-\sigma)-1}{4\sigma(1-\sigma)}},$$

as is readily found from the formula for X—namely, the real part of formula (4). The formula for Crit. r shows that the σ-range in which

X, regarded as a function of r, has a maximum within the transmitting band $(0<r<1)$ is

$$(\sqrt{2}-1)/2\sqrt{2}<\sigma<(\sqrt{2}+1)/2\sqrt{2},$$

that is, approximately,

$$0.146<\sigma<0.854.$$

For values of σ outside of this range, X has no maximum within the transmitting band; but X has then its largest value at $r=0$, decreasing from 1 at $r=0$ to 0 at $r=1$. When $\sigma=1/2$, Crit. $r=1$;

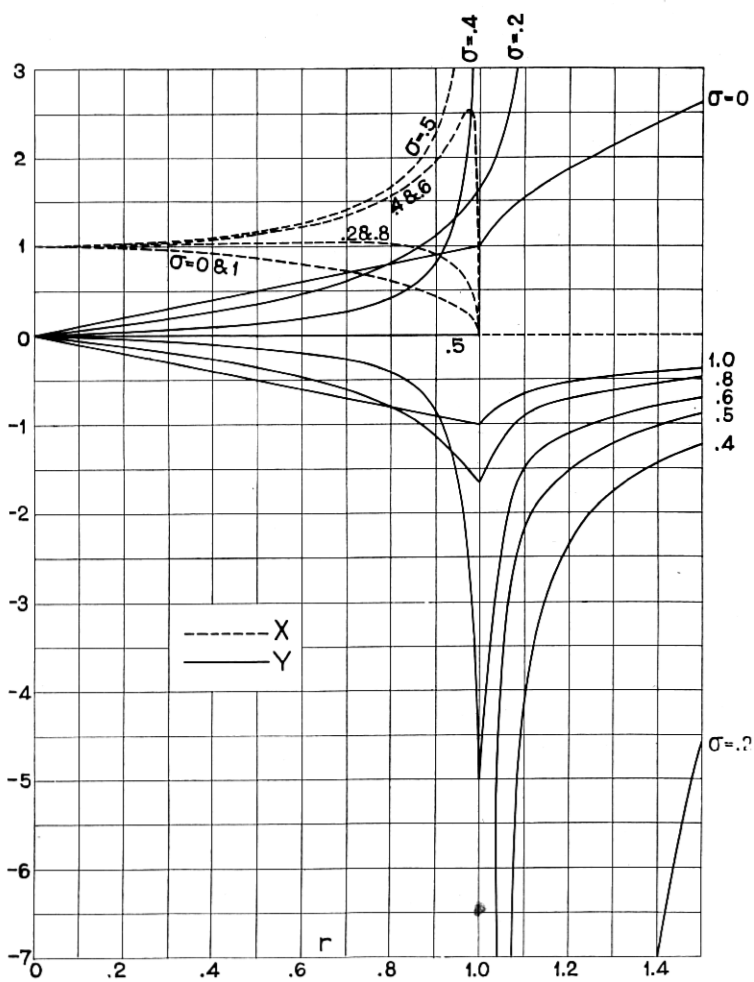

Fig. 4—Components of the σ-Section Relative Impedance $Z=X+iY$

when σ ranges from $1/2$ to either of its extreme values appearing in the foregoing inequality for σ, Crit. r decreases from 1 to 0.

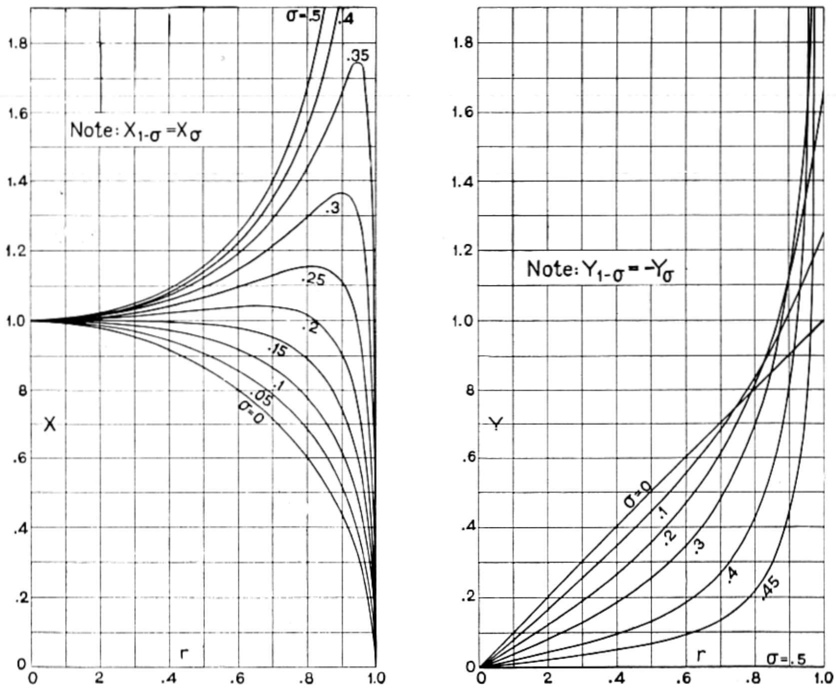

Fig. 5—Components of the σ-Section Relative Impedance $Z = X + iY$ in the Transmitting Band

PART III

IMPEDANCE OF NON-DISSIPATIVE LOADED LINES WITH DISTRIBUTED INDUCTANCE

Disposition of the Transmitting and the Attenuating Bands

It will be recalled that a loaded line without distributed inductance has only one transmitting band and only one attenuating band. In contrast, a loaded line (Fig. 1) with distributed inductance L has (as shown in Appendix A) an infinite sequence of alternate transmitting and attenuating bands; beginning with a transmitting band extending upward from zero frequency to the first transition frequency which, because of its special practical importance in being the upper boundary frequency of the first or principal transmitting band, is termed the "critical frequency" to distinguish it from the other transition frequencies. The critical frequency will be denoted by f_c; also by f_1—particularly when regarded as the first transition frequency. The relative frequency will be denoted by r, that is,

$$r = f/f_c = f/f_1. \tag{13}$$

Evidently $r_1 = 1$. General formulas for all of the transition frequencies are furnished a little further on. For the case of no distributed inductance ($L=0$), there is only one transition frequency—the critical frequency—and it has the value expressed by equation (3). When necessary for distinction, the critical frequency for the case of no distributed inductance will be denoted by f_c', also by f_1'; thus,

$$f_c' = f_1' = 1/\pi\sqrt{L'C}. \tag{14}$$

The ratio of the critical frequency of any loaded line to the critical frequency of the same loaded line without distributed inductance ($L=0$) will be denoted by p; that is,

$$p = f_c/f_c' = f_1/f_1'. \tag{15}$$

p can be evaluated by means of formula (22).

It is convenient to employ the term "compound band" to denote the band consisting of a transmitting band and the succeeding attenuating band. It is shown in Appendix A that, for any specific loaded line, the widths of all the compound bands are equal; though the transmitting bands become continually narrower with increasing

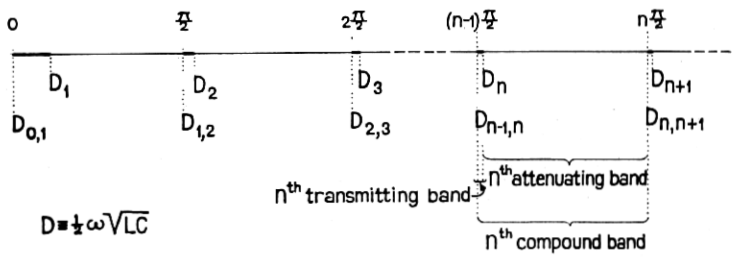

Fig. 6—Scale Showing the Disposition of the Transmitting and the Attenuating Bands of a Periodically Loaded Line (Fig. 1) with Distributed Inductance

frequency, while the attenuating bands become continually wider. These facts are represented on the D-scale in Fig. 6, D being proportional to the frequency f. Fundamentally D denotes the quantity $\frac{1}{2}\omega\sqrt{LC}$; but, by the substitution of $\lambda = L/L'$, and of r and p defined by (13) and (15), D can be written in the following four identically equivalent forms:

$$D = \tfrac{1}{2}\omega\sqrt{LC} = \tfrac{1}{2}\omega\sqrt{\lambda L'C} = rp\sqrt{\lambda} = rD_1. \tag{16}$$

It is of some interest to note that $D=\frac{1}{2}\omega\sqrt{LC}$ is equal to one-half the "phase constant" ("wave-length constant") of each section of line (L, C) between loads. In Fig. 6 the compound bands are numbered 1, 2, 3, . . . , n, . . . Thus D_n denotes the transition value of D within the nth compound band; that is, D_n is the value of D at the transition point between the nth transmitting band and the nth attenuating band. $D_{n,n+1}$ denotes the transition value of D between the nth and $(n+1)$th compound bands; and hence the transition value of D between the nth attenuating band and the $(n+1)$th transmitting band. The corresponding values of f and of ω would be correspondingly subscripted. By (16),

$$D_n=\tfrac{1}{2}\omega_n\sqrt{LC}=\tfrac{1}{2}\omega_n\sqrt{\lambda L'C}=r_n p\sqrt{\lambda}=r_n D_1; \tag{17}$$

and similarly for $D_{n-1,n}$ and $D_{n,n+1}$. In particular, $D_1=p\sqrt{\lambda}$, since $r_1=1$. As shown in Appendix A,

$$D_{n-1,n}=(n-1)\pi/2, \qquad D_{n,n+1}=n\pi/2. \tag{18}$$

Thus the D-width of each compound band is $\pi/2$, that is,

$$D_{n,n+1}-D_{n-1,n}=\pi/2; \tag{19}$$

and hence, by (16), the f-width has the value

$$f_{n,n+1}-f_{n-1,n}=1/2\sqrt{LC}=1/2\sqrt{\lambda L'C}=\pi f_1'/2\sqrt{\lambda}. \tag{20}$$

If τ_n denotes the D-width of the nth transmitting band,—that is, $\tau_n=D_n-D_{n-1,n}$,—then the f-width has the value

$$f_n-f_{n-1,n}=\tau_n/\pi\sqrt{LC}=\tau_n/\pi\sqrt{\lambda L'C}=\tau_n f_1'/\sqrt{\lambda}. \tag{20.1}$$

With regard to the nth compound band it will be noted that there are two kinds of transition points—namely, the internal transition point D_n, and the boundary transition points $D_{n-1,n}$ and $D_{n,n+1}$. This distinguishing terminology will be found convenient in connection with the transition frequencies also.

As indicated by Fig. 6, the widths of all the compound bands are equal; but with increasing n the width of the nth transmitting band continually decreases toward a width of 0, while the nth attenuating band continually increases toward a D-width of $\pi/2$; so that the infinitely remote compound bands are pure attenuating bands, the infinitely remote transmitting bands being vanishingly narrow.

The situation of the critical value D_n of D within the nth compound band has no such simple expressions as have the boundary points $D_{n-1,n}$ and $D_{n,n+1}$; for D_n is a root of a transcendental equation and can be expressed only by an infinite series of terms or of operations. In Appendix A a power series formula has been derived for D_n in terms of $\lambda = L/L'$ and $D_{n-1,n} = (n-1)\pi/2$; if, for brevity, the somewhat cumbersome (though expressive) symbol $D_{n-1,n}$ is denoted by d_n, this power series is

$$D_n = d_n + \frac{\lambda}{d_n} - \frac{1}{d_n}\left(\frac{\lambda}{d_n}\right)^2 + \left(\frac{2}{d_n^2} - \frac{1}{3}\right)\left(\frac{\lambda}{d_n}\right)^3 - \left(\frac{5}{d_n^3} - \frac{4}{3d_n}\right)\left(\frac{\lambda}{d_n}\right)^4$$
$$+ \left(\frac{14}{d_n^4} - \frac{5}{d_n^2} + \frac{1}{5}\right)\left(\frac{\lambda}{d_n}\right)^5 - \left(\frac{42}{d_n^5} - \frac{56}{3d_n^3} + \frac{23}{15d_n}\right)\left(\frac{\lambda}{d_n}\right)^6 + \ldots, \quad (21)$$

valid for $n = 2,3,4, \ldots$ but not for $n = 1$. For $n = 1$, so that $D_n = D_1$, it is shown in Appendix A that the appropriate formula is [7]

$$D_1 = \sqrt{\lambda}\left(1 - \frac{\lambda}{6} + \frac{11\lambda^2}{360} - \frac{17\lambda^3}{5040} - \frac{281\lambda^4}{604800} + \frac{44029\lambda^5}{119750400} \cdots\right). \quad (22)$$

Since, by (16), $p = D_1/\sqrt{\lambda}$, the series for p is the series in the parenthesis; see also (23-A) in Appendix A. Alternative series-formulas for evaluating D_1 and D_n are derived in Appendix A—formulas (23-A) and (23.1-A) for D_1, and (20.2-A) for D_n. It may be observed that $D_n - d_n < \lambda/d_n$, that $D_1 < \sqrt{\lambda}$, and that $1 - p < \lambda/6$.

The smaller λ, the more convergent are these formulas. Formula (22) is highly convergent, even when λ is as large as unity or even somewhat larger. The convergence of formula (21) depends very much on d_n and hence on n: when n is large, (21) is satisfactorily convergent even for fairly large values of λ; but when n is small, (21) is satisfactorily convergent only for rather small values of λ.

As a supplement to or as an alternative to formulas (21) and (22) there will now be given a widely applicable formula of successive approximation for D_n, valid for all the values of n—including $n = 1$—and suitable even for large values of λ. With $D_n - d_n$ (the D-width of the nth transmitting band) denoted by τ_n, this formula (derived by Newton's general method of approximation) is:

$$\tau_n'' = \frac{\lambda\tau_n' + \lambda \sin \tau_n' \cos \tau_n' - d_n \sin^2 \tau_n'}{\lambda + \sin^2 \tau_n'}, \quad (22.1)$$

wherein τ_n' is some approximate known value of τ_n, and τ_n'' is a more accurate approximate value yielded by the formula. τ_n'', in turn, is to be used in the formula to compute a still more accurate approximate value τ_n'''; and so on, through as many cycles as may be

[7] From the sequence of signs in this formula, namely $-+--+$, the sign of the next term is not evident. A similar remark applies to formulas (23-A) and (23.1-A) in Appendix A.

necessary—usually not more than two or three, though occasionally four. First-approximation values for τ_n are:

$$\tau_n' = \frac{\lambda}{d_n}\left(1 - \frac{\lambda}{d_n^2}\right) \text{ when } n \neq 1,$$

$$\tau_n' = \sqrt{\lambda}\,(1 - \lambda/6) \text{ when } n = 1,$$

as can be seen from (21) and (22) respectively. When $n=1$, $\tau_n = D_1$' since $d_1 = 0$ by the first of (18).

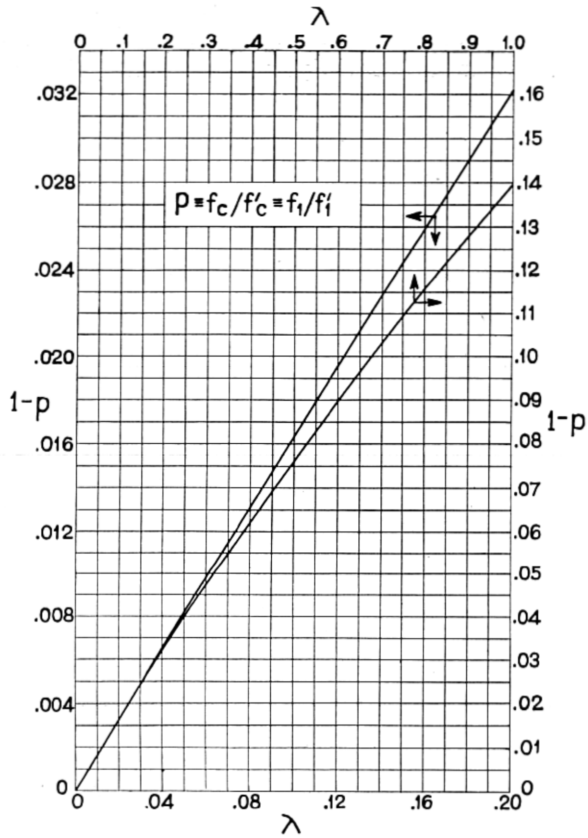

Fig. 7—Graphs of $1-p$ Representing the Fractional Lowering of the Critical Frequency by Distributed Inductance

D_n having been evaluated, the transition frequency f_n between the nth transmitting band and the nth attenuating band is calculable immediately from

$$f_n = \frac{D_n}{\pi\sqrt{LC}} = \frac{D_n}{\pi\sqrt{\lambda L'C}} = \frac{D_n f_1'}{\sqrt{\lambda}}, \tag{23}$$

derived from (17) supplemented by (14). Formula (23) is valid also when $n=1$, with D_1 evaluated from one of its appropriate formulas; the resulting formula for the critical frequency $f_1=f_c$ reduces to

$$f_1=f_c=p\sqrt{\lambda}/\pi\sqrt{LC}=p/\pi\sqrt{L'C}=pf_c'=pf_1', \tag{24}$$

because $D_1=p\sqrt{\lambda}$, by (16); it is seen that (24) is consistent with (15).

For use in (24) and for certain other purposes to be met later, Fig. 7 gives graphs of $1-p$, calculated by (22) and also (22.1), for a wide range of λ. Up to the present time the largest value of λ occurring in practical applications in the Bell System is about 0.12; Fig. 7 covers

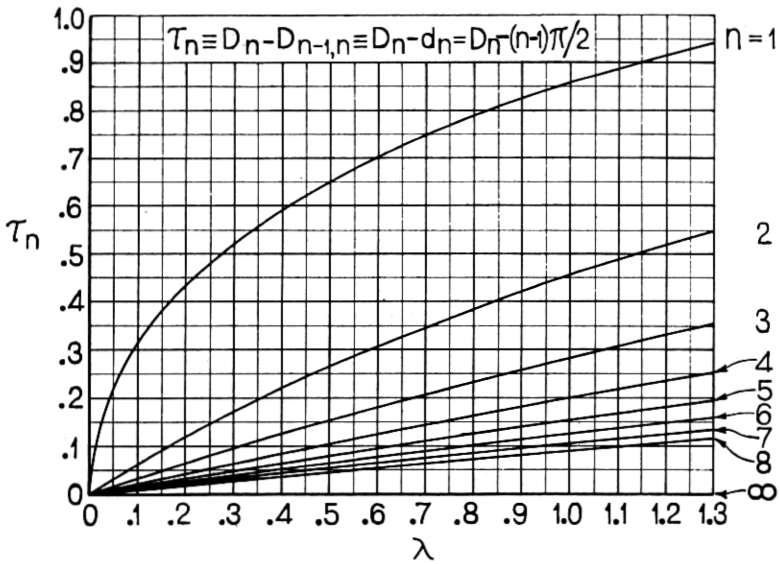

Fig. 7.1—Graphs for Finding the Widths of the Transmitting Bands

about eight times this range. Inspection of it shows that the graph of $1-p$ is sensibly a straight line up to values of r somewhat larger than even 0.12; and that $1-p$ is only slightly less than $\lambda/6$, which is merely the first term in the power series formula for $1-p$ obtained from (22).

The graphs in Fig. 7.1—constructed by means of formulas (22.1), (22), (21)—represent directly the dependence of the D-width $\tau_n=D_n-D_{n-1,n}$ of the nth transmitting band on λ and n, for a wide range of λ and the first eight values of n. The f-width is then obtainable immediately from (20.1); and f_n from (23), since $D_n = \tau_n + (n-1)\pi/2$. In particular, the graph for $n=1$ is a graph of D_1; but D_1—and hence f_1—can be evaluated much more precisely by means of Fig. 7 described in the preceding paragraph.

The boundary transition frequencies $f_{n-1,n}$ and $f_{n,n+1}$ of the nth compound band (any compound band) depend on only one parameter (besides n)—namely, the product LC. The internal transition frequency f_n depends on two independent parameters (besides n)—namely, the product LC and the ratio $\lambda = L/L'$. Hence, fixing LC fixes all of the boundary frequencies of the compound bands; fixing LC and λ fixes all of the transition frequencies—boundary and internal. Fixing any one boundary frequency fixes LC and thereby fixes all of the remaining boundary frequencies; fixing any two transition frequencies of which at least one is an internal transition frequency fixes LC and λ and thereby fixes all of the remaining transition frequencies—boundary and internal.

The relative widths of all the transmitting and attenuating bands depend on only one parameter—namely, the ratio $\lambda = L/L'$. Hence, fixing λ fixes the relative widths of all these bands; fixing the ratio of the widths of any two bands not both of which are compound bands fixes λ and thereby fixes the relative widths of all the transmitting and attenuating bands.

The effect of increasing λ, when $L'C$ is fixed, is to lower the critical frequency $f_c = f_1$, the critical frequency approaching zero when λ approaches infinity. But for even the largest values of λ met in practice the critical frequency is not much lower than for $\lambda = 0$; the fractional decrease $(f_c' - f_c)/f_c'$ produced in the critical frequency by increasing λ from 0 to any value λ is exactly equal to $1-p$ and hence for any ordinary value of λ is, by (22), closely equal to $\lambda/6$ (which is only 0.02 for $\lambda = 0.12$). It is interesting to note that the nominal impedance—defined by equation (1)—is increased about three times as much as the critical frequency is decreased; for the fractional increase in the nominal impedance is exactly $\sqrt{1+\lambda} - 1$, and hence approximately $\lambda/2$.

All the transition frequencies are reduced by increasing λ, when $L'C$ is fixed. The transition frequencies bounding the compound bands, and hence the widths of the compound bands, decrease in direct proportion to an increase of $\sqrt{\lambda}$. But the values of the internal transition frequencies do not decrease so rapidly; for the ratio of transmitting band width to attenuating band width increases with increasing λ.

The effect of adding distributed inductance L to a loaded line (L', C) having originally none is to replace the previous single compound band of infinite width by an infinite number of compound bands each of finite width. The larger L the narrower are the compound f-bands, and the further to the left they are situated. Although, as already noted, increasing L decreases the critical frequency, it increases the relative width of each transmitting band—namely, the ratio of the width of each transmitting band to the compound band of which it is a constituent. Thus, when L becomes very large (so that LC and λ become very large) there are within even a moderate frequency-range a very large number of compound bands whose transmitting constituents are very wide compared with the attenuating constituents.

The effect of applying lumped loading to a given smooth line (L, C) is to introduce into the previous transmitting band of infinite width an infinite number of attenuating bands whose upper boundary points are equidistant and whose widths continually decrease toward the lower frequencies. When the inductance L' of the loads is continually increased the attenuating bands continually increase in width as a consequence of their lower boundary points moving downward to lower frequencies, so that ultimately the attenuating bands fill the entire frequency scale from zero to infinity. An alternative but equivalent statement regarding the effect of applying lumped loading is that the previous pure transmitting bands, each of D-width equal to $\pi/2$, become compound bands whose attenuating constituents continually increase in width when L' is increased.

(The four preceding paragraphs are based on the last five paragraphs of Appendix A.)

In Fig. 6 the transmitting bands are represented as being relatively narrow compared with the attenuating bands. In existing loaded lines this is indeed the case, but it is not an inherent relation: for any number of the transmitting bands can be made wider than the associated attenuating bands by so designing the loading (lumped or smooth or both) as to secure a sufficiently large value of the ratio $\lambda = L/L'$. (However, for any fixed loading and hence a fixed value of λ, there is some frequency beyond which the transmitting bands are narrower than the associated attenuating bands.)

There will now be given two examples illustrating the relations represented in Fig. 6, and illustrating also the applications of certain of the foregoing formulas and graphs.

The first example pertains to a heavily loaded open-wire line of No. 12 N. B. S. gauge, having loading coils of inductance $L' = 0.241$

henry at a spacing of $s=7.88$ miles. The line has a capacity of $.00835\times10^{-6}$ farad and an inductance of .00367 henry, each per mile; whence, for each line-segment between loads, $C=.0658\times10^{-6}$ farad and $L=.0289$ henry. Therefore $\lambda=0.12$. With λ known, the internal transition frequencies f_n (with $n=1, 2, 3, 4, \ldots$) can be readily evaluated from (23) through the values of D_n obtainable from Fig. 7.1. However, when particularly high accuracy is desired for the first transition frequency f_1—the critical frequency—this can be attained by resort to formula (22) or to (22.1), or else to Fig. 7; it is thus found that $1-p=.0196$, whence $p=0.9804$, and then $f_1=2479$ cycles per second, by (24). The f-width of each compound band is 11464, by (20). The following table shows the locations and widths of the first five ($n=1, 2, 3, 4, 5$) transmitting bands and associated attenuating bands of this loaded line. The numbers in the columns headed $f_{n-1,n}$ and f_n are the transition frequencies constituting, respectively, the lower and upper boundary points of the transmitting bands; and the numbers in the column headed $f_n-f_{n-1,n}$ are therefore the widths of the transmitting bands. The next to the last column shows the relative widths of the transmitting bands, referred to the first or principal transmitting band—whose width is $f_1-0=f_1=2479$, the critical frequency being 2479. Similarly, the last column shows the relative widths of the attenuating bands.

n	τ_n	$f_{n-1,n}$	f_n	$f_n-f_{n-1,n}$	$(f_n-f_{n-1,n})/f_1$	$(f_{n,n+1}-f_n)/f_1$
1	.3396	0	2,479	2,479	1.000	3.625
2	.0729	11,464	11,996	532	.215	4.410
3	.0377	22,928	23,203	275	.111	4.514
4	.0253	34,392	34,577	185	.074	4.551
5	.0190	45,856	45,995	139	.056	4.569

It will be observed that the transmitting bands decrease rapidly in width at first, then more and more slowly; and that the associated attenuating bands are relatively very wide. For instance, the second transmitting band (0.215) is only about one-fifth the width of the first (1.000), and the second attenuating band (4.410) is more than twenty times the width of the second transmitting band (0.215).

The second example pertains to a hypothetical, though not necessarily impracticable, loaded line. Before loading, the line is the same as in the first example; but it is very lightly loaded—namely, with loading coils of inductance $L'=.0578$ henry at a spacing of $s=15.76$ miles. Hence, $C=0.1316\times10^{-6}$ farad and $L=.0578$ henry. Therefore

$\lambda = 1$. The following table shows the locations and widths of the first eight transmitting bands and attenuating bands. The critical frequency is $f_1 = 3140$, and the f-width of each compound band is 5732.

n	τ_n	$f_{n-1,n}$	f_n	$f_n - f_{n-1,n}$	$(f_n - f_{n-1,n})/f_1$	$(f_{n,n+1} - f_n)/f_1$
1	.8604	0	3,140	3,140	1.000	.826
2	.4579	5,732	7,403	1,671	.532	1.294
3	.2840	11,464	12,500	1,036	.330	1.496
4	.2008	17,196	17,929	733	.234	1.592
5	.1541	22,928	23,490	562	.179	1.647
6	.1247	28,660	29,115	455	.145	1.681
7	.1046	34,392	34,774	382	.122	1.704
8	.0900	40,124	40,452	328	.105	1.721

Comparison of this table with that of the first example brings out the great diversity between the two examples: the minor transmitting bands in the second example are relatively and absolutely much wider and situated at much lower frequencies than in the first example. In the second example the first or principal transmitting band is somewhat wider than the first attenuating band.

A further application of the foregoing formulas and graphs is to obtain a precise and explicit solution of the important practical problem of loading a given smooth line with lumped loading to secure specified values of the critical frequency f_1 and nominal impedance k. The design-problem consists in determining the requisite values of the load inductance L' and load spacing s in terms of f_1 and k and the known values of the inductance and capacity, L'' and C'', per unit length of the given smooth line. Since $L = sL''$ and $C = sC''$, the solution can be obtained as follows: Substituting $L' = sL''/\lambda$ into (1) and solving for λ gives

$$\lambda = \frac{L''/C''}{k^2 - L''/C''}.$$

Then D_1 becomes known by means of Fig. 7 or Fig. 7.1 or formula (22) or (22.1). Next, s becomes known from (23) or (24):

$$s = D_1/\pi f_1 \sqrt{L''C''}.$$

Finally, from these formulas for λ and s together with the relation $L' = sL''/\lambda$, it follows that

$$L' = \frac{D_1(k^2 - L''/C'')}{\pi f_1 \sqrt{L''/C''}}.$$

The Relative Impedances

The formulas for the impedances and admittances of a non-dissipative periodically loaded line (Fig. 1) with any amount of distributed inductance L will next be set down, and discussed somewhat, with particular regard to the transmitting and the attenuating bands of the loaded line.

As before, it is convenient to deal with the relative impedances Z, Z' and the relative admittances W, W' defined by equations (2). Special attention is given to the particular values $Z_{.5}$, $Z'_{.5}$, $W_{.5}$, $W'_{.5}$ corresponding to mid-point terminations.

It is found that Z, Z', W, W' can be expressed in terms of three independent quantities—namely, the relative frequency $r=f/f_c$, the inductance ratio $\lambda=L/L'$, and the relative termination σ or σ'. For most applications the quantity $r=f/f_c$ is more significant than any other quantity proportional to the frequency f, and on that score it would be desirable to employ it explicitly in the formulas for the impedances and admittances. However, the formulas are rendered considerably more compact by employing the quantity D defined by equation (16). Whenever desired, D can be expressed in terms of r, λ, and p by means of (16); and thence in terms of r and λ by means of (22).

Because of their special importance the formulas for the mid-point relative impedances and relative admittances will be set down first. From Appendix D these formulas are found to be

$$Z_{.5}=\frac{1}{W_{.5}}=\sqrt{\frac{\lambda}{\lambda+1}}\sqrt{\frac{\lambda+D\cot D}{\lambda-D\tan D}}, \tag{25}$$

$$Z'_{.5}=\frac{1}{W'_{.5}}=\sqrt{\frac{(\lambda+D\cot D)\,(\lambda-D\tan D)}{\lambda(\lambda+1)}}, \tag{26}$$

$$=\sqrt{\frac{\lambda^2+2\lambda D\cot 2D-D^2}{\lambda(\lambda+1)}}. \tag{26.1}$$

From these formulas it can be verified that $Z_{.5}$ and $Z'_{.5}$ are pure imaginary throughout every attenuating band, and it can be seen that they are pure real throughout every transmitting band.

A study of equations (25) and (26) brings out also the following facts regarding the variation of $Z_{.5}$ and $Z'_{.5}$ in the transmitting and the attenuating bands, with increasing frequency:

In the first transmitting band, $Z_{.5}$ ranges from 1 to ∞, but in all of the other odd transmitting bands it ranges from ∞ to ∞, through finite intervening values; in the even transmitting bands it ranges from 0 to 0, through finite intervening values. In the odd attenuating bands it ranges from $-i\infty$ to $-i0$; and in the even attenuating bands it ranges from $+i0$ to $+i\infty$.

In the first transmitting band, $Z'_{.5}$ ranges from 1 to 0, but in all of the other transmitting bands it ranges from ∞ to 0. In all of the attenuating bands it ranges from $+i0$ to $+i\infty$.

These facts are illustrated by Fig. 8, which gives graphs of $Z_{.5}$ and $Z'_{.5}$ over a range of three compound bands, as functions of $r=f/f_1=D/D_1$, with $\lambda=0.12$; also with $\lambda=0$, for comparison. On the scale there used, the curves for the two values of λ are indistinguishable

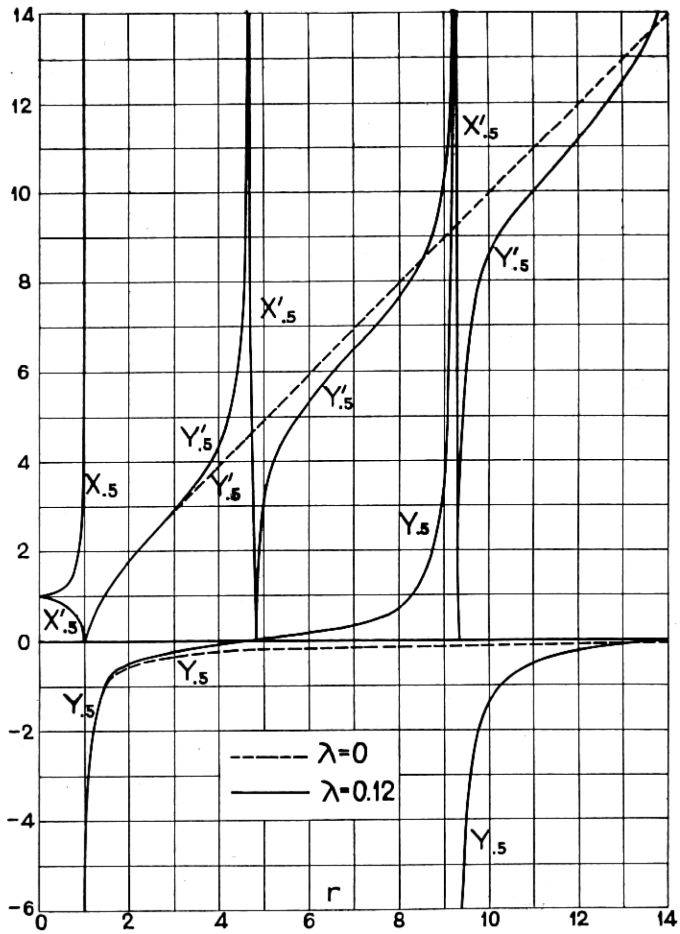

Fig. 8—Mid-Section Relative Impedance $Z_{.5}=X_{.5}+iY_{.5}$ and Mid-Load Relative Impedance $Z'_{.5}=X'_{.5}+iY'_{.5}$ Over a Range of Three Compound Bands

throughout the first transmitting band ($0<r<1$) and a considerable part of the succeeding attenuating band; but depart widely beyond.

The exact formulas for Z, W and Z', W' for any terminations σ and σ' can be written in the forms

$$Z=\frac{1}{W}=\frac{Z_{.5}\cot(2\sigma-1)D+i\sqrt{\lambda/(1+\lambda)}}{\cot(2\sigma-1)D+iZ_{.5}\sqrt{(1+\lambda)/\lambda}}, \tag{27}$$

$$Z'=\frac{1}{W'}=Z'_{.5}+i\frac{(2\sigma'-1)D}{\sqrt{\lambda(1+\lambda)}}. \tag{28}$$

These equations are not restricted to values of σ and σ' less than unity; they are valid for any (real) values of these quantities. When $\lambda=0$, they reduce immediately to (4) and (5) respectively.

From (27) and (28) it is readily verified that Z and Z' are pure imaginary throughout every attenuating band, and it can be easily seen that they are complex throughout every transmitting band; because $Z_{.5}$ and $Z'_{.5}$ are pure imaginary throughout every attenuating band, and pure real throughout every transmitting band.

It is seen from (27) and (28) that, throughout every transmitting band, each of the quantities Z, W, Z', W' changes merely to its conjugate when σ is changed to $1-\sigma$. Thus the conjugate property expressed by equations (8) is not limited to loaded lines without distributed inductance but holds when there is any amount of distributed inductance. Thus it continues to be true that complementary characteristic impedances are mutually conjugate—throughout every transmitting band. For Z' and W', these facts are readily seen from physcial considerations also; though not so readily for Z and W.

From physical considerations, as well as from equation (28), it is readily seen that Z' continues to possess the property expressed by the second of equations (9); on the other hand, W no longer possesses the property expressed by the first of (9).

We shall now return to the important formulas (25) and (26) for the mid-point relative impedances in order to discuss them for small values of λ such as occur in practice, and particularly for a frequency-range not greatly exceeding that of the first transmitting band. For this purpose it is advantageous to write these formulas in the following forms, notwithstanding some sacrifice of compactness:

$$Z_{.5}=\frac{1}{W_{.5}}=\sqrt{\frac{\lambda+D\cot D}{\lambda+1}}\Bigg/\sqrt{1-\frac{D\tan D}{D_1\tan D_1}}, \tag{29}$$

$$Z'_{.5}=\frac{1}{W'_{.5}}=\sqrt{\frac{\lambda+D\cot D}{\lambda+1}}\;\sqrt{1-\frac{D\tan D}{D_1\tan D_1}}. \tag{30}$$

For the discussion of these it should be recalled that $D=\frac{1}{2}\omega\sqrt{\lambda L'C}$ and $r=D/D_1=f/f_1=f/f_c$; also that $D_1 \tan D_1=\lambda$, whence D_1 is approximately equal to $\sqrt{\lambda}$ when λ is small.

Equations (29) and (30) are in such form as to exhibit the manner in which $Z_{.5}$ and $Z'_{.5}$ approach their simple limiting values for $\lambda=0$, represented by equations (6) and (7) respectively. For when λ approaches 0, $D \cot D$ and $D \tan D$ approach 1 and D^2 respectively; and for values of λ even larger than the largest (about 0.12) occurring in practice, $D \cot D$ and $D \tan D$ respectively are at least roughly equal to 1 and to D^2 throughout even more than the first transmitting band.

The expression for $Z_{.5}$ reduces immediately to $1/\sqrt{1-r^2}$ when λ is zero. When λ is not zero, $Z_{.5}$ is less than $1/\sqrt{1-r^2}$ for all values of r in the first transmitting band $(0<r<1)$; when r increases from 0 to 1, $Z_{.5}$ increases from 1 to ∞.

The expression for $Z'_{.5}$ reduces immediately to $\sqrt{1-r^2}$ when λ is zero. Even when λ is several tenths, $Z'_{.5}$ is very closely equal to $\sqrt{1-r^2}$ for all values of r in the first transmitting band; when r increases from 0 to 1, $Z'_{.5}$ decreases from 1 to 0.

Effects of Distributed Inductance; the "Simulative Loaded Line"

The above-described relations are exemplified in Fig. 9, which gives graphs of $Z_{.5}$ and $Z'_{.5}$ over the first transmitting band and part of the succeeding attenuating band, as functions of r, with λ as parameter equal to 0.12 and to 0. It is seen that the curves of $Z_{.5}$ for the two values of λ do not differ much in the transmitting band $(0<r<1)$; and that the curves of $Z'_{.5}$ for the two values of λ are indistinguishable—on the scale there used.

In order to indicate more precisely to what extent the forms of $Z_{.5}$ and $Z'_{.5}$ are affected by the presence of distributed inductance, as specified by $\lambda=L/L'$, Fig. 10 has been prepared. This gives a graph of the ratio of the values of $Z_{.5}$ for $\lambda=0.12$ and $\lambda=0$; and likewise of $Z'_{.5}$. That is, formulated in functional notation, it gives graphs of $Z_{.5}(r, \lambda)/Z_{.5}(r, 0)$ and $Z'_{.5}(r, \lambda)/Z'_{.5}(r, 0)$. From these it is seen that, in the transmitting band, the mid-section ratio (first ratio) and the mid-load ratio (second ratio) do not differ from unity by more than four per cent. and one-tenth of one per cent., respectively. These observations—particularly the second—suggest that, at least over the whole of the first transmitting band, the impedance of a non-dissipative periodically loaded line with small distributed inductance can be rather closely simulated by a periodically loaded line without distributed inductance but with suitably chosen load-inductance L_0' and section-capacity C_0. The utility of this observation resides

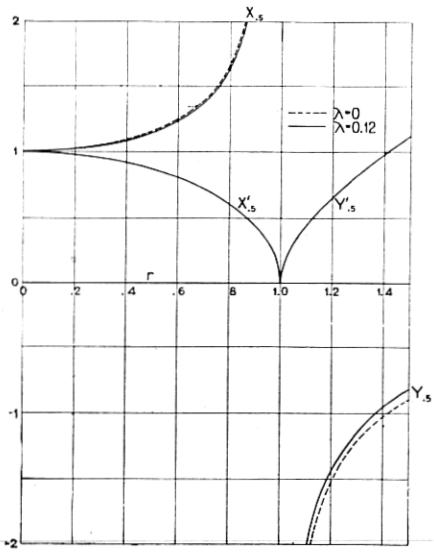

Fig. 9—Mid-Section and Mid-Load Relative Impedances $Z_{.5}$ and $Z'_{.5}$ Over the First Transmitting Band and Part of the Succeeding Attenuating Band

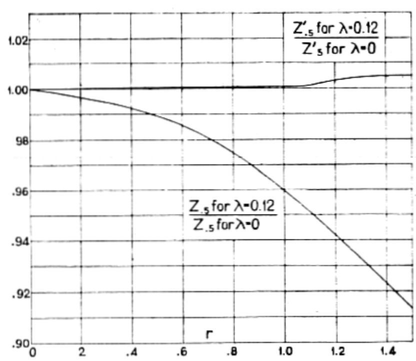

Fig. 10—Ratio Curves Showing Effects of Distributed Inductance on the Forms of the Curves of $Z_{.5}$ and $Z'_{.5}$

ultimately in the fact that the formulas for loaded lines without distributed inductance are much simpler than those for loaded lines with distributed inductance.

For mid-section or for mid-load termination the simulation of the effects of distributed inductance described in the preceding paragraph can be made exact at two different frequencies simultaneously, and the requisite values of the load-inductance L_0' and section-capacity C_0 of the simulating loaded line thereby determined. This simulating loaded line will be termed the "simulative loaded line" corresponding to the two particular frequencies contemplated.

In many applications a suitable simulation can be attained by imposing the conditions that the simulating loaded line (L_0', C_0) shall have the same nominal impedance k and critical frequency f_c as the actual loaded line (L', L, C). The particular simulating loaded line so determined will be called the "principal simulative loaded line"; evidently its load-inductance L_0' and section-capacity C_0 are determined in terms of k and f_c and also in terms of L', L, C by the pair of equations

$$k=\sqrt{(L'+L)/C}=\sqrt{L_0'/C_0}, \tag{31}$$

$$f_c=p/\pi\sqrt{L'C}=1/\pi\sqrt{L_0'C_0}, \tag{32}$$

of which (31) corresponds to (1), and (32) to (15) and (14) combined or to (24). The solution of the pair of equations (31) and (32) is the pair of values

$$L_0'=L'(\sqrt{1+\lambda})/p=k/\pi f_c, \tag{33}$$

$$C_0=C/p\sqrt{1+\lambda}=1/\pi f_c k. \tag{34}$$

In conjunction with (22), these formulas show that $L_0'>L'$ and $C_0<C$; in fact they show that $L_0'/L'=1+2\lambda/3$ and $C_0/C=1-\lambda/3$, as first approximations; precise values of these ratios can be readily calculated by substituting for p the power series contained in equation (22).

The simulative precision of the "principal simulative loaded line" depends on the value of the relative termination (σ or σ'). The simulation is far more precise for mid-load termination ($\sigma'=0.5$) than for mid-section termination ($\sigma=0.5$); this can be seen by developing in power series the functions involved; for $\lambda=0.12$ the fact is illustrated by Fig. 10 already cited. The simulative precision for other terminations will not be discussed here, beyond remarking that the "principal simulative loaded line" terminating at σ'-load could not exactly simulate the actual loaded line terminating at σ'-load, even if the simulation were exact at 0.5-load; for the excess-inductances $(\sigma'-0.5)L_0'$ and $(\sigma'-0.5)L'$ are not exactly equal, the former being slightly the larger—as shown by equation (33). However, the smallness of the impedance-departure between the "principal simulative loaded line" and the actual loaded line when both lines terminate at mid-load can be identically preserved for any other load-point termination of either line by so choosing the load-point termination of the other line that the excess inductance of its end-load beyond half load has the same value. This fact should be kept in mind when designing simulating and compensating networks, particularly such as pertain to a loaded line that terminates with a fractional load; also when choosing the relative termination σ' of the fractional load.

Some idea as to the simulative precision of the propagation constant $\Gamma = A + iB$ of the "principal simulative loaded line" can be obtained from Fig. 22 in Appendix *A*. For the present purpose the graphs for $\lambda = 0$ can be regarded as pertaining exactly to the "principal simulative loaded line" corresponding to any non-dissipative periodically loaded line having any amount of distributed inductance, while the graphs for $\lambda = 0.12$ are for any non-dissipative loaded line having the particular inductance ratio $\lambda = 0.12$. Of course, A is zero in the range $0 < r < 1$.

PART IV

Networks for Simulating and for Compensating the Impedance of Loaded Lines—General Considerations

The remainder of the paper relates to the simulation and the compensation of the impedance of periodically loaded lines by means of the simulating and the compensating networks devised by the writer, as mentioned in the latter part of the Introduction.

The term "compensating network" requires at least a tentative definition. The compensating networks dealt with in the present paper are of two types: reactance-compensators, and susceptance-compensators. For the present they may be defined—rather narrowly—with reference to the first transmitting band of non-dissipative loaded lines, as follows: a reactance-compensator is a network that neutralizes the characteristic reactance of the line and hence simulates its complementary characteristic reactance; a susceptance-compensator is a network that neutralizes the characteristic susceptance of the line and hence simulates its complementary characteristic susceptance.

As actually worded, this division (Part IV) of the paper pertains mainly to the simulation of loaded lines; but with appropriate slight changes of wording most of it pertains also to compensation. Compensation is dealt with explicitly in portions of Parts V and VIII of the paper.

The simulating and the compensating networks were devised from purely theoretical studies of the characteristic impedance and admittance of periodically loaded lines as dependent on the frequency and on the relative termination, in somewhat the same way as the previously described [1] networks for smooth lines were devised from purely theoretical studies of the characteristic impedance of smooth lines as dependent on the frequency.

Building-out Structures, Basic Networks, and Excess-Simulators

Although the characteristic impedance of a periodically loaded line depends greatly on its relative termination (σ or σ'), yet there is no need of attempting to devise various independent networks corresponding to various relative terminations of the line. For any network that will simulate the line-impedance at any particular relative termination can be "extended" or "built-out" to simulate it at any other relative termination by merely supplementing the network with an "extension network" or "building-out structure" in the nature of an artificial line structure corresponding as closely as may be necessary to the portion of actual line structure included between the two relative terminations contemplated. Simulation can be attained also by building-out the line instead of the network, or by building-out both the line and the network to any common relative termination; but in practice these alternatives are not usually permissible, the usual requirement being the simulation of a given fixed line. (In present practice, the line is terminated usually at mid-section [$\sigma=0.5$], or as closely thereto as practicable.)

The term "basic network" will be used to denote a network which simulates the characteristic impedance of a non-dissipative periodically loaded line without the network's containing in its structure any building-out elements. Regarding the loaded line, the particular relative termination to which the basic network pertains will be termed the "basic relative termination" of the loaded line, and will be denoted by σ_b or σ_b' whenever a symbol is needed for it. (For the kinds of basic networks thus far devised, σ_b and σ_b' lie between about 0.1 and about 0.2, that range having been found to include the relative terminations most favorable to the design of those kinds of basic networks.) The foregoing terms, when used in connection with a dissipative loaded line, will be understood to refer to the corresponding non-dissipative loaded line. A considerable number of kinds of basic networks will be described in Part V supplemented by Part VI.

The amount by which the characteristic impedance of any periodically loaded line exceeds the impedance of the corresponding non-dissipative loaded line will be termed the "excess impedance" (or, more fully, the "excess characteristic impedance"); and a network for simulating it will be termed an "excess-simulator." Excess-simulators for loaded lines will be considered very briefly in Part VII.

(In passing, it may be noted that the foregoing definition of the "excess impedance" of a periodically loaded line properly includes the definition already given [1] of the excess impedance of a smooth

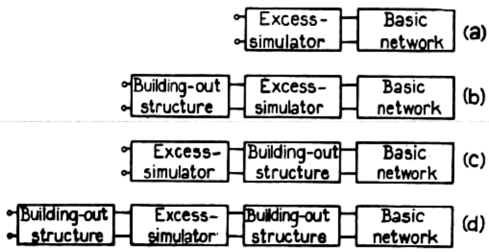

Fig. 11—Abstract Diagrams of Complete Networks for Simulating Characteristic Impedance of Loaded Line

line; for the "nominal impedance" of any smooth line was defined [1] as the impedance of the corresponding non-dissipative smooth line. A similar statement is applicable to the terms "excess simulator" and "basic network" previously defined [1] for smooth lines.)

The foregoing considerations and definitions have prepared the way for Fig. 11, which indicates in an abstract manner how the impedance of any loaded line having any relative termination can be simulated by combinations of basic networks, excess simulators, and building-out structures.

Fig. 11a corresponds to the simple but unusual case in which the loaded line has the basic relative termination: its impedance then can be simulated by the corresponding basic network and excess simulator, without any building-out structure.

When, as usual, the given line does not have the basic relative termination, there are available the two natural alternatives represented by Figs. 11b and 11c. Fig. 11b shows the whole network of Fig. 11a built-out to the relative termination of the given line by means of the requisite building-out structure, which for the highest precision must be dissipative to correspond to the actual line. In Fig. 11c the basic network is built-out to the relative termination of the given line with a non-dissipative building-out structure; and then the resulting network, which simulates the impedance that the actual line would have if non-dissipative, is supplemented with an excess-simulator such as to simulate the excess impedance of the actual line.

Since the excess impedance depends somewhat on the relative termination it can be simulated more easily at certain relative terminations than at others. This fact is utilized in the arrangement represented by Fig. 11d. Here the basic network is built-out to some relative termination that is particularly favorable for the design of an excess-simulator; the excess-simulator is applied; and then is applied the building-out structure, which for the highest precision must be dissipative to correspond to the actual line.

The simulation-range of the basic networks described in this paper is a little less than the first transmitting band of the loaded line; but after a basic network has been built-out, its simulation-range may extend a little way into the succeeding attenuating band, omitting the immediate neighborhood of the critical frequency. The compensation-range of the compensating-networks is somewhat less than the first transmitting band of the loaded line.

PART V

Networks for Non-Dissipative Loaded Lines without Distributed Inductance

In this Part will be described a considerable number of kinds of "basic networks" for simulating the characteristic impedance of non-dissipative loaded lines without distributed inductance; and two types of compensating networks for such lines. The modifications necessary when the lines have small distributed inductance will be indicated in Part VI.

The various kinds of basic networks here described may be regarded as of two different types corresponding to the terminations of the loaded lines to which they pertain; there may be several varieties of each type. The two types correspond to fractional-section and to fractional-load terminations respectively; that is, to the relative terminations σ_b and σ_b' respectively. (It has been stated already, in Part IV, that σ_b and σ_b' lie between about 0.1 and about 0.2.) It will appear below that these two types are inverse types, in the sense that the impedance of a network of one type is of the same functional form as the admittance of the corresponding network of the other type, when the frequency is regarded as the independent variable. In particular, for equal relative terminations ($\sigma_b=\sigma_b'$), the ratio of the impedance and the admittance of any two corresponding inverse networks is independent of frequency. This corresponds to the relations $Z/W'=1$ and $Z'/W=1$, holding for the loaded line itself, according to equations (4) and (5). Hence the two types of networks will sometimes be distinguished as impedance type and admittance type. More specifically, the simulating networks of the two types will be distinguished as impedance-simulators and admittance-simulators, respectively; and the compensating networks as reactance-compensators and susceptance-compensators, respectively.

By being built out to the requisite extent, either type of network evidently can be employed with a loaded line terminating at any point in either a section or a load; but, depending on such termination, one type will require less building-out than the other, and hence will be somewhat preferable on that score. For instance, for simulating the impedance of a loaded line terminating at mid-section ($\sigma = 0.5$), a basic network of the fractional-section type of termination will require less building-out than one of the fractional-load type of termination.

The Basic Networks

The various basic networks mentioned will now be described briefly, by aid of circuit diagrams which show the forms of the networks and which include explicit design-formulas for the proportioning. Mutually corresponding networks of inverse types will be described together or in sequence, in order to exhibit clearly their correlation.

In the design-formulas the requisite values for the network-elements will be expressed in terms of the load-inductance L' and the section-capacity C of the given loaded line; but when desired they can instead be readily expressed in terms of the nominal impedance k and critical frequency f_c, by means of the relations

$$L' = k/\pi f_c, \qquad C = 1/\pi k f_c.$$

Of course, the design-formulas involve also the relative terminations σ and σ'.

Figs. 12 and 13 show two rather simple networks which simulate very well, over most of the transmitting band, the σ-section character-

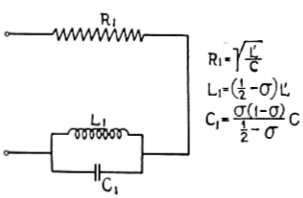

Fig. 12 — Impedance-Simulator for a Loaded Line Terminating at σ-Section, with σ in the Neighborhood of 0.2

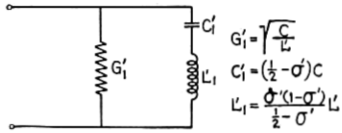

Fig. 13 — Admittance-Simulator for a Loaded Line Terminating at σ'-Load, with σ' in the Neighborhood of 0.2

istic impedance and the σ'-load characteristic admittance, respectively, of a non-dissipative loaded line, when σ and σ' are in the neighborhood of 0.2. The theoretical bases of these two networks and of their proportioning are outlined in Appendix B. (See also Patent No. 1124904 and No. 1437422, respectively.)

Figs. 14 and 15 show two networks which are considerably less simple than those of Figs. 12 and 13 but possess a substantially wider

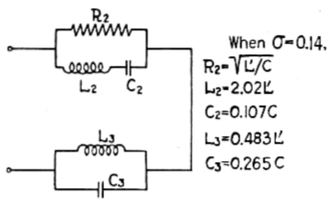

Fig. 14—Impedance-Simulator for a Loaded Line Terminating at σ-Section, with σ about 0.14

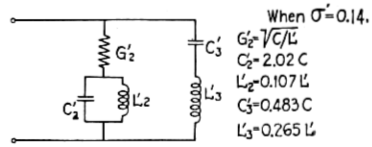

Fig. 15—Admittance-Simulator for a Loaded Line Terminating at σ'-Load, with σ' about 0.14.

frequency-range of simulation; for them the best value of σ and of σ' is about 0.14. The theoretical bases of these two networks are indicated below in the descriptions of the networks in Figs. 20 and 21, respectively. (See also Patent No. 1167693 and No. 1437422, respectively.)

Fig. 16 shows a network called a reactance-compensator, for a non-dissipative loaded line terminating at σ-section. When proportioned

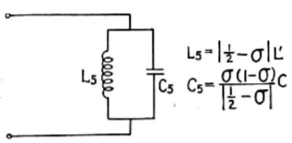

Fig. 16—Reactance-Compensator for a Loaded Line Terminating at σ-Section: Reactance-Simulator when $0<\sigma<1/2$ Reactance-Neutralizer when $1/2<\sigma<1$

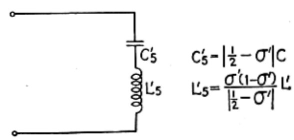

Fig. 17—Susceptance-Compensator for a Loaded Line Terminating at σ'-Load: Susceptance-Simulator when $0<\sigma'<1/2$ Susceptance-Neutralizer when $1/2<\sigma'<1$

in accordance with the design-formulas there given, this network possesses the following two-fold property with reference to the σ-section characteristic reactance of the loaded line: When σ has any fixed value between 0 and 1/2, the network exactly simulates the σ-section reactance, and exactly neutralizes the $(1-\sigma)$-section reactance; or, what is equivalent, when σ has any fixed value between 1/2 and 1, the network exactly neutralizes the σ-section reactance and exactly simulates the $(1-\sigma)$-section reactance.

Fig. 17 shows a network called a susceptance-compensator, for a non-dissipative loaded line terminating at σ'-load. When proportioned in accordance with the design-formulas there given, this network possesses the following two-fold property with reference to the σ'-load characteristic susceptance of the loaded line: When σ' has any fixed value between 0 and 1/2, the network exactly simulates the σ'-load susceptance, and exactly neutralizes the $(1-\sigma')$-load susceptance; or, what is equivalent, when σ' has any fixed value between 1/2 and 1, the network exactly neutralizes the σ'-load susceptance and exactly simulates the $(1-\sigma')$-load susceptance.

It may be noted that the resonant frequency f_r of the compensators in Figs. 16 and 17 is never less than the resonant frequency f_c of the loaded line; for when $\sigma=\sigma'$ the two types of compensators have the same value of f_r, and

$$f_r/f_c=1/2\sqrt{\sigma(1-\sigma)}.$$

This ratio has a minimum value of unity, when $\sigma=1/2$; and becomes infinite when $\sigma=0$ and when $\sigma=1$. It is equal to 1.25 when $\sigma=0.2$ and when $\sigma=0.8$.

The compensators in Figs. 16 and 17 are evidently inverse networks; the theoretical principles underlying them are outlined together in Appendix C. (See also Patent No. 1243066 and No. 1475997, respectively.)

With σ and σ' each in the neighborhood of 0.2 or of 0.8, the σ-section characteristic reactance and the σ'-load characteristic conductance of a non-dissipative loaded line are simulated pretty well by the constant resistance R_1 and the constant conductance G_1' of Figs. 12 and 13, respectively, as pointed out in Appendix B.

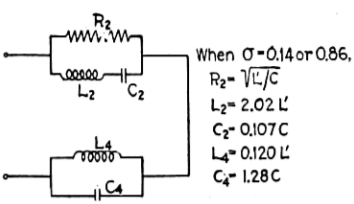

Fig. 18—Resistance-Simulator for a Loaded Line Terminating at σ-Section, with σ about 0.14 or about 0.86

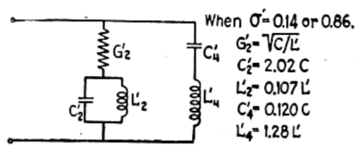

Fig. 19—Conductance-Simulator for a Loaded Line Terminating at σ'-load, with σ' about 0.14 or about 0.86

Simulation of the σ-section resistance and of the σ'-load conductance can be accomplished over a substantially wider frequency-range than in the foregoing paragraph, by means of the networks of Figs. 18 and 19, respectively; for them the best value of σ and of σ' is about 0.14.

These networks must not be confused with those of Figs. 14 and 15: they are like the latter in form but differ in the values of certain of their elements, as will be seen on close examination; they differ also in their functions, the networks of Figs. 14 and 15 simulating the σ-section impedance and the σ'-load admittance, respectively, whereas the networks of Figs. 18 and 19 simulate merely the resistance and the conductance components of these, respectively. In Fig. 18 the reactance of the L_4C_4-portion neutralizes that of the $R_2L_2C_2$-portion; and in Fig. 19 the susceptance of the $L_4'C_4'$-portion neutralizes that of the $G_2'C_2'L_2'$-portion. (See also Patent No. 1167693 and No. 1437422, respectively.)

By combining the resistance-simulator of Fig. 18 and the reactance-simulator of Fig. 16 there results the impedance-simulator of Fig. 20.

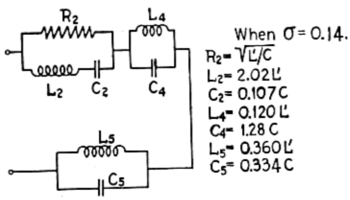

Fig. 20—Impedance-Simulator for a Loaded Line Terminating at σ-Section, with σ about 0.14. (This figure indicates the synthesis of the network in Fig. 14.)

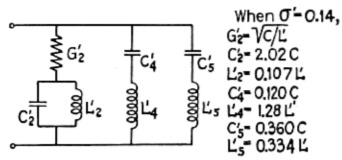

Fig. 21—Admittance-Simulator for a Loaded Line Terminating at σ'-Load, with σ' about 0.14. (This figure indicates the synthesis of the network in Fig. 15.)

But it is found that the L_4C_4-portion and the L_5C_5-portion can be combined, without appreciable sacrifice of simulative precision, into the single L_3C_3-portion of Fig. 14—whose synthesis is thereby indicated. (See also Patent No. 1167693.)

By combining the conductance-simulator of Fig. 19 and the susceptance-simulator of Fig. 17 there results the admittance-simulator of Fig. 21. But it is found that the $L_4'C_4'$-portion and the $L_5'C_5'$-portion can be combined, without appreciable sacrifice of simulative precision, into the single $L'_3C'_3$-portion of Fig. 15—whose synthesis is thereby indicated. (See also Patent No. 1437422.)

PART VI

Networks for Non-Dissipative Loaded Lines with Distributed Inductance

From the latter portion of Part III it will be recalled that the approximate effect of small distributed inductance is to alter slightly the nominal impedance and the critical frequency of the loaded line without much affecting the relative impedance when expressed as a function of the relative frequency, over the first transmitting band and the lower part of the succeeding attenuating band. Thus an approximate way of taking account of the effects of small distributed inductance is to deal with the constants L_0' and C_0 of the corresponding "principal simulative loaded line"; since this line has no distributed inductance it is seen that the networks described in Part V for loaded lines without distributed inductance are adequate for loaded lines with small distributed inductance; the design-formulas remain unchanged beyond substituting L_0' for L' and C_0 for C; however, the simulative precision of the networks is altered slightly.

A slightly better approximation may be secured by working not only with L_0' and C_0 but also with fictitious values of σ and σ', say σ_0 and σ_0', slightly different from those which would be best if there were no distributed inductance.

Owing to the presence of a certain amount of distributed inductance in all transmission lines (even in cables), simulation of the σ'-load impedance ($\sigma' > \sigma_b'$) by means of a fractional-load (σ_b') type of basic network built out to σ'-load is slightly more precise than simulation of the σ-section impedance ($\sigma = \sigma'$) by means of a fractional-section (σ_b) type of basic network built out to σ-section. This is evident from the latter portion of Part III of this paper.

(Regarding the effects of small distributed inductance in loaded lines, Patent No. 1167693 may be of some interest.)

PART VII

Networks for Dissipative Loaded Lines

A natural first-approximation network for simulating the impedance of a dissipative loaded line is the network for the corresponding non-dissipative loaded line, the excess impedance thus being neglected; in the case of a high grade loaded line this is a good approximation except at very low frequencies. Various forms and types of networks for non-dissipative loaded lines having the basic relative terminations were described in Parts V and VI; those networks ("basic networks") can be built-out readily to any relative terminations by means of simple non-dissipative building-out structures.

When the excess impedance of the loaded line is not negligible an excess-simulator is required. A first-approximation excess-simulator for a loaded line is the excess-simulator for the corresponding smooth line.[1] This is a good approximation over about the lower half or two-thirds of the transmitting band; but to be adequate in the upper part of the transmitting band it requires some modification in its proportioning or even in its form, according to several circumstances, such as the relative termination, the amount and distribution of the dissipation, and the ratio of the highest contemplated frequency to the critical frequency. The immediate neighborhood of the critical frequency is here disregarded, as having thus far been unimportant in practice; modification of the networks to extend their range of simulation right up to the critical frequency appears to present much greater difficulties.

PART VIII

Applications of the Simulating and the Compensating Networks

In this Part a considerable number of applications of the above-described networks will be outlined. (For some details and further applications, reference may be made to the patents cited in Part V—namely, Patent No. 1124904, No. 1167693, and No. 1437422, pertaining to the simulating networks; and No. 1243066 and No. 1475997 pertaining to the compensating networks.)

Applications of the Simulating Networks

Foremost of the uses of the simulating networks is their employment for balancing purposes in connection with 22-type repeaters, already spoken of in the Introduction.

Another application of a simulating network is for terminating an actual loaded line in the field or an artificial loaded line in the laboratory in such a way as to avoid reflection effects. For this purpose the proper terminating impedance is evidently one equal to the complementary characteristic impedance of the loaded line. Such a terminating impedance is often needed in the making of electrical tests or electrical measurements on a loaded line.

Furthermore, in making certain tests on apparatus normally associated with a loaded line, such line may be represented conveniently by the appropriate simulating network.

Applications of the Compensating Networks

The compensating networks have a wide variety of uses as neutralizing networks and also as simulating networks. These uses depend mainly on the fact that a compensating network when used as a neutralizer enables the impedance of a loaded line to simulate approximately the impedance of a smooth line and hence to simulate at least roughly a constant resistance, and when used as a simulator enables the impedance of a smooth line to simulate approximately the impedance of a loaded line.

Foremost of the uses of the compensating networks is their employment for properly connecting together a loaded line and a smooth line, to reduce reflection effects at the junction. This may be accomplished either by means of the reactance compensator (Fig. 16) or by means of the susceptance compensator (Fig. 17) by adopting a suitable relative termination for the loaded line in each method. In describing these two methods, it will be assumed at first that the loaded line and the smooth line are non-dissipative and have equal nominal impedances. In the first method of compensation the loaded line is terminated at σ-section with σ in the neighborhood of 0.8, where its curve of characteristic resistance is nearly flat; and a reactance-compensator (Fig. 16) is inserted in series between the two lines. This compensator, by neutralizing the reactance of the given loaded line, makes that line appear like a smooth line; while, by simulating the complementary characteristic reactance of the loaded line, it makes the smooth line appear complementary to the given loaded line. In the second method of compensation the loaded line is terminated at σ'-load with σ' in the neighborhood of 0.8, where its curve of characteristic conductance is nearly flat; and a susceptance-compensator (Fig. 17) is inserted in shunt between the two lines at their junction. This compensator, by neutralizing the susceptance of the given loaded line, makes that like appear like a smooth line; while, by simulating the characteristic susceptance of the complementary loaded line, it makes the smooth line appear complementary to the given loaded line.

When, as actually, the lines are dissipative, the compensator continues to make the loaded line appear approximately like a smooth line, and to make the smooth line appear approximately like a loaded line; but now, unless the lines happen to be about equally dissipative, there will exist at their junction an irregularity arising chiefly from inequality in their "excess-impedances." This irregularity can be largely prevented from occurring when the gage of either or both of the lines is at the disposal of the designer; when this is not the case and the irregularity is seriously large, resort may be had to special equalizers termed "excess-impedance equalizers."

When the nominal impedances of the two lines are unequal, adjustment in that respect can be made by means of a transformer of suitable ratio.

Some other uses for the compensators are as follows: (a) to properly connect a loaded. ine to a repeater system whose impedance is nearly constant resistance; (b) to connect a loaded line type of filter (low-pass filter) to an amplifying element whose impedance is nearly constant resistance; (c) to connect a loaded line to terminal apparatus whose impedance is nearly constant resistance; (d) to convert the impedance of a loaded line to that of the corresponding smooth line and thereby enable it to be simulated (or to be balanced) by a smooth-line type of simulating network; (e) to convert the impedance of a smooth line to that of a loaded line and thereby enable it to be simulated (or to be balanced) by a loaded-line type of simulating network; (f) to neutralize the characteristic reactance of an approximately non-dissipative loaded line, thereby enabling the resulting nearly pure resistance impedance to be closely simulated (or to be closely balanced) by the network (Fig. 18) simulating the characteristic resistance of the loaded line; or—though somewhat less closely—by a mere resistance element; (g) to neutralize the characteristic susceptance of an approximately non-dissipative loaded line, thereby enabling the resulting nearly pure conductance admittance to be closely simulated (or to be closely balanced) by the network (Fig. 19) simulating the characteristic conductance of the loaded line; or—though somewhat less closely—by a mere conductance element.

In applications (a), (b), (c) the irregularity at the junction can be still further reduced by the addition of an excess simulator for simulating the excess impedance of the loaded line.

APPENDIX A

The Transmitting and the Attenuating Bands of a Non-Dissipative Loaded Line with Distributed Inductance

This Appendix contains the derivations of the formulas in Part III pertaining to the disposition of the transmitting and the attenuating bands; and also several alternative formulas; it outlines six graphical methods for studying the bands; and it discusses, more comprehensively than in the body of the paper, the salient properties of the bands and the effects produced by varying certain of the parameters.

Disposition of the Transmitting and the Attenuating Bands

The propagation constant $\Gamma = A + iB$ of a non-dissipative loaded line (per periodic interval) can be expressed in terms of $\lambda = L/L'$ and the quantity D defined by equation (16). From Appendix D,

$$\cosh \Gamma = \cos 2D - \frac{D}{\lambda} \sin 2D, \tag{1-A}$$

$$\sinh^2 \Gamma = (\sin^2 2D)(D \tan D - \lambda)(D \cot D + \lambda)/\lambda^2 \tag{2-A}$$

$$= (\sin^2 2D)(D^2 - \lambda^2 - 2\lambda D \cot 2D)/\lambda^2 \tag{3-A}$$

$$= (-\sin^2 2D)(1 + 1/\lambda) Z'^2_{.5}. \tag{3.1-A}$$

Thus, for a non-dissipative loaded line, $\cosh \Gamma$ and $\sinh^2 \Gamma$ are both pure real.

When $\cosh \Gamma$ is known, A and B can be evaluated by means of the identity

$$\cosh \Gamma = \cosh (A + iB) = \cosh A \cos B + i \sinh A \sin B. \tag{4-A}$$

In particular, when $\cosh \Gamma$ is pure real—as for a non-dissipative loaded line—the values of A and B must evidently be such as to satisfy the pair of equations

$$\sinh A \sin B = 0, \quad (5\text{-A}) \qquad \cosh A \cos B = \cosh \Gamma; \quad (6\text{-A})$$

with, of course, the added restriction that A must be real and positive, and B real. Thence it is readily found that:

When $\cosh^2 \Gamma < 1$, that is, $\sinh^2 \Gamma < 0$,
then $A = 0$ and $B = \cos^{-1} \cosh \Gamma$; (7-A)

When $\cosh^2 \Gamma > 1$, that is, $\sinh^2 \Gamma > 0$,
then $A = \cosh^{-1} |\cosh \Gamma|$ and $B = q\pi$; (8-A)

$\cosh \Gamma$ being real, and q being an even or an odd integer according as $\cosh \Gamma$ is positive or negative, respectively.

Before continuing with the general case ($\lambda \neq 0$) it seems worth while to digress long enough to apply the preceding general formulas to the limiting case where $\lambda = 0$. For it, formula (1-A) reduces to

$$\cosh \Gamma = 1 - 2r^2, \tag{9-A}$$

where $r = f/f_c = D/D_c$, and f_c is given by (3). Application of (7-A) and (8-A) to (9-A) shows that:

When $0 < r < 1$, then $A = 0$ and $B = 2 \sin^{-1} r$; (10-A)

When $r > 1$, than $A = 2 \cosh^{-1} r$ and $B = q\pi$, (11-A)

where q is an odd integer.

For illustrative purposes, Fig. 22 gives graphs of A and B throughout the first transmitting band $(0<r<1)$ and part of the succeeding attenuating band, for a non-dissipative loaded line, with $\lambda=0$ and with $\lambda=0.12$. Of course, A is zero in the range $0<r<1$.

Returning now to the general case $(\lambda\neq0)$, we see that the transmitting bands $(A=0)$ are characterized by the inequality $\sinh^2\Gamma<0$,

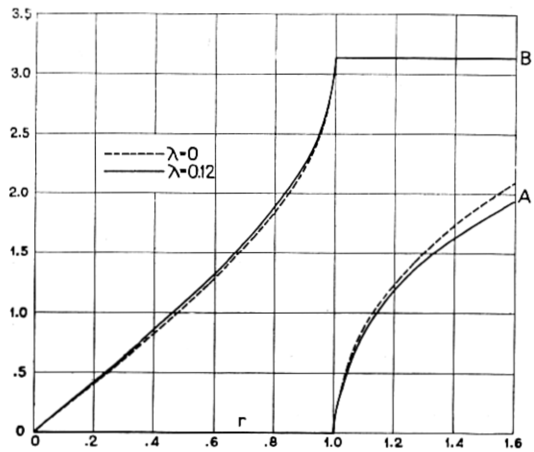

Fig. 22—Propagation Constant $\Gamma=A+iB$ in the First Transmitting Band $(0<r<1)$ and in Part of the Succeeding Attenuating Band, of a Non-Dissipative Loaded Line with $\lambda=0$ and with $\lambda=0.12$

and the attenuating bands $(A\neq0)$ by the inequality $\sinh^2\Gamma>0$; and hence the transition points between the two kinds of bands are characterized by the equation $\sinh^2\Gamma=0$.

We seek the transition values of D, that is, the values of D where $\sinh^2\Gamma=0$; and we seek the transmitting and the attenuating ranges of D, that is, the ranges of D where $\sinh^2\Gamma<0$ and $\sinh^2\Gamma>0$, respectively.

The transition values of D are perhaps most readily found from the equation for $\sinh^2\Gamma$ when written in the form (2-A). They are the zeros of the first three factors in the right-hand member of that equation. The zeros of the factor $\sin^2 2D$ are at $D=m\pi/2$, with $m=0, 1, 2, 3, \ldots$; thus they subdivide the D-scale into segments of width $\pi/2$ each, as represented by Fig. 6; and they have the values represented by (18). The zeros of the factors $D\tan D-\lambda$ and $D\cot D+\lambda$ are situated in the odd and even numbered segments, respectively, because, λ is positive; there is one and only one zero in each segment. Thus, if D_n denotes the zero of $\sinh^2\Gamma$ situated in the nth segment, then

$$(n-1)\frac{\pi}{2} < D_n < n\frac{\pi}{2}. \qquad (12\text{-A})$$

Either analytically or graphically it is readily seen that, when λ is small, D_n is only slightly greater than $(n-1)\pi/2$; it approaches that value as a limit when n approaches infinity, for all finite values of λ. The power series formula (21) for D_n is derived at a little later point in this Appendix.

Formulated analytically, with the arguments of the trigonometric functions reduced to the smallest positive values that preserve the values of the functions, the transition values of D are the values of $D_{n,n+1}$ and D_n satisfying the equations

$$\sin^2 2\left(D_{n,n+1} - n\frac{\pi}{2}\right) = 0, \qquad (13\text{-A})$$

$$D_n \tan\left(D_n - [n-1]\frac{\pi}{2}\right) = \lambda, \qquad (14\text{-A})$$

with $n=0, 1, 2, 3, \ldots$ in (13-A) and $n=1, 2, 3, \ldots$ in (14-A). Equation (13-A) is equivalent to $\sin^2 2D=0$. With n odd and with n even, (14-A) is equivalent respectively to $D\tan D - \lambda = 0$ and to $D\cot D + \lambda = 0$. An equivalent of (14-A) is obtainable from the second factor of (3-A). By (3.1-A), still another equivalent is $Z'_{.5}=0$; that is, the values of D_n are the zeros of the mid-load relative impedance $Z'_{.5}$, and hence of the mid-load impedance $K'_{.5}$.

With $(n-1)\pi/2$ denoted by d_n, equation (14-A) shows that

$$D_n - d_n < \lambda/d_n, \quad (n=2, 3, 4, \ldots) \qquad D_1 < \sqrt{\lambda}.$$

By inspection of (2-A) it can be readily verified that $\sinh^2\Gamma$ is negative when $D_{n-1,n} < D < D_n$ and positive when $D_n < D < D_{n,n+1}$; and hence that these two ranges of D are a transmitting band and an attenuating band, respectively, the corresponding compound band thus being the range $D_{n-1,n} < D < D_{n,n+1}$. In this connection it may be of some academic interest to note that, strictly speaking, $D=0$ is not a transition value of D between a transmitting and an attentuating band. For (2-A) shows that $\sinh^2\Gamma$ does not change sign when D passes through 0; on the contrary, $\sinh^2\Gamma$ is entirely unchanged when D is changed to $-D$. Thus, $D=0$ is a point of symmetry, but not a transition point.

The values of D_n, namely, the roots of (14-A), cannot be written down directly or expressed exactly. But they can be found to any desired degree of approximation by first developing the left side of (14-A) into a power series involving D_n; and then, by successive approximation or by undetermined coefficients, solving the resulting equation so as to express D_n as a power series in λ (that is, "reverting" the first series to obtain the second).

Digression on the Reversion of Power Series

Since there will be several occasions here for reverting a power series it seems worth while to digress sufficiently to furnish the requisite general formulas for the reversion of power series:[8]

Given $y=F(x)$ developed as a convergent power series in x,

$$y=x+a_2x^2+a_3x^3+a_4x^4+\ldots. \qquad (15\text{-A})$$

The coefficient of x has been assumed to be unity because the formulation of the reversion is much simplified thereby without any real sacrifice of generality; for, if the coefficient of x were a_1, the equation could be reduced immediately to the form (15-A), either by treating a_1x as the independent variable, or by dividing through by a_1 and then treating y/a_1 as the dependent variable.

The given equation (15-A) expresses y as a power series in x. It is required to revert this relation, that is, to express x as a power series in y. In the present work this was done originally by successive approximation, and was verified later by the method of undetermined coefficients. Evidently the first approximation to the solution of (15-A) is merely $x_1=y$, and thence the second approximation is $x_2=y-a_2x_1{}^2=y-a_2y^2$. But the higher approximations cannot be written down thus directly; indeed the labor of obtaining them increases rapidly. The work was carried through the sixth approximation, with the result:

$$\begin{aligned}x=&\,y+(-a_2)y^2+(2a_2^2-a_3)y^3+(-5a_2^3+5a_2a_3-a_4)y^4\\&+(14a_2^4-21a_2^2a_3+6a_2a_4+3a_3^2-a_5)y^5\\&+(-42a_2^5+84a_2^3a_3-28a_2^2a_4-28a_2a_3^2+7a_2a_5+7a_3a_4-a_6)y^6+\ldots \qquad (16\text{-A})\end{aligned}$$

[8] Cf., for instance, Bromwich, "Theory of Infinite Series"; Goursat-Hedrick, "Mathematical Analysis"; Wilson, "Advanced Calculus"; Chrystal, "Text Book of Algebra." But in none of these references is the reversion carried far enough; moreover, the formulas there obtained do not apply directly to a series containing only even powers—one of the cases in the present application. At considerable labor, by two independent methods, I remedied both of these lacks. Somewhat later I came upon a valuable article by C. E. Van Orstrand, "The Reversion of Power Series" (*Phil. Mag.*, March, 1910), where the reversion is carried to no less than thirteen terms, but is not directly applicable to series containing only even powers.

This was verified by the method of undetermined coefficients, consisting in assuming

$$x=y+b_2y^2+b_3y^3+b_4y^4+\ \ldots$$

and then substituting this expression for x into (15-A) to evaluate the b's by treating the resulting equation as an identity.

In the degenerate case where only even powers of x are present in (15-A) the formula (16-A) when applied directly does not correctly express the solution (for reasons appearing below). However, the given equation, containing only even powers of x, say

$$y=x^2+c_2x^4+c_3x^6+c_4x^8+\ \ldots, \tag{17-A}$$

can be correctly solved for (x^2) by direct application of (16-A), with $a_s=c_s$; and then the value of x can be expressed as a power series in y by extracting the square root of the power series representing (x^2). In that way the solution of (17-A) was found to be

$$\frac{x}{\sqrt{y}}=1+\left(-\frac{1}{2}c_2\right)y+\left(\frac{7}{8}c_2^2-\frac{1}{2}c_3\right)y^2+\left(-\frac{33}{16}c_2^3+\frac{9}{4}c_2c_3-\frac{1}{2}c_4\right)y^3$$
$$+\left(\frac{715}{128}c_2^4-\frac{143}{16}c_2^2c_3+\frac{11}{4}c_2c_4+\frac{11}{8}c_3^2-\frac{1}{2}c_5\right)y^4+\left(-\frac{4199}{256}c_2^5\right.$$
$$\left.+\frac{1105}{32}c_2^3c_3-\frac{195}{16}c_2^2c_4-\frac{195}{16}c_2c_3^2+\frac{13}{4}c_2c_5+\frac{13}{4}c_3c_4-\frac{1}{2}c_6\right)y^5+\ \ldots. \tag{18-A}$$

This result was verified by the method of undetermined coefficients, by writing x in the form

$$x=\sqrt{y}\,(1+e_1y+e_2y^2+e_3y^3+\ \ldots) \tag{18.1-A}$$

and then evaluating the e's by substituting (18.1-A) into (17-A). Still another method would be to extract the square root of (17-A) as the first step, thereby expressing $\sqrt{y}$ as a power series in x of the form (15-A); and then reverting by application of (16-A), thereby expressing x as a power series in $\sqrt{y}$ and thence of the form (18.1-A).

For use in this connection it may be noted that the square root of a power series having the form

$$y^2=1+h_1x+h_2x^2+h_3x^3+\ \ldots$$

will be of the form

$$y=1+k_1x+k_2x^2+k_3x^3+\ \ldots.$$

The k's can be evaluated by identifying the first equation with the square of the second; their values are found to be

$$k_1=\tfrac{1}{2}h_1,\qquad k_2=\tfrac{1}{2}h_2-\tfrac{1}{2}k_1^2,\qquad k_3=\tfrac{1}{2}h_3-k_1k_2,$$
$$k_4=\tfrac{1}{2}h_4-\tfrac{1}{2}k_2^2-k_1k_3,\qquad k_5=\tfrac{1}{2}h_5-k_1k_4-k_2k_3,$$
$$k_6=\tfrac{1}{2}h_6-\tfrac{1}{2}k_3^2-k_1k_5-k_2k_4.$$

Derivations of Formulas for the Transition Points

The above general formulas for the reversion of power series will now be applied in the derivation of the formulas (21) and (22) for D_n and D_1, in the body of the paper; and also in the derivation of certain other formulas, not included there.

To outline the derivation of the formula (21) for D_n, denote $(n-1)\pi/2$ by d_n and $D_n - d_n$ by τ_n, so that (14-A) becomes

$$(d_n+\tau_n)\tan\tau_n=\lambda. \tag{19-A}$$

Now replace $\tan\tau_n$ by its known power series expression, and divide both sides of the resulting equation by d_n; thus (19-A) becomes

$$\frac{\lambda}{d_n}=\tau_n+\frac{1}{d_n}\tau_n^2+\frac{1}{3}\tau_n^3+\frac{1}{3d_n}\tau_n^4+\frac{2}{15}\tau_n^5+\frac{2}{15d_n}\tau_n^6+\dots . \tag{20-A}$$

This is of the form (15-A), and hence can be reverted by direct application of (16-A); the result is (21).

An alternative formula for D_n can be obtained by starting from Gregory's series,

$$v=\tan v-\frac{\tan^3 v}{3}+\frac{\tan^5 v}{5}-\frac{\tan^7 v}{7}+\dots . \tag{20.1-A}$$

Application of this to (19-A) enables the left side of that equation to be expressed as a power series in $\tan\tau_n$; and when the resulting equation is reverted by means of (16-A) and then τ_n replaced by D_n-d_n the result is

$$\tan(D_n-d_n)=\frac{\lambda}{d_n}-\frac{1}{d_n}\left(\frac{\lambda}{d_n}\right)^2+\frac{2}{d_n^2}\left(\frac{\lambda}{d_n}\right)^3-\left(\frac{5}{d_n^3}-\frac{1}{3d_n}\right)\left(\frac{\lambda}{d_n}\right)^4$$
$$+\left(\frac{14}{d_n^4}-\frac{2}{d_n^2}\right)\left(\frac{\lambda}{d_n}\right)^5-\left(\frac{42}{d_n^5}-\frac{28}{3d_n^3}+\frac{1}{5d_n}\right)\left(\frac{\lambda}{d_n}\right)^6+\dots . \tag{20.2-A}$$

It has already been noted that (21) is not valid for $n=1$ and hence does not include the formula (22) for D_1. To obtain this formula for D_1, start with the equation

$$D_1\tan D_1=\lambda, \tag{21-A}$$

obtained by setting $n=1$ in (14-A). Then replace $\tan D_1$ by its known power series expansion, thus obtaining the equation

$$\lambda=D_1^2+\frac{1}{3}D_1^4+\frac{2}{15}D_1^6+\frac{17}{315}D_1^8+\frac{62}{2835}D_1^{10}+\frac{1382}{155925}D_1^{12}+\dots . \tag{22-A}$$

This is of the form (17-A), and hence can be reverted by direct application of (18-A); the result is (22).

It may be noted that (22-A), when regarded as a power series in (D_1^2), is of the form (15-A) and hence that (D_1^2) can be expressed as a power series in λ by direct application of (16-A); the result is[7]

$$D_1^2=\lambda-\frac{\lambda^2}{3}+\frac{4\lambda^3}{45}-\frac{16\lambda^4}{945}+\frac{16\lambda^5}{14175}+\frac{64\lambda^6}{93555}\cdots\cdot \qquad (23\text{-A})$$

In certain applications this formula for D_1^2 is more useful than formula (22) for D_1; though the two are ultimately equivalent. A formula for p^2 is obtainable by dividing both sides of (23-A) by λ; for $p^2=D_1^2/\lambda$, by (16).

An alternative formula for D_1 can be obtained by starting from Gregory's series (20.1-A). Application of this to (21-A) enables the left side of that equation to be expressed as a power series in $\tan D_1$; and when the resulting equation is reverted by means of (18-A) the result is[7]

$$\tan D_1=\sqrt{\lambda}\left(1+\frac{\lambda}{6}-\frac{\lambda^2}{360}-\frac{11\lambda^3}{5040}+\frac{1357\lambda^4}{1814400}\cdots\right). \qquad (23.1\text{-A})$$

Series that are even more convergent than (21) and (22), though much less simple, can be obtained by expanding the original function in the neighborhood of a value of the variable known to be an approximate solution of the equation to be solved, and then reverting the resulting series. To formulate the procedure analytically and generally, let u denote the variable, and $\psi(u)$ the function; and let the equation to be solved for u be

$$\psi(u)=q. \qquad (24\text{-A})$$

Then, if U is an approximate solution of this equation, application of Taylor's theorem leads to the following implicit equation for $u-U$:

$$\frac{q-\psi(U)}{\psi'(U)}=(u-U)+\frac{(u-U)^2}{2!}\frac{\psi''(U)}{\psi'(U)}+\frac{(u-U)^3}{3!}\frac{\psi'''(U)}{\psi'(U)}+\cdots. \qquad (25\text{-A})$$

The left side of this is known. The right side is a power series in $u-U$, with U known; the better the approximation represented by U, the more rapidly convergent is the series. This equation (25-A) in $u-U$ is of the form (15-A), with

$$y=\frac{q-\psi(U)}{\psi'(U)}, \qquad x=u-U, \qquad a_s=\frac{\psi^{(s)}(U)}{s!\,\psi'(U)}; \qquad (26\text{-A})$$

and thence (25-A) can be reverted by application of (16-A). so that $u-U$ will be expressed as a power series in $[q-\psi(U)]/\psi'(U)$.

To apply the above general method in order to obtain for D_n a series more convergent than (21), return to (19-A) and note that when λ is small a first approximation for τ_n is $\tau_n=\lambda/d_n$. Then apply (16-A), with y, x, and a_s having the values expressed by (26-A); and $q=\lambda$, $u=\tau_n$, $U=\lambda/d_n$, and $\psi(u)=(u+d_n)\tan u$. The formulas for the first few successive derivatives of $\psi(u)$ will be needed, of course.

Similarly, to obtain for D_1 a series more convergent than (22), return to (21-A) and note that when λ is small a first approximation for D_1 is $D_1=\sqrt{\lambda}$. Then apply (16-A), with y, x, and a_s having the values expressed by (26-A); and $q=\lambda$, $u=D_1$, $U=\sqrt{\lambda}$, and $\psi(u)=u \tan u$.

Graphical Methods for Locating the Tansition Points

The positions of the transition points D_n $(n=1, 2, 3, \ldots)$ on the D-scale can be determined also graphically, in several different ways corresponding to several different ways of writing the function $(D \tan D-\lambda)(D \cot D+\lambda)$ whose zeros are the values of D_n. To formulate such graphical methods concisely, let E denote any function of the variable D, so that, geometrically, E is the ordinate corresponding to the abscissa D. Six of the various possible graphical methods are then briefly but completely indicated by the following respective statements that the points D_n are the abscissas of the points of intersection of:

1. The horizontal straight line $E=\lambda$ with the curves $E=D \tan D$; the horizontal straight line $E=-\lambda$ with the curves $E=D \cot D$.
2. The straight line $E=D$ with the curves $E=\lambda \cot D$; the straight line $E=-D$ with the curves $E=\lambda \tan D$.
3. The straight line $E=D/\lambda$ with the cotangent curves $E=\cot D$; the straight line $E=-D/\lambda$ with the tangent curves $E=\tan D$.
4. The hyperbola $E=\lambda/D$ with the tangent curves $E=\tan D$; the hyperbola $E=-\lambda/D$ with the cotangent curves $E=\cot D$.
5. The parabola $E=D^2/\lambda-\lambda$ with the curves $E=2D \cot 2D$.
6. The curve $E=D/2\lambda-\lambda/2D$, compounded of the straight line $E=D/2\lambda$ and the hyperbola $E=-\lambda/2D$, with the cotangent curves $E=\cot 2D$.

In methods 1, 2, 3, 4. the first set of intersections is situated in the odd-numbered segments, the second set in the even numbered segments; each segment of width $\pi/2$.

Besides being susceptible of quantitative service, these graphical methods are useful for qualitative purposes. For instance, they show clearly that: one and only one transition value of D lies within each segment of width $\pi/2$; $\sinh^2\Gamma<0$ when $D_{n-1,n}<D<D_n$, and $\sinh^2\Gamma>0$ when $D_n<D<D_{n,n+1}$; the zeros of $\lambda-D\tan D$ and of $\lambda+D\cot D$ are situated in the odd and even numbered segments, respectively; with increasing D, the transmitting bands continually decrease in width and the attenuating bands continually increase in width, the change taking place rapidly at first and then more and more slowly; the mid-point relative impedances are pure imaginary throughout every attenuating band and pure real throughout every transmitting band, and, they have the ranges stated in the third and fourth paragraphs following equation (26.1). The graphical methods are useful also for showing the nature of the effects produced by varying the parameter λ.

Discussion of the Disposition of the Bands

The rest of this Appendix will be devoted to a discussion of the most salient properties of the compound bands and their constituent transmitting and attenuating bands.

The ratio of transmitting band width to compound band width continually decreases with increasing D and becomes zero when D becomes infinite; that is, the transmitting bands vanish and the compound bands become pure attenuating bands. These facts can be seen graphically, or analytically from equation (14-A).

The ratio of transmitting band width to compound band width continually increases with increasing λ; this ratio ranging from zero when λ is zero to unity when λ is infinite. These facts can be seen graphically, or from equation (14-A). When λ approaches zero the f-width of each compound band approaches infinity; the f-width of each transmitting band approaches zero, except for the first transmitting band, whose width approaches a value equal to $f'_1=f'_c$—for equation (14-A) shows that $D_n(D_n-D_{n-1,n})/\lambda$ approaches unity, and hence that $f_n(f_n-f_{n-1,n})$ approaches $1/\pi^2L'C=f'^2_1$, whence $f_n-f_{n-1.n}$ approaches zero for $n\neq1$ and approaches f'_1 for $n=1$.

The effects of varying the parameter λ will now be outlined briefly, in the next two paragraphs, for the cases respectively of $L'C$ fixed and LC fixed. The conclusions reached depend partly on the equation $D=\frac{1}{2}\omega\sqrt{LC}=\frac{1}{2}\omega\sqrt{\lambda L'C}$ defining D; partly on the fact already deduced that the D-width of each compound band is an absolute constant $(\pi/2)$; and partly on equation (14-A).

When $L'C$ is fixed, increasing λ reduces all of the transition frequencies. The transition frequencies bounding the compound bands, and hence the widths of the compound bands, decrease in direct proportion to increase of $\sqrt{\lambda}$. The internal transition frequencies, however, do not decrease so rapidly; for the ratio of transmitting band width to attenuating band width increases with increasing λ. When λ approaches infinity each compound band approaches a width of zero, but the ratio of transmitting band width to compound band width approaches unity; so that when λ becomes infinite there are within any finite frequency range an infinite number of compound bands which are pure transmitting bands. On the other hand, when λ approaches zero the compound bands approach infinite width and hence move out toward infinity, except that the left end-point of the first band is fixed at $f=0$. When λ has become zero the first compound band has expanded to an infinite width; and its critical value f_1 of f has become equal to the limiting value $f'_1=1/\pi\sqrt{L'C}$ —as can be seen from (14-A) by putting $n=1$ and then applying the relation $D/\sqrt{\lambda}=\frac{1}{2}\omega\sqrt{L'C}$.

When LC is fixed the f-widths and locations of the compound bands are independent of λ, but the widths of the constituent attenuating and transmitting bands depend on λ; that is, the boundary points $f_{n-1,n}$ and $f_{n,n+1}$ of the nth compound band are independent of λ, but the internal transition point f_n depends on λ. With increasing λ the attenuating bands become continually narrower, and vanish when λ becomes infinite, the transmitting bands thereby coalescing to form a pure transmitting band extending from zero to infinity. With decreasing λ the transmitting bands become continually narrower, and vanish when λ becomes zero, the attenuating bands thereby coalescing to form a pure attenuating band extending from zero to infinity.

APPENDIX B

Theoretical Bases of the Simulating Networks in Figs. 12 and 13

The Impedance-Simulator in Fig. 12

This network takes advantage of the fact, depicted in Fig. 5, that the graph of the σ-section characteristic resistance of a loaded line, for values of σ in the neighborhood of 0.2, is nearly flat over most of the transmitting band and hence can be approximately simulated by a mere constant resistance chosen approximately equal to the nominal impedance $\sqrt{L'/C}$. This is the basis for the R_1-portion of the network in Fig. 12. The basis for the L_1C_1-portion is the fact (proved in Appendix C) that, in the transmitting band, the σ-section characteristic reactance can be exactly simulated (for any fixed value of σ between 0 and 1/2) by the network in Fig. 16.

The Admittance-Simulator in Fig. 13

This network takes advantage of the fact, depicted in Fig. 5, that the graph of the σ'-load characteristic conductance of a loaded line, for values of σ' in the neighborhood of 0.2, is nearly flat over most of the transmitting band and hence can be approximately simulated by a mere constant conductance chosen approximately equal to the nominal admittance $\sqrt{C/L'}$. This is the basis for the G_1'-portion of the network in Fig. 13. The basis for the $L'_1C'_1$-portion is the fact (proved in Appendix C) that, in the transmitting band, the σ'-load characteristic susceptance can be exactly simulated (for any fixed value of σ' between 0 and 1/2) by the network in Fig. 17.

APPENDIX C

Derivations of the Design-Formulas for the Compensating Networks in Figs. 16 and 17

The Reactance-Compensator in Fig. 16

For any values of C_5 and L_5 the reactance T of this network is

$$T=\frac{\omega L_5}{1-\omega^2 L_5 C_5}.$$

By equation (4) the characteristic reactance N of the loaded line within its transmitting band is

$$N=\frac{k(1-2\sigma)\omega/\omega_c}{1-4\sigma(1-\sigma)\omega^2/\omega_c^2}.$$

Comparison of these two equations shows that T and N are of the same functional form in ω; and that the conditions for T to be identically equal to $\pm N$ are

$$L_5=\pm k(1-2\sigma)/\omega_c, \qquad L_5C_5=4\sigma(1-\sigma)/\omega_c^2,$$

whence

$$C_5=\pm 4\sigma(1-\sigma)/(1-2\sigma)k\omega_c,$$

the upper and the lower sign of $\pm$ corresponding to the use of the compensator as a reactance-simulator and a reactance-neutralizer, respectively. These values of L_5 and C_5 are equivalent to those appearing in Fig. 16, because $k=\sqrt{L'/C}$ and $\omega_c=2\pi f_c=2/\sqrt{L'C}$.

For positive values of L_5 the equation for L_5 shows that $\sigma \lesseqgtr 1/2$, corresponding to $\pm$; and then the equation for C_5 shows that $\sigma \genfrac{}{}{0pt}{}{>0}{<1}$, corresponding to $\pm$. Hence $0<\sigma<1/2$ for $T=+N$, and $1/2<\sigma<1$ for $T=-N$.

The Susceptance-Compensator in Fig. 17

For any values of C_5' and L_5' the susceptance S' of this network is

$$S' = \frac{\omega C_5'}{1-\omega^2 L_5' C_5'}.$$

By equation (5) the characteristic susceptance Q' of the loaded line within its transmitting band is

$$Q' = \frac{h(1-2\sigma')\omega/\omega_c}{1-4\sigma'(1-\sigma')\omega^2/\omega_c^2}.$$

Thus S' and Q' are of the same functional form in ω; and the conditions for S' to be identically equal to $\pm Q'$ are that

$$C_5' = \pm h(1-2\sigma')/\omega_c,$$

$$L_5' = \pm 4\sigma'(1-\sigma')/(1-2\sigma')h\omega_c,$$

the upper and the lower sign of $\pm$ corresponding to the use of the compensator as a susceptance-simulator and a susceptance-neutralizer respectively. These values of C_5' and L_5' are equivalent to those appearing in Fig. 17, because $h=\sqrt{C/L'}$ and $\omega_c=2/\sqrt{L'C}$.

The equations for C_5' and L_5' show that $0<\sigma'<1/2$ for $S'=+Q'$, and that $1/2<\sigma'<1$ for $S'=-Q'$.

APPENDIX D

General Formulas for the Characteristic Impedances and the Propagation Constant of Loaded Lines

For reference purposes this Appendix gives the general formulas for the mid-section ($\sigma=0.5$) and mid-load ($\sigma'=0.5$) characteristic impedances $K_{.5}$ and $K'_{.5}$ and the propagation constant Γ of a periodically loaded line (of the series type).

The symbols have the following meanings: d denotes the impedance of each load. g and γ pertain to the line before loading; g denotes the characteristic impedance, and γ denotes the propagation constant of a segment whose length is equal to the distance between adjacent loads after the line is loaded.

The formulas for the mid-section and mid-load characteristic impedances $K_{.5}$ and $K'_{.5}$ are [9]

$$K_{.5}=g\sqrt{\frac{1+\frac{d}{2g}\coth\frac{\gamma}{2}}{1+\frac{d}{2g}\tanh\frac{\gamma}{2}}}, \tag{1-D}$$

$$K'_{.5}=g\sqrt{\left(1+\frac{d}{2g}\coth\frac{\gamma}{2}\right)\left(1+\frac{d}{2g}\tanh\frac{\gamma}{2}\right)} \tag{2-D}$$

$$=g\sqrt{1+\frac{1}{4}\left(\frac{d}{g}\right)^2+\frac{d}{g}\coth\gamma}. \tag{3-D}$$

Several mutually equivalent formulas for the propagation constant Γ (per periodic interval) are:

$$\cosh\Gamma=\cosh\gamma+\frac{d}{2g}\sinh\gamma, \tag{4-D}$$

$$\sinh\Gamma=\frac{K'_{.5}}{g}\sinh\gamma, \tag{5-D}$$

$$\tanh\tfrac{1}{2}\Gamma=\frac{K_{.5}}{g}\tanh\tfrac{1}{2}\gamma. \tag{6-D}$$

The sending-end impedance J of any smooth line, of characteristic impedance g_1 and total propagation constant γ_1, whose distant end is closed through any impedance J_1, has the formula

$$J=g_1\frac{J_1/g_1+\tanh\gamma_1}{1+(J_1/g_1)\tanh\gamma_1}. \tag{7-D}$$

This enables the formula for the σ-section characteristic impedance K_σ of a loaded line to be established by starting with the formula (1-D) for the mid-section characteristic impedance $K_{.5}$.

[9] Formulas (2-D) and (3-D) for $K'.5$ and formula (4-D) for cosh Γ are given by G. A. Campbell in his paper on loaded lines (*Phil. Mag.*, March, 1903) cited in footnote 2.