The Building-up of Sinusoidal Currents in Long Periodically Loaded Lines

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IMPORTANT information regarding the excellence of a signal transmission system is deducible from a knowledge of the mode in which sinusoidal currents "build-up" in response to suddenly applied sinusoidal electromotive forces, since on the character and duration of the "building-up" process depend the speed and fidelity with which the circuit transmits rapid signal fluctuations.[1] The object of this note is to disclose and discuss general formulas and curves which describe the building-up phenomena, as a function of the line characteristics and the frequency of the applied e.m.f., in the extremely important case of long periodically loaded lines. The formulas in question are approximate but give accurate engineering information and are applicable to all types of periodic loading under two restrictions: (1) the line must be fairly long, that is, comprise at least 100 loading sections, and (2) it must be approximately equalized, as regards *absolute* steady-state values of the received current, in the neighborhood of the applied frequency. Fortunately these conditions are usually satisfied in practice in those cases where the building-up phenomena are of practical engineering importance. Furthermore, the formulas to be discussed supply a means for the accurate and rapid comparison of different types of loading in correctly engineered lines.

The building-up process may be precisely defined and formulated as follows: Suppose that an e.m.f., $E \cos \omega t$, is suddenly applied, at reference time $t=o$, to a network of transfer impedance

$$Z(i\omega)=|Z(i\omega)|\cdot \exp[iB(\omega)]. \tag{1}$$

The resultant current, $I(t)$, may be written as

$$I(t)=\frac{1}{2}\frac{E}{|Z(i\omega)|}\{(1+\rho)\cos[\omega t-B(\omega)]+\sigma \sin[\omega t-B(\omega)]\}, \tag{2}$$

$$=\frac{1}{2}\sqrt{(1+\rho)^2+\sigma^2}\,\frac{E}{|Z(i\omega)|}\cos[\omega t-B(\omega)+\theta], \tag{3}$$

where

$$\theta=\tan^{-1}(\sigma/\rho).$$

Evidently the functions ρ and σ must be -1 and o respectively for negative values of t, and approach the limits $+1$ and o as $t \doteq \infty$.

[1] For published discussions of the "building-up" of sinusoidal currents in loaded lines, see Clark, *Journ. A.I.E.E.*, Jan., 1923, Kupfmuller, Telegraphen u. Fernsprech-Technik, Nov., 1923; Carson, Trans. A.I.E.E., 1919.

In an engineering study of the building-up process we are principally concerned with the *envelope* of the oscillations, which, by (3), is proportional to

$$\frac{1}{2}\sqrt{(1+\rho)^2+\sigma^2}.$$

The problem is therefore to determine the functions ρ and σ and to examine the effect of the applied frequency $\omega/2\pi$ and the characteristics of the circuit on their rate of building-up and mode of approach to their ultimate steady values.

Two propositions will now be stated which cover the building-up process in the practically important cases. Since the line is assumed to be approximately equalized, as regards the absolute value of the received current in the neighborhood of the applied frequency $\omega/2\pi$, the building-up process depends only on the total phase angle $B(\omega)$. The successive derivatives of the phase angle with respect to ω will be denoted by $B'(\omega)$, $B''(\omega)$, $B'''(\omega)$, $B^{iv}(\omega)$, etc.

Case I. $B''(\omega)\neq O$ and $\sqrt{B''(\omega)/2!}$ large compared with $\sqrt[3]{B'''(\omega)/3!}$.

The envelope of the oscillations in response to an e.m.f. $E\cos\omega t$ *applied at time* $t=o$, *is proportional to*

$$\frac{1}{2}\sqrt{(1+\rho)^2+\sigma^2} \tag{4}$$

where

$$\rho=C(x^2)+S(x^2), \tag{5}$$

$$\sigma=C(x^2)-S(x^2), \tag{6}$$

$$x=\frac{t-B'(\omega)}{\sqrt{2B''(\omega)}}=\frac{t'}{\sqrt{2B''(\omega)}}, \tag{7}$$

and $C(x)$, $S(x)$ *are Fressel's Integrals to argument* x.

The envelope therefore reaches 50 per cent. *of its ultimate steady value at time* $t=\tau=B'(\omega)$ *and its rate of building-up is inversely proportional to* $\sqrt{B''(\omega)}$.

The curve of Fig. 1 is a plot of the envelope function $\frac{1}{2}\sqrt{(1+\rho)^2+\sigma^2}$ to the argument x and is therefore applicable to all types of loading and lengths of line, subject to the restrictions noted above.

Case II. $B''(\omega)=O$: $B'''(\omega)\neq O$ and $\sqrt[3]{B'''(\omega)/3!}$ large compared with $\sqrt[4]{B^{iv}(\omega)/4!}$.

The envelope of the oscillations is proportional to

$$\frac{1}{3}+\frac{1}{2}\int_0^y A(\mu)d\mu \tag{8}$$

where $A(\mu)$ is Airey's Integral [2] *and*

$$y=\left(\frac{2}{\pi}\right)^{2/3}\frac{t-B'(\omega)}{\sqrt[3]{B'''(\omega)/3!}} \tag{9}$$

$$=t'\sqrt[3]{\frac{24}{\pi^2 B'''(\omega)}}. \tag{10}$$

At time $t=B'(\omega)$ the envelope N has reached 1/3 of its ultimate steady value and its rate of building-up is inversely proportional to $\sqrt[3]{B'''(\omega)}$.

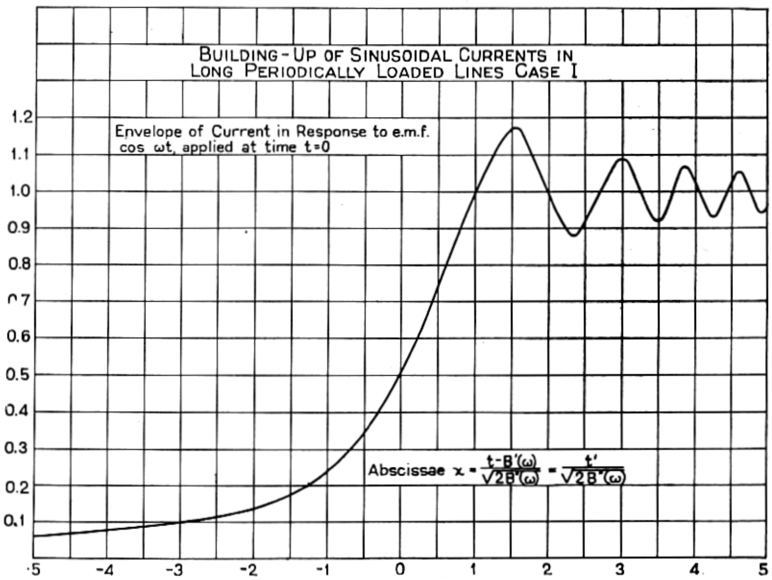

Fig. 1

The curve of Fig. 2 is a plot of the envelope function $\frac{1}{3}+\frac{1}{2}\int_0^y A(\mu)d\mu$ to the argument y and is therefore of general applicability under the circumstances where case II obtains.

The practical value of the foregoing propositions resides in the fact that they enable us to calculate two important criteria of the transmission properties of the line: (1) the variation with respect to frequency of the time interval τ required for the current to build-up to

[2] See Watson, Theory of Bessel Functions, p. 190.

its proximate steady-state value: and (2) its rate of building-up at time $t=\tau$.

As will be seen in connection with the proof given below, the formulas of the foregoing propositions are approximate. Provided, however, that the lines to which they are applied are long and provided that the

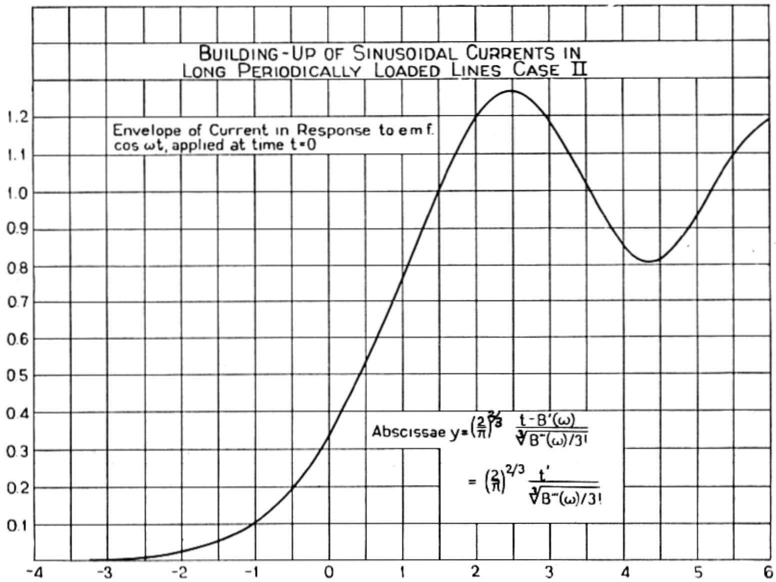

Fig. 2

applied frequency is such that the restrictions underlying either I or II are satisfied, their accuracy is quite sufficient for engineering purposes, such as the design of loading systems, or a study of the comparative merit of different types of loading.

Before proceeding with the mathematical proof, the formulas will be applied to the interesting and important case of an ideal non-dissipative periodically coil-loaded line of N sections in length and cut-off frequency $\omega_c/2\pi$. For this line it is easy to show that [3]

$$B'(\omega)=\frac{2N}{\omega_c}\frac{1}{\sqrt{1-w^2}}=N\beta'(\omega),$$

$$B''(\omega)=\frac{2N}{\omega_c^2}\frac{w}{(1-w^2)^{3/2}}=N\beta''(\omega),$$

$$B'''(\omega)=\frac{2N}{\omega_c^3}\frac{1+2w^2}{(1-w^2)^{5/2}}=N\beta'''(\omega),$$

[3] The following formulas assume that the line is closer to its characteristic impedance. $\beta(\omega)$ is then the phase angle per loading section of the line.

where w denotes ω/ω_c. It follows that

$$t' = t - \frac{2N}{\omega_c}\frac{1}{\sqrt{1-w^2}}$$

and that the oscillations build-up to the proximate steady-state in a time interval [4] $\tau = 2N/\omega_c\sqrt{1-w^2}$ after the voltage is applied.

Case I, it will be observed, does not hold for $\omega = o$ since $B''(o) = o$. The condition that Case I shall apply is that

$$\sqrt[6]{18N}\cdot(1-w^2)^{1/12}\frac{\sqrt{w}}{(1+2w^2)^{1/3}}$$

shall be substantially greater than unity. Hence Case I applies only when $1/\sqrt[3]{18N} < w < 1$. This however, includes the important part of the signalling frequency range in properly designed lines, provided that they are long ($N \geq 100$).

In the range of applied frequencies, therefore, corresponding to $1/\sqrt[3]{18N} < w < 1$, the current reaches 50 per cent. of its ultimate steady value in a time interval $\frac{2N}{\omega_c}\frac{1}{\sqrt{1-w^2}}$ after the voltage is applied and its rate of building-up at this time is proportional to

$$\frac{\omega_c}{\sqrt{4N}}\frac{(1-w^2)^{3/4}}{\sqrt{w}}.$$

For the non-dissipative coil-loaded line $B''(\omega) = o$ when $\omega = o$, and Case II applies. Consequently when $\omega = o$, the oscillations reach 1/3 of the ultimate steady value at time $t = 2N/\omega_c$, at which time their rate of building-up is proportional to

$$\omega_c\sqrt[3]{\frac{12}{\pi^2 N}}.$$

The foregoing formulas have been shown to be in good agreement with experimental results, and have been applied to the design of loaded lines in the Bell System.

Mathematical Discussion

The functions ρ and σ of equations (2) and (3) can be formulated as the Fourier integrals

[4] It will be noted that this formula breaks down at $\omega = \omega_c$ or $w = 1$.

$$\rho=\frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\sin t\lambda.\ [P_\omega(\lambda)+P_\omega(-\lambda)]$$
$$-\frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\cos t\lambda\,[Q_\omega(\lambda)-Q_\omega(-\lambda),] \tag{11}$$

$$\sigma=\frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\sin t\lambda\,[Q_\omega(\lambda)+Q_\omega(-\lambda)]$$
$$+\frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\cos t\lambda\,[P_\omega(\lambda)-P_\omega(-\lambda)], \tag{12}$$

where

$$P_\omega(\lambda)=\frac{A(\omega+\lambda)}{A(\omega)}\cos[B(\omega+\lambda)-B(\omega)], \tag{13}$$

$$Q_\omega(\lambda)=\frac{A(\omega+\lambda)}{A(\omega)}\sin[B(\omega+\lambda)-B(\omega)], \tag{14}$$

and $A(\omega)=1/|Z(i\omega)|$.

These formulas are directly deducible from the fact that the applied e.m.f., defined as zero for negative values of t and $E\cos\omega t$ for $t\geq o$, can itself be expressed as

$$\frac{E}{2}\cos\omega t\left[1+\frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\sin t\lambda\right].$$

In the practically important case where $B'(\omega)$ is finite, it is of advantage to introduce the transformation $t'=t-B'(\omega)$, and to write:

$$\rho=\frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\sin t'\lambda\,[U_\omega(\lambda)+U_\omega(-\lambda)]$$
$$-\frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\cos t'\lambda\,[V_\omega(\lambda)-V_\omega(-\lambda)], \tag{15}$$

$$\sigma=\frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\sin t'\lambda\,[V_\omega(\lambda)+V_\omega(-\lambda)]$$
$$+\frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\cos t'\lambda\,[U_\omega(\lambda)-U_\omega(-\lambda)], \tag{16}$$

where

$$U_\omega(\lambda)=\frac{A(\omega+\lambda)}{A(\omega)}\cos[B(\omega+\lambda)-B(\omega)-\lambda B'(\omega)], \tag{17}$$

$$V_\omega(\lambda)=\frac{A(\omega+\lambda)}{A(\omega)}\sin[B(\omega+\lambda)-B(\omega)-\lambda B'(\omega)]. \tag{18}$$

The foregoing formulas for ρ and σ are exact subject to certain restrictions on the impedance function $Z(i\omega)$ which are satisfied in the case of periodically loaded lines. Their useful application to the problem under consideration depends, however, on the following approximations.

First it will be assumed that the line is approximately equalized, as regards *absolute* value of steady state received currents in the neighborhood of the impressed frequency $\omega/2\pi$. By virtue of this assumption, which is more or less closely realized in practice, the ratio $A(\omega+\lambda)/A(\omega)$ may be replaced by unity in the integrals (15) and (16), and in equations (17) and (18). It is further assumed that the function

$$B(\omega+\lambda)-B(\omega)-\lambda B'(\omega)$$

admits of power series expansion, so that

$$U_\omega(\lambda)=\cos\left[(h_2\lambda)^2+(h_3\lambda)^3+\ .\ .\right], \tag{19}$$

$$V_\omega(\lambda)=\sin\left[(h_2\lambda)^2+(h_3\lambda)^3+\ .\ .\right], \tag{20}$$

where

$$h_n{}^n=\frac{1}{n!}\frac{d^n}{d\omega^n}\,B(\omega)=\frac{1}{n!}\,B^{(n)}(\omega).$$

By virtue of the foregoing ρ and σ are given by

$$\rho\doteq\frac{2}{\pi}\int_0^\infty\frac{d\lambda}{\lambda}\sin\left[t'\lambda-(h_3\lambda)^3-(h_5\lambda)^5\ .\ .\right]\cdot\cos\left[(h_2\lambda)^2+(h_4\lambda)^4+\ .\ .\right], \tag{21}$$

$$\sigma\doteq\frac{2}{\pi}\int_0^\infty\frac{d\lambda}{\lambda}\sin\left[t'\lambda-(h_3\lambda)^3-(h_5\lambda)^5\ .\ .\right]\cdot\sin\left[(h_2\lambda)^2+(h_4\lambda)^4+\ .\ .\right]. \tag{22}$$

Now if the line is very long the integrals (11) and (12) may be replaced by the approximations

$$\rho\doteq\frac{2}{\pi}\int_0^\infty\frac{d\lambda}{\lambda}\sin\left[t'\lambda-(h_3\lambda)^3\right]\cdot\cos\,(h_2\lambda)^2, \tag{23}$$

$$\sigma\doteq\frac{2}{\pi}\int_0^\infty\frac{d\lambda}{\lambda}\sin\left[t'\lambda-(h_3\lambda)^3\right]\cdot\sin\,(h_2\lambda)^2. \tag{24}$$

In other words we retain only the leading terms in the expansion of the function

$$B(\omega+\lambda)-B(\omega)-\lambda B'(\omega).$$

The justification for this procedure depends on arguments similar to those underlying the Principle of Stationary Phase (see Watson, Theory of Bessel Functions, p. 229). Furthermore the upper limit

∞ may be retained without serious error, even when the line cuts off at a frequency $\omega_c/2\pi$, provided the line is sufficiently long, and the frequency $\omega/2\pi$ not too close to the cut-off frequency $\omega_c/2\pi$.

The formal solutions of the infinite integrals (23) and (24) can be written down by virtue of the following known relations:

$$\frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\sin t'\lambda\cdot\cos(h_2\lambda)^2 = C(x^2)+S(x^2), \tag{25}$$

$$\frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\sin t'\lambda\cdot\sin(h_2\lambda)^2 = C(x^2)-S(x^2), \tag{26}$$

where $C(x^2)$ and $S(x^2)$ are Fresnel's Integrals to argument x^2, and $x=t'/2h_2$.

$$\frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\sin\left[t'\lambda-(h_3\lambda)^3\right] = -\frac{1}{3}+\int_0^y A(y)dy \tag{27}$$

where $A(y)$ denotes Airey's Integral (see Watson, Theory of Bessel Functions) and $y=(2/\pi)^{2/3}(t'/h_3)$.

By aid of the preceding.

$$\rho = \left\{1+\frac{\mu^3}{1!}\frac{d^3}{dx^3}+\frac{\mu^6}{2!}\frac{d^6}{dx^6}+\,..+..\right\}\cdot\left\{C(x)^2+S(x^2)\right\}, \tag{28}$$

$$\sigma = \left\{1+\frac{\mu^3}{1!}\frac{d^3}{dx^3}+\frac{\mu^6}{2!}\frac{d^6}{dx^6}+\,..+..\right\}\cdot\left\{C(x^2)-S(x^2)\right\}, \tag{29}$$

where $\mu=(h_3/2h_2)$.

This is the appropriate form of solution when (h_3/h_2) is less than unity.

On the other hand when (h_3/h_2) is greater than unity, the appropriate form of solution is

$$\rho = \left\{1-\frac{\nu^4}{2!}\frac{d^4}{dy^4}+\frac{\nu^8}{4!}\frac{d^8}{dy^8}+\,..\right\}\cdot\left\{-\frac{1}{3}+\int_0^y A(y)dy\right\}, \tag{30}$$

$$\sigma = \left\{\frac{\nu^2}{1!}\frac{d^2}{dy^2}-\frac{\nu^6}{3!}\frac{d^6}{dy^6}+\,..\right\}\cdot\left\{-\frac{1}{3}+\int_0^y A(y)dy\right\}, \tag{31}$$

where $\nu=\left(\frac{2}{\pi}\right)^{\frac{2}{3}}\left(\frac{h_3}{h_2}\right)$.

While no thorough investigation has been made, it appears probable that for all values of the ratio h_3/h_2, either (28), (29) or (30), (31) will be convergent. However, in practice it is sufficient for present purposes to deal only with the cases where h_3/h_2 is either small or large compared with unity, and to use the following approximations:

(1) (h_3/h_2) *small compared with unity.*

$$\rho = C(x^2) + S(x^2),$$
$$\sigma = C(x^2) - S(x^2),$$
$$x = (t'/2h_2)^2,$$
$$t' = t - B'(\omega).$$

(2) (h_3/h_2) *large compared with unity.*

$$\rho = -\frac{1}{3} + \int_0^y A(y)dy,$$
$$\sigma = O,$$
$$y = (2/\pi)^{2/3}(t'/h_3).$$