Transmission Characteristics of Electric Wave-Filters

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SYNOPSIS: The transmission loss characteristic of a transmitting network as a function of frequency is an index of the network's steady-state selective properties. Methods of calculation heretofore employed to determine these characteristics for composite wave-filters are long and tedious. This paper gives a method for such determinations which greatly simplifies and shortens the calculations by the introduction of a system of charts. Account is taken of the effects of both wave-filter dissipation and terminal conditions. The method is based upon formulae containing new parameters, called "image parameters," which are the natural ones to use with composite wave-filters.

A detailed illustration of the use of this chart calculation method is given and the transmission losses so obtained are found to agree, except for differences which in practice are negligible, with those obtained by long direct computation.

In the Appendix are derived two sets of corresponding formulae which are applicable to a linear transducer of the most general type, namely, an active, dissymmetrical one; the one set contains image parameters and the other set recurrent parameters. An impedance relation is found to exist between the four open-circuit and short-circuit impedances of a linear transducer even in the most general case. Reduction of these formulae to the more usual case of a passive linear transducer is also made, those containing the image parameters being especially applicable to the case of composite wave-filters.

I. INTRODUCTION

ELECTRIC wave-filter characteristics and systematic methods of deriving them have been considered in previous numbers of this Journal.[1] This paper deals with a simple and rapid method of calculating the steady-state transmission losses of wave-filter networks over both the transmitting and attenuating frequency bands, including the effects of dissipation and wave-filter terminal conditions. Such transmission loss determinations are essential in showing the selective characteristics of these networks and serve as important guides in meeting given design requirements.

General formulae for any dissymmetrical linear transducer are derived in terms of new parameters, called image parameters. One of the formulae is fundamental to the solution of the present problem and is particularly well adapted to calculations in composite wave-filter structures. These parameters of such a composite structure, being readily obtainable from those of its parts, are the natural parameters to use in this case. The formula possesses, among others,

[1] Physical Theory of the Electric Wave-Filter, G. A. Campbell, B. S. T. J., Nov., 1922; Theory and Design of Uniform and Composite Electric Wave-Filters, O. J. Zobel, B. S. T. J., Jan., 1923; Transient Oscillations in Electric Wave-Filters, J. R. Carson and O. J. Zobel, B. S. T. J., July, 1923.

the advantage over other formulae and calculation methods of requiring for every alteration in a composite wave-filter only a partial recalculation rather than a more or less complete one. In addition, by its use much of the otherwise necessary calculation can be eliminated through the means of graphical representation.

The main object of this paper is to present this chart calculation method of determining composite wave-filter transmission losses, giving its theory, the necessary charts, and an application of its use.

Structure of Wave-Filter Networks

The ladder type of recurrent network having physical series and shunt impedances z_1 and z_2, respectively, as shown in Fig. 1, is the

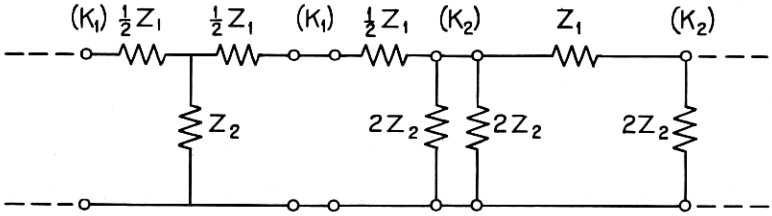

Fig. 1—Ladder Type Recurrent Network

one most frequently employed for wave-filters. Also any passive transducer having two pairs of terminals can theoretically be reduced to the form of the ladder type. Hence, in what follows the ladder type terminology will be used understanding, however, that other structural types may also be included; for example, such as are derivable from the ladder type by the substitution of an equivalent transformer with mutual impedance for T or Π connected inductances, or the lattice type. The figure illustrates, from left to right, one mid-series section, one mid-half section (a dissymmetrical half section terminated at mid-series and mid-shunt points), and one mid-shunt section all connected so as to give a uniform structure.[2] The characteristic impedances of the ladder type at mid-series and at mid-shunt points are K_1 and K_2, respectively.

The majority of wave-filter networks are not uniform throughout their length but have a composite structure designed as given in the paper (B. S. T. J., Jan., 1923) already mentioned. That is, the interior or *mid-part* of a composite wave-filter consists of mid-series,

[2] The same network may also be considered as made up in other ways; for example, two mid-series and one mid-half sections, one mid-half and two mid-shunt sections, or five mid-half sections.

mid-shunt, and mid-half sections, usually dissimilar, so connected serially and of such types that at any junction the terminations of the two adjacent types correspond to an equivalent image impedance. The use of dissimilar sections gives a resultant selective characteristic different from that possible with a uniform type. At the terminals of the network there need not be complete full or half sections; this

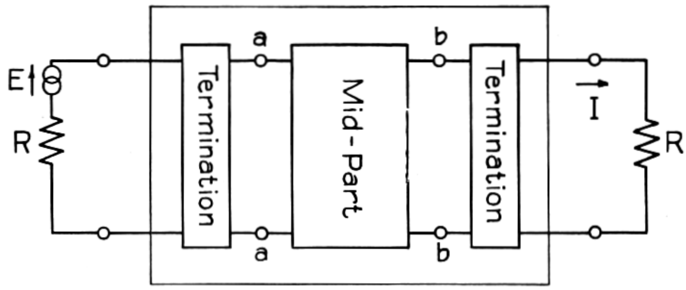

Fig. 2—General Composite Wave-Filter Network

is represented in Fig. 2 by the wave-filter parts external to the ***mid-part*** which latter is included between terminals a and b.

The terminations of wave-filter networks specifically considered in detail here include all terminations which have been found to be practical. In any particular class of wave-filter they are all closely related to the "constant k" wave-filter ($z_{1k}z_{2k}=k^2=$constant) of that class and are of the following four types,[3] being designated by their characteristic impedances in corresponding ladder type structures.

[3] It is assumed that the reader is familiar with the terms and notation used in the paper, B.S.T.J., Jan., 1923.

If z_{1k} and z_{2k} are the series and shunt impedances of the "constant k" wave-filter, the corresponding series and shunt impedances of the mid-series "constant k" equivalent M-type are expressible as

$$z_{11}=mz_{1k},$$

and

$$z_{21}=\frac{1-m^2}{4m}z_{1k}+\frac{1}{m}z_{2k};$$

and of the mid-shunt "constant k" equivalent M-type

$$z_{12}=\frac{1}{\dfrac{1}{mz_{1k}}+\dfrac{1}{\dfrac{4m}{1-m^2}z_{2k}}},$$

and

$$z_{22}=\frac{1}{m}z_{2k}.$$

Here the condition $0<m\leqq 1$ is sufficient for a physical structure in all cases.

1, mid-shunt of a mid-series "constant k" equivalent M-type, ($K_{21}(m)$);
2, mid-series of a mid-shunt "constant k" equivalent M-type, ($K_{12}(m)$);
3, x-shunt of the "constant k" wave-filter, (K_{x2}); and
4, x-series of the "constant k" wave-filter, (K_{x1}).

The terminations in $K_{21}(m)$ and $K_{12}(m)$ are employed, as stated in a previous paper,[4] when it is desirable to obtain certain selective characteristics and to minimize reflection losses at the important frequencies to be transmitted, a minimum for the latter occurring where $m=.6$ approximately. The x-shunt "constant k" termination, designated by the characteristic impedance K_{x2}, is a "constant k" type termination in a shunt element, whose admittance is x times (x from 0 to 1) that of a full shunt "constant k" admittance, $\frac{1}{z_{2k}}$; that is, a shunt element whose impedance is $\frac{z_{2k}}{x}$. Similarly, the x-series "constant k" type termination corresponding to the characteristic impedance K_{x1} ends in a series element of impedance xz_{1k}. In the usual case where two or more wave-filter networks having different transmitting bands are associated together, either termination 1 or 2 is suitable for the unconnected terminals, while terminations 3 and 4 are adapted to the terminals connected in series or in parallel, respectively. For two complementary wave-filters, thus connected, minimum reflection losses occur at their junction with a transmission line if $x=.8$ approximately. A relation between this case and termination $K_{12}(m)$ and $K_{21}(m)$ has previously been pointed out, namely, that the series or parallel connected wave-filters have a combined impedance in the transmitting band of either wave-filter approximately like that of $K_{12}(m)$ or $K_{21}(m)$, respectively.

Where the termination is x-shunt or x-series we shall consider that the *mid-part* of the wave-filter begins at the mid-shunt or the mid-series point, respectively, irrespective of whether x is greater or less than .5. Also the *mid-part* need not here necessarily begin in the "constant k" type, but in any wave-filter having an equivalent characteristic impedance.

Transmission Loss

In the design of a wave-filter network the magnitude of k for the corresponding "constant k" wave-filter has been taken equal to the

[4] B. S. T. J., Jan., 1923, page 18, gives a diagram for the non-dissipative case of $R/K_{21}(m)$ and $K_{12}(m)/R$ in the transmitting band.

mean resistance, R, of the line with which the network is to be associated. If the network is closed at each end by a resistance of magnitude R, as in Fig. 2, we have not only a circuit arrangement which approximates more or less closely actual operating conditions,[5] but also a simple test circuit in which to determine the transmission loss of the network over the desired frequency range.

The transmission loss of a wave-filter network, defined with reference to Fig. 2, is the natural logarithm, with negative sign, of the ratio of the absolute value of the current transmitted from a source of resistance R to a receiving resistance R when the latter are connected through the network, to that transmitted when they are connected directly. Let E represent the electromotive force of the source, I the current transmitted to R through the network, and $E/2R$ that transmitted by direct connection. Then the *transmission loss L*, thus defined, is

$$L = -\log_e \left| \frac{I}{E/2R} \right|, \tag{1}$$

and

$$e^{-L} = |2RI/E|. \tag{2}$$

The unit in which L is expressed, the *attenuation unit*,[6] is the natural unit to use here and from the above relations it is seen that one attenuation unit of transmission loss corresponds to an absolute value of current ratio of $1/e$. The method of determining the transmission loss under various possible conditions will be presented in the next part of this paper.

II. Theory of Chart Calculation Method

The principles given here are basic and apply to composite wave-filters having any terminations. However, in all practical cases, as previously stated, the terminations belong to the four types: 1, mid-shunt M-type; 2, mid-series M-type; 3, x-shunt "constant k;" and 4, x-series "constant k," all related to the "constant k" wave-filter.

[5] It should be clearly borne in mind that the unique selective properties of a wave-filter of freely transmitting currents in continuous frequency bands and of attenuating others are those for the wave-filter terminated in its characteristic impedance. It is practical to have approximately such a termination in the transmitting band only, as when connecting the wave-filter to a transmission line, in which case the general properties still persist. *Correct termination* rather than *number of sections* is what brings out these properties although the degree of selectivity is naturally increased by the addition of sections.

[6] A synonym sometimes used is the *Napier.* One attenuation unit is equivalent to 9.174 "800-cycle miles of standard cable," and to 8.686 TU. The TU (transmission unit) is that unit which designates a power ratio of $10^{.1}$, and the number of TU is ten times the common logarithm of the power ratio.

These four cases will be developed in detail and equivalence relations for certain sets of terminal combinations shown.

Fundamental Formula

The formula which is general and fundamental to what follows is the one giving the current received through a passive transducer in terms of the sending electromotive force, the terminal impedances,

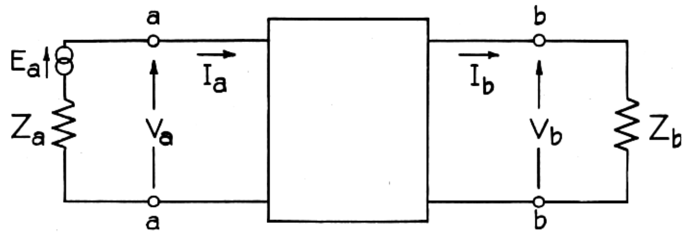

Fig. 3—General Linear Transducer

and the transfer constant [7] and image impedances of the transducer. Referred to Fig. 3 the received current is

$$I_b = \frac{2E_a \sqrt{W_a W_b}\, e^{-T}}{(W_a+Z_a)(W_b+Z_b)(1-r_a r_b e^{-2T})}, \tag{3}$$

where

E_a = sending electromotive force,

Z_a, Z_b = sending and receiving impedances,

$T = D + iS$ = transfer constant of the transducer,

D, S = diminution constant and angular constant, defined as the real and imaginary parts of the transfer constant,

W_a, W_b = image impedances of the transducer at terminals a and b,

r_a, r_b = current reflection coefficients at terminals a and b,

$$r_a = \frac{W_a - Z_a}{W_a + Z_a},$$

and

$$r_b = \frac{W_b - Z_b}{W_b + Z_b}.$$

[7] The terms *transfer constant*, T and *image impedances*, W_a and W_b, as applied to a dissymmetrical passive transducer, are defined in the Appendix. These three parameters are to be distinguished from another set, the propagation constant, Γ, and characteristic impedances, K_a and K_b. In a symmetrical structure $T=\Gamma$ and $W_a = W_b = K_a = K_b$.

Another form obtained by suitable transformation is

$$I_b = \frac{E_a \sqrt{W_a W_b}}{(W_a W_b + Z_a Z_b) \sinh T + (W_a Z_b + W_b Z_a) \cosh T}. \tag{4}$$

Formula (3), derived in the Appendix with several general transducer formulae and relations, is especially useful when applied to composite wave-filter networks, since, as we shall see, it contains the natural parameters for such structures. Upon comparing Fig. 2, which represents such a general network, with Fig. 3 we find that the two can be made to correspond exactly if the mid-part of the wave-filter, between terminals *a* and *b* in Fig. 2, is considered to be the transducer of Fig. 3, and if the wave-filter terminations combined with the resistances R are considered to be the terminal impedances Z_a and Z_b of Fig. 3. The relation between the electromotive force, E, applied in R and that, E_a, acting through Z_a depends upon the particular wave-filter termination at terminals *a*. Similarly, the relation between the currents, I and I_b, transmitted to R and Z_b, respectively, depends upon the termination at terminals *b*.

As already stated, the mid-part of the composite wave-filter consists in general of mid-series, mid-shunt, and mid-half sections, properly combined as to their impedance relations at the junction points. *The method of combination employed in a composite wave-filter consists in connecting two sections whose image impedances at their junction are equal.* (An analogy which might be given is the matching of dominoes in a line by the corresponding ends, numbers referring to image impedances.)

Let us assume for the moment that the mid-part, as thus made up, is terminated by impedances respectively equal to its image impedances. There is then an "image condition" for the impedances measured in the two directions not only at each of these terminal points but also at each junction point throughout the network; and in this case each section transmits under the "image condition" of its terminating impedances. As a result we obviously obtain the following properties for the mid-part.

1. *The transfer constant of the mid-part of a composite wave-filter, consisting of mid-series, mid-shunt, and mid-half sections, is the sum of the transfer constants of all the individual sections.*

2. *The image impedances of the mid-part of a composite wave-filter are the external image impedances of the two end sections.*

In addition we have the following important relations between the transfer constant and image impedances of a single section, and the propagation constant and mid-point characteristic impedances of the corresponding ladder network.

3. *The transfer constant of a symmetrical mid-series or mid-shunt section is equal to the propagation constant of the corresponding ladder type; that of a dissymmetrical mid-half section (having mid-series and mid-shunt terminations) is equal to one-half the above propagation constant.*

4. *The image impedance of a mid-point section at a mid-series or mid-shunt point termination is equal to the mid-series,* K_1, *or mid-shunt,* K_2, *characteristic impedance, respectively, of the corresponding ladder network.*

Formula (3) is for the present purpose superior to the well known formula for transmitted current (derived for comparison in the Appendix) which contains the transducer recurrent parameters in the form of its propagation constant, Γ, and characteristic impedances, K_a and K_b. The reason for this is that in a dissymmetrical composite wave-filter where K_a differs from K_b, the usual case, no simple relations exist between these latter parameters of the transducer and the corresponding parameters of the individual sections comprising the network. In the special case of symmetrical networks, however, the latter formula becomes identical with (3) which follows from what has already been said.

Another method of obtaining the transmitted current, which may be termed the "section-by-section elimination method," consists in calculating by the aid of the Kirchhoff laws the current ratios and total impedances from section to section back through the entire network beginning at the receiving impedance. From the standpoint of time economy certain objections may be raised to the possible use here of this general long hand method of calculation. The method carries with it the determination of the phase as well as the amplitude of the transmitted current; but since the amplitude only is required in the transmission loss formula, this method does more than is necessary. Again, an alteration in the composite network structure requires a more or less complete recalculation when this method is employed, whereas by the application of (3) it will be found that this is not necessary. However, this method is useful where irregularities exist in the network, or where the particular method of design which had been followed in obtaining the composite structure cannot readily be found, but its impedance elements and R are known.

General Form of Transmission Loss Formula

Formulae (2) and (3) corresponding to Figs. 2 and 3 may be combined. If (3) is written in the general form

$$2RI/E = F_t\, F_a\, F_b\, F_r, \tag{5}$$

we obtain with (2)

$$e^{-L} = |\,2RI/E\,| = e^{-(L_t+L_a+L_b+L_r)}, \tag{6}$$

where the four factors comprising the current ratio $2RI/E$ are

$F_t = e^{-T} =$ the transfer factor between terminals a and b;

$F_a = \dfrac{2\sqrt{W_a R}\, E_a}{(W_a+Z_a)\, E} =$ the terminal factor at terminals a;

$F_b = \dfrac{2\sqrt{W_b R}\, I}{(W_b+Z_b)\, I_b} =$ the terminal factor at terminals b;

$F_r = \dfrac{1}{1-r_a r_b e^{-2T}} =$ the interaction factor due to repeated reflections

at terminals a and b where the current reflection coefficients are

$$r_a = \frac{W_a - Z_a}{W_a + Z_a} \text{ and } r_b = \frac{W_b - Z_b}{W_b + Z_b};$$

and the transmission losses corresponding to the absolute values of these factors are called, respectively,

$L_t =$ the transfer loss;

$L_a,\ L_b =$ the terminal losses at terminals a and b;

and $L_r =$ the interaction loss.

The total transmission loss is the sum of these four losses, thus,

$$L = L_t + L_a + L_b + L_r. \tag{7}$$

The relative importance of the three types of losses, transfer, terminal, and interaction, is usually in the order given. Hence, as a first approximation the transmission loss of a composite wave-filter is given by the transfer loss, L_t, but the error due to the omission of the other losses is often considerable. A second approximation is obtained by including the terminal losses, L_a and L_b, and for many purposes this is sufficiently accurate. The final step for accuracy is the further addition of the interaction loss, L_r, whose effect on the total transmission loss is usually appreciable in the transmitting band of a wave-filter near the critical frequencies.

The three types of losses will now be considered separately and in detail.

1. *Transfer Losses*

The transfer loss, L_t, is by (6) equal to D, the diminution constant, which is the real part of the transfer constant, T, of the wave-filter mid-part taken between mid-points.

We have previously established the following:

> (1) *T is the sum of the transfer constants of all the individual sections, i.e., $T=\Sigma T_j$; and* (2) *the transfer constant of a mid-series or mid-shunt section is equal to the propagation constant, $\Gamma = A+iB$ per full section, of the corresponding ladder type; that of a mid-half section is $\Gamma/2$.*

Hence, to get the transfer loss we need to know only the attenuation constant, A, of each full mid-section, the half or whole of which forms a part of the composite wave-filter structure. However, since the interaction factor which is to be discussed later requires a knowledge of the phase constant, B, as well, we shall consider both parts of the propagation constant at this point.

Propagation Constant of Ladder Type Network. The relation between the propagation constant $\Gamma = A+iB$, and the series and shunt impedances, z_1 and z_2, respectively, of the ladder type in Fig. 1 is known to be

$$\cosh \Gamma = 1 + \tfrac{1}{2}\frac{z_1}{z_2}. \tag{8}$$

This applies as well to any recurrent structure if z_1 and z_2 correspond to the analytically equivalent ladder type.

Let us introduce two variables U and V by making the substitution

$$\frac{z_1}{4z_2} = U + iV. \tag{9}$$

The reason for this choice is that this ratio appears frequently in impedance formulae. Then in non-dissipative wave-filters, where $V=O$, the transmitting bands include all frequencies at which U satisfies the relation

$$-1 \leqq U \leqq O. \tag{10}$$

By (8) and (9)

$$\cosh(A+iB)=\cosh A\cos B+i\sinh A\sin B=1+2U+i2V, \quad (11)$$

whence

$$\cosh A\cos B=1+2U,$$

and $$\qquad\qquad (12)$$

$$\sinh A\sin B=2V.$$

The solution of this pair of simultaneous equations leads to separate relations for A and B,

$$\left(\frac{1+2U}{\cosh A}\right)^2+\left(\frac{2V}{\sinh A}\right)^2=1, \quad (13)$$

and

$$\left(\frac{1+2U}{\cos B}\right)^2-\left(\frac{2V}{\sin B}\right)^2=1. \quad (14)$$

As is well known from (13) equal attenuation constant loci are represented in the U, V plane by confocal ellipses with foci at $U=-1$, $V=O$ and $U=O$, $V=O$, thus having symmetry about the U-axis. *The locus for $A=O$, the limiting case, is a straight line between the foci and it corresponds to the transmitting band in a non-dissipative wave-filter.* Similarly from (14) equal phase constant loci are represented by confocal hyperbolas which have the same foci as above and are orthogonal to the equal attenuation constant ellipses. It will be assumed that the phase constant, B, lies between $-\pi$ and $+\pi$, which amounts to neglecting multiples of 2π. Then from (12) B has the same sign as V, so that loci in the upper half of the plane correspond to a positive phase constant while those in the lower half correspond to a negative one.

It is possible, however, to represent all this in just the upper half of the plane using coordinates U and $|V|$. Put

$$V=c\,|V|, \quad (15)$$

where $c=\pm 1$, the sign being that of V. The attenuation constant is independent of the sign of V, i.e., of c. But for the phase constant we get from (12)

$$\sin cB=\frac{2|V|}{\sinh A}, \quad (16)$$

and

$$O\leqq cB\leqq +\pi.$$

Thus, as here considered, *the product cB, where $c=\pm 1$ has the sign of V, is always positive with a value less than or equal to π.*

Explicit formulae for A and B from (13) and (14) are

$$A=\sinh^{-1}\sqrt{2\left[\left|\sqrt{(U+U^2+V^2)^2+V^2}\right|+(U+U^2+V^2)\right]}, \tag{17}$$

and

$$cB=\sin^{-1}\sqrt{2\left[\left|\sqrt{(U+U^2+V^2)^2+V^2}\right|-(U+U^2+V^2)\right]}. \tag{18}$$

The above formulae are general and applicable to any ladder type structure or its equivalent.

In the case of wave-filters certain approximate formulae are often useful. At frequencies in the attenuating bands away from the critical frequencies and the frequencies of maximum attenuation, and *wherever* V^2 *is negligible compared with* $(U+U^2)>O$,

$$A=\sinh^{-1} 2\sqrt{U+U^2},$$

and (19)

$$cB=O \text{ or } \pi.$$

At the critical frequencies and the frequencies of maximum attenuation, *where* $(U+U^2)$ *is negligible compared with* V^2,

$$A=\cosh^{-1}(\sqrt{1+V^2}+|V|),$$

and (20)

$$cB=\cos^{-1}\pm(\sqrt{1+V^2}-|V|).$$

In the latter the positive sign applies to a critical frequency at which $U=O$, and the negative sign to one at which $U=-1$.

U and V for "Constant k" and M-type Wave-Filters. Since the wave-filter structures under consideration have "constant k" or derived M-type terminations, the U and V variables corresponding to these wave-filters will always be required. Hence, formulae for the variables are given here, limiting them to the four lowest wave-filter classes generally used.

Resistance in an inductance coil of inductance, L_1, is taken into account by expressing the total coil impedance as

$$(d+i)\ L_1 2\pi f,$$

where d, the "coil dissipation constant," is the ratio of coil resistance to coil reactance. The value of d is ordinarily between $d=.004$ and $d=.04$, and it does not vary rapidly with frequency. Similarly, dissipation in a condenser of capacity C_1 can be included by expressing the total condenser admittance as $(d'+i)\ C_1 2\pi f$, but since d' is usually negligible in practice it will here be omitted.

The formulae derived from (9) are based upon those given in this

Journal, Jan., 1923, pages 39 to 41, and contain the critical frequencies and frequencies of maximum attenuation. Subscripts $_k$ and $_m$ will be used to denote the "constant k" and M-type U and V variables. The "constant k" formulae for the four classes follow.

Low Pass.

$$U_k = -\left(\frac{f}{f_2}\right)^2,$$

and (21)

$$V_k = d\left(\frac{f}{f_2}\right)^2.$$

High Pass.

$$U_k = -\left(\frac{f_1}{f}\right)^2 / (1+d^2),$$

and (22)

$$V_k = -d\left(\frac{f_1}{f}\right)^2 / (1+d^2).$$

Low-and-High Pass.

$$U_k = -\frac{(f_1-f_0)^2}{f_0 f_1} \frac{\left[\frac{f_0 f_1}{f^2} - (1+d^2)(2 - \frac{f^2}{f_0 f_1})\right]}{\left[\frac{f_0 f_1}{f^2} + (1+d^2)\frac{f^2}{f_0 f_1} - 2\right]^2},$$

and (23)

$$V_k = d\,\frac{(f_1-f_0)^2}{f_0 f_1} \frac{\left[\frac{f_0 f_1}{f^2} - (1+d^2)\frac{f^2}{f_0 f_1}\right]}{\left[\frac{f_0 f_1}{f^2} + (1+d^2)\frac{f^2}{f_0 f_1} - 2\right]^2}.$$

Band Pass.

$$U_k = -\frac{f_1 f_2}{(f_2-f_1)^2}\left[\frac{f_1 f_2}{(1+d^2)f^2} + \frac{f^2}{f_1 f_2} - 2\right],$$

and (24)

$$V_k = -d\,\frac{f_1 f_2}{(f_2-f_1)^2}\left[\frac{f_1 f_2}{(1+d^2)f^2} - \frac{f^2}{f_1 f_2}\right].$$

At the mid-frequency, $\sqrt{f_1 f_2}$, the point of confluency of two bands in the transmitting band of this wave-filter, we obtain approximately from (19), when d is small,

$$A = 2d\,\frac{\sqrt{f_1 f_2}}{f_2 - f_1},$$

and (25)

$$B = O.$$

The derived M-type variables of any class are given directly in terms of the "constant k" variables of that class and the parameter m by the general relations

$$U_m=\frac{m^2[U_k+(1-m^2)(U_k^2+V_k^2)]}{[1+(1-m^2)U_k]^2+(1-m^2)^2V_k^2},$$

and (26)

$$V_m=\frac{m^2V_k}{[1+(1-m^2)U_k]^2+(1-m^2)^2V_k^2}.$$

This assumes that the M-type has the same grade of coils and condensers as its "constant k" prototype. The parameter m has a different formula determining its value for each class, the general relation being (neglecting dissipation)

$$m=\sqrt{1+\left(\frac{1}{U_k}\right)_{f_\infty}}, \tag{27}$$

where f_∞ is a frequency of maximum attenuation of the M-type. The particular relations for the above four classes follow.

Low Pass

$$m=\sqrt{1-\frac{f_2^2}{f_{2\infty}^2}}. \tag{28}$$

High Pass

$$m=\sqrt{1-\frac{f_{1\infty}^2}{f_1^2}}. \tag{29}$$

Low-and-High Pass

$$m=\frac{\sqrt{\left(1-\frac{f_0^2}{f_{1\infty}^2}\right)\left(1-\frac{f_{1\infty}^2}{f_1^2}\right)}}{1-\frac{f_0}{f_1}}. \tag{30}$$

Band Pass

$$m=\frac{\sqrt{\left(1-\frac{f_1^2}{f_{2\infty}^2}\right)\left(1-\frac{f_2^2}{f_{2\infty}^2}\right)}}{1-\frac{f_1f_2}{f_{2\infty}^2}}. \tag{31}$$

2. *Terminal Losses*

The general terminal losses L_a and L_b are determined by (6) from the absolute values of the terminal factors F_a and F_b, which factors we have assumed apply to the sending and receiving ends, respectively.

That either factor is dependent only upon its own type of termination and not upon its position at the sending or receiving end, can readily be shown. By the reciprocal theorem the product $F_tF_aF_bF_r$ is independent of the direction of current propagation, and from the forms of F_t and F_r the latter are also, whence the product $F_a\ F_b$ is independent of direction. Since in addition F_a and F_b are independent of each other they cannot depend upon position. This is equivalent to the statement that the ratios E_a/E and I/I_b which any particular termination would give at the sending and receiving ends, respectively, are equal. It will then be sufficient to consider the factor for a given termination at either end, say the receiving end.

The four terminations found practical give terminal losses which are reducible to two, namely, L_m and L_x now to be derived.

Terminal Losses, L_m, with Mid-M-type Terminations. These terminations, already mentioned, are

1, mid-shunt of a mid-series "constant k" equivalent M-type, $(K_{21}(m))$; and

2, mid-series of a mid-shunt "constant k" equivalent M-type, $(K_{12}(m))$.

The relations between the M-type characteristic impedances $K_{21}(m)$ and $K_{12}(m)$, the parameter m, and the variables U_k and V_k of the "constant k" prototype are, from formulae [8] in a previous paper

$$\frac{R}{K_{21}(m)}=\frac{K_{12}(m)}{R}=\frac{\pm\sqrt{1+U_k+iV_k}}{1+(1-m^2)(U_k+iV_k)}. \tag{32}$$

Since $K_{12}(m)\cdot K_{21}(m)=R^2$, $K_{12}(m)$ and $K_{21}(m)$ are inverse networks of impedance product R^2. As either of these terminations is at a mid-point, it forms an end for the wave-filter mid-part and in the terminal factor F_b, arbitrarily chosen, $Z_b=R$ and $I/I_b=1$, leaving

$$F_b=\frac{2\sqrt{W_bR}}{W_b+R}. \tag{33}$$

In this factor the image impedance W_b is either $K_{21}(m)$ or $K_{12}(m)$, depending upon the type of termination. By (32) the factor is the same for both types provided they have the same parameter m, so

[8] The radicals which occur in this and succeeding formulae are proportional to physical impedances with positive resistance components. Hence, in each case the double sign is to be interpreted such as to make the real part of the radical positive.

that we may put for either of them *the single terminal loss* L_m defined by (6) as

$$e^{-L_m}=\left|\frac{2\sqrt{K_{21}(m)R}}{K_{21}(m)+R}=\frac{2\sqrt{K_{12}(m)R}}{K_{12}(m)+R}\right|$$

which upon the substitution of (32) gives

$$L_m=\log_e\left(\frac{1}{2}\left|1\pm\frac{\sqrt{1+U_k+iV_k}}{1+(1-m^2)\,(U_k+iV_k)}\right|\cdot\left|\frac{1+(1-m^2)\,(U_k+iV_k)}{\sqrt{1+U_k+iV_k}}\right|^{\frac{1}{2}}\right).\quad(34)$$

Terminal Losses, L_x, *with x-"constant k" Terminations.* The terminations are

3, x-shunt of the "constant k" wave-filter, (K_{x2}); and

4, x-series of the "constant k" wave-filter, (K_{x1}).

The x-shunt and x-series characteristic impedances, K_{x2} and K_{x1}, are related by the formulae

$$\frac{R}{K_{x2}}=\frac{K_{x1}}{R}=\frac{K_{1k}+(x-.5)z_{1k}}{R}=\pm\sqrt{1+U_k+iV_k}\pm(2x-1)\sqrt{U_k+iV_k},\quad(35)$$

and

$$K_{x1}K_{x2}=K_{1k}K_{2k}=z_{1k}z_{2k}=R^2,$$

where K_{2k} and K_{1k} are the mid-shunt and mid-series values corresponding to $x=.5$. With either termination K_{x2} or K_{x1} it is assumed that the mid-part of the wave-filter begins at the mid-point, i.e., at the position corresponding to K_{2k} or K_{1k}, respectively, even when x is less than .5. In the latter case an impedance is theoretically added which is sufficient to "build-out" the wave-filter to the mid-point, and an equal impedance is similarly subtracted from the terminal impedance.

For termination 3, that is K_{x2}, the elements of factor F_b in (6) have the values

$$W_b=K_{2k},$$

$$Z_b=z_{2k}\,R/(z_{2k}+(x-.5)R),\quad(36)$$

and

$$I/I_b=z_{2k}/(z_{2k}+(x-.5)R).$$

For termination 4, K_{x1}, they are

$$W_b=K_{1k},$$

$$Z_b=R+(x-.5)z_{1k},\quad(37)$$

and

$$I/I_b=1.$$

The substitution of (36) or (37) in F_b gives an identical result, as shown by relations (35), provided x is the same in both. *A single terminal loss L_x may then apply to either*, which is defined from (6) as

$$e^{-L_x}=\left|\frac{2\sqrt{(R^2/K_{2k})R}}{R^2/K_{x2}+R}=\frac{2\sqrt{K_{1k}R}}{K_{x1}+R}\right|,$$

giving by (35)

$$L_x=\log_e\left(\frac{1}{2}\left|1\pm\sqrt{1+U_k+iV_k}\pm(2x-1)\sqrt{U_k+iV_k}\right|\cdot\left|\frac{1}{1+U_k+iV_k}\right|^{\frac{1}{4}}\right).(38)$$

A comparison of (34) and (38) shows that when $m=1$ and $x=.5$, $L_m=L_x$ as should be the case.

3. *Interaction Losses*

The interaction loss defined in (6) is expressible in its general form as

$$L_r=\log_e|1-r_ar_be^{-2T}|. \tag{39}$$

It depends not only upon the transfer constant T, including both diminution and angular constants, but also upon the complex reflection coefficients, r_a and r_b, at the two ends. That is, it is a function both of the internal structure and of the terminations of the wave-filter. For this reason its determination offers the most complexity of all the three types of losses and, in fact, requires a knowledge of the transfer loss. On the other hand, it is usually the least important part of the total transmission loss and may usually be omitted except at frequencies within a transmitting band and near a critical frequency.

The transfer constant $T=D+iS$ is given by the relations and formulae developed when considering the transfer loss.

The multiplication of the reflection coefficients and the square of the transfer factor is simplified to a problem in addition by expressing each of these coefficients in the exponential form,

$$r_a=e^{-G_a-iH_a},$$

and

$$r_b=e^{-G_b-iH_b}.$$

Then, putting $r_ar_be^{-2T}=e^{-P-iQ}$,

$$L_r=\tfrac{1}{2}\log_e(1+e^{-2P}-2e^{-P}\cos Q), \tag{40}$$

where $P=G_a+G_b+2D,$

and $Q=H_a+H_b+2S.$

The subscripts, as before, merely refer to the terminations. The G and H expressions which correspond to the reflection coefficient with each of the four particular types of terminations, 1, 2, 3, and 4, follow.

Reflection Coefficients, r_{m2} and r_{m1}, with Mid-M-type Terminations. For termination 1 arbitrarily assumed at b we have $W_b=K_{21}(m)$ and $Z_r=R$. Introducing for this case the subscript $_{m2}$, signifying M-type and mid-shunt, it follows by (6) and (32) that

$$r_{m2}=\frac{K_{21}(m)-R}{K_{21}(m)+R},$$

and its equivalent

$$e^{-G_{m2}-iH_{m2}}=r_{m2}=\frac{1+(1-m^2)(U_k+iV_k)-(\pm\sqrt{1+U_k+iV_k})}{1+(1-m^2)(U_k+iV_k)\ \pm\ \sqrt{1+U_k+iV_k}}. \tag{41}$$

With termination 2, $W_b=K_{12}(m)$ and $Z_b=R$, so that by (32) the corresponding coefficient r_{m1} becomes

$$r_{m1}=-r_{m2},$$

or (42)

$$e^{-G_{m1}-iH_{m1}}=-e^{-G_{m2}-iH_{m2}}.$$

Since $-1=e^{-i\pi}$,

$$G_{m1}=G_{m2},$$

and (43)

$$H_{m1}=H_{m2}+\pi.$$

Reflection Coefficients, r_{x2} and r_{x1}, with x-"constant k" Terminations. In the case of the x-shunt termination 3, K_{x2}, relations (36) give

$$r_{x2}=\frac{K_{2k}-z_{2k}R/(z_{2k}+(x-.5)R)}{K_{2k}+z_{2k}R/(z_{2k}+(x-.5)R)}.$$

Introducing (35) this is

$$e^{-G_{x2}-iH_{x2}}=r_{x2}=\frac{1\pm(2x-1)\sqrt{U_k+iV_k}-(\pm\sqrt{1+U_k+iV_k})}{1\pm(2x-1)\sqrt{U_k+iV_k}\pm\sqrt{1+U_k+iV_k}}. \tag{44}$$

The x-series termination 4, K_{x1}, has a coefficient r_{x1} determined by (37) which is related to r_{x2} through (35) as

$$r_{x1}=-r_{x2}. \tag{45}$$

It follows from the corresponding exponential expressions that

$$G_{x1}=G_{x2},$$

and (46)

$$H_{x1}=H_{x2}+\pi.$$

Hence, the two members of each pair of reflection coefficients, r_{m2}, r_{m1}, and r_{x2}, r_{x1}, differ only in sign so that their G's are the same but their H's differ by π.

4. *Wave-Filter Structures Having Equivalent Transmission Losses*

There are six groups of possible wave-filter networks involving the four terminations above, each group of which is made up of pairs having equivalent current ratios $2RI/E$ and hence equivalent transmission losses. By (5) this means that the members of such a pair have products for their four factors, $F_t F_a F_b F_r$, which are equal. It may readily be shown from preceding relations that these groups, represented symbolically by brackets enclosing the transfer constants of their mid-parts and the terminations, are the following:

$$\begin{aligned}
&(a)\ [T,\ K_{21}(m),\ K_{21}(m')] = [T,\ K_{12}(m),\ K_{12}(m')],\\
&(b)\ [T,\ K_{21}(m),\ K_{12}(m')] = [T,\ K_{12}(m),\ K_{21}(m')],\\
&(c)\ [T,\ K_{21}(m),\ K_{x2}] = [T,\ K_{12}(m),\ K_{x1}],\\
&(d)\ [T,\ K_{21}(m),\ K_{x1}] = [T,\ K_{12}(m),\ K_{x2}],\\
&(e)\ [T,\ K_{x2},\ K_x{}'_2] = [T,\ K_{x1},\ K_x{}'_1],\\
&(f)\ [T,\ K_{x2},\ K_x{}'_1] = [T,\ K_{x1},\ K_x{}'_2].
\end{aligned} \tag{47}$$

This symbolic representation in (*c*), for example, means that a composite wave-filter whose mid-part has a transfer constant, T, and whose terminations are those designated by $K_{21}(m)$ and K_{x2}, will give the same current ratio $2RI/E$ as another wave-filter whose mid-part has the same transfer constant, T, but whose terminations are those designated by $K_{12}(m)$ and K_{x1} where m and x are respectively the same in both networks.

III. Charts for Determining Transmission Losses

The accompanying charts apply to the three groups of transmission losses, transfer, terminal, and interaction, and are derived from the general formulae already given. The curves represent constant parameter loci for A, cB, L_m, L_x, G_{m2}, cH_{m2}, G_{x2}, cH_{x2}, and L_r as functions of several variables and include the most practical range; where further extension is required the original formulae may be consulted. The U and V variables for the ladder type of recurrent network (or its equivalent) which form the basis of this chart calculation method are to be found as a function of frequency, in the general case from formula (9),

$$z_1/4z_2 = U + iV,$$

and in the lower class "constant k" and M-type wave-filters from formulae (21) to (31). Owing to the large number of intermediate equations which it was necessary to obtain before direct computations could suitably be made for the charts, these equations will not be given here, but only a brief designation of the resulting charts together with the approximations involved, if any.

The units employed throughout are the *attenuation unit* and the *radian*. The former unit applies to A, D, L_m, L_x, G_{m2}, G_{x2}, P, L_r and L, and the latter unit to B, S, H_{m2}, H_{x2} and Q.

Transfer Loss

This is determined through the propagation constant, $\Gamma = A + iB$.

Charts 1, 2, *and* 3.—*A and cB in and about transmitting band*;
$c = \pm 1$ has the sign of V.

Chart 4.—*A in attenuating band*;
V^2 negligible compared with $(U + U^2) > O$.

Chart 5.—*A at maximum attenuation*;
$(U + U^2)$ negligible compared with V^2.

Terminal Losses, L_m and L_x

Chart 6.—*L_m in transmitting band*;
V_k neglected.

Chart 7.—*L_m at critical frequency*;
$U_k = -1$.

Chart 8.—*L_m in attenuating band*;
V_k neglected.

Chart 9.—*L_m at maximum attenuation of M-type*;
$$U_k = -\frac{1}{1 - m^2}.$$

Chart 10.—*L_x in transmitting band*;
V_k neglected.

Chart 11.—*L_x at critical frequency*;
$U_k = -1$.

Chart 12.—*L_x in attenuating band*;
V_k neglected.

Reflection Coefficients

Note that

$$G_{m1}=G_{m2},$$

and $$H_{m1}=H_{m2}+\pi;$$

also that

$$G_{x1}=G_{x2},$$

and $$H_{x1}=H_{x2}+\pi.$$

Chart 13.—*G_{m2} and H_{m2} in transmitting band*;
V_k neglected.

Chart 14.—*G_{m2} and cH_{m2} at critical frequency*;
$U_k=-1$ and $c=\pm 1$ has the sign of V_k.

Chart 15.—*G_{m2} and cH_{m2} in attenuating band*;
V_k neglected.

Chart 16.—*G_{x2} and cH_{x2} in transmitting band*;
V_k neglected and $c=\pm 1$ has the sign of V_k.

Chart 17.—*G_{x2} and cH_{x2} at critical frequency*;
$U_k=-1$.

Chart 18.—*G_{x2} and cH_{x2} in attenuating band*;
V_k neglected.

Interaction Loss, L_r

Note that $T=D+iS=$transfer constant of mid-part of wave-filter:

$$P=G_a+G_b+2D,$$

and $$Q=H_a+H_b+2S,$$

where a and b refer to the terminations.

Chart 19.—*L_r as a function of P and Q.*

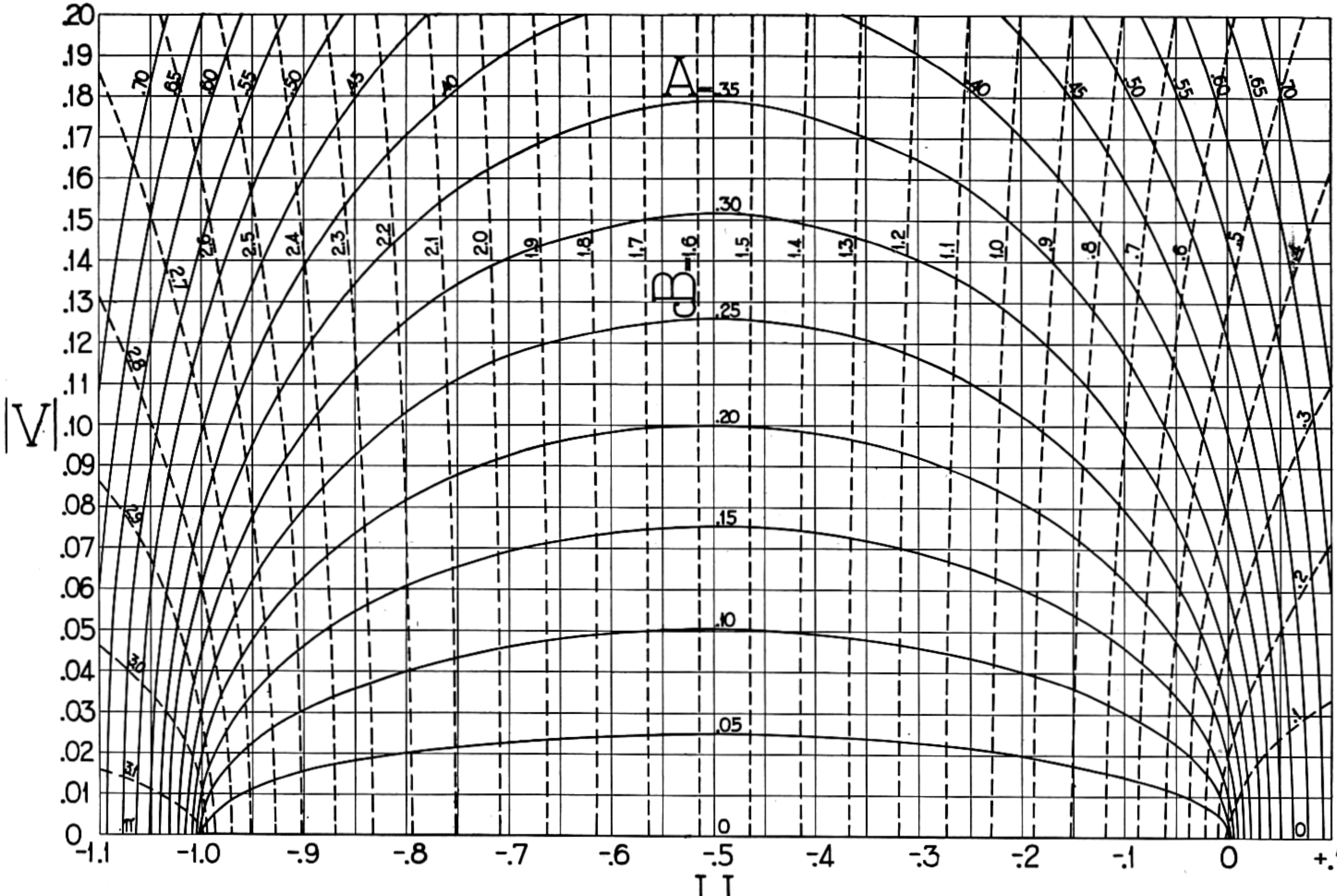

Chart 1.—A and cB in and about transmitting band;
$c = \pm 1$ has the sign of V.

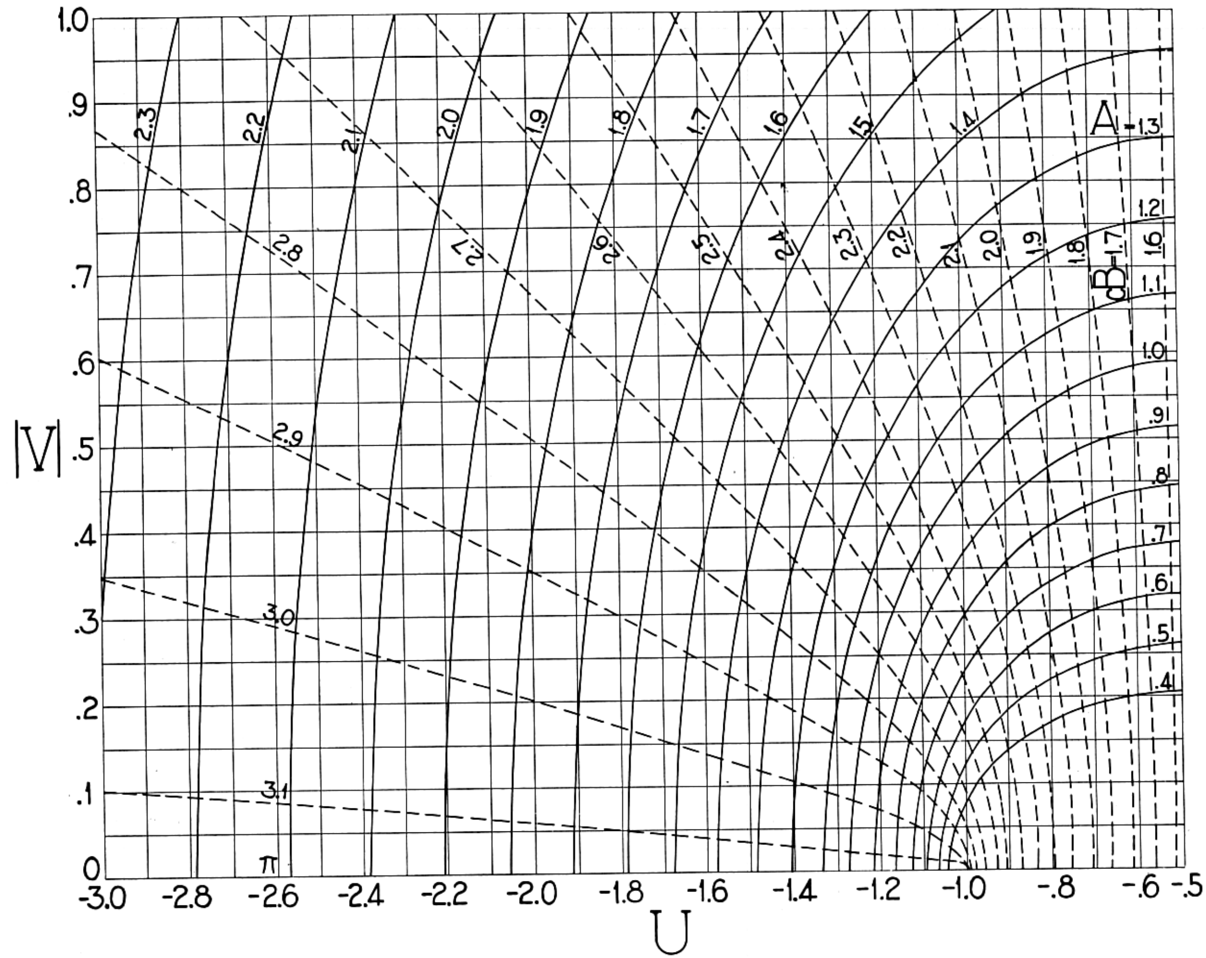

Chart 2.—A and cB in and about transmitting band;
$c = \pm 1$ has the sign of V.

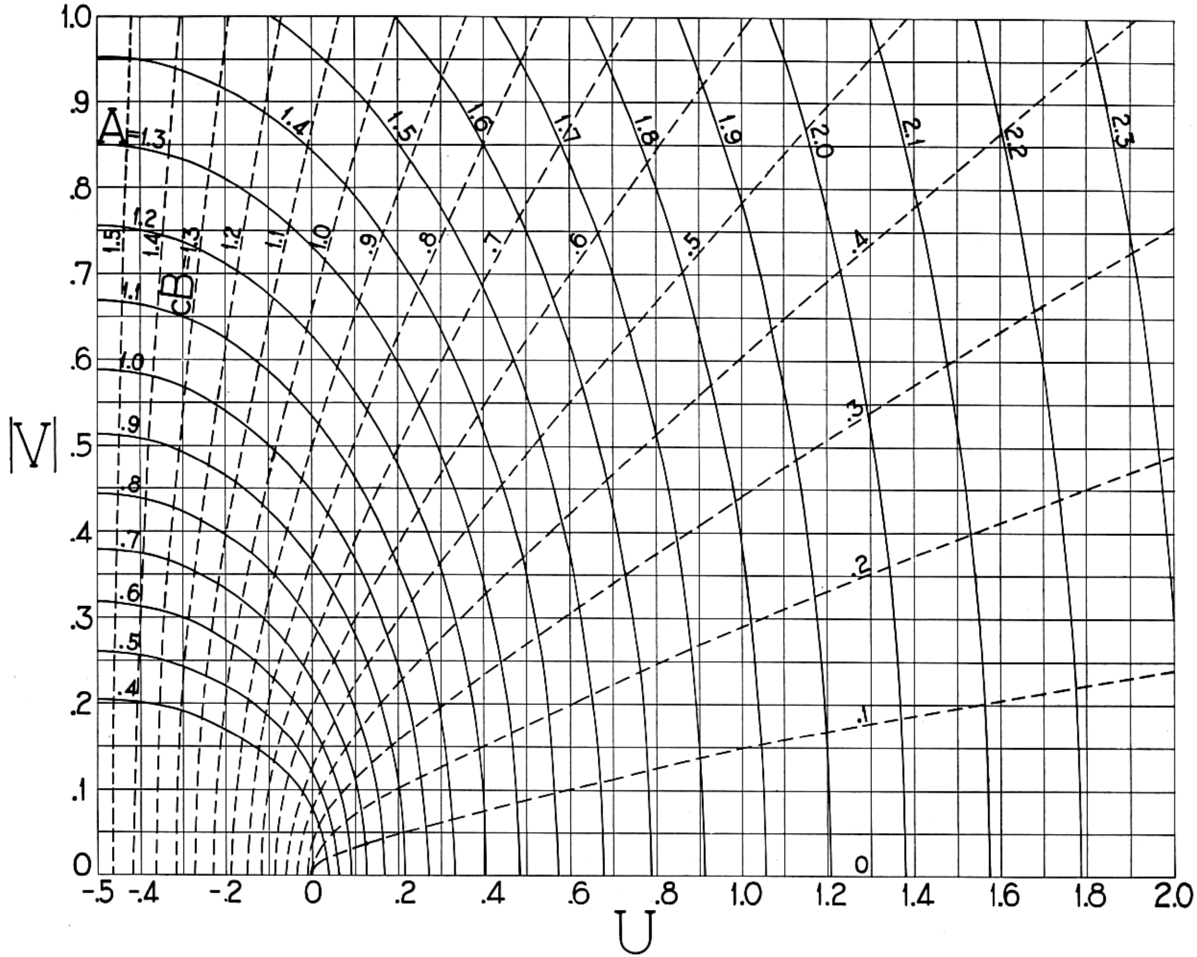

Chart 3.—A and cB in and about transmitting band;
$c = \pm 1$ has the sign of *V*.

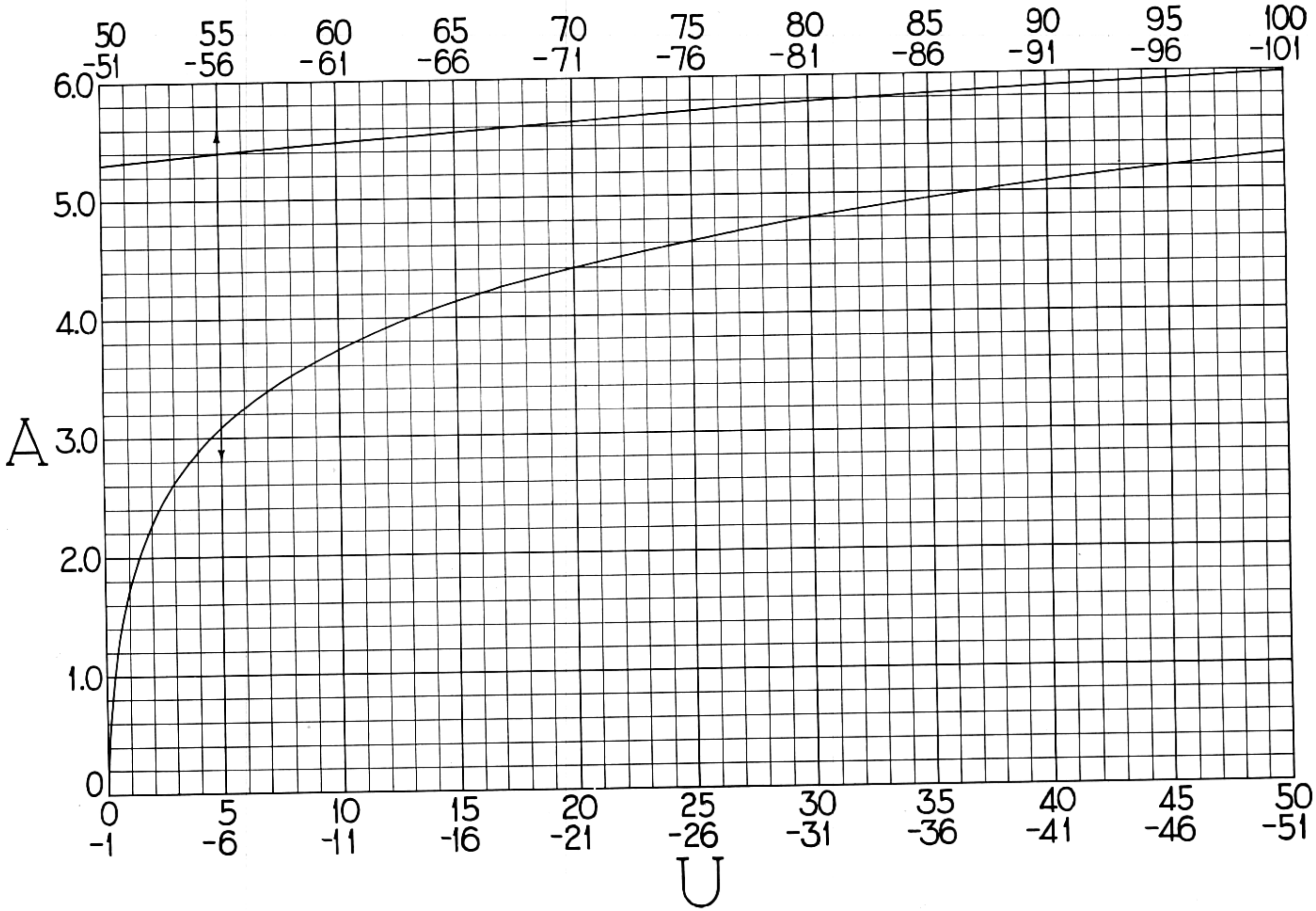

Chart 4.—A in attenuating band;
V^2 negligible compared with $(U+U^2)>O$.

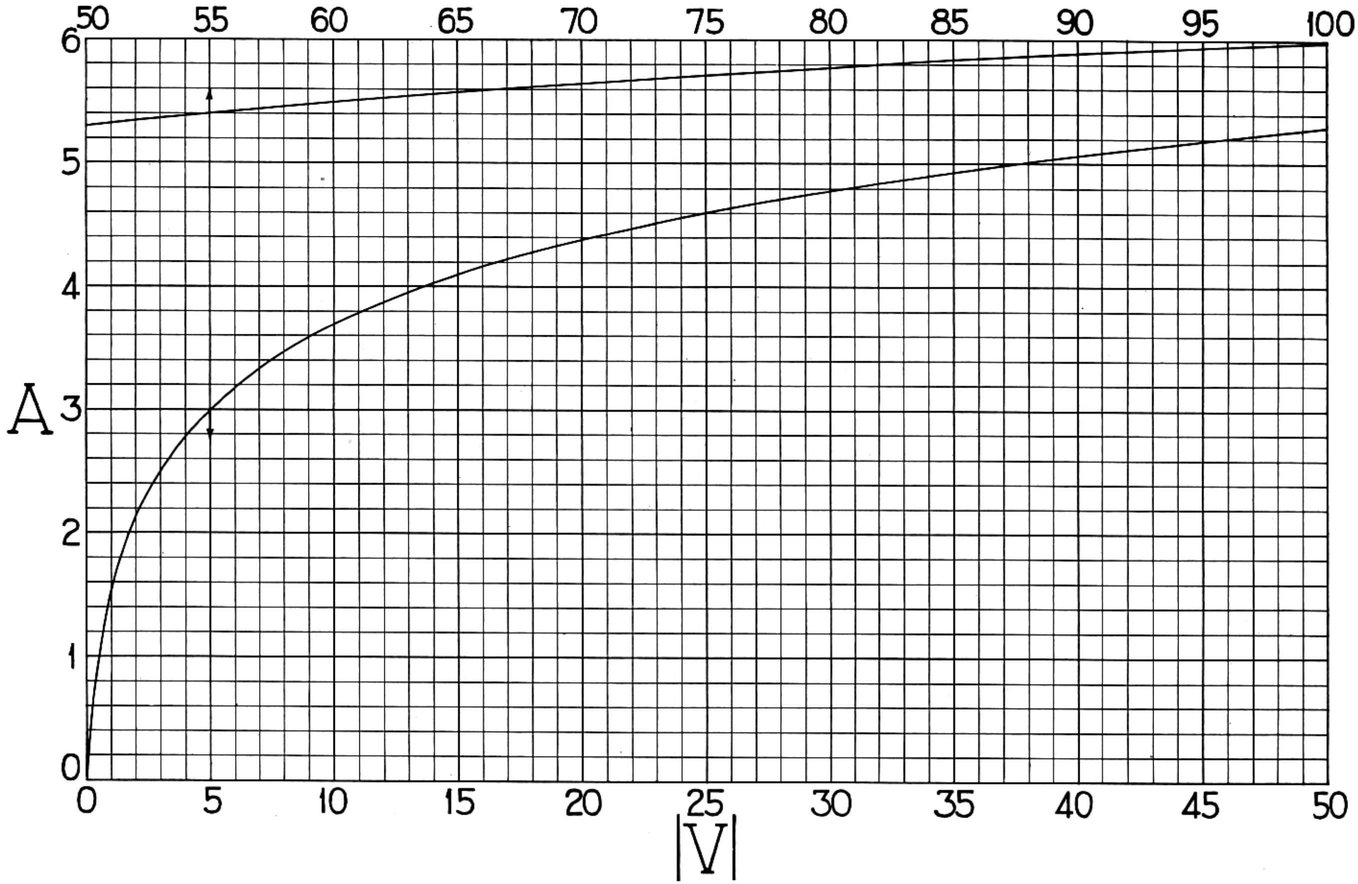

Chart 5.—*A at maximum attenuation;*
$(U+U^2)$ negligible compared with V^2.

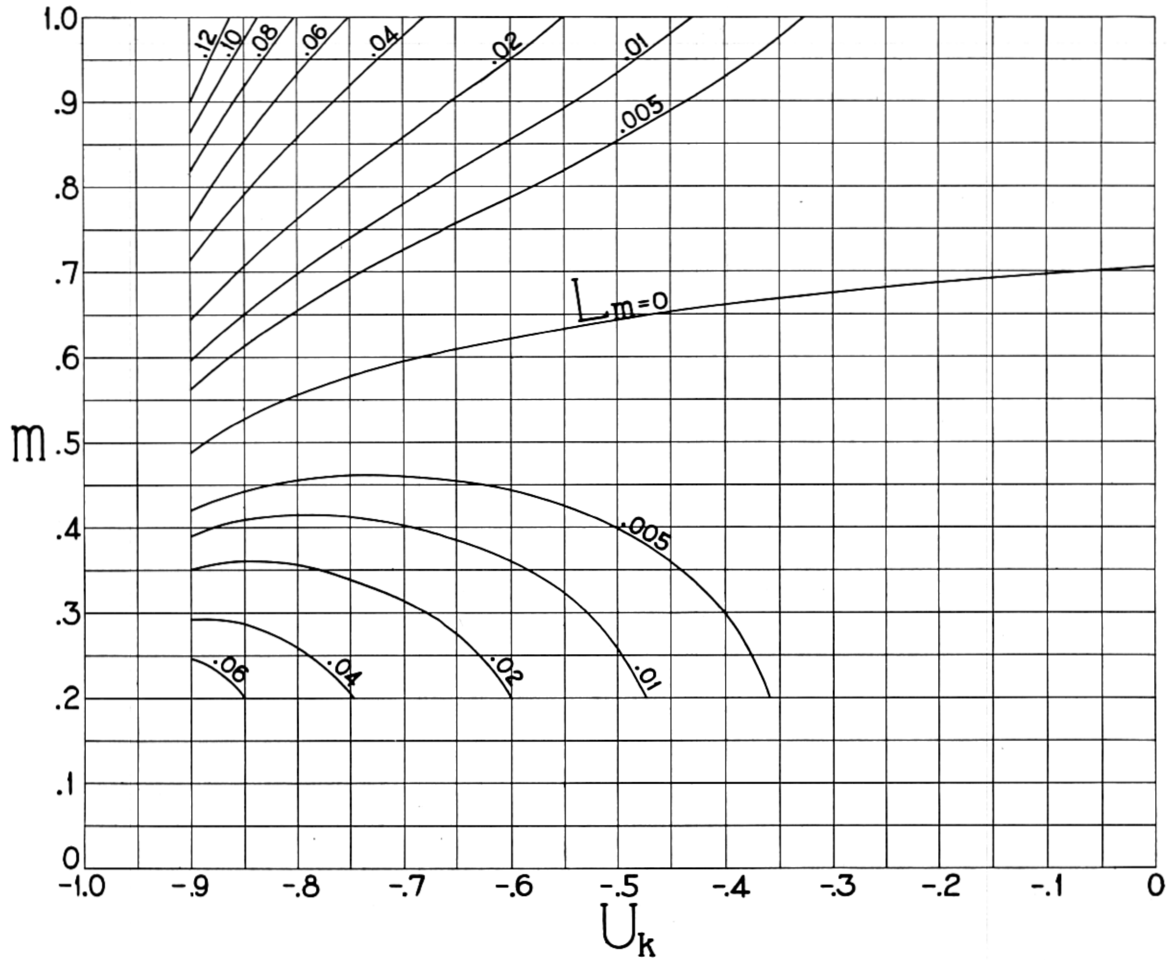

Chart 6.—L_m in transmitting band;
V_k neglected.

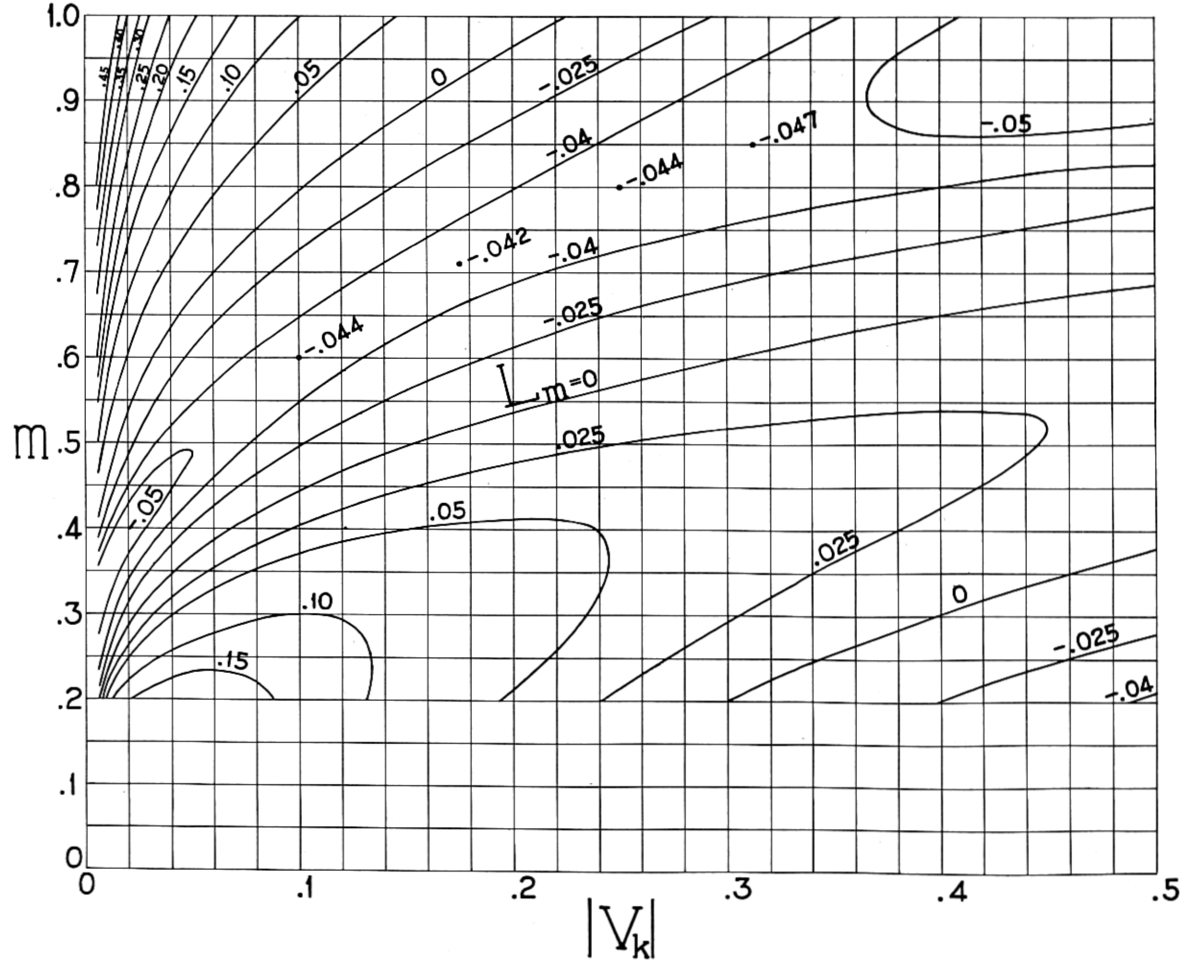

Chart 7.—L_m at critical frequency;
$U_k = -1.$

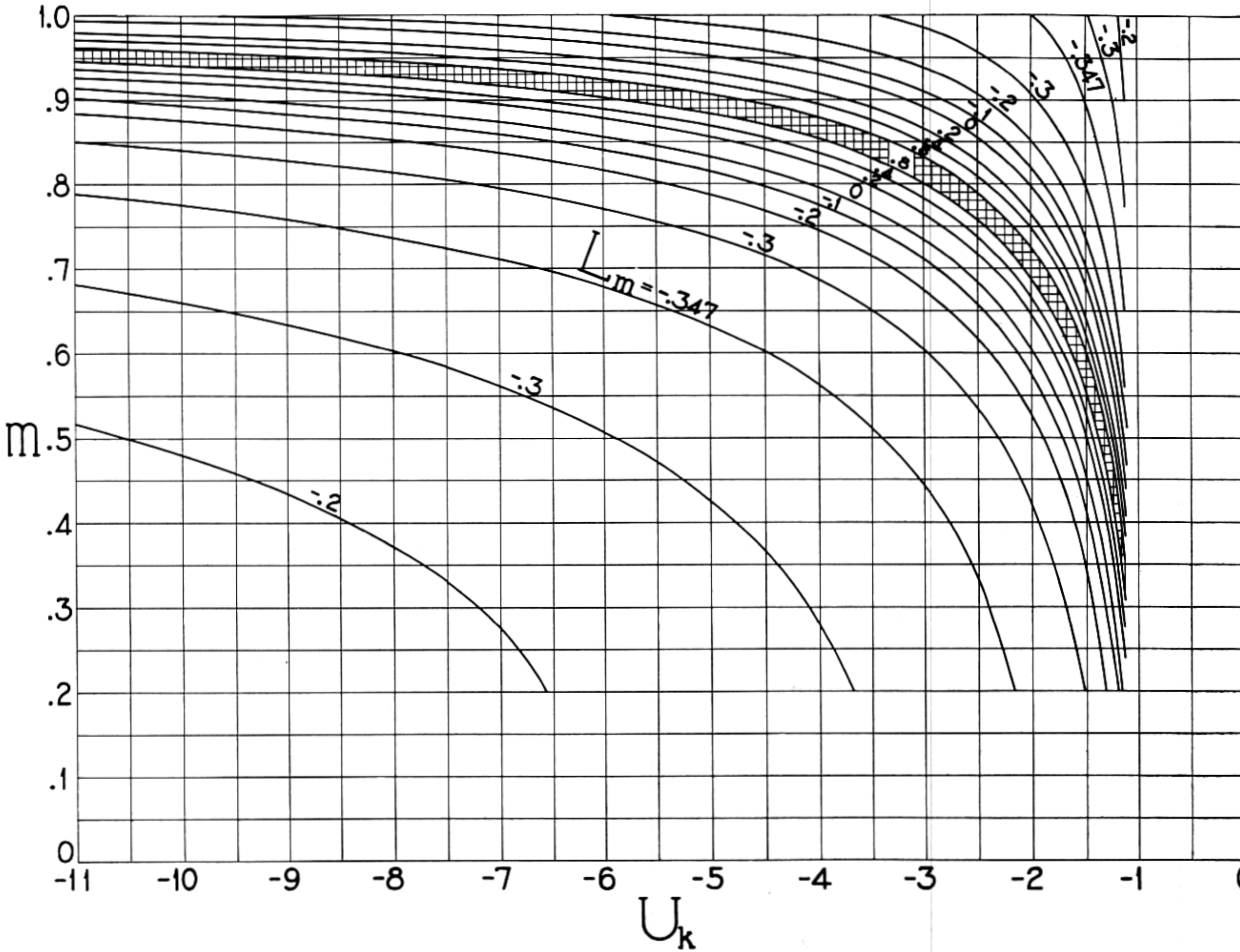

Chart 8.—L_m in attenuating band; V_k neglected.

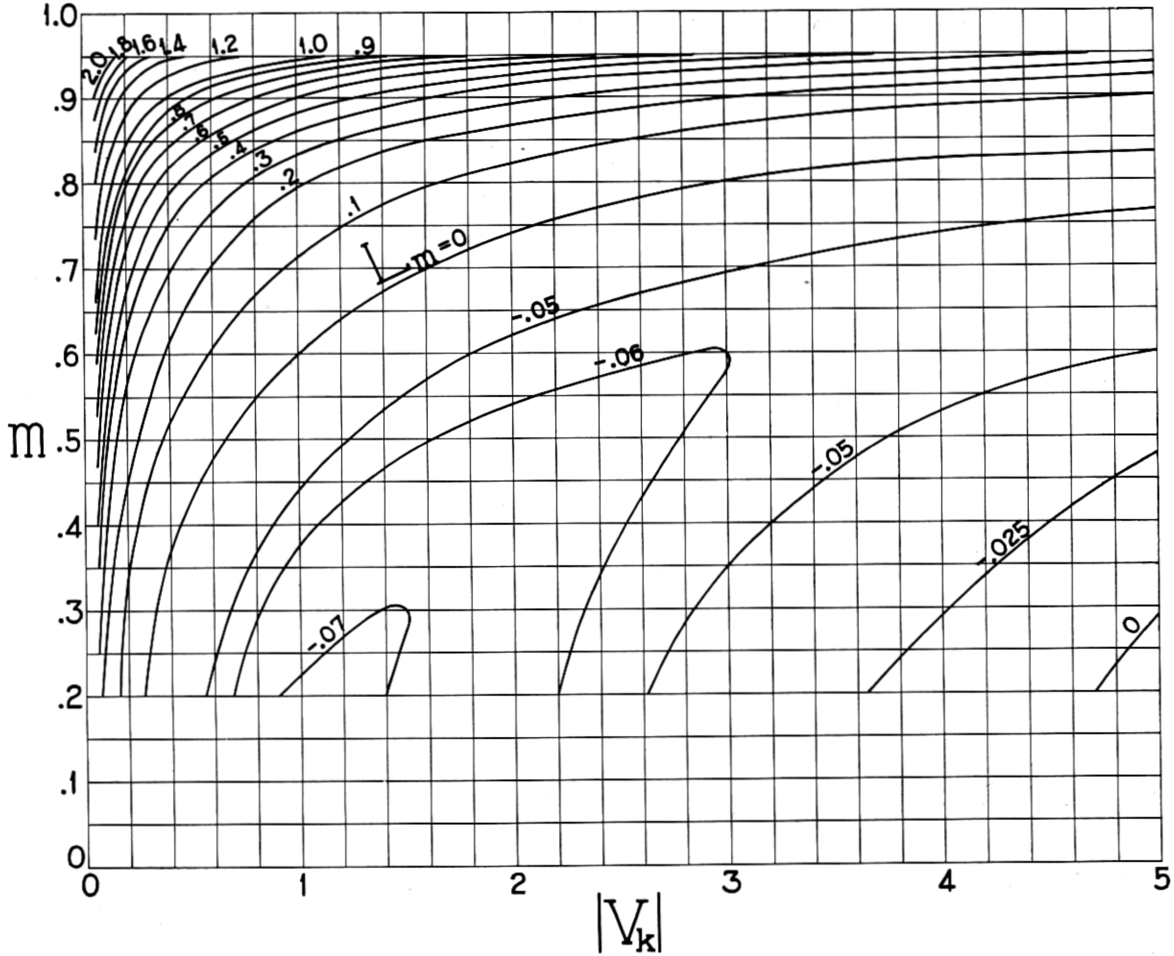

Chart 9.—L_m at maximum attenuation of M-type;

$$U_k = -\frac{1}{1-m^2}.$$

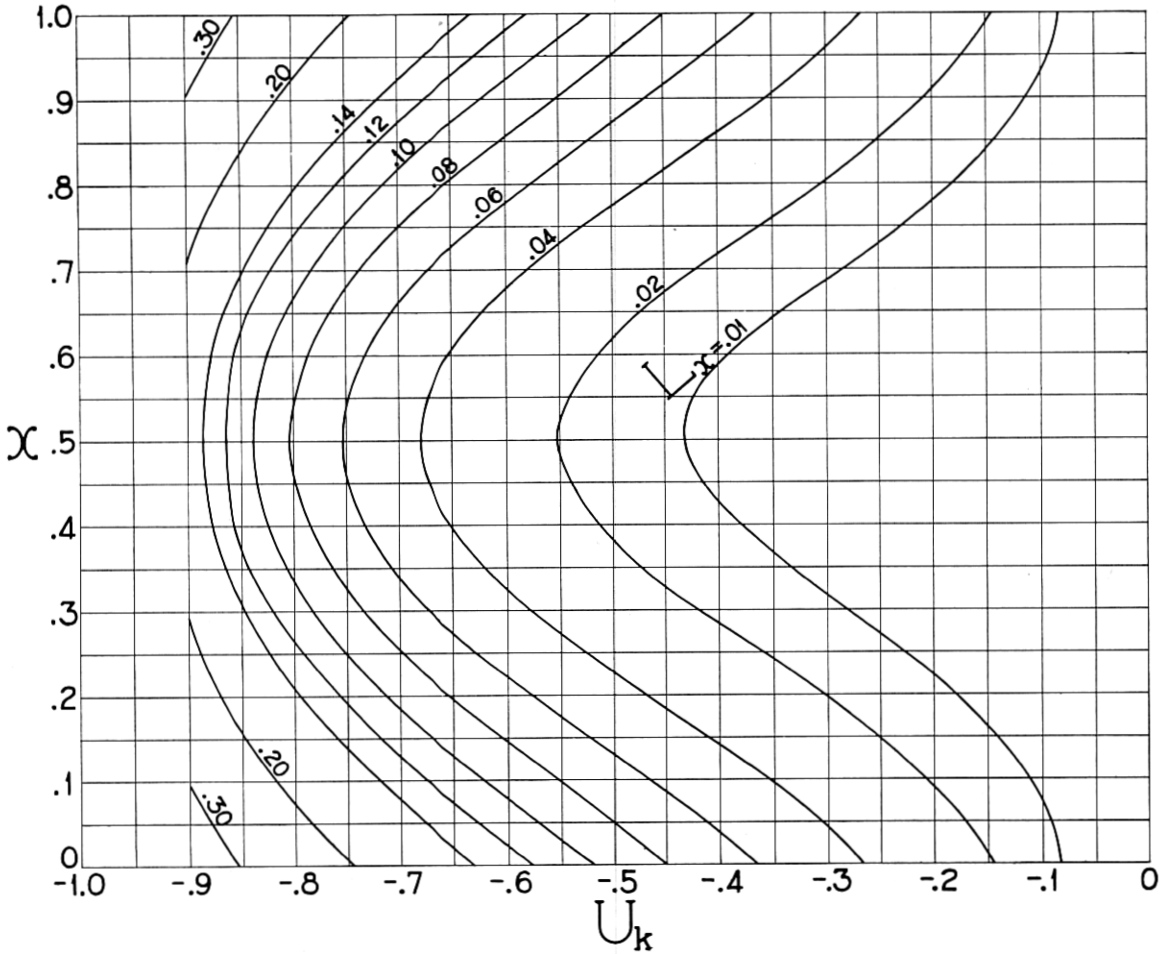

Chart 10.—L_x *in transmitting band;*
V_k neglected.

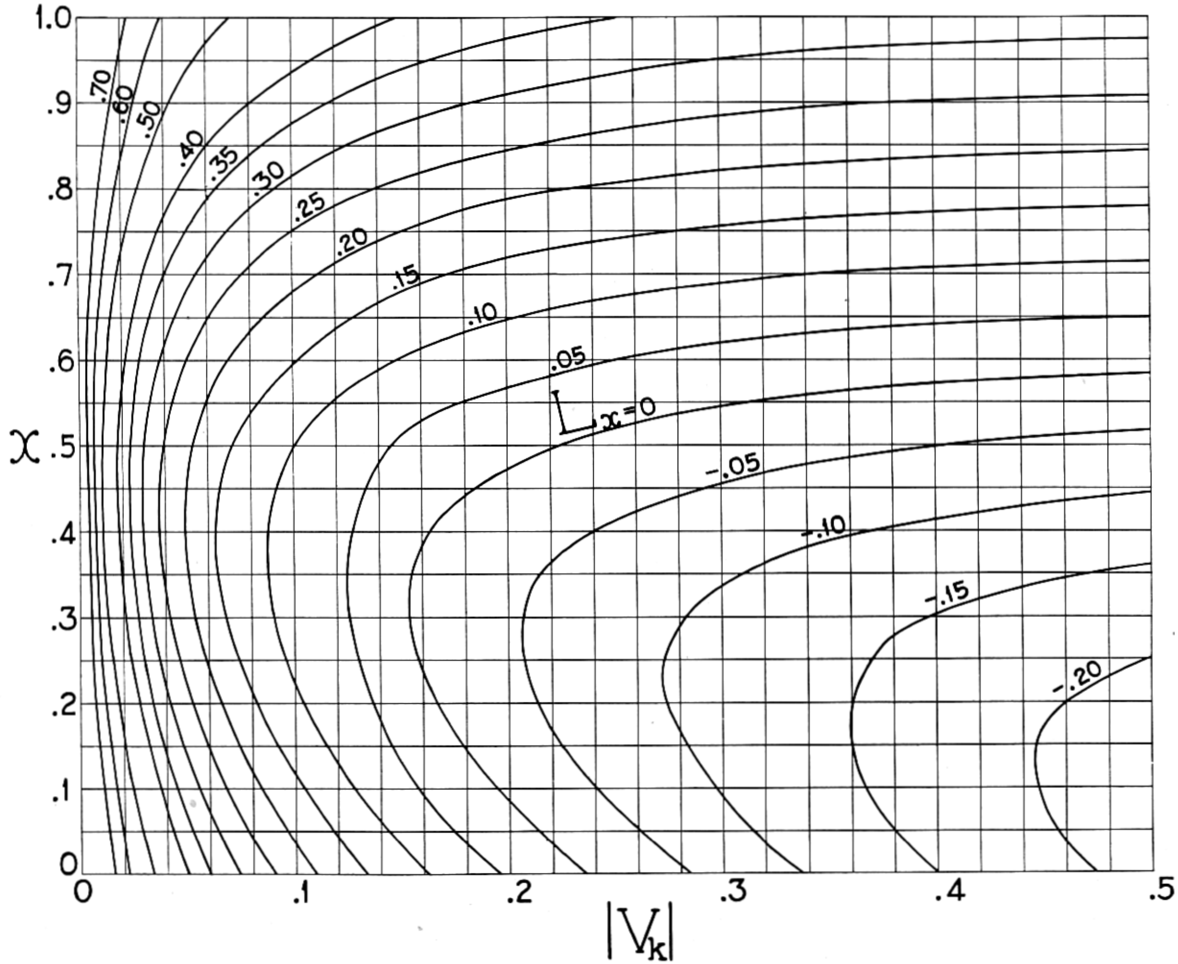

Chart 11.—L_x *at critical frequency;*
$U_k = -1$.

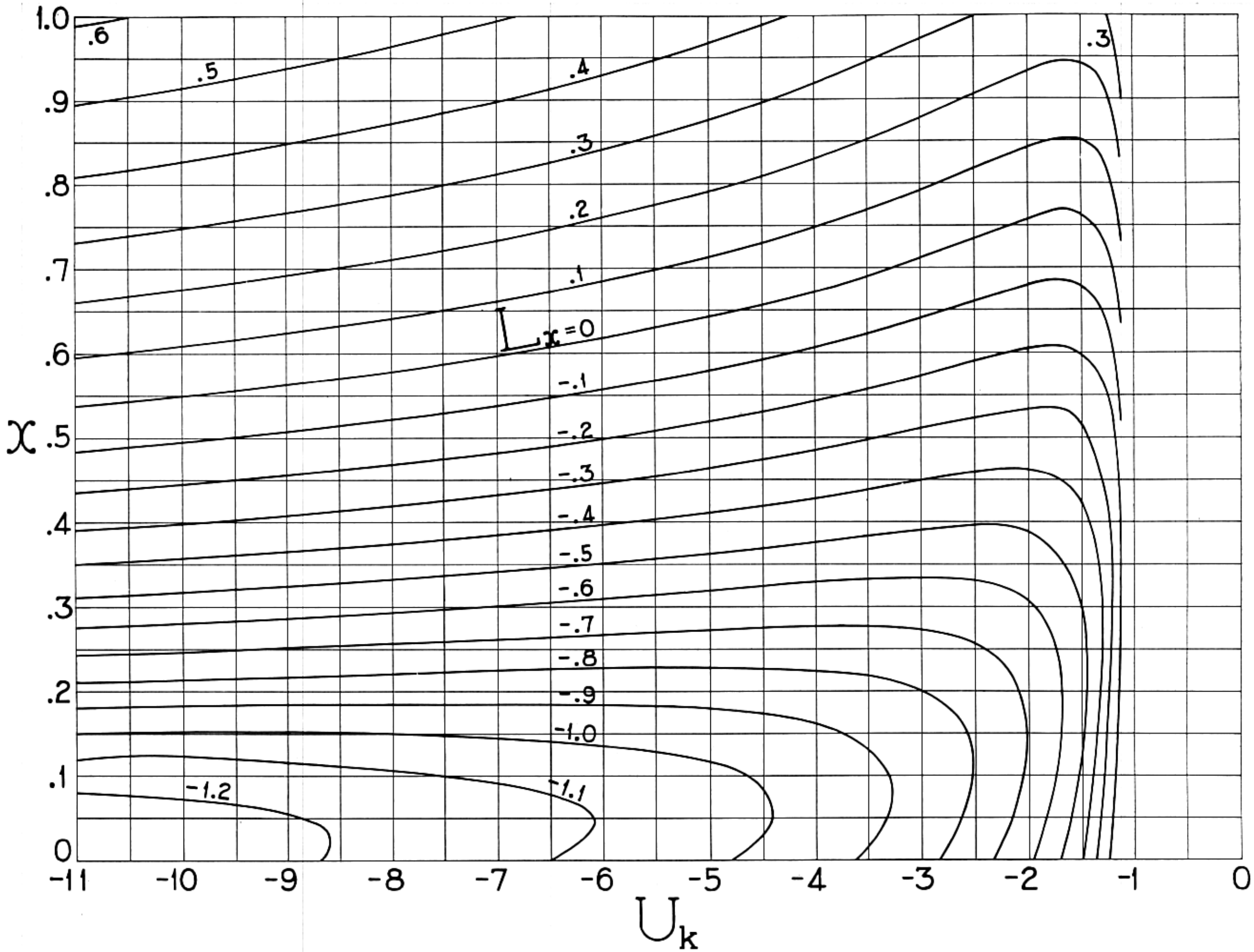

Chart 12.—L_x in attenuating band;
V_k neglected.

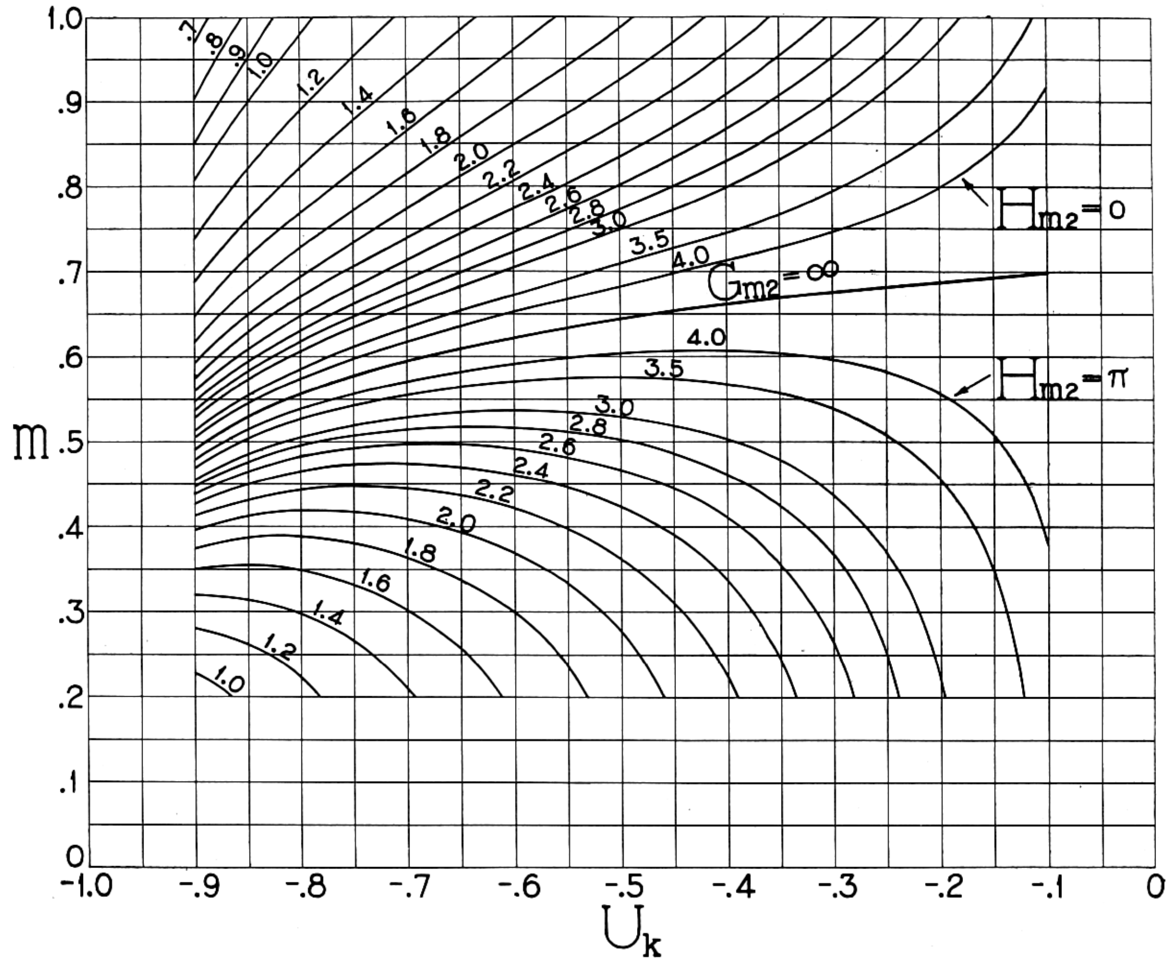

Chart 13.—G_{m2} *and* H_{m2} *in transmitting band;*
V_k neglected.

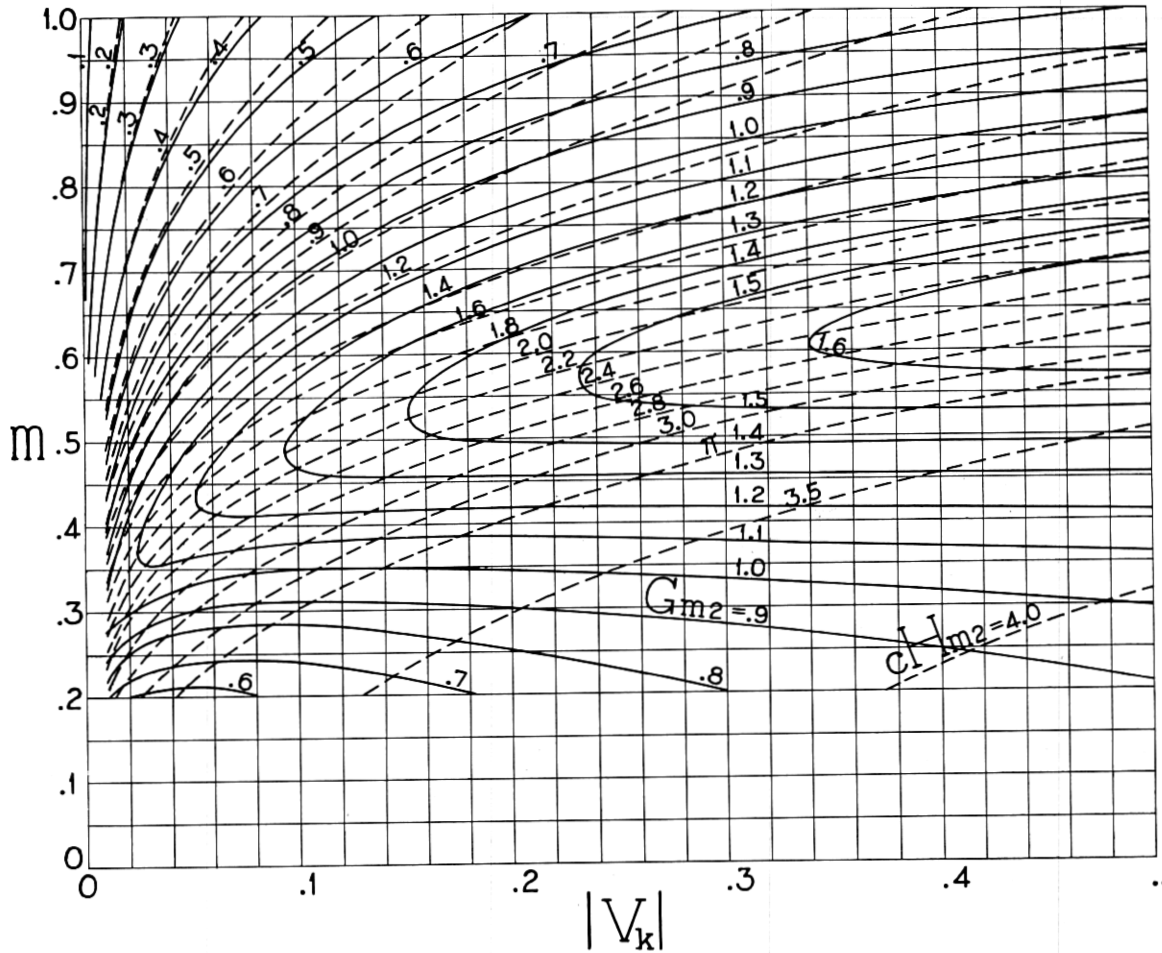

Chart 14.—G_{m2} and cH_{m2} at critical frequency;
$U_k = -1$ and $c = \pm 1$ has the signo f V_k.

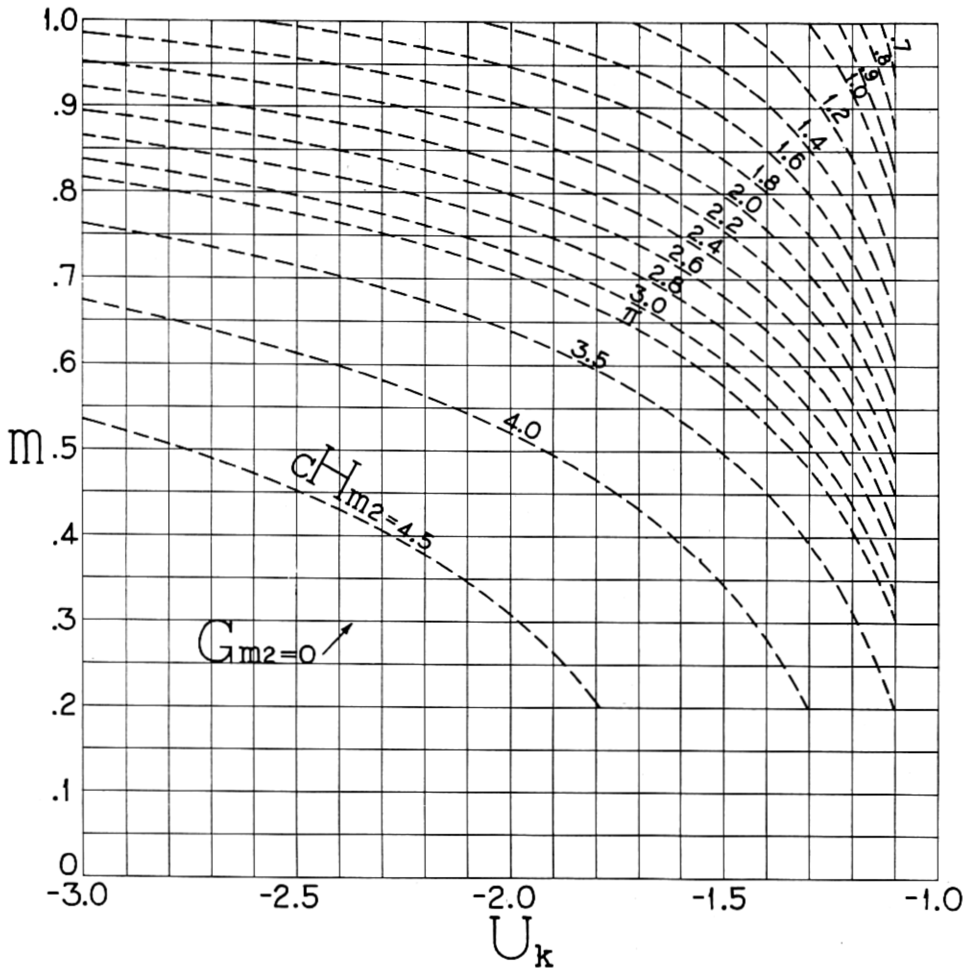

Chart 15.—G_{m2} *and* cH_{m2} *in attenuating band;*
V_k neglected.

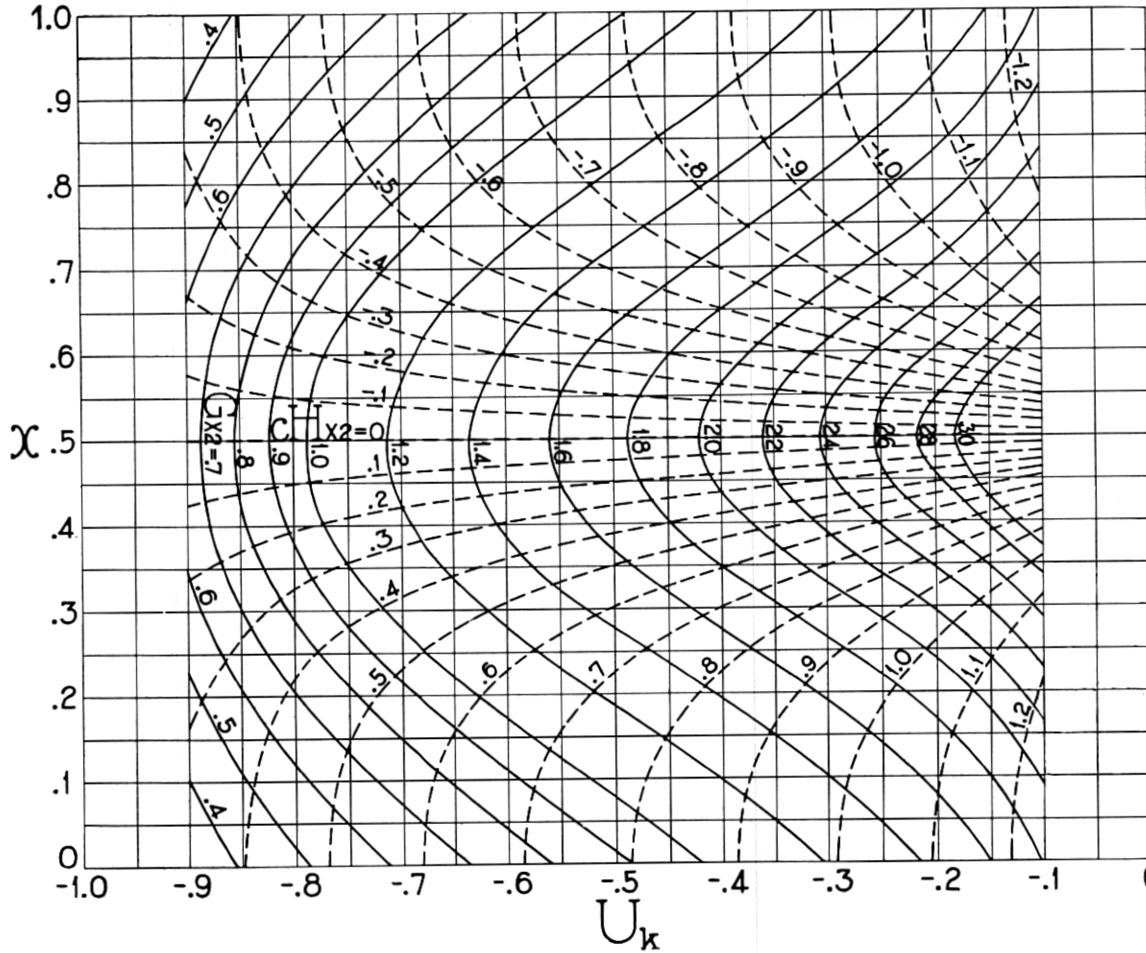

Chart 16.—G_{x2} and cH_{x2} in transmitting band;
V_k neglected and $c = \pm 1$ has the sign of V_k.

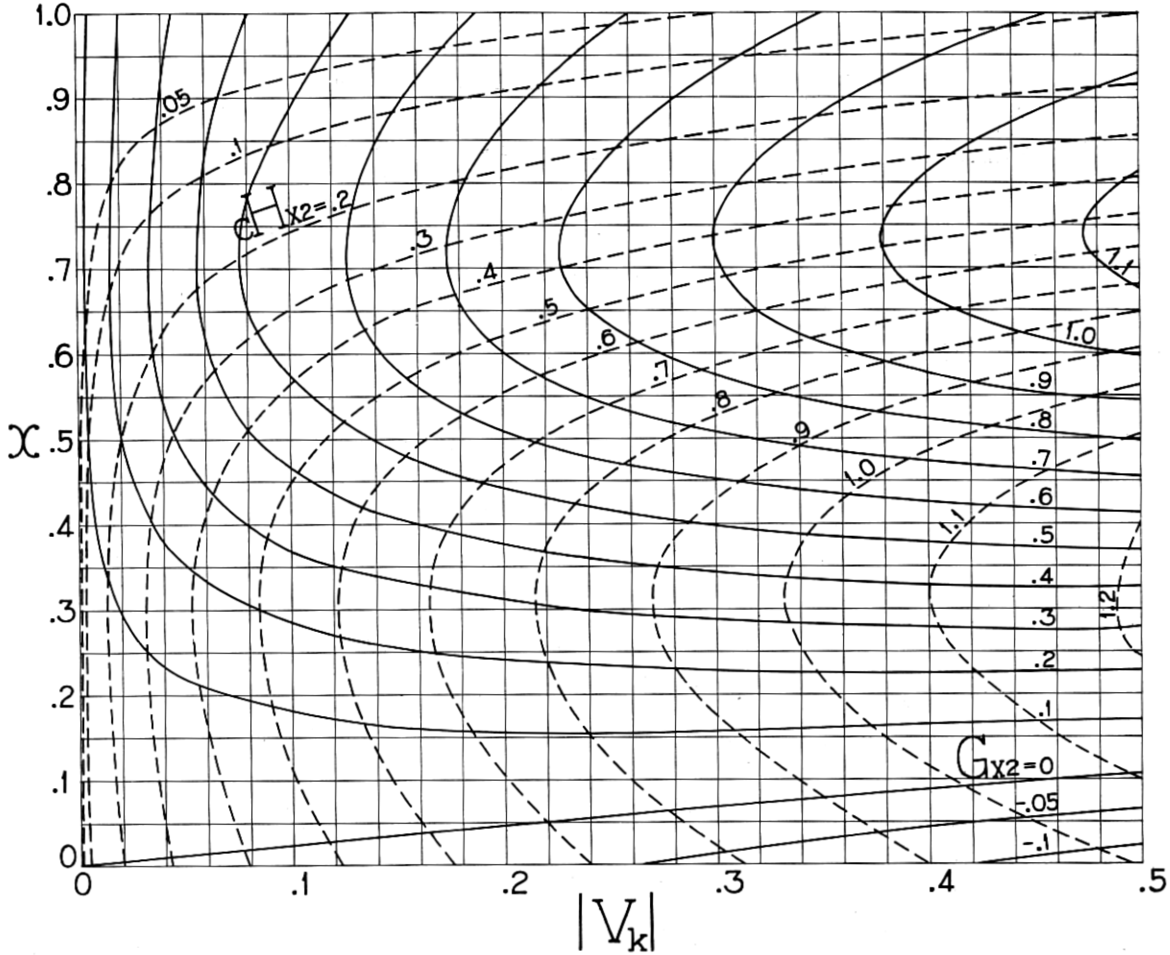

Chart 17.—G_{x2} and cH_{x2} at critical frequency; $U_k = -1$.

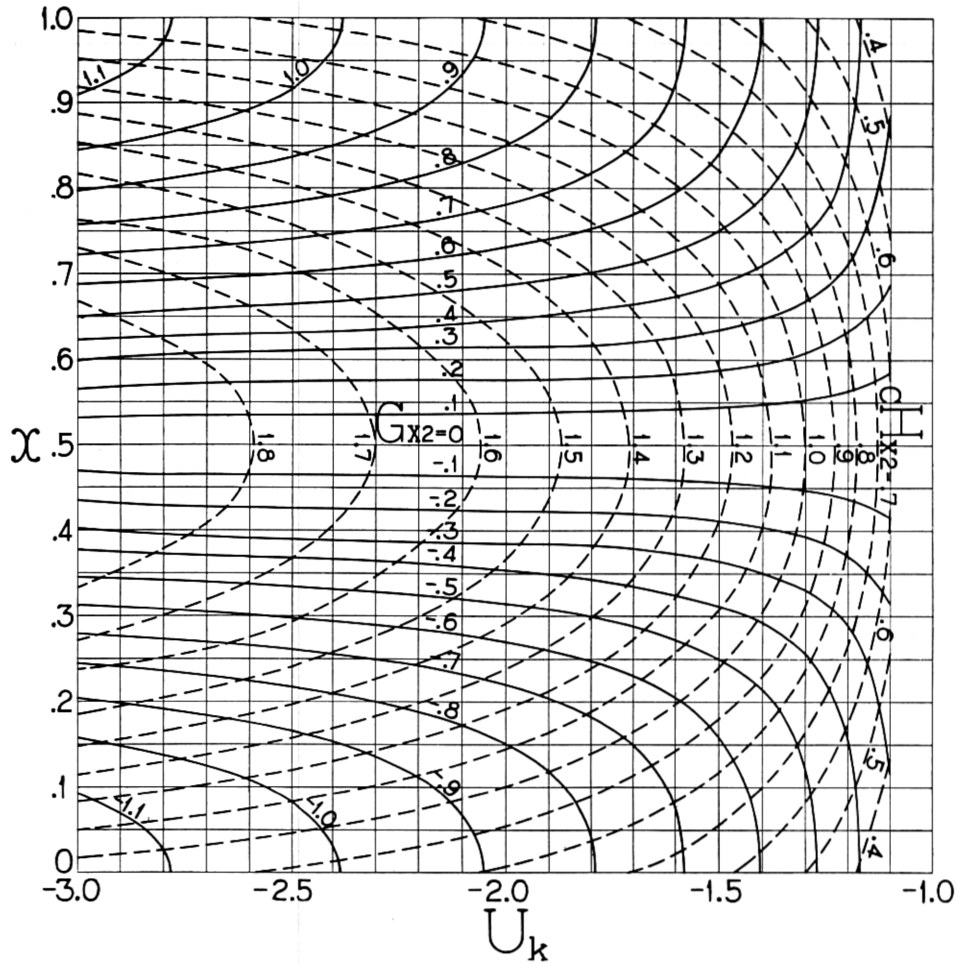

Chart 18.—*G_{x2} and cH_{x2} in attenuating band;*
V_k neglected.

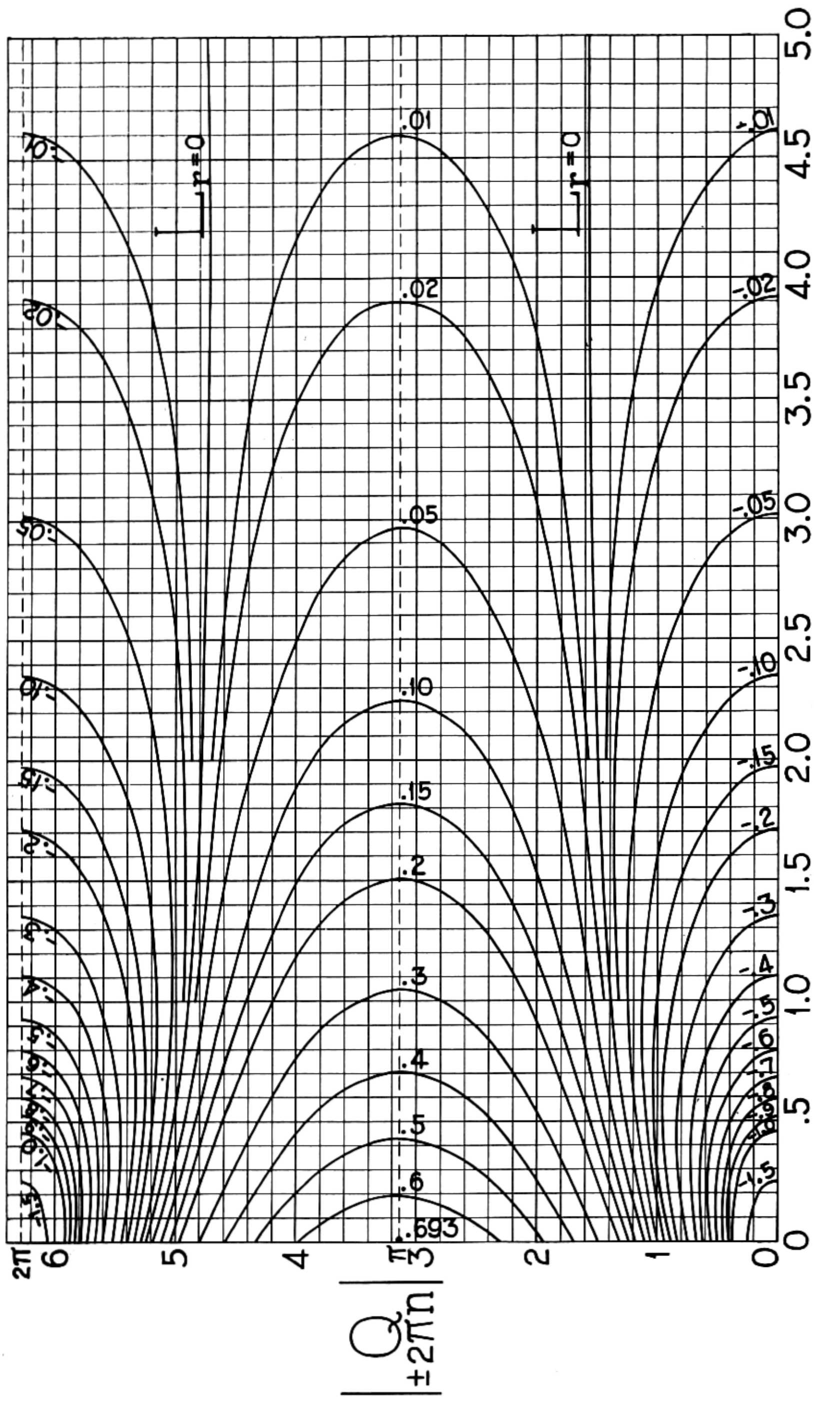

Chart 19.—L_r as a function of P and Q.

IV. Illustrative Application of the Method

In the following illustration a small number of band pass wave-filter sections having different characteristics is chosen purposely so as to allow an appreciable interaction factor and the use of all the charts.

The mid-part of the composite wave-filter is made up of one mid-series section of type VI_1 and one mid-half section of M-type IV_1, the designations being those of a previous paper. The termination at one end is made K_{x1} by adding $(x-.5)z_{1k}$ in series with type VI_1 and at the other is $K_{21}(m)$, as is diagrammatically represented at the top of Fig. 4. The values of all the parameters were chosen as follows:

$R=600$ ohms, $x=.80$,

$f_1=4{,}000\sim$, M-type, $f_{2\infty}=8{,}000\sim$,

$f_2=7{,}000\sim$,

and $d=.01$ (assumed constant for computation purposes).

With these values the magnitudes and locations of inductances and capacities are as shown in the center of Fig. 4, where the series impedance parts have been merged together.

The variables U_k and V_k for the "constant k" band pass wave-filter as well as U_m, V_m, and m of the M-type are given by formulae (24), (26), and (31). In the 3-element type VI_1

$$U_1+iV_1=\frac{1-(f/f_1)^2}{(f_2/f_1)^2-1}+id\,\frac{(f/f_1)^2}{(f_2/f_1)^2-1}. \tag{48}$$

These variables have been computed in the present case for frequencies on both sides of the transmitting band and are tabulated below. The other tables including that of transmission losses are based upon this table and the charts.

The next to the last, and the last columns give the total transmission losses as obtained by this chart method and by direct network computation, respectively. Comparison shows that there is a very satisfactory agreement between them, the differences at all frequencies being negligible in practice. The greatest differences of approximately .05 attenuation units at frequencies 3750 and 7500 cycles per second, just outside the transmitting band, can readily be explained as due to the omission of dissipation in the two terminal loss factors and the reflection coefficients. The transmission loss is shown graphically at the bottom of Fig. 4.

It is believed that the use of this chart method will result in considerable time economy with calculations of this kind.

TABLE I

U and V Variables

f Cycles/sec.	"Constant k"		Type VI_1		M-type, m = .7454	
	U_k	V_k	U_1	V_1	U_m	V_m
1000	−81.0	−.870	.455	.0003	1.29	− .0004
1500	−32.7	−.385	.417	.0007	1.34	− .0012
2000	−16.0	−.213	.364	.0012	1.45	− .0032
2500	− 8.41	−.132	.296	.0019	1.71	− .0098
3000	− 4.46	−.0868	.212	.0027	2.53	− .050
3250	− 3.20	−.0707	.165	.0032	4.21	− .220
3500	− 2.25	−.0575	.114	.0037	.36	−48.9
3750	− 1.53	−.0463	.0586	.0042	−2.67	− .253
4000	− 1.00	−.0367	0	.0049	− .997	− .0659
4250	− .607	−.0282	− .0625	.0055	− .461	− .0293
4500	− .329	−.0205	− .129	.0061	− .214	− .0156
5292	.00031	0	− .364	.0085	.00017	0
6500	− .534	.0263	− .796	.0128	− .389	.0252
6750	− .752	.0315	− .897	.0138	− .627	.0395
7000	− 1.00	.0367	−1.00	.0148	− .998	.0659
7500	− 1.58	.0470	−1.22	.0170	−2.90	.290
8000	− 2.25	.0575	−1.46	.0194	.85	48.9
8500	− 3.01	.0682	−1.70	.0219	4.92	.329
9000	− 3.85	.0792	−1.97	.0246	3.00	.0866
10000	− 5.76	.102	−2.55	.0303	2.05	.0234
11000	− 7.94	.127	−3.18	.0367	1.75	.0110
12000	−10.4	.154	−3.88	.0436	1.60	.0065

TABLE II

Transfer Constants

f Cycles/sec.	Mid-series Type VI_1		Mid-half M-type IV_1		Mid-part of Wave-filter	
	$T_1 = A_1 + iB_1$		$T_m = \frac{1}{2}(A_m + iB_m)$		$T = T_1 + T_m = D + iS$	
1000	1.26	—	.97	—	2.23	—
1500	1.21	—	.99	—	2.20	—
2000	1.14	—	1.02	—	2.16	—
2500	1.04	—	1.08	—	2.12	—
3000	.89	—	1.23	—	2.12	—
3250	.79	—	1.46	—	2.25	—
3500	.66	—	2.64	—	3.30	—
3750	.480	$+i$ .02	1.077	$-i$1.51	1.557	$-i$1.49
4000	.100	$+i$ .10	.181	$-i$1.39	.281	$-i$1.29
4250	.025	$+i$ .51	.029	$-i$ .75	.054	$-i$ .24
4500	.019	$+i$ .73	.019	$-i$ .48	.038	$+i$ .25
5292	.018	$+i$1.30	.013	$+i$ 0	.031	$+i$1.30
6500	.032	$+i$2.20	.026	$+i$ .67	.058	$+i$2.87
6750	.043	$+i$2.48	.040	$+i$ .92	.083	$+i$3.40
7000	.173	$+i$2.97	.181	$+i$1.39	.354	$+i$4.36
7500	.910	$+i$3.10	1.125	$+i$1.51	2.035	$+i$4.61
8000	1.27	—	2.64	—	3.91	—
8500	1.52	—	1.53	—	3.05	—
9000	1.74	—	1.31	—	3.05	—
10000	2.09	—	1.15	—	3.24	—
11000	2.36	—	1.09	—	3.45	—
12000	2.59	—	1.06	—	3.65	—

TABLE III

Reflection Coefficients and Interaction Factor

f Cycles/sec.	$x=.80$		$m=.7454$		P	Q
	G_{x1}	H_{x1}	G_{m2}	H_{m2}		
3750	.58	2.18	0	−2.31	3.69	−3.11
4000	.30	3.06	.49	− .50	1.35	− .02
4250	1.04	3.75	2.58	0	3.73	3.27
4500	1.57	4.04	3.85	0	5.50	4.54
5292	∞	—	∞	—	∞	—
6500	1.17	2.45	2.90	0	4.19	8.19
6750	.78	2.67	1.92	0	2.87	9.47
7000	.30	3.22	.49	.50	1.50	12.44
7500	.60	4.14	0	2.39	4.67	15.75

TABLE IV

Transmission Losses

f Cycles/sec.	Transfer	Terminal		Interaction	Total $=L$	
	L_t	L_x	L_m	L_r	ΣL_j	Network Computation
1000	2.23	.88	.02	—	3.13	3.13
1500	2.20	.65	−.18	—	2.67	2.68
2000	2.16	.48	−.30	—	2.34	2.35
2500	2.12	.33	−.35	—	2.10	2.11
3000	2.12	.19	−.25	—	2.06	2.08
3250	2.25	.12	−.01	—	2.36	2.37
3500	3.30	.06	1.21	—	4.57	4.59
3750	1.557	.042	−.190	.025	1.434	1.487
4000	.281	.443	.082	−.300	.506	.508
4250	.054	.067	.004	.024	.149	.154
4500	.038	.023	.001	.001	.063	.068
5292	.031	.000	.000	.000	.031	.036
6500	.058	.052	.003	.005	.118	.127
6750	.083	.118	.011	.055	.267	.276
7000	.354	.443	.082	−.250	.629	.632
7500	2.035	.038	−.150	.009	1.932	1.987
8000	3.91	.06	1.21	—	5.18	5.19
8500	3.05	.11	.06	—	3.22	3.24
9000	3.05	.16	−.18	—	3.03	3.05
10000	3.24	.24	−.31	—	3.17	3.18
11000	3.45	.31	−.34	—	3.42	3.43
12000	3.65	.38	−.33	—	3.70	3.70

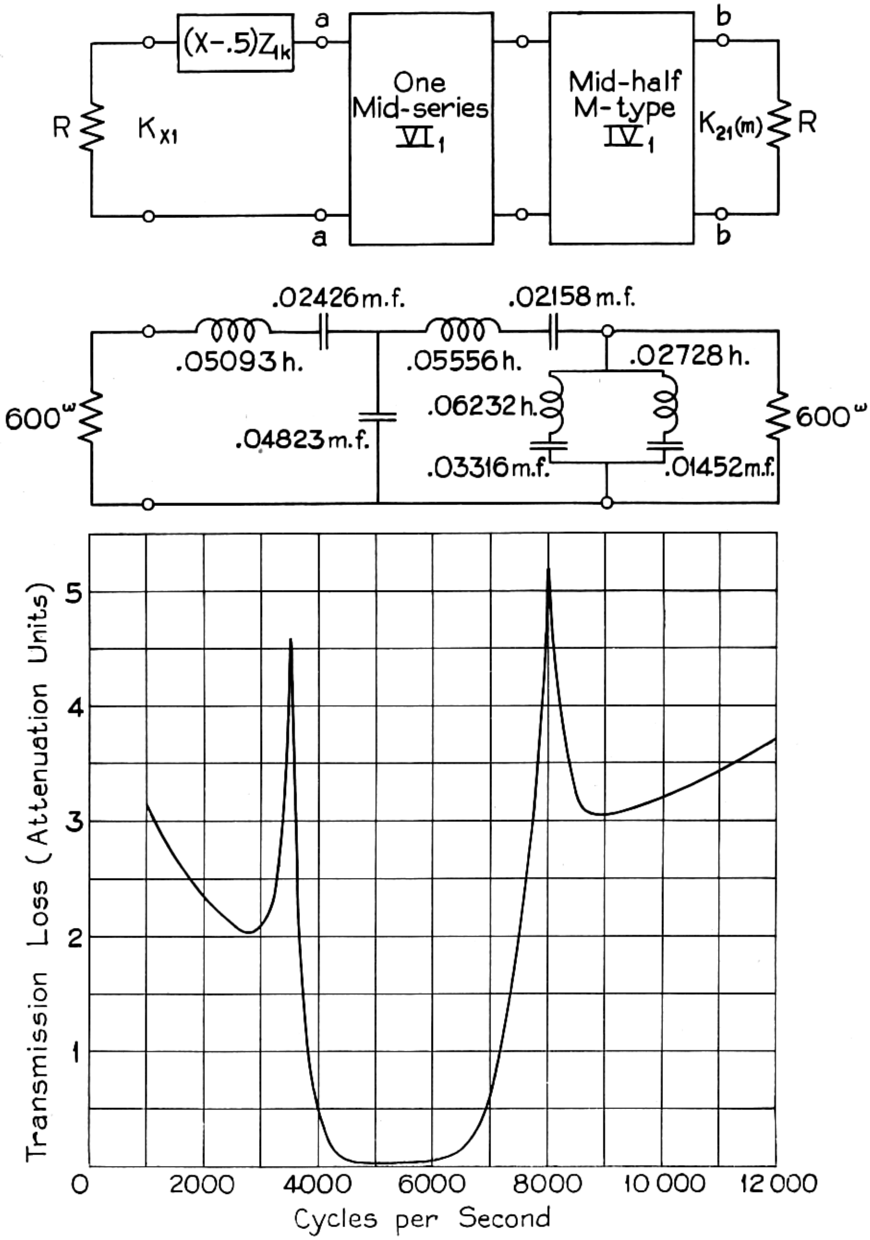

Fig. 4—Transmission Loss of Composite Band Pass Wave-Filter

APPENDIX

Derivation of Linear Transducer Formulae

The formula used in the text for a dissymmetrical composite wave-filter structure contains the image parameters [9] and is a special case of a general formula which is applicable to any linear transducer, active or passive. This general formula is derived here together with other useful ones.

A linear transducer will be defined as an electrical network which has two input and two output terminals and a structure such that so far as these terminals are concerned the currents are linear functions of the potential differences and therefore the principle of superposition holds. The structure may contain sources as well as sinks of energy; that is, the transducer may be active or passive. In the most general case, that of an active dissymmetrical linear transducer, four independent parameters are necessary to specify its electrical properties. Two sets of such parameters will be considered in deriving corresponding formulae, the image parameters and the recurrent parameters.

I. Image Parameters

1. *General Linear Transducer.* The parameters in this case are defined with reference to the single transducer in Fig. 3. Let the terminal impedances in this figure be so chosen that the impedances in the two directions from terminals a are equal, that is, the latter impedances are the "image" of each other, and at the same time a similar "image condition" holds with reference to terminals b. With the transducer so terminated, its *directional transfer constants* are here defined as $T_{ab}=\log_e(I_a/I_b)$ when transmitting from terminals a to terminals b, and $T_{ba}=\log_e(I_b/I_a)$ when transmitting from terminals b to terminals a. The *image impedance* W_a of the transducer is the impedance across terminals a in either direction, and the *image impedance* W_b is similarly defined at terminals b. In general, T_{ab} and T_{ba} are different, as are also W_a and W_b.

The transducer is now to be terminated by the general impedances Z_a and Z_b with an electromotive force E_a applied in series with Z_a.

[9] The relations among five other distinct sets of parameters for a transducer (such as a passive one) which can be specified by three complex parameters were given by G. A. Campbell in Cisoidal Oscillations, Trans. A. I. E. E., Vol. XXX, Part II, Table I, p. 885, 1911. The different sets correspond to the four normal networks designated as the T, the Π, the transformer, and the artificial line, and to the simple circuit one-point and two-point impedances. A sixth set, one-point and two-point admittances, was used in Appendix I of my paper in the B. S. T. J., Jan. 1923.

It is desired to obtain, among others, expressions for the sending end and receiving end currents I_a and I_b which contain the image parameters.

Each terminal impedance will be considered as equivalent to the image impedance at that end plus another impedance whose potential drop is to be replaced in the usual manner by an equal opposing electromotive force. In effect this equivalent electromotive force substitution reduces the system to one in which the transducer is terminated by its image impedances and in which determinate electromotive forces are acting at *both* ends. From this viewpoint, the total effective electromotive forces acting at the ends a and b of the transducer terminated by its image impedances W_a and W_b are, respectively,

$$E_a+(W_a-Z_a)I_a,$$

and

$$(W_b-Z_b)I_b. \tag{49}$$

Superposing the currents due to these electromotive forces at both ends we may write the current expressions immediately from the definitions of the parameters involved.

Thus

$$I_a=\frac{E_a+(W_a-Z_a)I_a}{2W_a}+\frac{(W_b-Z_b)I_b}{2W_b}e^{-T_{ba}},$$

and

$$I_b=\frac{E_a+(W_a-Z_a)I_a}{2W_a}e^{-T_{ab}}+\frac{(W_b-Z_b)I_b}{2W_b}. \tag{50}$$

Their solution gives the explicit formulae [10]

$$I_a=\frac{E_a}{W_a+Z_a}\,\frac{(1+r_b e^{-(T_{ab}+T_{ba})})}{(1-r_a r_b e^{-(T_{ab}+T_{ba})})},$$

and

$$I_b=\frac{E_a}{W_a+Z_a}\,\frac{(1+r_b)e^{-T_{ab}}}{(1-r_a r_b e^{-(T_{ab}+T_{ba})})}, \tag{51}$$

where r_a and r_b, the current reflection coefficients at terminals a and b, are

$$r_a=\frac{W_a-Z_a}{W_a+Z_a},$$

and

$$r_b=\frac{W_b-Z_b}{W_b+Z_b}.$$

[10] These formulæ may also be derived synthetically by the current reflection method.

Although the transducer has four independent parameters, it will be seen that the sending end current involves but three effective transducer parameters, the sum $(T_{ab}+T_{ba})$, W_a, and W_b. As a result, the four one-point impedance measurements which can be made upon the transducer itself, the open-circuit and short-circuit driving-point impedances at both ends, must have a relation between them. Let X_a and Y_a denote the driving-point impedances across terminals a when terminals b are open-circuited and short-circuited, respectively. Then if in (51) $Z_a=0$ and terminals b are open-circuited by putting $Z_b=\infty$, the impedance at terminals a, the *open-circuit impedance*, is

$$X_a=\frac{E_a}{I_a}=W_a \coth \tfrac{1}{2}(T_{ab}+T_{ba}). \tag{52}$$

Similarly for the *short-circuit impedance*, when $Z_a=0$ and $Z_b=0$,

$$Y_a=W_a \tanh \tfrac{1}{2}(T_{ab}+T_{ba}). \tag{53}$$

For the other end we get by interchanging subscripts

$$X_b=W_b \coth \tfrac{1}{2}(T_{ab}+T_{ba}), \tag{54}$$

and

$$Y_b=W_b \tanh \tfrac{1}{2}(T_{ab}+T_{ba}). \tag{55}$$

These give the necessary relation as

$$\frac{X_a}{Y_a}=\frac{X_b}{Y_b}. \tag{56}$$

Hence, *in the most general linear transducer the ratio of the open-circuit to short-circuit impedances at one end is equal to the corresponding ratio at the other end.*

Other important derived formulae are

$$T_{ab}+T_{ba}=2 \tanh^{-1}\sqrt{\frac{Y_a}{X_a}}=2 \tanh^{-1}\sqrt{\frac{Y_b}{X_b}}, \tag{57}$$

$$W_a=\sqrt{X_a Y_a}, \tag{58}$$

$$W_b=\sqrt{X_b Y_b}, \tag{59}$$

$$W_a-W_b=\sqrt{(X_a-X_b)\,(Y_a-Y_b)}, \tag{60}$$

and

$$W_a W_b=X_a Y_b=X_b Y_a. \tag{61}$$

Thus the open-circuit and short-circuit impedance measurements determine the *sum* of the directional transfer constants and both of the image impedances.

To obtain the separate values of T_{ab} and T_{ba}, it is necessary to make at least one two-point measurement, as seen from the formula for I_b which contains four distinct transducer parameters. For example, to find T_{ab} perhaps the simplest method is to terminate with the image impedance at terminals b, whence

$$T_{ab} = \log_e(I_a/I_b), \text{ where } Z_b = W_b. \tag{62}$$

The constant T_{ba} is the difference between the sum $(T_{ab}+T_{ba})$, obtained from (57), and T_{ab}; it may also be determined by a two-point measurement similar to the above for transmission in the opposite direction.

For some purposes it is convenient to have formulae involving the potential differences V_a and V_b across the two pairs of terminals, rather than the terminal impedances Z_a and Z_b and the series applied electromotive force E_a. Such formulae in combination with the above can be used to advantage in determining the currents and potential differences at points within a composite transducer. They are derived readily by making the substitutions in the above,

$$Z_a = \frac{E_a - V_a}{I_a},$$

and

$$Z_b = V_b/I_b. \tag{63}$$

For current transmission from terminals a to terminals b

$$I_a = I_b\left(\frac{e^{T_{ab}}+e^{-T_{ba}}}{2}\right) + \frac{V_b}{W_b}\left(\frac{e^{T_{ab}}-e^{-T_{ba}}}{2}\right),$$

and

$$V_a = V_b\frac{W_a}{W_b}\left(\frac{e^{T_{ab}}+e^{-T_{ba}}}{2}\right) + I_b W_a\left(\frac{e^{T_{ab}}-e^{-T_{ba}}}{2}\right). \tag{64}$$

Also,

$$I_b = I_a\left(\frac{e^{T_{ba}}+e^{-T_{ab}}}{2}\right) - \frac{V_a}{W_a}\left(\frac{e^{T_{ba}}-e^{-T_{ab}}}{2}\right),$$

and

$$V_b = V_a\frac{W_b}{W_a}\left(\frac{e^{T_{ba}}+e^{-T_{ab}}}{2}\right) - I_a W_b\left(\frac{e^{T_{ba}}-e^{-T_{ab}}}{2}\right). \tag{65}$$

Interchanging the subscripts and changing the signs of the currents in (64) will also lead to (65).

2. *Passive Linear Transducer.* Since the reciprocal theorem holds here one relation exists between the four parameters leaving three

independent ones. This relation is given directly by the theorem in the case where $Z_a = W_a$ and $Z_b = W_b$, the equivalent transfer currents being

$$\frac{e^{-T_{ab}}}{2W_a} = \frac{e^{-T_{ba}}}{2W_b}. \tag{66}$$

Although any three of these parameters might be assumed as independent, it is convenient to take as the independent parameters T, W_a, and W_b, where

$$T = D + iS = \tfrac{1}{2}(T_{ab} + T_{ba}) \tag{67}$$

is thus defined for the *passive transducer* as the *transfer constant. The transfer constant is the arithmetic mean of the two directional transfer constants.* The real and imaginary parts of T, namely D and S, will be called the *diminution constant* and the *angular constant* to distinguish them from the attenuation constant and the phase constant of the ordinary propagation constant to which they reduce in the case of a symmetrical transducer. Then these parameters are given by the formulae

$$T = \tanh^{-1}\sqrt{\frac{Y_a}{X_a}} = \tanh^{-1}\sqrt{\frac{Y_b}{X_b}},$$

$$W_a = \sqrt{X_a Y_a}, \tag{68}$$

and

$$W_b = \sqrt{X_b Y_b},$$

and are completely determined by the open-circuit and short-circuit driving-point impedances.

With these parameters the current formulae become

$$I_a = \frac{E_a}{W_a + Z_a} \frac{(1 + r_b e^{-2T})}{(1 - r_a r_b e^{-2T})},$$

and

$$I_b = \frac{E_a}{W_a + Z_a}\sqrt{\frac{W_a}{W_b}} \frac{(1 + r_b) e^{-T}}{(1 - r_a r_b e^{-2T})}, \tag{69}$$

$$= \frac{2E_a \sqrt{W_a W_b}\, e^{-T}}{(W_a + Z_a)(W_b + Z_b)(1 - r_a r_b e^{-2T})},$$

the latter being the one used in the text. Other forms are

$$I_a = \frac{E_a (Z_b \sinh T + W_b \cosh T)}{(W_a W_b + Z_a Z_b) \sinh T + (W_a Z_b + W_b Z_a) \cosh T},$$

and

$$I_b = \frac{E_a \sqrt{W_a W_b}}{(W_a W_b + Z_a Z_b) \sinh T + (W_a Z_b + W_b Z_a) \cosh T}. \tag{70}$$

Introducing the potential differences, for current transmission from terminals a to terminals b

$$I_a = I_b\sqrt{\frac{W_b}{W_a}}\cosh T + \frac{V_b}{\sqrt{W_a W_b}}\sinh T,$$

and (71)

$$V_a = V_b\sqrt{\frac{W_a}{W_b}}\cosh T + I_b\sqrt{W_a W_b}\sinh T.$$

Also,

$$I_b = I_a\sqrt{\frac{W_a}{W_b}}\cosh T - \frac{V_a}{\sqrt{W_a W_b}}\sinh T,$$

and (72)

$$V_b = V_a\sqrt{\frac{W_b}{W_a}}\cosh T - I_a\sqrt{W_a W_b}\sinh T.$$

II. Recurrent Parameters

1. *General Linear Transducer.* Here four parameters [11] of the transducer in Fig. 3 are defined in terms of its properties when it is one section of an infinite recurrent structure which is made up of identical sections, similarly oriented. With such terminal conditions for the transducer, its *directional propagation constants* are defined as follows: $\Gamma_{ab} = \log_e (I_a/I_b)$ when transmitting from terminals a to terminals b, and $\Gamma_{ba} = \log_e (I_b/I_a)$ when transmitting from terminals b to terminals a. The *characteristic impedance* K_a is the impedance across terminals a in the direction from a to b, and the *characteristic impedance* K_b is similarly defined for the impedance across terminals b in the opposite direction.

Terminating the transducer by the general impedances Z_a and Z_b and applying an electromotive force E_a in series with Z_a, the current formulae containing the recurrent parameters may be derived in a manner analogous to that used with the image parameters. In this case the total effective electromotive forces acting at the ends a and b of the transducer terminated by its characteristic impedances K_b and K_a are, respectively,

$$E_a + (K_b - Z_a)I_a,$$

and (73)

$$(K_a - Z_b)I_b.$$

[11] These parameters may also be designated in the general case as those of a generalized artificial line.

Hence,

$$I_a=\frac{E_a}{K_a+Z_a}\ \frac{(1+\rho_b e^{-(\Gamma_{ab}+\Gamma_{ba})})}{(1-\rho_a\rho_b e^{-(\Gamma_{ab}+\Gamma_{ba})})},$$

and (74)

$$I_b=\frac{E_a}{K_a+Z_a}\ \frac{(1+\rho_b)e^{-\Gamma_{ab}}}{(1-\rho_a\rho_b e^{-(\Gamma_{ab}+\Gamma_{ba})})},$$

where the current reflection coefficients at terminals a and b are

$$\rho_a=\frac{K_b-Z_a}{K_a+Z_a},$$

and

$$\rho_b=\frac{K_a-Z_b}{K_b+Z_b}.$$

Introducing the open-circuit and short-circuit driving-point impedances X_a, X_b and Y_a, Y_b of the transducer it follows that

$$\Gamma_{ab}+\Gamma_{ba}=2\ \tanh^{-1}\frac{\sqrt{(X_a-X_b)^2+2(X_aY_b+X_bY_a)}}{X_a+X_b}, \tag{75}$$

$$\left.\begin{matrix}K_a\\K_b\end{matrix}\right\}=\frac{1}{2}\Big[\sqrt{(X_a-X_b)^2+2(X_aY_b+X_bY_a)}\pm(X_a-X_b)\Big], \tag{76}$$

$$K_a-K_b=X_a-X_b, \tag{77}$$

and

$$K_aK_b=X_aY_b=X_bY_a. \tag{78}$$

Any three of these measured impedances are sufficient, because of relation (56), to obtain the *sum* $(\Gamma_{ab}+\Gamma_{ba})$, K_a, and K_b.

A directional propagation constant may be obtained separately from one two-point measurement; thus

$$\Gamma_{ab}=\log_e\ (I_a/I_b),\ \text{where}\ Z_b=K_a. \tag{79}$$

The current and potential difference at one pair of terminals in terms of those at the other are given by the following.

For current transmission from terminals a to terminals b

$$I_a=I_b\left(\frac{K_be^{\Gamma_{ab}}+K_ae^{-\Gamma_{ba}}}{K_a+K_b}\right)+\frac{V_b}{K_a+K_b}\ (e^{\Gamma_{ab}}-e^{-\Gamma_{ba}}),$$

and (80)

$$V_a=V_b\left(\frac{K_ae^{\Gamma_{ab}}+K_be^{-\Gamma_{ba}}}{K_a+K_b}\right)+I_b\frac{K_aK_b}{K_a+K_b}\ (e^{\Gamma_{ab}}-e^{-\Gamma_{ba}}).$$

Also,

$$I_b = I_a\left(\frac{K_a e^{\Gamma_{ba}} + K_b e^{-\Gamma_{ab}}}{K_a+K_b}\right) - \frac{V_a}{K_a+K_b}\,(e^{\Gamma_{ba}} - e^{-\Gamma_{ab}}),$$

and (81)

$$V_b = V_a\left(\frac{K_b e^{\Gamma_{ba}} + K_a e^{-\Gamma_{ab}}}{K_a+K_b}\right) - I_a\frac{K_a K_b}{K_a+K_b}\,(e^{\Gamma_{ba}} - e^{-\Gamma_{ab}}).$$

2. *Passive Linear Transducer.* Because of the reciprocal theorem the directional propagation constants become equal giving a single *propagation constant*,

$$\Gamma = A + iB = \Gamma_{ab} = \Gamma_{ba}, \tag{82}$$

which is obtainable from the general formula (75). Here A is the *attenuation constant* and B is the *phase constant.*

The current formulae become

$$I_a = \frac{E_a}{K_a+Z_a}\,\frac{(1+\rho_b e^{-2\Gamma})}{(1-\rho_a\rho_b e^{-2\Gamma})},$$

and (83)

$$I_b = \frac{E_a}{K_a+Z_a}\,\frac{(1+\rho_b)e^{-\Gamma}}{(1-\rho_a\rho_b e^{-2\Gamma})}.$$

In the other form they are

$$I_a = \frac{E_a[(-K_a+K_b+2Z_b)\sinh\Gamma + (K_a+K_b)\cosh\Gamma]}{[(2(K_aK_b+Z_aZ_b)-(K_a-K_b)(Z_a-Z_b))\sinh\Gamma + (K_a+K_b)(Z_a+Z_b)\cosh\Gamma]},$$

and (84)

$$I_b = \frac{E_a(K_a+K_b)}{[(2(K_aK_b+Z_aZ_b)-(K_a-K_b)(Z_a-Z_b))\sinh\Gamma + (K_a+K_b)(Z_a+Z_b)\cosh\Gamma]}.$$

Introducing the terminal potential differences, when transmitting from terminals a to terminals b

$$I_a = I_b\left(\cosh\Gamma - \frac{K_a-K_b}{K_a+K_b}\sinh\Gamma\right) + \frac{V_b}{K_a+K_b}\,2\sinh\Gamma,$$

and (85)

$$V_a = V_b\left(\cosh\Gamma + \frac{K_a-K_b}{K_a+K_b}\sinh\Gamma\right) + I_b\frac{K_aK_b}{K_a+K_b}\,2\sinh\Gamma;$$

and at the other terminals

$$I_b = I_a\left(\cosh \Gamma + \frac{K_a - K_b}{K_a + K_b}\sinh \Gamma\right) - \frac{V_a}{K_a + K_b}2\sinh \Gamma,$$

and (86)

$$V_b = V_a\left(\cosh \Gamma - \frac{K_a - K_b}{K_a + K_b}\sinh \Gamma\right) - I_a\frac{K_a K_b}{K_a + K_b}2\sinh \Gamma.$$

Comparison shows that the general formulae for the currents I_a and I_b given by (51) and (74) in terms of the two sets of parameters are of the same functional form involving their respective reflection coefficients; the latter are of slightly different functional forms. This similarity is what one expects when deriving the formulae synthetically by the current reflection method.

In all cases by (61) and (78)

$$W_a W_b = K_a K_b\,. \tag{87}$$

The sum $(T_{ab}+T_{ba})$, W_a, and W_b of any transducer are obviously also equal to the propagation constant and respective characteristic impedances of the two symmetrical transducers which can individually be formed with two such identical transducers.

If $T_{ab}=T_{ba}$, the reciprocal theorem holds only when $W_a=W_b$, for which case the transducer is symmetrical. On the other hand if $\Gamma_{ab}=\Gamma_{ba}$, this theorem holds irrespective of the values of K_a and K_b. In each of these cases which satisfies the theorem the transducer may be active or passive.

In an electrically symmetrical transducer, whether active or passive, two parameters specify its properties
where

$$T_{ab}=T_{ba}=\Gamma_{ab}=\Gamma_{ba},$$

and (88)

$$W_a=W_b=K_a=K_b,$$

in which case the corresponding formulae are identical in the parameters. Structural symmetry is not necessary here as may be seen, for example, in the case of a composite wave-filter made up of different mid-series sections whose characteristic impedances are equivalent.

In a passive dissymmetrical transducer the formulae containing hyperbolic functions are of simpler form with the parameters T, W_a, and W_b than with the parameters Γ, K_a, and K_b. The image parameter formulae are readily applicable where the transducer is made up of parts whose image impedances at the junctions are equivalent, as in the present case of a composite wave-filter. Simple relations exist here between these parameters of the transducer and of its parts, as shown in the text, which is not true with the other parameters. The recurrent parameter formulae, on the other hand, apply more naturally when dealing with a succession of identical dissymmetrical sections, or of different dissymmetrical sections whose characteristic impedances in one direction are equivalent, in which cases the propagation constant of the transducer is equal to the sum of the progagation constants of the parts. In conclusion, it is seen that the set of parameters most suitable for use in any case depends upon the particular structure of the transducer.