Theorems Regarding the Driving-Point Impedance of Two-Mesh Circuits*

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SYNOPSIS: The necessary and sufficient conditions that a driving-point impedance be realizable by means of a two-mesh circuit consisting of resistances, capacities, and inductances are stated in terms of the four roots and four poles (including the poles at zero and infinity) of the impedance. The roots and the poles are the time coefficients for the free oscillations of the circuit with the driving branch closed and opened, respectively. For assigned values of the roots, the poles are restricted to a certain domain, which is illustrated by figures for several typical cases; the case of real poles which are not continuously transformable into complex poles is of special interest. All driving-point impedances satisfying the general conditions can be realized by any one of eleven networks, each consisting of two resistances, two capacities, and two self-inductances with mutual inductance between them; these are the only networks without superfluous elements by which the entire range of possible impedances can be realized; the three remaining networks of this type give special cases only. For each of these eleven networks, formulas are given for the calculation of the values of the elements from the assigned values of the roots and poles.

1. STATEMENT OF RESULTS

THE object of this paper is, first, to determine the necessary and sufficient conditions that a driving-point impedance [1] be realizable by means of a two-mesh circuit consisting of resistances, capacities, and inductances, and second, to determine the networks [2] realizing any specified driving-point impedance staisfying these conditions.

These necessary and sufficient conditions are stated in the form of the following theorem:

Theorem I. Any driving-point impedance S of a two-mesh circuit consisting of resistances, capacities, and inductances is a function of the time coefficient $\lambda = ip$ of the form

$$S = H\frac{(\lambda-\alpha_1)(\lambda-\alpha_2)(\lambda-\alpha_3)(\lambda-\alpha_4)}{\lambda(\lambda-\beta_2)(\lambda-\beta_3)} \tag{1a}$$

$$= \frac{a_0\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4}{b_1\lambda^3+b_2\lambda^2+b_3\lambda}, \tag{1b}$$

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[1] The driving-point impedance of a circuit is the ratio of an impressed electromotive force at a point in a branch of the circuit to the resulting current at the same point.

[2] The networks considered in this paper consist of any arrangement of resistances, capacities, and inductances with two accessible terminals such that, if the two terminals are short-circuited, the resulting circuit has two independent meshes. Thus the impedance measured between the terminals of the network is the same as the driving-point impedance of the corresponding two-mesh circuit. Throughout the paper this distinction will be made in the use of the terms "network" and "circuit."

where $$H \geq 0,\ \alpha_1+\alpha_2 \leq 0,\ \alpha_1\alpha_2 \geq 0,\ \alpha_3+\alpha_4 \leq 0,\ \alpha_3\alpha_4 \geq 0,$$
$$\beta_2+\beta_3 \leq 0,\ \beta_2\beta_3 \geq 0, \tag{2}$$

and $$b_1^2(a_3^2-4a_4d)+b_2^2[(a_2-d)^2-4a_0a_4]+b_3^2(a_1^2-4a_0d)$$
$$-2b_1b_2[a_3(a_2-d)-2a_1a_4]-2b_1b_3[a_1a_3-2d(a_2-d)]$$
$$-2b_2b_3[a_1(a_2-d)-2a_0a_3]=0, \tag{3}$$

for all values of $d \geq 0$, *provided*
$$-a_4b_2^2+a_3b_2b_3-db_3^2 \geq 0, \tag{4}$$
$$-a_0b_3^2+(a_2-d)b_3b_1-a_4b_1^2 \geq 0, \tag{5}$$
$$-db_1^2+a_1b_1b_2-a_0b_2^2 \geq 0, \tag{6}$$

and, conversely, any impedance S of the form (1) *satisfying these conditions* (2)–(6) *can be realized as the driving-point impedance of a two-mesh circuit consisting of resistances, capacities, and inductances.*

Theorem I thus gives the most general form of this type of impedance, showing that it is a rational function of the time coefficient,[3] completely determined, except for a constant factor, by assigning four roots and two poles, in addition to the poles at zero and infinity, subject to certain conditions. The assigned roots and poles are the time coefficients for the free oscillations of the circuit with the driving branch closed and opened, respectively. That is, the roots and poles correspond to the resonant and anti-resonant points of the impedance.

The conditions are as follows: The real part of each root and pole is negative or zero; the roots and poles occur in pairs of real or conjugate complex quantities; certain additional restrictions must be satisfied, as stated in terms of the symmetric functions of the roots and poles by formulas (3)–(6).

By virtue of these restrictions, the pair of poles, for assigned values of the two pairs of roots, is limited to a certain domain of values. This domain is conveniently illustrated by plotting, in the upper half of the complex plane, the locus of one pole, the other pole being its conjugate. For real poles, a device is used to indicate pairs of points on the real axis. Figs. 3–5 show the domain of the poles, plotted in this manner, for several typical cases.

Provided the roots are not all real, this domain consists of a connected region of values, so that it is possible to pass from one pair of poles to any other pair satisfying the same conditions by a continuous transformation. In the case of four real roots, however, the domain consists, in general, of two non-connected regions, as illustrated in Fig. 5. Under these circumstances there is a region of real poles which are not continuously transformable into complex poles.

The networks realizing any specified driving-point impedance are

[3] All electrical oscillations considered in this paper are of the form $e^{\lambda t}$, where the time coefficient $\lambda = ip$ may have any value, real or complex.

Table I

Networks	1	2	3	4	5	6	7	8	9	10	11
M	$\frac{a_0T_1^2-dT_2^2-a_4T_3^2-a_3T_2T_3}{b_1T_1^2}.$	$\frac{T_1T_3}{b_2^2b_3}$	$\frac{(a_3b_2-db_3)T_3}{b_2^2T_1}$	$\frac{(cb_3-a_4b_1)T_1+(a_3b_3-a_4b_2)T_2}{b_3^2T_1}$	$\frac{T_1T_2}{b_2b_3^2}$	$\frac{(a_3b_3-a_4b_2)T_2}{b_3^2T_1}$	$\frac{(a_1b_2-db_1)T_1+(a_3b_2-db_3)T_3}{b_2^2T_1}$	$\frac{-b_1U_1^2\pm(a_1b_3-cb_2)U_1}{2T_1^2}$	$\frac{(db_3-a_3b_2)(a_3b_1+cb_2-a_1b_3)}{2b_2T_1^2}$	$\frac{(a_3b_3-a_4b_2)(-a_3b_1+cb_2-a_1b_3)}{2b_3T_1^2}$	$\frac{a_0b_2b_3-T_2T_3}{b_1b_2b_3}$
L_1	0	$\frac{T_3^2+a_0b_2^2}{b_1b_2^2}$	$\frac{T_3^2+a_0b_2^2}{b_1b_2^2}$	0	$\frac{T_2^2+a_0b_3^2}{b_1b_3^2}$	$\frac{cb_3-a_4b_1}{b_3^2}$	0	$\frac{\mp U_1(a_1b_3-cb_2)+c(a_3b_2-2db_3)+a_1(a_3b_3-2a_4b_2)}{2T_1^2}$	$\frac{c(a_3b_2-db_3)-a_4(a_1b_2-db_1)}{T_1^2}$	$\frac{d(a_4b_1-cb_3)-a_1(a_4b_2-a_3b_3)}{T_1^2}$	0
L_2	$\frac{U_1(a_1b_3-cb_2)+c(a_3b_2-2db_3)+a_1(a_3b_3-2a_4b_2)}{2T_1^2}$	$\frac{b_1T_1^2}{b_2^2b_3^2}$	$\frac{b_1(a_3b_2-db_3)^2}{b_2^2T_1^2}$	$\frac{d(a_4b_1-cb_3)+a_1(a_3b_3-a_4b_2)}{T_1^2}$	$\frac{b_1T_1^2}{b_2^2b_3^2}$	$\frac{b_1(a_3b_3-a_4b_2)^2}{b_3^2T_1^2}$	$\frac{c(a_3b_2-db_3)-a_4(a_1b_2-db_1)}{T_1^2}$	$\frac{b_1U_1^2}{T_1^2}$	$\frac{b_1(a_3b_2-db_3)^2}{b_2^2T_1^2}$	$\frac{b_1(a_4b_2-a_3b_3)^2}{b_3^2T_1^2}$	$\frac{a_0b_3^2+T_2^2}{b_1b_3^2}$
L_3	$\frac{-U_1(a_1b_3-cb_2)+c(a_3b_2-2db_3)+a_1(a_3b_3-2a_4b_2)}{2T_1^2}$	0	0	$\frac{cb_3-a_4b_1}{b_3^2}$	0	0	$\frac{a_1b_2-db_1}{b_2^2}$	0	0	0	$\frac{a_0b_2^2+T_3^2}{b_1b_2^2}$
R_1	0	$\frac{d}{b_2}$	$\frac{d}{b_2}$	0	$\frac{d}{b_2}$	0	$\frac{d}{b_2}$	0	$\frac{d}{b_2}$	0	$\frac{d}{b_2}$
R_2	$\frac{b_2U_1^2-(a_3b_2-2db_3)U_1}{2T_1^2}$	$\frac{T_1^2}{b_2b_3^2}$	$\frac{(a_3b_2-db_3)^2}{b_2T_1^2}$	$\frac{d(a_3b_3-a_4b_2)}{T_1^2}$	0	$\frac{d(a_3b_3-a_4b_2)}{T_1^2}$	$\frac{(a_3b_2-db_3)^2}{b_2T_1^2}$	$\frac{b_2U_1^2\mp(a_3b_2-2db_3)U_1}{2T_1^2}$	0	$\frac{a_3b_3-a_4b_2}{b_3^2}$	$\frac{T_1^2}{b_2b_3^2}$
R_3	$\frac{b_2U_1^2+(a_3b_2-2db_3)U_1}{2T_1^2}$	0	0	$\frac{a_3b_3-a_4b_2}{b_3^2}$	$\frac{T_1^2}{b_2b_3^2}$	$\frac{a_3b_3-a_4b_2}{b_3^2}$	0	$\frac{b_2U_1^2\pm(a_3b_2-2db_3)U_1}{2T_1^2}$	$\frac{(a_3b_2-db_3)^2}{b_2T_1^2}$	$\frac{d(a_3b_3-a_4b_2)}{T_1^2}$	0
C_1	∞	$\frac{b_3}{a_4}$	∞	$\frac{b_3}{a_4}$	$\frac{b_3}{a_4}$	$\frac{b_3}{a_4}$	∞	∞	∞	$\frac{b_3}{a_4}$	$\frac{b_3}{a_4}$
C_2	$\frac{2T_1^2}{b_3U_1^2+(a_3b_3-2a_4b_2)U_1}$	∞	$\frac{T_1^2}{a_4(a_3b_2-db_3)}$	$\frac{b_3T_1^2}{(a_3b_3-a_4b_2)^2}$	$\frac{b_2^2b_3}{T_1^2}$	$\frac{b_3T_1^2}{(a_3b_3-a_4b_2)^2}$	$\frac{T_1^2}{a_4(a_3b_2-db_3)}$	$\frac{2T_1^2}{b_3U_1^2\pm(a_3b_3-2a_4b_2)U_1}$	$\frac{b_2^2}{a_3b_2-db_3}$	∞	∞
C_3	$\frac{2T_1^2}{b_3U_1^2-(a_3b_3-2a_4b_2)U_1}$	$\frac{b_2^2b_3}{T_1^2}$	$\frac{b_2^2}{a_3b_2-db_3}$	∞	∞	∞	$\frac{b_2^2}{a_3b_2-db_3}$	$\frac{2T_1^2}{b_3U_1^2\mp(a_3b_3-2a_4b_2)U_1}$	$\frac{T_1^2}{a_4(a_3b_2-db_3)}$	$\frac{b_3T_1^2}{(a_3b_3-a_4b_2)^2}$	$\frac{b_2^2b_3}{T_1^2}$

$$s = \frac{a_0\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4}{b_1\lambda^3+b_2\lambda^2+b_3\lambda}, \quad \lambda = ip,$$

$$d^2(b_2^2-4b_1b_3)-2d(2a_4b_1^2+a_2b_2^2+2a_0b_3^2-a_3b_1b_2-2a_2b_1b_3-a_1b_2b_3)+\left[a_3^2b_1^2+(a_2^2-4a_0a_4)b_2^2+a_1^2b_3^2-2(a_2a_3-2a_1a_4)b_1b_2-2a_1a_3b_1b_3-2(a_1a_2-2a_0a_3)b_2b_3\right]=0,$$

$$c = a_2-d,$$

$$T_1 = \pm\sqrt{-a_4b_2^2+a_3b_2b_3-db_3^2}, \qquad U_1 = \sqrt{a_3^2-4a_4d},$$

$$T_2 = \pm\sqrt{-a_0b_3^2+cb_3b_1-a_4b_1^2}, \qquad U_2 = \sqrt{c^2-4a_0a_4},$$

$$T_3 = \pm\sqrt{-db_1^2+a_1b_1b_2-a_0b_2^2}, \qquad U_3 = \sqrt{a_1^2-4a_0d}.$$

with signs chosen so that

$$b_1T_1+b_2T_2+b_3T_3 = 0,$$

determined by the arrangement and magnitudes of the elements, as given by the following theorem:

Theorem II. All driving-point impedances satisfying the necessary and sufficient conditions, as stated in Theorem I, can be realized by any one of the eleven networks shown by Fig. 1, *upon assigning to the elements of each network the values given by Table I. These eleven networks are the only networks without superfluous elements by which the entire range of possible impedances can be realized.*

By Theorem II, any network obtained from a two-mesh circuit

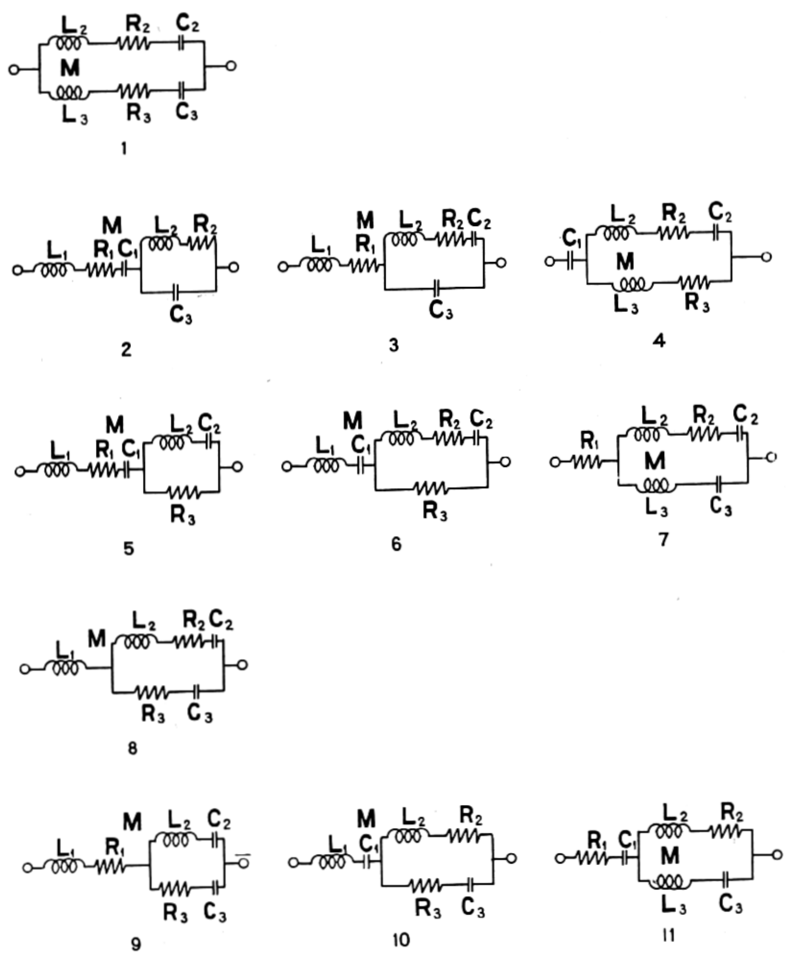

Fig. 1—Networks realizing any driving-point impedance of a two-mesh circuit consisting of resistances, capacities, self-inductances, and mutual inductances

consisting of resistances, capacities, and inductances can be replaced, in so far as the impedance between terminals is concerned, by any one of the eleven networks shown by Fig. 1, upon assigning the proper values to the elements. Each of these networks consists of two resistances, two capacities, and two self-inductances with mutual inductance between them.

Each of these eleven networks realizes impedances with arbitrarily assigned roots and with poles anywhere in the entire domain of possibilities, subject to the general conditions stated in Theorem I. Special cases of these networks realize, for arbitrarily assigned roots, only critical lines and points in the domain. All these special cases are listed in Table III, with a specification of the lines or points in the domain realizable by each, as illustrated by Figs. 4 and 5.

Certain limited regions of the domain can be realized by networks which contain no mutual inductance and which are not special cases of the networks given by Theorem II. These networks are given by the following theorem:

Theorem III. Any driving-point impedance of a two-mesh circuit consisting of resistances, capacities, and self-inductances can be realized by at least three and not more than five of the twelve networks shown by Fig. 2, upon assigning to the elements of each network the values given by Table II. These twelve networks are the only networks without mutual inductance and without superfluous elements by which any impedance can, in general, be realized.

These twelve networks, taken together, cover that portion of the domain realizable without mutual inductance. Networks with mutual inductance are needed in order to cover the entire domain. These twelve are the only networks, without superfluous elements, realizing limited regions in the domain. Each of these networks consists of two resistances, two capacities, two self-inductances, and one additional resistance, capacity, or self-inductance. The twelve networks, with their special cases, are all listed in Table III, with a specification of the regions, lines, or points realizable by each.

In addition to the specific formulas for the networks of Figs. 1 and 2, it is convenient to have general formulas for the computation of all networks meeting the given conditions, including those networks with superfluous elements as well as all special cases. The most general two-mesh circuit is shown by Fig. 6; accordingly, the most general network under consideration is that shown by Fig. 7. Formulas

Table II

Networks	12	13	14	15	16	17
L_1	$\frac{a_0T_1^2-dT_2^2-a_4T_3^2-a_3T_2T_3}{b_1T_1^2}$	$\frac{a_0b_2b_3-T_2T_3}{b_1b_2b_3}$	$\frac{(a_1b_2-db_1)T_1+(a_3b_2-db_3)T_3}{b_2^2T_1}$	$\frac{(cb_3-a_4b_1)T_1+(a_3b_3-a_4b_2)T_2}{b_3^2T_1}$	0	$\frac{a_0}{b_1}$
L_2	$\frac{b_1U_1^2+(a_1b_3-cb_2)U_1}{2T_1^2}$	$-\frac{T_1T_2}{b_2b_3^2}$	$\frac{(a_3b_2-db_3)(a_3b_1+cb_2-a_1b_3)}{2b_2T_1^2}$	$\frac{(a_3b_3-a_4b_2)(a_3b_1-cb_2+a_1b_3)}{2b_3T_1^2}$	$\frac{b_1U_2^2-(cb_1-2a_0b_3)U_2}{2T_2^2}$	$\frac{T_2^2}{b_1b_3^2}$
L_3	$\frac{b_1U_1^2-(a_1b_3-cb_2)U_1}{2T_1^2}$	$-\frac{T_1T_3}{b_2^2b_3}$	$\frac{(db_3-a_3b_2)T_3}{b_2^2T_1}$	$\frac{(a_4b_2-a_3b_3)T_2}{b_3^2T_1}$	$\frac{b_1U_2^2+(cb_1-2a_0b_3)U_2}{2T_2^2}$	0
R_1	0	$\frac{d}{b_2}$	$\frac{d}{b_2}$	0	$\frac{dT_2^2-a_4T_3^2-a_0T_1^2-cT_1T_3}{b_2T_2^2}$	$\frac{db_1b_3-T_1T_3}{b_1b_2b_3}$
R_2	$\frac{b_2U_1^2-(a_3b_2-2db_3)U_1}{2T_1^2}$	$\frac{T_1^2}{b_2b_3^2}$	$\frac{(a_3b_2-db_3)^2}{b_2T_1^2}$	$\frac{d(a_3b_3-a_4b_2)}{T_1^2}$	$\frac{b_2U_2^2+(a_1b_3-a_3b_1)U_2}{2T_2^2}$	$-\frac{T_1T_2}{b_1b_3^2}$
R_3	$\frac{b_2U_1^2+(a_3b_2-2db_3)U_1}{2T_1^2}$	0	0	$\frac{a_3b_3-a_4b_2}{b_3^2}$	$\frac{b_2U_2^2-(a_1b_3-a_3b_1)U_2}{2T_2^2}$	$-\frac{T_2T_3}{b_1^2b_3}$
C_1	∞	$\frac{b_3}{a_4}$	∞	$\frac{b_3}{a_4}$	∞	$\frac{b_3}{a_4}$
C_2	$\frac{2T_1^2}{b_3U_1^2+(a_3b_3-2a_4b_2)U_1}$	∞	$\frac{T_1^2}{a_4(a_3b_2-db_3)}$	$\frac{b_3T_1^2}{(a_3b_3-a_4b_2)^2}$	$\frac{2T_2^2}{b_3U_2^2+(cb_3-2a_4b_1)U_2}$	∞
C_3	$\frac{2T_1^2}{b_3U_1^2-(a_3b_3-2a_4b_2)U_1}$	$\frac{b_2^2b_3}{T_1^2}$	$\frac{b_2^2}{a_3b_2-db_3}$	∞	$\frac{2T_2^2}{b_3U_2^2-(cb_3-2a_4b_1)U_2}$	$\frac{b_1^2b_3}{T_2^2}$

Networks	18	19	20	21	22	23
L_1	$\frac{a_0}{b_1}$	0	0	$\frac{a_0}{b_1}$	$\frac{a_0}{b_1}$	0
L_2	$\frac{(cb_1-a_0b_3)^2}{b_1T_2^2}$	$\frac{a_0(cb_3-a_4b_1)}{T_2^2}$	$\frac{b_1U_3^2+(a_1b_1-2a_0b_2)U_3}{2T_3^2}$	$\frac{T_3^2}{b_1b_2^2}$	$\frac{(a_1b_1-a_0b_2)^2}{b_1T_3^2}$	$\frac{a_0(a_1b_2-db_1)}{T_3^2}$
L_3	0	$\frac{cb_3-a_4b_1}{b_3^2}$	$\frac{b_1U_3^2-(a_1b_1-2a_0b_2)U_3}{2T_3^2}$	0	0	$\frac{a_1b_2-db_1}{b_2^2}$
R_1	$\frac{(a_1b_1-a_0b_2)T_2+(cb_1-a_0b_3)T_3}{b_1^2T_2}$	$\frac{(cb_3-a_4b_1)T_1+(a_3b_3-a_4b_2)T_2}{b_3^2T_2}$	0	$\frac{d}{b_2}$	0	$\frac{d}{b_2}$
R_2	$\frac{(cb_1-a_0b_3)(cb_2+a_3b_1-a_1b_3)}{2b_1T_2^2}$	$\frac{(cb_3-a_4b_1)(-a_3b_1+cb_2+a_1b_3)}{2b_3T_2^2}$	$\frac{b_2U_3^2-(a_1b_2-2db_1)U_3}{2T_3^2}$	0	$\frac{d(a_1b_1-a_0b_2)}{T_3^2}$	$\frac{(a_1b_2-db_1)^2}{b_2T_3^2}$
R_3	$\frac{(a_0b_3-cb_1)T_3}{b_1^2T_2}$	$\frac{(a_4b_1-cb_3)T_1}{b_3^2T_2}$	$\frac{b_2U_3^2+(a_1b_2-2db_1)U_3}{2T_3^2}$	$\frac{T_3^2}{b_1^2b_2}$	$\frac{a_1b_1-a_0b_2}{b_1^2}$	0
C_1	∞	$\frac{b_3}{a_4}$	$\frac{b_3T_3^2}{a_4T_3^2-dT_2^2-a_0T_1^2-a_1T_1T_2}$	$\frac{b_1b_2b_3}{a_4b_1b_2-T_1T_2}$	$\frac{b_1^2T_3}{(cb_1-a_0b_3)T_3+(a_1b_1-a_0b_2)T_2}$	$\frac{b_2^2T_3}{(a_3b_2-db_3)T_3+(a_1b_2-db_1)T_1}$
C_2	$\frac{T_2^2}{a_4(cb_1-a_0b_3)}$	$\frac{b_3T_2^2}{(cb_3-a_4b_1)^2}$	$\frac{2T_3^2}{b_3U_3^2+(a_3b_1-cb_2)U_3}$	$-\frac{b_1b_2^2}{T_1T_3}$	$\frac{2b_1T_3^2}{(a_1b_1-a_0b_2)(a_1b_3-cb_2+a_3b_1)}$	$\frac{2b_2T_3^2}{(a_1b_2-db_1)(a_1b_3+cb_2-a_3b_1}$
C_3	$\frac{b_1^2}{cb_1-a_0b_3}$	∞	$\frac{2T_3^2}{b_3U_3^2-(a_3b_1-cb_2)U_3}$	$-\frac{b_1^2b_2}{T_2T_3}$	$\frac{b_1^2T_3}{(a_0b_2-a_1b_1)T_2}$	$\frac{b_2^2T_3}{(db_1-a_1b_2)T_1}$

for the computation of the elements of this general network can be stated in the form of the following theorem:

Theorem IV. Any driving-point impedance satisfying the necessary and sufficient conditions, as stated in Theorem I, can be realized

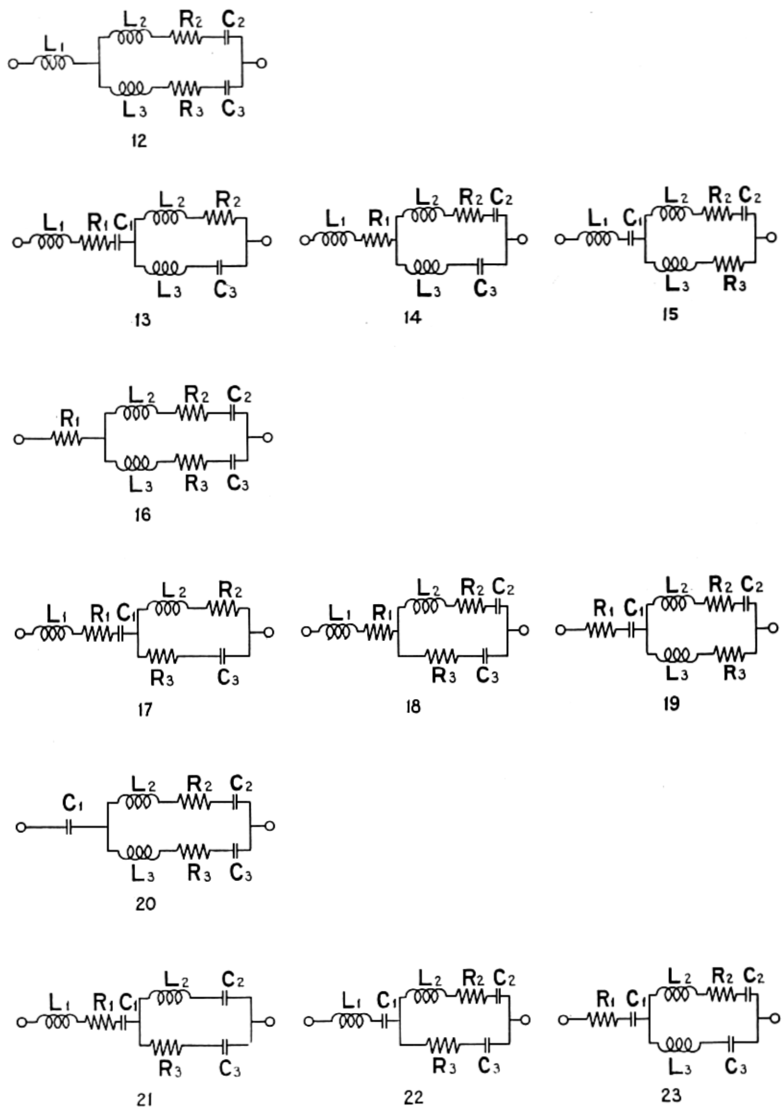

Fig. 2—Networks without mutual inductance realizing any driving-point impedance of a two-mesh circuit consisting of resistances, capacities, and self-inductances.

by any network of the form of Fig. 7, provided the elements of the network satisfy the following relations:

$$L_1'L_2'+L_1'L_3'+L_2'L_3'=a_0k^2, \quad (7)$$

$$R_1R_2+R_1R_3+R_2R_3=dk^2, \quad (8)$$

$$D_1D_2+D_1D_3+D_2D_3=a_4k^2, \quad (9)$$

$$L_2'+L_3'=b_1k^2, \quad (10)$$

$$R_2+R_3=b_2k^2, \quad (11)$$

$$D_2+D_3=b_3k^2, \quad (12)$$

$$R_2D_3-R_3D_2=\pm k^3(-a_4b_2^2+a_3b_2b_3-db_3^2)^{1/2}, \quad (13)$$

$$D_2L_3'-D_3L_2'=\pm k^3[-a_0b_3^2+(a_2-d)b_3b_1-a_4b_1^2]^{1/2}, \quad (14)$$

$$L_2'R_3-L_3'R_2=\pm k^3(-db_1^2+a_1b_1b_2-a_0b_2^2)^{1/2}, \quad (15)$$

where

$$D_1=C_1^{-1},\ D_2=C_2^{-1},\ D_3=C_3^{-1}, \quad (16)$$

and

$$L_1'=L_1+M_{12}+M_{13}+M_{23}, \quad (17)$$

$$L_2'=L_2+M_{12}-M_{13}-M_{23}, \quad (18)$$

$$L_3'=L_3-M_{12}+M_{13}-M_{23}, \quad (19)$$

the positive directions in Fig. 7 all being assigned arbitrarily to the right. The signs of (13)–(15) *are chosen so as to satisfy the identity*

$$(R_2D_3-R_3D_2)(L_2'+L_3')+(D_2L_3'-D_3L_2')(R_2+R_3) +(L_2'R_3-L_3'R_2)(D_2+D_3)=0. \quad (20)$$

The value of d is given by equation (3), *which may be written in the form*

$$d^2(b_2^2-4b_1b_3)-2d(2a_4b_1^2+a_2b_2^2+2a_0b_3^2-a_3b_1b_2-2a_2b_1b_3-a_1b_2b_3) +[a_3^2b_1^2+(a_2^2-4a_0a_4)b_2^2+a_1^2b_3^2-2(a_2a_3-2a_1a_4)b_1b_2-2a_1a_3b_1b_3 -2(a_1a_2-2a_0a_3)b_2b_3]=0. \quad (21)$$

The parameter k may have any real value other than zero.

In these formulas the value of k is independent of the impedance, but can be chosen so as to give particular forms of the network. If the necessary and sufficient conditions as stated by Theorem I are satisfied, the values of the elements given by these formulas are positive or zero, and the values of the inductances satisfy the usual restrictions. The formulas of Tables I and II, for example, can all be computed by means of Theorem IV.

2. The Driving-Point Impedance of a Two-Mesh Circuit

Previous investigations of the two-mesh circuit have been directed, for the most part, toward the determination of the free periods (resonant frequencies and associated damping constants) of the circuit from the known values of the elements. This problem is intimately related to the determination of the driving-point impedance of the circuit, since the free periods of the circuit can be found by setting the driving-point impedance in any one mesh equal to zero.[4] By this method the free periods are found as the roots of an equation of the fourth degree,[5] the exact solution of which involves, in general, cumbersome formulas. In order to obtain formulas which are better adapted to numerical computation, various approximations are usually made.[6]

This electrical problem of the free oscillations of a circuit is formally the same as the dynamical problem of the small oscillations of a system about a position of equilibrium. The determination of the free periods of a circuit can be made directly from the solution of this dynamical problem.[7]

The first part of this paper treats a much more general problem than the determination of the driving-point impedance of a particular circuit from the given values of the elements, namely, the determination of the entire range of possibilities, together with the inherent limitations, of such an impedance. The method employed is to find the general form of the impedance as a function of the time coefficient, and then to investigate the restrictions which must be satisfied by a function of this character in order that it may represent an impedance realizable by means of a circuit consisting of resistances, capacities, and inductances. In the present paper, this investigation is limited to the driving-point impedance of a two-mesh circuit; the driving-point impedance of an n-mesh circuit will be treated in a future paper.

The driving-point impedance of any circuit containing no resistances has been investigated in a previous paper,[8] where it has been shown that any such impedance is a pure reactance with a number of resonant and anti-resonant frequencies which alternate with each other, and

[4] G. A. Campbell, *Transactions of the A. I. E. E.*, 30, 1911, pages 873–909.

[5] An exhaustive discussion of this fourth degree equation has been given by J. Sommer, *Annalen der Physik*, fourth series, 58, 1919, pages 375–392.

[6] For typical methods of solution see the papers of L. Cohen, *Bulletin of the Bureau of Standards*, 5, 1908–9, pages 511–541; B. Macků, *Jahrbuch der drahtlosen Telegraphie und Telephonie*, 2, 1909, pages 251–293; V. Bush, *Proceedings of the I. R. E.*, 5, 1917, pages 363–382.

[7] Representative investigations of this dynamical problem are those of Lord Rayleigh, *Proceedings of the London Mathematical Society*, 4, 1873, pages 357–368, *Philosophical Magazine*, fifth series, 21, 1886, pages 369–381, and sixth series, 3, 1902, pages 97–117 ("Scientific Papers," I, 170–181, II, 475–485, and V, 8–26); E. J. Routh, "Advanced Rigid Dynamics," sixth edition, 1905, pages 232–243; A. G. Webster, "Dynamics," second edition, 1912, pages 157–164.

[8] R. M. Foster, *Bell System Technical Journal*, 3, 1924, pages 259–267.

that any such impedance may be realized by a network consisting of a number of simple resonant elements (inductance and capacity in series) in parallel or a number of simple anti-resonant elements (inductance and capacity in parallel) in series.

With resistances added to the circuit, the impedance is, in general, complex; that is, it has both resistance and reactance components. For a two-mesh circuit the impedance is expressed as a function of the time coefficient by Theorem I.

Formula (1) gives the driving-point impedance of a two-mesh circuit for any electrical oscillation of the form $e^{\lambda t}$, where the time coefficient λ may have any value, real or complex. The time coefficients for the free oscillations of the circuit with the driving branch closed are the roots of the numerator (α_1, α_2, α_3, α_4), as given by (1a); the free periods of the circuit with the driving branch opened are the roots of the denominator (β_2, β_3), that is, the poles of the impedance function. For a complex value of the time coefficient, $\lambda=\lambda_1+i\lambda_2$, λ_1 is the damping factor and λ_2 is the frequency multiplied by 2π.

The two forms of formula (1) are equivalent, but each has its special advantages. Sometimes one, sometimes the other, form is more convenient; they will be used interchangeably throughout the paper.

Formula (1a) gives the impedance directly in terms of the roots and poles. Formula (1b) gives the impedance in terms of the symmetric functions of the roots and poles, with the addition of an arbitrary factor. Thus, without changing the impedance, all the coefficients of the numerator and denominator of (1b) may be multiplied by the same constant factor having any value other than zero. Formulas stated in terms of the coefficients of (1b) are in homogeneous and symmetrical form, and have the added advantage of involving real quantities only.

The special case of one root equal to zero is obtained by setting $\alpha_1=0$ in (1a) and $a_4=0$ in (1b). For one root infinite, however, in (1a) it is necessary to set $\alpha_1=\infty$ and $H=0$, with the provision that $H\alpha_1$ be finite; whereas in (1b) it is simply necessary to set $a_0=0$.

It is sometimes convenient to add the notation $\beta_1=0$ and $\beta_4=\infty$, corresponding to the poles at zero and infinity. In formula (1b) the corresponding addition to the notation consists of the coefficients $b_0=0$ and $b_4=0$.

By the general restrictions (2) the constant H is positive or zero, and the roots and poles are arranged in three pairs, (α_1, α_2), (α_3, α_4), and (β_2, β_3), each pair being the roots of a quadratic equation with positive real coefficients. Thus each pair of the roots and poles is either a pair of conjugate complex quantities or a pair of real quantities, with the added provision that the real part of each root and pole is negative or zero.

Stated in terms of (1b), these general restrictions (2) require all the coefficients to be real and to have the same sign. Throughout this paper these signs will always be taken positive; thus all the a's and b's are positive or zero. In order to provide that the real part of each root be negative or zero, the coefficients of the numerator must satisfy the additional requirement

$$-a_4a_1^2+a_1a_2a_3-a_0a_3^2\geq 0, \tag{22}$$

and also

$$a_2^2-4a_0a_4\geq 0. \tag{23}$$

The second condition (23) is satisfied automatically by virtue of the first condition (22), unless both a_1 and a_3 are zero; in that case (23) is required. These are precisely the necessary and sufficient conditions that the numerator of (1b) be factorable into two real quadratic factors with positive coefficients.

In addition to the general restrictions (2) upon the individual roots and poles, there are certain additional conditions which must be satisfied by all the roots and poles together. These conditions are more conveniently stated in terms of the coefficients by prescribing a certain domain of values of the eight coefficients (a_0, a_1, a_2, a_3, a_4, b_1, b_2, b_3) such that the coefficients of any driving-point impedance of a two-mesh circuit lie in this domain, and, conversely, any set of values in this domain can be realized as the coefficients of a driving-point impedance of a two-mesh circuit.

By a realizable circuit is understood a circuit consisting of resistances, capacities, and self-inductances, with positive or zero values, together with mutual inductances with values such that every principal minor of the determinant of the inductances is positive or zero. In the case of two self-inductances with mutual inductance between them, this reduces to the well known condition $L_1L_2-M^2\geq 0$.

The domain is defined analytically by formulas (3)–(6), in terms of a parameter d. This parameter is intimately related to the resistances in the circuit, as will be shown later. In order that this domain may contain real values, the following relation must be satisfied:

$$-d^3+2a_2d^2-(a_1a_3+a_2^2-4a_0a_4)d+(-a_4a_1^2+a_1a_2a_3-a_0a_3^2)\geq 0, \tag{24}$$

or in equivalent form,

$$-[d-a_0(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)][d-a_0(\alpha_1+\alpha_3)(\alpha_2+\alpha_4)]$$
$$[d-a_0(\alpha_1+\alpha_4)(\alpha_2+\alpha_3)]\geq 0. \tag{25}$$

Provided there is one pair of conjugate complex roots of the numerator of the impedance, α_1 and α_2, the value of d is restricted to the range from zero to the smallest real root of (24), that is,

$$0 \leq d \leq a_0(\alpha_1+\alpha_2)(\alpha_3+\alpha_4). \tag{26}$$

In the case of four real roots, $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$, the parameter d is restricted to the values

$$\left.\begin{aligned} 0 \leq d \leq a_0(\alpha_1+\alpha_2)(\alpha_3+\alpha_4), \\ a_0(\alpha_1+\alpha_3)(\alpha_2+\alpha_4) \leq d \leq a_0(\alpha_1+\alpha_4)(\alpha_2+\alpha_3). \end{aligned}\right\} \tag{27}$$

Thus there are, in general, two distinct ranges for the value of d in this case. The corresponding domain of values of the roots and poles consists of two non-connected regions, so that it is impossible to pass by a continuous transformation from a set of values in one region to a set in the other.

Formulas (3)–(6) are symmetrical in three different respects, since they remain unaltered upon interchanging certain pairs of elements, which may be any one of the three following sets:

$$\left.\begin{aligned} &\text{(a) } b_1 \text{ and } b_2,\ a_0 \text{ and } d,\ a_3 \text{ and } (a_2-d), \\ &\text{(b) } b_1 \text{ and } b_3,\ a_0 \text{ and } a_4,\ a_1 \text{ and } a_3, \\ &\text{(c) } b_2 \text{ and } b_3,\ a_4 \text{ and } d,\ a_1 \text{ and } (a_2-d). \end{aligned}\right\} \tag{28}$$

These three sets correspond to interchanging resistances and inductances, inductances and capacities, and resistances and capacities, respectively.

Since d is always positive or zero, formulas (4)–(6) lead to simple necessary conditions, namely,

$$a_3b_3 - a_4b_2 \geq 0, \tag{29}$$

$$-a_4b_1^2 + a_2b_1b_3 - a_0b_3^2 \geq 0, \tag{30}$$

$$a_1b_1 - a_0b_2 \geq 0. \tag{31}$$

The first and third of these conditions are conveniently interpreted in terms of the roots and poles: the sum of the reciprocals of the poles is algebraically greater than or equal to the sum of the reciprocals of the roots; and the sum of the poles is algebraically greater than or equal to the sum of the roots.

3. Domain of Poles for Assigned Roots

The conditions (2)–(6) define a domain of values for the roots and poles without distinguishing in any way those roots and poles which may be chosen independently. For many purposes it is convenient to specialize the problem to the extent of assigning definite values to the roots, subject, of course, to the restrictions (2), and then to investigate the domain of the poles which can be associated with these assigned roots.

For the mathematical analysis of the problem it is convenient to assign values of the coefficients $a_0 \ldots . a_4$, subject to the restrictions stated in the preceding section, and then to plot the domain for the coefficients b_1, b_2, b_3,—treating the latter as homogeneous coordinates [9] in the plane, with $x = b_2/b_1$ and $y = b_3/b_1$.

With this method of representation, equation (3) is, for any fixed value of d, the equation of a conic. Considering d as a variable parameter, (3) represents a one-parameter family of conics. Each curve of this family is tangent to the four lines

$$\alpha_j^2 b_1 + \alpha_j b_2 + b_3 = 0, \qquad (j = 1, 2, 3, 4). \tag{32}$$

These lines are real lines in the plane if, and only if, the corresponding roots are real. They are all tangent to the parabola

$$b_2^2 - 4b_1 b_3 = 0, \tag{33}$$

which is the limiting case of the conic (3) as d becomes infinite. This parabola is a critical curve for the poles; every point in the plane above the parabola corresponds to a pair of conjugate complex poles, every point below the curve to a pair of real and distinct poles, and every point on the curve to a pair of real and equal poles.

The complete family of conics, that is, the set of curves for all real values of d, might be defined as the family of conics tangent to these four lines, which are the four lines tangent to the critical parabola (33) corresponding to the four roots of the impedance.

Not all the curves of this family lie in the domain of poles, however, since the conditions (4)–(6) must also be satisfied. For any fixed value of d, each of the three equations (4)–(6) is a degenerate conic, that is, a pair of straight lines. The six lines defined by these conditions are all tangent to the conic (3) corresponding to this same value of d. The inequalities (4)–(6) thus demand, in general, that the domain of poles lie within the area bounded by these six lines. Thus only those conics of the family (3) which are real ellipses, or their limiting cases, lie within the domain.

The condition that the conic (3) be an ellipse is precisely the necessary restriction on the value of d already stated, formula (24). Ellipses are obtained for all negative values of d, but these are not in the

[9] For some purposes the other choices of x and y might be used; this choice is more convenient here inasmuch as $-x$ is the sum and y the product of the poles.

domain, since by the conditions of the electrical problem d must be positive or zero. Ellipses for values of d from zero up to the smallest real root of the equation (24) are in the domain. If the roots of the impedance are all complex, equation (24) has three real roots, and thus there is a range of values of d from the second to the third root, arranged in the order of magnitude, for which the curves are ellipses, but these ellipses are imaginary, that is, there are no real points on them; thus there is only the one range of d which gives points in the domain. If two roots of the impedance are real and two complex, equation (24) has only the one real root, and thus there is only the one range of d. If all four roots of the impedance are real, however, equation (24) has again three real roots, and both ranges of d give real ellipses. In this case the two sets of ellipses are separate and distinct.

For the limiting values of d, that is, for the roots of equation (24), the corresponding conic (3) degenerates into a pair of coincident straight lines. Only those segments of these lines which satisfy the corresponding inequalities (4)–(6) are in the domain. Such segments are the limiting cases of the real ellipses for values of d above or below the critical values, as the case may be.

The domain of poles, plotted in terms of the coefficients in the manner described, consists of that domain covered by these real ellipses for $d \geq 0$, a domain bounded by the envelope of the curves. The env elopeconsists of the conic for $d=0$ and the four lines (32). For the case of four complex roots of the impedance, therefore, the domain consists simply of the region bounded by the ellipse (3) for $d=0$. For two complex and two real roots, the domain consists of the region bounded by the ellipse with the addition of the corner bounded by the ellipse and the two tangent lines to the ellipse corresponding to the two real roots. For four real roots, the domain consists of the region bounded by the ellipse together with the two corners bounded by the ellipse and the tangent lines, one by the two lines corresponding to the two smallest roots and the other the two largest roots; and a second region consisting of the quadrilateral bounded by the four tangent lines.

All points in the domain lying on or above the critical parabola lie on a single curve of the family of conics composing the domain, points below the parabola on two curves of the family. The corner regions and the quadrilateral are entirely below the critical parabola. Where there is a corner region, the ellipse goes below the parabola, otherwise not.

The foregoing discussion has all been for the general case of unrestricted roots. For special cases of zero, pure imaginary, or infinite roots, the corresponding domains are the limiting cases of the general domain, described above. Such limiting cases may reduce to a single segment or to a region bounded in part by the line at infinity. The homogeneous coordinates employed are very useful in dealing with these special cases.

4. Figures Illustrating the Domain of Poles

The preceding section presented a discussion of the domain of the poles associated with any four assigned roots, the domain being plotted in terms of the coefficients of the denominator of the impedance, that is, in terms of symmetric functions of the poles. In order to show the mutual relations between the actual values of the roots and the poles, it is convenient to plot, in the upper half of the complex plane, the domain of one pole, the other pole being its conjugate. This provides a complete representation for the case of complex poles. In order to include the domain of real poles, an auxiliary graph can be provided to indicate pairs of points on the real axis.

The mathematical analysis for this form of representation can be obtained from that of the preceding section by substituting $\beta_2+\beta_3=-b_2/b_1$ and $\beta_2\beta_3=b_3/b_1$. For complex poles, $\beta_2=u+iv$ and $\beta_3=u-iv$, this transformation from the x, y plane to the u, v plane is simply $2u=-x$ and $u^2+v^2=y$. Thus a conic in the x, y plane becomes, in general, a curve of the fourth degree in the u, v plane. The analysis of the curves obtained in the u, v plane is not so simple as in the other plane, but there is a decided advantage in the interpretation of the results in this plane, since the coordinate u, the real part of the pole, corresponds to the damping factor, and the coordinate v, the imaginary part of the pole, corresponds to the frequency factor.

In the complex plane, the necessary conditions (29)–(31) require the domain of complex poles to lie entirely within the region bounded by the vertical axis, a vertical line to the left of the axis, two circles about the origin as center, and a circle through the origin with its center on the real axis. Furthermore, the boundary curve of the domain must be tangent to each of these lines and circles, since the corresponding conic (3) for $d=0$ is tangent to the corresponding lines (4)–(6) for $d=0$.

For the special case of one root a positive pure imaginary, the second root being its conjugate, the domain in the upper half of the complex plane reduces merely to the points on an arc of a circle with its center on the real axis. If the third root is complex with a positive imaginary part, the fourth root being its conjugate, the domain is the circular arc extending from the first root to the third root. For a pure imaginary value of the third root the radius of the circle becomes infinite, and the domain is the segment of the vertical axis between the first and third roots. This is precisely the result already obtained for the resistanceless circuit.

For the limiting case of the third root real, with the fourth root equal to it, the domain is the circular arc extending from the root on the imaginary axis to the double root on the real axis. When the third and fourth roots are real and distinct, the domain is the circular arc from the first root to the point on the real axis midway between the two real roots. The complete domain also includes real poles in the segment between the two real roots, equally spaced about the midpoint of the segment.

This case of one pair of roots on the axis of imaginaries is illustrated by Fig. 3a, with the first root fixed at the point a, and the third root lying on any one of the family of circular arcs drawn through a, the fourth root being its conjugate; or the third and fourth roots lying on the real axis equally spaced about the end-point of one of the arcs.

Starting with one pair of roots on the axis of imaginaries, it is interesting to investigate the changes made in the domain by moving this pair of roots off the axis. The domain broadens out into a region lying about the circular arc, as shown by Fig. 3b for four typical cases. The first case is for the third root also near the axis ($\alpha_1=-0.5+i3$, $\alpha_3=-0.5+i9$); and the second case is for the third root some distance from the axis ($\alpha_1=-0.1+i3$, $\alpha_3=-5+i8$). The third section of Fig. 3b shows the domain when the third and fourth roots are real and equal ($\alpha_1=-0.1+i3$, $\alpha_3=\alpha_4=-9$); in this case the region has a cusp at this double root. The fourth section shows the domain of complex poles when the third and fourth roots are real and distinct ($\alpha_1=-0.1+i3$, $\alpha_3=-6$, $\alpha_4=-10$); in this case the region of complex poles terminates along a segment of the real axis lying in the interval between the two real roots, there is also a domain of real poles which is not shown.

It is interesting to note that, when both pairs of roots are near the axis of imaginaries, that is, for small damping, the frequency factor of the pole may always be taken outside the range of the frequency factors of the roots; whereas for zero damping the pole must lie between the roots, as noted above.

Fig. 3c shows the domain of the poles for two pairs of equal roots. If the first and third roots are equal, the second and fourth roots being their conjugates and thus also equal, the domain is bounded by a circle tangent to the vertical axis with its center vertically above the double root. If, for example, the double root describes a circle about the origin through the point a on the vertical axis, the corresponding circle is tangent to the vertical axis at a. Thus in Fig. 3c,

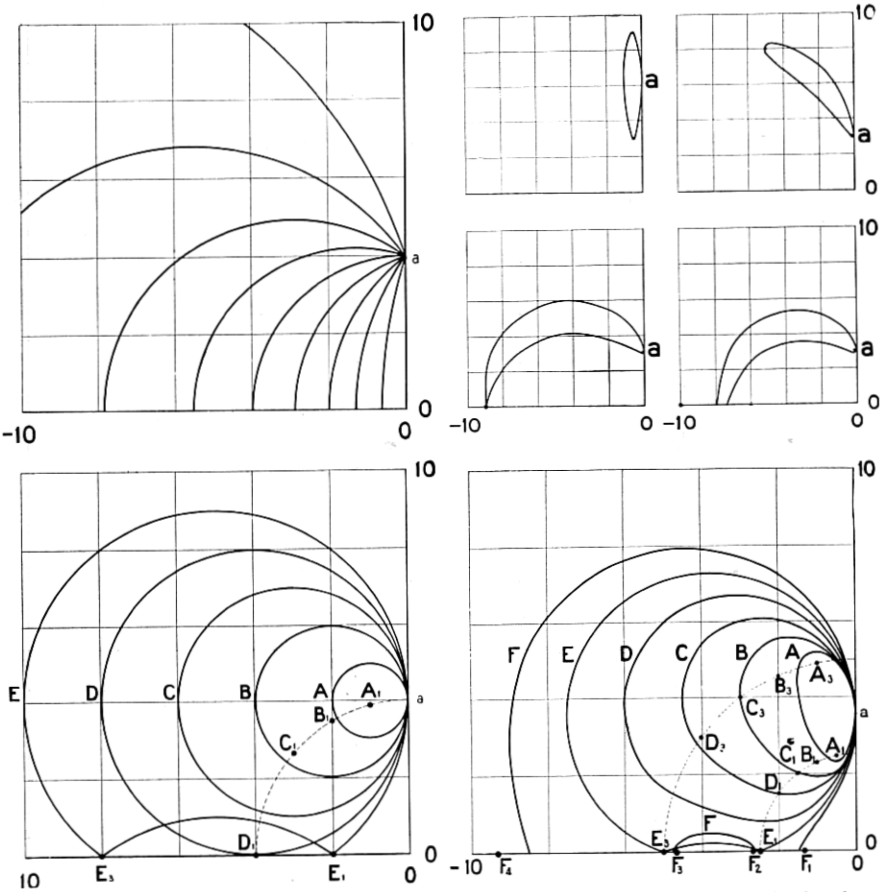

Fig. 3—Domain of the poles of the driving-point impedance of a two-mesh circuit with (a) one pair of roots on the axis of imaginaries, (b) one pair of roots near the axis of imaginaries, (c) two pairs of equal roots, and (d) two pairs of roots with equal angles.

for double roots at A_1, B_1, C_1, the corresponding domain is bounded by the circles A, B, C, respectively. The centers of these circles are all on the horizontal line through a, and the double roots are selected so as to space the centers uniformly. If all four roots are real and equal, the domain is bounded by a circle D tangent to the vertical axis at a and to the horizontal axis at this fourfold root D_1. If the roots are all real and equal in pairs the domain is bounded by a circle E, tangent to the vertical axis and passing through the two double roots, E_1 and E_3, and by the reflection of this circle in the real axis. Thus the domain has cusps at the double roots. For two pairs of

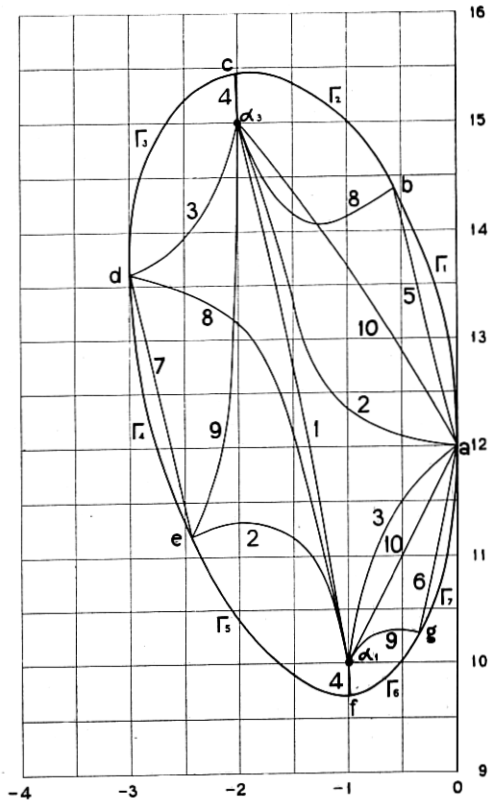

Fig. 4—Domain of the poles of the driving-point impedance of a two-mesh circuit with two pairs of complex roots, showing the portions of the domain realizable by each network listed in Table III.

equal roots, whether real or complex, the distance Oa is the geometrical mean value of all four roots.

Another kind of special case is shown by Fig. 3d, the case of two pairs of roots with equal angles. The first and third roots are on a line with the origin, so that the second and fourth roots, being their conjugates, are also on a line with the origin. Fig. 3d shows the boundary curves ($A \ldots E$) for five sets of roots ($A_1, A_3 \ldots E_1, E_3$) satisfying these conditions and with the same absolute values of the roots in each set, so that the roots lie on two circles about the origin.

The fifth set of roots (E_1, E_3) has a domain of the same type as the corresponding set of roots on Fig. 3c, since this set, being on the real axis, is a double set. The sixth curve F is the boundary of the domain for four real roots so chosen that $F_1F_3=E_1^2$ and $F_2F_4=E_3^2$. This is the same type of domain as will be described later under Fig. 5. The curves of Fig. 3d are all tangent to the vertical axis at the same point a; for each of these sets of roots the distance Oa is the geometrical mean value of all four roots.

The general case of four complex roots is illustrated by Fig. 4 for the numerical values $\alpha_1=-1+i10$, $\alpha_2=-1-i10$, $\alpha_3=-2+i15$, $\alpha_4=-2-i15$. For all complex roots the poles must also be complex; the pole with positive imaginary part must lie in the region bounded by the curve $\Gamma=\Gamma_1+\Gamma_2+\ldots+\Gamma_7$. This curve is tangent to the vertical axis at the point a, and tangent to a vertical line at the left at the point d. The largest absolute value of any point in the domain occurs at the point c, and the smallest at f; these two points are the points of tangency of the curve Γ with circles about the origin as center. The curve Γ is tangent at the point e to a circle through the origin having its center on the real axis. The coordinates of these points are all given in Table V.

The general case of four real roots is illustrated by Fig. 5 for the numerical values $\alpha_1=-1$, $\alpha_2=-2$, $\alpha_3=-5$, $\alpha_4=-7$. The domain of complex poles is bounded by the curve Γ, with the critical points defined and labeled as in Fig. 4. The domain of complex poles is bounded in part by two segments on the real axis, one lying in the interval between α_1 and α_2, the other between α_3 and α_4. Approximately, these segments are from -1.13 to -1.93 and from -5.13 to -6.70, for this numerical example. The points on these segments are in the domain of poles, corresponding to double real poles. The domain of real poles is shown by the graph below the axis, each point of this graph representing two real values, the two points on the real axis reached by following the $\pm 45°$ lines through the point. The domain of real poles is bounded by the continuation of the curve Γ and the tangent lines corresponding to the four roots. This gives two corners associated with the two segments on the real axis, and an isolated rectangle. Corresponding to the points in the rectangle, one pole may be chosen anywhere in the range from α_1 to α_2, and the second pole anywhere in the range from α_3 to α_4. Both poles may be chosen in the range from α_1 to α_2, or in the range from α_3 to α_4, with certain restrictions as shown by the figure, since the curve Γ cuts off the points of the triangles. The two corners and the rectangle are shown by Fig. 5a on a larger scale, with greater accuracy.

In some respects, the case illustrated by Fig. 5 is the most general case, from which all other cases can be obtained by a continuous transformation of the roots. Two of the adjacent real roots may be brought together to a single double root; the corresponding boundary curve then shrinks to a cusp at this point on the real axis, and the rectangle

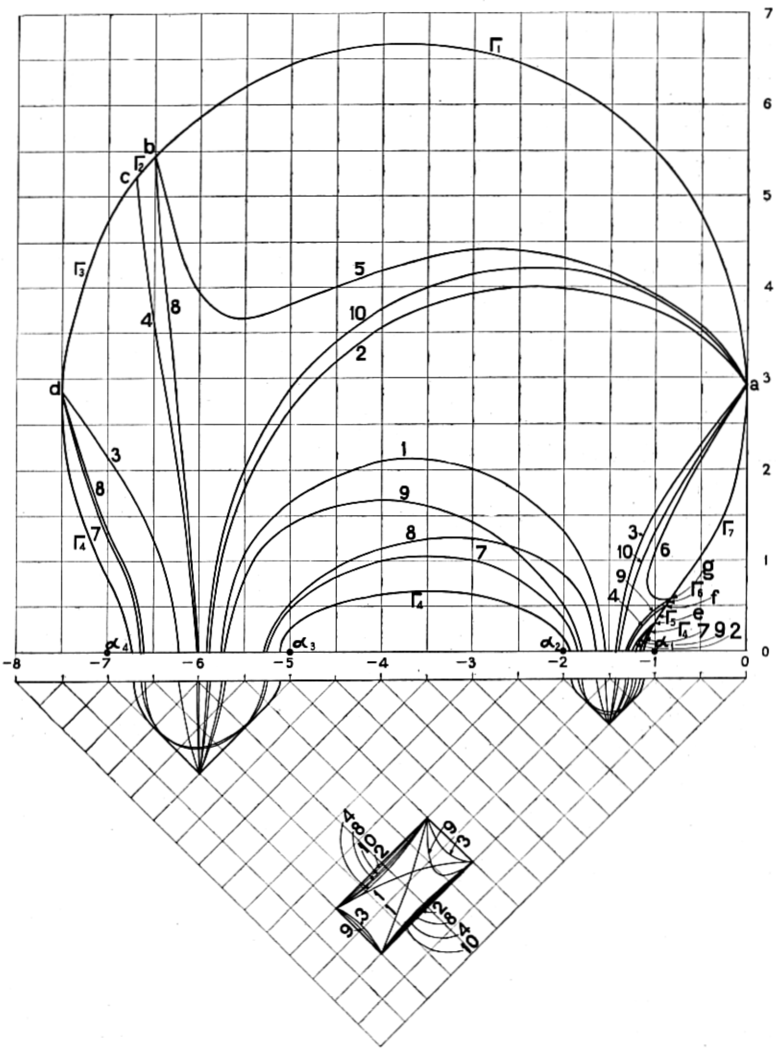

Fig. 5—Domain of the poles of the driving-point impedance of a two-mesh circuit with four real roots, showing the portions of the domain realizable by each network listed in Table III.

in the auxiliary diagram narrows down to a single line segment. Then if the other two real roots are brought together, the boundary curve has a second cusp and the domain in the auxiliary diagram shrinks to a single isolated point. If, now, one of the pairs of equal real roots is separated into a pair of conjugate imaginary roots, the

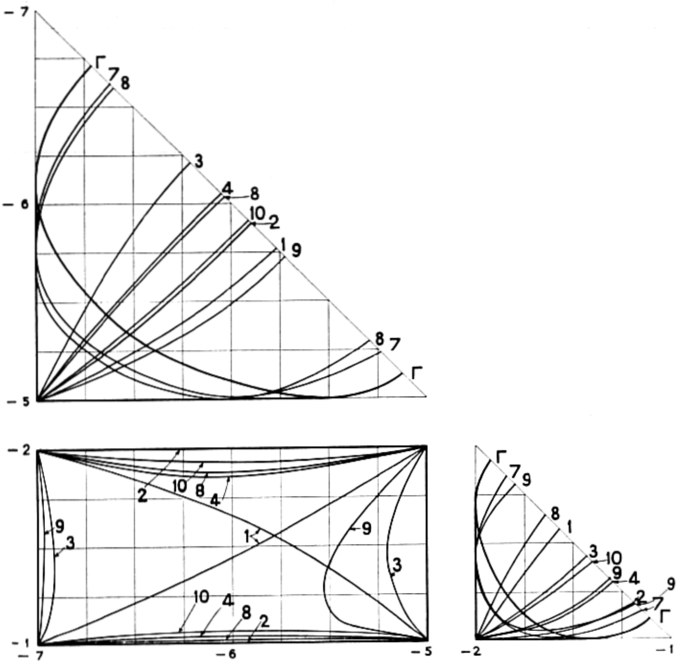

Fig. 5a—Domain of real poles of Fig. 5, on larger scale.

corresponding cusp is rounded off away from the axis, and the point in the auxiliary diagram vanishes. When the other pair of equal real roots separates into conjugate complex roots, the case illustrated by Fig. 4 is obtained. As one pair of complex roots approaches the imaginary axis, the domain narrows until, for one pair of roots on the vertical axis, the domain shrinks to a circular arc as illustrated by Fig. 3a. This sort of transformation may be followed through in different ways in order to obtain any desired distribution of the roots.

The complete domains are unique, that is, any one domain is given by only one set of roots.

Every domain includes the points corresponding to the roots for which the domain is defined. For these points, that is, for a pole coinciding with a root, the impedance expression has a common factor in numerator and denominator. When both poles coincide with roots the corresponding impedance expression can be obtained by means of a one-mesh circuit.

5. Two-Mesh Circuits and Associated Networks

The second object of this paper is the determination of the networks realizing any specified driving-point impedance which satisfies the conditions established in the first part of the paper. It is necessary to find the number, character, and arrangement of the elements in these networks, as well as to find the values of these elements.

Thus the problem met in this investigation differs from the usual network problem in that it calls for the determination of the elements of a network which has a certain specified impedance, instead of calling for the determination of the impedance of a network which has certain specified elements.

The most general two-mesh circuit has three branches connected in parallel, each branch containing resistance, capacity, and self-

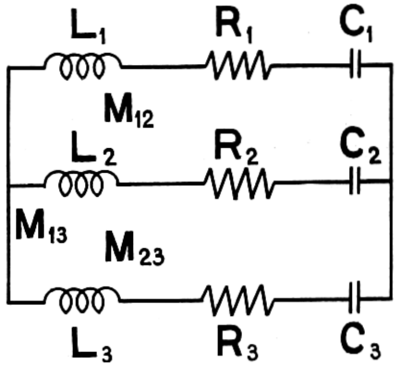

Fig. 6—Most general two-mesh circuit consisting of resistances, capacities, and inductances.

inductance, with mutual inductance between each pair of branches, as shown by Fig. 6.

The most general network under consideration is, therefore, the network obtained by opening one branch of this two-mesh circuit, as shown by Fig. 7. All the networks considered are special cases of this general network, obtained by making a sufficient number of the elements either zero or infinite. If, in particular, all the elements in one branch are replaced by a short circuit, the network splits up into two separate sections connected essentially only by mutual inductance, as shown by Fig. 7a.

It is convenient to limit this investigation to the determination of those networks which, without superfluous elements, realize any driving-point impedance having arbitrarily assigned roots. A network is considered to have superfluous elements if there exist other

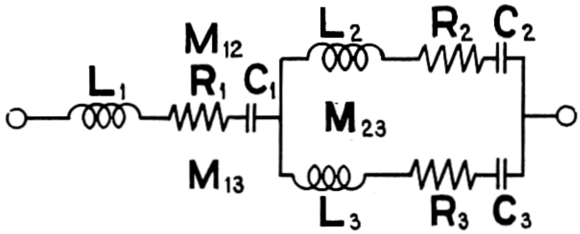

Fig. 7—Most general network obtained by opening one branch of a two-mesh circuit.

networks with fewer elements which, individually or collectively, realize the same range of possible impedances. Impedances with zero, pure imaginary, or infinite roots can be realized by the limiting cases of these networks.

A network realizing an impedance with arbitrarily assigned roots must consist of at least five elements,—one resistance, two capacities,

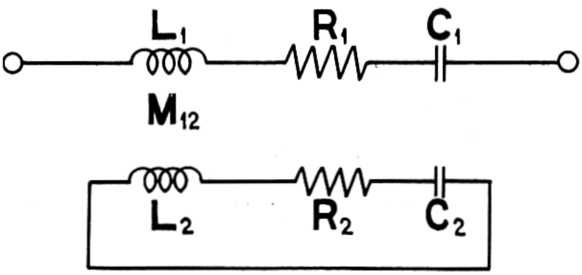

Fig. 7a—Special case of Fig. 7, obtained by replacing the elements of one branch by a short circuit

and two self-inductances, in order that the numerator of the impedance expression (1b) may contain odd powers of λ, a constant term, and a term in λ^4, respectively.

Since the general expression for the driving-point impedance contains essentially seven constants which may be assigned arbitrarily, subject to the restrictions already established, it is to be expected that the entire range of possible impedances can be realized by one or more networks consisting of seven elements only. This proves to be the case. Hence all networks with more than seven elements contain superfluous elements. It is also to be expected that one additional condition must be satisfied by the roots and poles in order that an impedance may be realized by a six-element network, and two additional conditions for a five-element network.

Accordingly, a census has been made of all networks consisting of not more than seven elements, each network containing at least one resistance, two capacities, and two self-inductances. This census is shown by Table III.

Each two-mesh circuit meeting these requirements as to the number of elements is represented in symbolical form in Table III. The letters L, R, and C, printed in the first, second, or third lines of the symbol, indicate the presence of self-inductance, resistance, and capacity in the first, second, or third branches of the circuit, respectively. The letter M is printed in the two lines of the symbol corresponding to the two branches which are connected by a mutual inductance. Thus the first circuit in the table is represented by the symbol

$$\begin{array}{l} LRCM \\ L\ \ CM \\ L \end{array}$$

which indicates self-inductance, resistance, and capacity in the first branch, self-inductance and capacity in the second branch, and self-inductance in the third branch, with mutual inductance between the first two branches.

Three networks, in general, are obtained from each of these circuits by opening each of the three branches. If two of these branches are alike, only two distinct networks are obtained. If one branch of a circuit is a short-circuit, there being no elements assigned to that branch, the network obtained by opening one of the other branches is of the type shown by Fig. 7a; if the short-circuited branch is opened, the network consists simply of the parallel combination of the other two branches.

With circuits represented in this symbolical manner, there is, opposite each line of the symbol, a reference to the domain of poles indicating the portion of the domain realizable by the network obtained by opening the corresponding branch. Two like branches in a circuit are bracketed together with a single reference mark, since they each give the same network. The entire domain is indicated by Σ; the boundary curve of the domain by Γ, this being divided into seven segments, Γ_1, Γ_2, . . . , Γ_7; ten critical lines in the domain by the

TABLE III

Networks Realizing the Driving-Point Impedance of a Two-Mesh Circuit

1.	*LRCM*	((*a*))
	L CM	((*e*))
	L	(Γ)
2.	*LRCM*	((*a*))
	L C	((*e*))
	L M	(Γ)
3.	*LRC*	((*a*))
	L CM	((*e*))
	L M	(Γ)
4.	*LR M*	((*a*))
	L CM	(Γ)
	L C	(Γ)
5.	*LR*	((*a*))
	L CM }	(Γ)
	L CM }	
6.	*LRCM* }	(2)
	LRCM }	
	—	Σ
7.	*LRCM*	Σ
	LR M	Σ
	C	Σ
8.	*LRCM*	Σ
	L CM	Σ
	R	Σ
9.	*LRCM*	(2)
	L M	Σ
	RC	(2)
10.	*LR M*	Σ
	L CM	Σ
	RC	Σ
11.	*LRCM*	((*a*))
	L CM	(Γ)
	C	(Γ)
12.	*L CM* }	(Γ)
	L CM }	
	RC	((*a*))
13.	*LRC* }	(2)
	LRC }	
	L	$1-8-\Gamma_2-\Gamma_3$
14.	*LRC*	$3-4-6-\Gamma_3-\Gamma_6$
	LR	$3-5-10-\Gamma_2-\Gamma_3$
	L C	$4-7-9-\Gamma_3-\Gamma_6$
15.	*LRC*	((*a*))
	L C }	$(\Gamma_2, \Gamma_3, \Gamma_5, \Gamma_6)$
	L C }	
16.	*LRC* }	(4)
	LRC }	
	R	$1-10$
17.	*LRC*	$2-3-7$
	LR	$3-5-8$
	RC	$2-6-9$
18.	*LRC* }	(3)
	LRC }	
	C	$1-9-\Gamma_5-\Gamma_6$
19.	*LRC*	$2-4-5-\Gamma_2-\Gamma_5$
	L C	$4-7-8-\Gamma_2-\Gamma_5$
	RC	$2-6-10-\Gamma_5-\Gamma_6$
20.	*LRCM*	(*a*)
	L CM	(*e*)
	—	Γ
21.	*LRCM*	(*a*)
	L M	Γ
	C	(*e*)
22.	*LR M*	(*a*)
	L CM	Γ
	C	Γ
23.	*L CM* }	Γ
	L CM }	
	R	(*a*)
24.	*L CM*	(*e*)
	L M	Γ
	RC	(*a*)
25.	*LRC*	(*a*)
	L C	(*e*)
	L	Γ_2, Γ_3
26.	*LR*	(*a*)
	L C }	Γ_3, Γ_6
	L C }	
27.	*LRC* }	*
	LRC }	
	—	1
28.	*LRC*	3
	LR	3
	C	9
29.	*LRC*	4
	L C	4
	R	10
30.	*LRC*	2
	L	8
	RC	2
31.	*LR*	5
	L C	7
	RC	6
32.	*LRC*	(*a*)
	L C	(*d*)
	C	Γ_5, Γ_6
33.	*L C* }	Γ_2, Γ_5
	L C }	
	RC	(*a*)
34.	*LRC*	*
	L C	*
	—	**
35.	*LRC*	*a*
	L	*d*
	C	*e*
36.	*LR*	*a*
	L C	*d*
	C	*g*
37.	*L C* }	*c*,
	L C }	
	R	*a*
38.	*L C*	*e*
	L	*b*
	RC	*a*

* Equivalent to a single-mesh network.

** Network by which only impedances having one pair of imaginary roots can be realized.

( ) The impedances realized by this network can be realized by other networks with one less element.

(( )) The impedances realized by this network can be realized by other networks with two less elements.

numbers 1, 2, . . . , 10; and seven critical points by the letters *a*, *b*, . . . , *g*, as illustrated by Figs. 4 and 5.

Networks with superfluous elements are indicated by placing parentheses around the corresponding reference mark, single parentheses for one superfluous element and double parentheses for two. In order that a seven-element network may contain no superfluous elements it must give the entire domain or a region in it, a six-element network a critical line, and a five-element network a critical point.

That is, an impedance with arbitrarily assigned roots, and with a pole chosen arbitrarily in the domain corresponding to these assigned roots, can be realized with the minimum number of elements only by a seven-element network. If the pole is chosen so as to satisfy one additional condition, namely, chosen at a point on one of the critical lines of the domain (including the boundary curve), the impedance can be realized by the six-element network giving that line. If the pole is chosen so as to satisfy two additional conditions, namely, chosen at one of the critical points, the impedance can be realized by the corresponding five-element network.

The conditions for the critical lines and for the critical points are given by Tables IV and V, respectively, in terms of the coefficients of the impedance.

TABLE IV

Critical Lines

Γ. $a_3^2b_1^2+(4a_1a_4-2a_2a_3)b_1b_2-2a_1a_3b_1b_3-(4a_0a_4-a_2^2)b_2^2+(4a_0a_3-2a_1a_2)b_2b_3+a_1^2b_3^2=0.$

1. $(8a_1a_4^2-4a_2a_3a_4+a_3^3)b_1^3-(16a_0a_4^2+2a_1a_3a_4-4a_2^2a_4+a_2a_3^2)b_1^2b_2$
$+(8a_0a_3a_4-4a_1a_2a_4+a_1a_3^2)b_1^2b_3+(8a_0a_3a_4-4a_1a_2a_4+a_1a_3^2)b_1b_2^2$
$-6(a_0a_3^2-a_1^2a_4)b_1b_2b_3-(8a_0a_1a_4-4a_0a_2a_3+a_1^2a_3)b_1b_3^2$
$-(a_0a_3^2-a_1^2a_4)b_2^3-(8a_0a_1a_4-4a_0a_2a_3+a_1^2a_3)b_2^2b_3+(16a_0^2a_4+2a_0a_1a_3$
$-4a_0a_2^2+a_1^2a_2)b_2b_3^2-(8a_0^2a_3-4a_0a_1a_2+a_1^3)b_3^3=0.$

2. $2a_4b_1b_2b_3-a_3b_1b_3^2-a_4b_2^3+a_3b_2^2b_3-a_2b_2b_3^2+a_1b_3^3=0.$

3. $a_3b_1^3-a_2b_1^2b_2-a_1b_1^2b_3+a_1b_1b_2^2+2a_0b_1b_2b_3-a_0b_2^3=0.$

4. $a_4b_1^2b_2-a_3b_1^2b_3+a_1b_1b_3^2-a_0b_2b_3^2=0.$

5. $a_3a_4b_1^3b_2+a_3^2b_1^3b_3+(a_2a_4-a_3^2)b_1^2b_2^2-(a_1a_4-a_2a_3)b_1^2b_2b_3$
$-2a_1a_3b_1^2b_3^2-a_1a_4b_1b_2^3+a_1a_3b_1b_2^2b_3+(a_0a_3-a_1a_2)b_1b_2b_3^2$
$+a_1^2b_1b_3^3+a_0a_4b_2^4-a_0a_3b_2^3b_3+a_0a_2b_2^2b_3^2-a_0a_1b_2b_3^3=0.$

6. $a_3a_4b_1^3b_2-a_3^2b_1^3b_3-a_2a_4b_1^2b_2^2-(a_1a_4-a_2a_3)b_1^2b_2b_3+2a_1a_3b_1^2b_3^2$
$+a_1a_4b_1b_2^3-a_1a_3b_1b_2^2b_3+(a_0a_3-a_1a_2)b_1b_2b_3^2-a_1^2b_1b_3^3-a_0a_4b_2^4$
$+a_0a_3b_2^3b_3-(a_0a_2-a_1^2)b_2^2b_3^2-a_0a_1b_2b_3^3=0.$

7. $a_3a_4b_1^3b_2-a_3^2b_1^3b_3-a_2a_4b_1^2b_2^2+(a_1a_4+a_2a_3)b_1^2b_2b_3-2a_1a_3b_1^2b_3^2$
$+a_1a_4b_1b_2^3-a_1a_3b_1b_2^2b_3+(a_0a_3+a_1a_2)b_1b_2b_3^2-a_1^2b_1b_3^3-a_0a_4b_2^4$
$+a_0a_3b_2^3b_3-a_0a_2b_2^2b_3^2+a_0a_1b_2b_3^3=0.$

8. $(8a_1a_4^2-4a_2a_3a_4+a_3^3)b_1^5-(8a_0a_4^2+2a_1a_3a_4-4a_2^2a_4+a_2a_3^2)b_1^4b_2$
$+(4a_0a_3a_4+a_1a_3^2-4a_1a_2a_4)b_1^4b_3+(2a_0a_3a_4+a_1a_3^2-4a_1a_2a_4)b_1^3b_2^2$
$-(4a_0a_2a_4-6a_1^2a_4)b_1^3b_2b_3+(4a_0a_1a_4-a_1^2a_3)b_1^3b_3^2-(a_0a_3^2-4a_0a_2a_4$
$-a_1^2a_4)b_1^2b_2^3-(8a_0a_1a_4+a_1^2a_3)b_1^2b_2^2b_3+(2a_0a_1a_3+a_1^2a_2)b_1^2b_2b_3^2$
$-a_1^3b_1^2b_3^3-2a_0a_1a_4b_1b_2^4+(2a_0^2a_4+2a_0a_1a_3)b_1b_2^3b_3$
$-(a_0^2a_3+2a_0a_1a_2)b_1b_2^2b_3^2+2a_0a_1^2b_1b_2b_3^3+a_0^2a_4b_2^5-a_0^2a_3b_2^4b_3$
$+a_0^2a_2b_2^3b_3^2-a_0^2a_1b_2^2b_3^3=0.$

9. $a_3a_4^2b_1^3b_2^2-2a_3^2a_4b_1^3b_2b_3+a_3^3b_1^3b_3^2-a_2a_4^2b_1^2b_2^3+(a_1a_4^2+2a_2a_3a_4)b_1^2b_2^2b_3$
$-(2a_1a_3a_4+a_2a_3^2)b_1^2b_2b_3^2-(4a_0a_3a_4-a_1a_3^2)b_1^2b_3^3+a_1a_4^2b_1b_2^4$
$-(2a_0a_4^2+2a_1a_3a_4)b_1b_2^3b_3+(8a_0a_3a_4+a_1a_3^2)b_1b_2^2b_3^2$
$+(4a_0a_2a_4-6a_0a_3^2)b_1b_2b_3^3-(4a_0a_1a_4-4a_0a_2a_3+a_1^2a_3)b_1b_3^4$
$-a_0a_4^2b_2^5+2a_0a_3a_4b_2^4b_3-(4a_0a_2a_4+a_0a_3^2-a_1^2a_4)b_2^3b_3^2$
$-(2a_0a_1a_4-4a_0a_2a_3+a_1^2a_3)b_2^2b_3^3+(8a_0^2a_4+2a_0a_1a_3-4a_0a_2^2+a_1^2a_2)b_2b_3^4$
$-(8a_0^2a_3-4a_0a_1a_2+a_1^3)b_3^5=0.$

10. $a_3^2a_4b_1^4b_2+a_3^3b_1^4b_3-a_3^3b_1^3b_2^2-2a_1a_3a_4b_1^3b_2b_3-3a_1a_3^2b_1^3b_3^2$
$+(4a_0a_4^2-a_2^2a_4+a_2a_3^2)b_1^2b_2^3+(4a_0a_3a_4+3a_1a_3^2-a_2^2a_3)b_1^2b_2^2b_3$
$-(a_0a_3^2-a_1^2a_4)b_1^2b_2b_3^2+3a_1^2a_3b_1^2b_3^3$
$-(4a_0a_3a_4-2a_1a_2a_4+a_1a_3^2)b_1b_2^4-(4a_0a_1a_4+3a_1^2a_3-a_1a_2^2)b_1b_2^2b_3^2$
$+2a_0a_1a_3b_1b_2b_3^3-a_1^3b_1b_3^4+(a_0a_3^2-a_1^2a_4)b_2^5+(4a_0a_1a_4-2a_0a_2a_3$
$+a_1^2a_3)b_2^4b_3-(4a_0^2a_4-a_0a_2^2+a_1^2a_2)b_2^3b_3^2+a_1^3b_2^2b_3^3-a_0a_1^2b_2b_3^4=0.$

TABLE V

Critical Points

Point	Coordinates	
	$\dfrac{b_2}{b_1}$	$\dfrac{b_3}{b_1}$
a	0	$\dfrac{a_3}{a_1}$
b	$\dfrac{a_0a_3^2+a_1^2a_4-a_1a_2a_3}{a_0(a_1a_4-a_2a_3)}$	$\dfrac{a_0a_3a_4+a_1a_2a_4-a_2^2a_3}{a_0(a_1a_4-a_2a_3)}$
c	$\dfrac{a_1}{2a_0}-\dfrac{2a_0a_3-a_1a_2}{2a_0\sqrt{a_2^2-4a_0a_4}}$	$\dfrac{1}{2a_0}(a_2+\sqrt{a_2^2-a_0a_4})$
d	$\dfrac{a_1}{a_0}$	$\dfrac{a_1a_2-a_0a_3}{a_0a_1}$
e	$\dfrac{a_3^2}{a_2a_3-a_1a_4}$	$\dfrac{a_3a_4}{a_2a_3-a_1a_4}$
f	$\dfrac{a_1}{2a_0}+\dfrac{2a_0a_3-a_1a_2}{2a_0\sqrt{a_2^2-4a_0a_4}}$	$\dfrac{1}{2a_0}(a_2-\sqrt{a_2^2-4a_0a_4})$
g	$\dfrac{a_0a_3^2+a_1^2a_4-a_1a_2a_3}{a_0a_1a_4+a_0a_2a_3-a_1a_2^2}$	$\dfrac{a_4(a_0a_3-a_1a_2)}{a_0a_1a_4+a_0a_2a_3-a_1a_2^2}$

These critical lines and points are illustrated, for numerical cases, by Figs. 4 and 5. The graph showing the domain of real poles in Fig. 5 is inaccurate to the extent that the critical lines have been spread somewhat apart from each other in order to show the sequence in which they occur. The actual curves are shown accurately drawn and on a larger scale in Fig. 5a. Even on this scale, Curve 2 cannot be distinguished from the side of the rectangle.

The diagrams for the domain of complex poles, as illustrated by Figs. 4 and 5, are approximately symmetrical with respect to the interchanging of inductances and capacities, with corresponding interchanges in all the curves and formulas. Thus b and g correspond, c and f, d and e, 2 and 3, 5 and 6, 8 and 9, α_1 and α_4, α_2 and α_3; while a, 1, 4, 7, and 10 remain unchanged. In the domain of real poles shown by Fig. 5, this symmetry does not appear. The explanation of this apparent discrepancy is as follows: Upon interchanging inductances and capacities, the values of the roots are changed to their reciprocals. Thus Fig. 5 is symmetrical with the corresponding figure drawn for the case of roots equal to -1, $-1/2$, $-1/5$, and $-1/7$, and thus symmetrical with the figure drawn for roots at -7, $-7/2$, $-7/5$, and -1, since the relative distribution of the roots is the same. This set of roots differs not very considerably from the original set of roots, in reverse order. In the main, therefore, the two figures may be expected to be approximately the same, that is, the original figure symmetrical with itself. In the rectangle, however, very small numerical changes in the constants make relatively large changes in the curves; so it is not surprising to find a lack of symmetry here. If the roots are assigned so that the product of two roots is equal to the product of the other two, there will be true symmetry in the corresponding diagram.

Table III lists 38 circuits, giving a total of 102 networks. Of these networks, three are essentially the equivalent of networks obtained from a one-mesh circuit, one realizes only those impedances which have one pair of pure imaginary roots, and, of the 98 remaining, 41 have superfluous elements. This leaves a total of 57 networks, of which 11 realize the entire domain as given by Theorem II, 12 realize regions in the domain as given by Theorem III, 23 realize critical lines in the domain, and 11 realize critical points.

The eleven networks of Theorem II are included in the first column of Table III and shown in detail by Fig. 1. Formulas for the computation of their elements are given by Table I. Thus the values of these elements can be computed directly in terms of the coefficients of the impedance expression as stated in the form (1b). The following method of computation is convenient:—First compute d as the root of the quadratic equation (21), which is repeated at the bottom of the table. Then find c by subtracting this value of d from a_2. Next compute T_1, T_2, and T_3, assigning signs so that the identity $b_1T_1 + b_2T_2 + b_3T_3 = 0$ is satisfied; this is possible since the equation for d was obtained by rationalizing this relation among the T's. There are, in general, two sets of signs for which this identity is satisfied; it is immaterial which set is chosen since the signs of all the T's may be changed without changing the values of any of the elements. Then compute U_1, U_2, and U_3, assigning positive values to each of these. With the values of all these quantities determined, the values of the elements of the networks can be calculated directly from the formulas given in the body of the table. If this solution turns out to be impossible, that is, if the value of an element is found to be negative or complex or if the value of a mutual inductance is found to be greater than the square root of the product of the associated self-inductances, it means that the conditions upon the roots and poles are not satisfied. If the conditions established in the first part of this paper are satisfied, the solution is possible.

These formulas give all the special cases of the eleven networks automatically, that is, the values of the appropriate elements will turn out to be zero or infinite, as the case may be. Since each of these eleven networks covers the entire domain, they are all mutually equivalent at all frequencies. These are the only networks without superfluous elements which cover the entire domain, that is, any network covering the entire domain must be one of these eleven or a network obtained from one of these by introducing additional elements. Each of the eleven contains just seven elements; thus the prediction that a seven-element network would cover the entire domain is verified. The three remaining networks of this same type, one from Circuit 6 and two from Circuit 9 of Table III give special cases only, in the sense that each of these can realize only those impedances which have a pole lying on Line 2; thus each of these three contains a superfluous element, since all the points on Line 2 can be realized by six-element networks, as shown in the fourth column of the table.

Network 1 of Fig. 1 is of particular interest since it consists simply of two branches in parallel, each containing resistance, capacity, and self-inductance, with mutual inductance between them.[10] By Theorem II, this network can be made equivalent to any network whatsoever obtained from a two-mesh circuit.

[10] It will be shown in a subsequent paper that any driving-point impedance of an n-mesh circuit can be realized by a network of n branches in parallel, each branch containing resistance, capacity, and self-inductance, with mutual inductance between each pair of branches.

The twelve networks of Theorem III are included in the second column of Table III and shown in detail by Fig. 2. Formulas for the computation of their elements are given by Table II. The values of the elements can be computed by the same rule as that given above for Table I.

Each of these twelve networks realizes those impedances which have poles lying in a certain restricted area or region of the entire domain of possibilities, as indicated for each network in the table by a specification of the boundary curves of the area. For each particular impedance in the domain various sets of these twelve networks are mutually equivalent. Some points in the domain cannot be realized by networks without mutual inductance. Of the remaining points, each is realizable, in general, by at least three, and by not more than five, of these twelve networks. This region of the domain which is realizable without mutual inductance is covered, with no overlapping, by each of the four following sets of networks: 13, 17, and 21; 13, 18, and 22; 14, 17, and 23; 15, 19, and 21; the numbers refer to the networks of Fig. 2.

That portion of the domain which cannot be realized by networks without mutual inductance comprises the three regions bounded by Γ_1 and 5, Γ_4 and 7, and Γ_7 and 6, respectively, as illustrated by Figs. 4 and 5.

The third and fourth columns of Table III show a total of 23 networks, each with six elements, realizing lines in the domain. Of these, eleven are derived as special cases of the networks of both Figs. 1 and 2, six as special cases of Fig. 1 but not of Fig. 2, and six as special cases of Fig. 2 alone. The fifth column of the table shows the eleven networks, each with five elements, realizing points in the domain.

6. Formulas for Calculation of General Network

Formulas for the calculation of the values of the elements of the general network of Fig. 7 are given in Theorem IV. These are given in the form of nine equations (7)–(15), inclusive, involving the twelve elements of the network and two parameters, d and k. The parameter d, however, is fixed by the impedance, since the left-hand members of equations (13)–(15) satisfy the identity (20). Upon substituting the right-hand members in the identity and rationalizing, equation (21) is obtained, this being a quadratic equation in d with coefficients which are functions of the known coefficients of the impedance. Since d is fixed in this way, there are essentially eight equations in thirteen variables,—the twelve elements and the arbitrary parameter k. In general, therefore, five of the elements may be specified, or five relations among the elements; whereupon the equations can be solved. Thus it is to be expected that a seven-element network will realize, in general, any specified driving-point impedance.

This method of solution is best illustrated by considering a particular case. Take, for example, the derivation of the formulas for Network 1 of Fig. 1, as given by Table I. This is the special case of the general network of Fig. 7 obtained by making $L_1=R_1=C_1^{-1}=M_{12}=M_{13}=0$. Substituting these values, together with the notation of Table I, equations (7)–(15) become

$$\begin{aligned} L_2L_3-M_{23}^2&=a_0k^2,\\ R_2R_3&=dk^2,\\ D_2D_3&=a_4k^2,\\ L_2+L_3-2M_{23}&=b_1k^2,\\ R_2+R_3&=b_2k^2,\\ D_2+D_3&=b_3k^2,\\ R_2D_3-R_3D_2&=T_1k^3,\\ D_2L_3-D_3L_2-(D_2-D_3)M_{23}&=T_2k^3,\\ L_2R_3-L_3R_2-(R_3-R_2)M_{23}&=T_3k^3. \end{aligned}$$

Eliminating R_2, R_3, D_2, and D_3 from the second, third, fifth, sixth, and seventh of these equations, the value of k is found to be equal to $\pm U_1/T_1$. Knowing the value of k, the equations may then be solved for the seven elements, obtaining the results given in Table I. The two sign choices for k in this example correspond to the possibility of interchanging branches 2 and 3 in the network. The values given in Table I are computed for k taken with the negative sign.

In the general solution, the parameter d is obtained from the quadratic equation (21). The explicit solution of this equation is

$$d=\frac{2a_4b_1^2+a_2b_2^2+2a_0b_3^2-a_3b_1b_2-2a_2b_1b_3-a_1b_2b_3\pm2\Delta}{b_2^2-4b_1b_3} \tag{34}$$

where

$$\begin{aligned} \Delta^2=&a_4^2b_1^4+a_0a_4b_2^4+a_0^2b_3^4-a_3a_4b_1^3b_2-(2a_2a_4-a_3^2)b_1^3b_3-a_1a_4b_1b_2^3\\ &-a_0a_3b_2^3b_3-(2a_0a_2-a_1^2)b_1b_3^3-a_0a_1b_2b_3^3\\ &+a_2a_4b_1^2b_2^2+(a_2^2+2a_0a_4-2a_1a_3)b_1^2b_3^2+a_0a_2b_2^2b_3^2\\ &+(3a_1a_4-a_2a_3)b_1^2b_2b_3-(4a_0a_4-a_1a_3)b_1b_2^2b_3\\ &+(3a_0a_3-a_1a_2)b_1b_2b_3^2, \end{aligned} \tag{35}$$

$$\begin{aligned} =&a_0^2(\alpha_1^2b_1+\alpha_1b_2+b_3)(\alpha_2^2b_1+\alpha_2b_2+b_3)\\ &(\alpha_3^2b_1+\alpha_3b_2+b_3)(\alpha_4^2b_1+\alpha_4b_2+b_3), \end{aligned} \tag{36}$$

$$\begin{aligned} =&a_0^2b_1^4(\alpha_1-\beta_2)(\alpha_1-\beta_3)(\alpha_2-\beta_2)(\alpha_2-\beta_3)\\ &(\alpha_3-\beta_2)(\alpha_3-\beta_3)(\alpha_4-\beta_2)(\alpha_4-\beta_3). \end{aligned} \tag{37}$$

In the case of real and distinct poles, formula (34) gives, in general, two positive values of d satisfying the necessary conditions (4)–(6), and thus two solutions for any particular network. For complex poles, only one such value of d is obtained, and there is thus a unique solution in each case. For real and equal poles, $b_2^2-4b_1b_3=0$, and so formula (34) does not apply directly; in this case, however, (21) reduces to a linear equation in d, so that the solution can be readily found.

An obvious necessary condition for a solution is that $\Delta^2 \geq 0$, for otherwise the value of d would be complex. This condition is satisfied for any choice of poles provided there is not an odd number of real roots lying between two real poles. Thus for the case of all complex roots or for the case of complex poles with any choice of roots this condition is automatically satisfied. It is interesting to note that an impedance expression with poles failing to satisfy this condition cannot be realized by any network with positive or negative resistances, capacities, and inductances; it can be realized only by a network with elements having complex values.

7. Networks with Negative Resistances

If negative resistances are allowed in the two-mesh circuit, the only change necessary in the statement of the results of this investigation, as given in Theorems I–IV, is the removal of the restrictions $\alpha_1+\alpha_2 \leq 0$, $\alpha_3+\alpha_4 \leq 0$, $\beta_2+\beta_3 \leq 0$, and $d \geq 0$. This removes the restriction of the real part of each root and pole to negative or zero values. The removal of the restriction on d adds to the domain of poles, considered in the x, y plane, all the ellipses of the family $-\infty < d < 0$, thus filling out the region above the critical parabola (33), together with the corners in the case of real roots. In the u, v plane the domain comprises the entire upper half of the complex plane and, in the auxiliary diagram, the complete triangular corners and the rectangle, with the provision that the rectangle is not included in the case of two roots positive and two negative.

By means of a two-mesh circuit employing negative resistances, any impedance expression of the form (1) can be realized, with roots arbitrarily assigned in conjugate pairs or in real pairs, subject only to the condition that the number of positive roots is even, and with any pair of complex poles or with a pair of real poles lying anywhere in the ranges from the first to the second real roots and from the third to the fourth real roots, arranged in order of magnitude, subject only to the condition that both poles must be positive or both negative.

The network diagrams and all the formulas for the calculation of the elements remain unchanged.

8. Mathematical Proof

The circuits treated in this investigation are special cases of the general circuit which has any number of terminals m connected in pairs by $m(m-1)/2$ branches, each of which consists of a self-inductance, a resistance, and a capacity in series, with mutual inductance between each pair of branches. The only restrictions imposed are those inherent in all electrical circuits, namely, that the magnetic energy, the dissipation, and the electric energy are each positive for any possible distribution of currents in the branches. Circuits with any arrangement of elements in series or in parallel or in separated meshes can be derived as limiting cases of this general circuit by making a sufficient number of the inductances, resistances, and capacities either zero or infinite.

This general circuit connecting m terminals or branch-points has $n=(m-1)(m-2)/2$ degrees of freedom, that is, n independent meshes. The discriminant [11] of the circuit is the determinant A having the element Z_{jk} in the jth row and kth column, Z_{jk} being the mutual impedance between meshes j and k (self-impedance when $j=k$), the determinant including n independent meshes of the circuit.

The driving-point impedance in the qth mesh S_q is equal to the ratio A/A_{qq}, where A_{qq} is the cofactor of the element in the qth row and qth column of the determinant A. In general, the cofactor of the product of the elements located at the intersection of rows $j, q, s, \ldots$ with columns $k, r, t, \ldots$, respectively, will be denoted by $A_{jk.qr.st\ldots}$.

The determinant A for the general circuit described above is of order n with the element

$$Z_{jk}=iL_{jk}p+R_{jk}+(iC_{jk}p)^{-1} \tag{38}$$

where L_{jk}, R_{jk}, and C_{jk} are the inductance, the resistance, and the capacity, respectively, common to the two meshes j and k. The inductance L_{jk} includes, therefore, the self-inductances of the branches common to the two meshes together with the mutual inductances connecting each branch of one mesh with each branch of the other mesh. The determinant is symmetrical, that is $Z_{jk}=Z_{kj}$, since $L_{jk}=L_{kj}$, $R_{jk}=R_{kj}$, and $C_{jk}=C_{kj}$.

[11] A complete discussion of the solution of circuits by means of determinants has been given by G. A. Campbell, *loc. cit.*, pages 883–886.

These coefficients L_{jk}, R_{jk}, and C_{jk} are subject to the energy conditions stated above, namely, that the magnetic energy, the dissipation, and the electric energy,

$$\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} L_{jk}i_ji_k,\ \sum_{j=1}^{n}\sum_{k=1}^{n} R_{jk}i_ji_k, \text{ and } \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{C_{jk}}\int i_jdt\int i_kdt, \qquad (39)$$

respectively, are each positive for any possible distribution of the currents (i_j, i_k, . . .) in the branches of the circuits.[12] In other words, the coefficients L_{jk}, R_{jk}, and $1/C_{jk}$ are subject to the condition that the three quadratic forms of which these are the coefficients must be positive for all real values of the variables. All the principal minors of the determinants

$$\begin{vmatrix} L_{11} & L_{12} & \ldots & L_{1n} \\ L_{21} & L_{22} & \ldots & L_{2n} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ L_{n1} & L_{n2} & \ldots & L_{nn} \end{vmatrix}, \begin{vmatrix} R_{11} & R_{12} & \ldots & R_{1n}' \\ R_{21} & R_{22} & \ldots & R_{2n} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ R_{n1} & R_{n2} & \ldots & R_{nn} \end{vmatrix}, \text{ and } \begin{vmatrix} \frac{1}{C_{11}} & \frac{1}{C_{12}} & \ldots & \frac{1}{C_{1n}} \\ \frac{1}{C_{21}} & \frac{1}{C_{22}} & \ldots & \frac{1}{C_{2n}} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \frac{1}{C_{n1}} & \frac{1}{C_{n2}} & \ldots & \frac{1}{C_{nn}} \end{vmatrix} \qquad (40)$$

are positive or zero by virtue of this condition.[13] This same condition holds for the inductances if the coefficients L_{jk} apply to branches instead of meshes.

By expanding the determinants in the numerator and denominator of the expression for the driving-point impedance given above, we find

$$S_q=\frac{A}{A_{qq}}=\frac{a_0(ip)^n+a_1(ip)^{n-1}+a_2(ip)^{n-2}+\ldots+a_{2n-1}(ip)^{-n+1}+a_{2n}(ip)^{-n}}{b_1(ip)^{n-1}+b_2(ip)^{n-2}+\ldots+b_{2n-1}(ip)^{-n+1}} \qquad (41)$$

[12] For a recent statement of the energy conditions in this form see L. Bouthillon, *Revue Générale de l'Electricité*, 11, 1922, pages 656–661.

[13] A necessary and sufficient condition that the real quadratic form in n variables

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_jx_k \qquad (a_{jk}=a_{kj}),$$

be positive for all real values of the variables is that each of the n determinants,

$$a_{11}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \ldots, \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix},$$

be positive. For a proof of this see, for example, H. Hancock, "Theory of Maxima and Minima," 1917, pages 82–91.

Upon substituting $\lambda = ip$, multiplying numerator and denominator by λ^n, and dropping the subscript q, formula (41) becomes

$$S = \frac{a_0\lambda^{2n} + a_1\lambda^{2n-1} + a_2\lambda^{2n-2} + \ldots + a_{2n-1}\lambda + a_{2n}}{b_1\lambda^{2n-1} + b_2\lambda^{2n-2} + \ldots + b_{2n-1}\lambda} \tag{42}$$

which may be taken as the most general form of a driving-point impedance. This formula, therefore, gives the impedance of the circuit for any electrical oscillations of the form $e^{\lambda t}$, where λ may have any value, real or complex. Formula (42) may be written in the alternative form

$$S = H\frac{(\lambda-\alpha_1)(\lambda-\alpha_2)(\lambda-\alpha_3)\ldots(\lambda-\alpha_{2n-1})(\lambda-\alpha_{2n})}{\lambda(\lambda-\beta_2)(\lambda-\beta_3)\ldots(\lambda-\beta_{2n-1})}. \tag{43}$$

Thus there are $2n$ roots of S, regarded as a function of λ, which are the $2n$ resonant points of the circuit. There are also $2n$ poles of S, which are the $2n$ anti-resonant points of the circuit, namely, zero, infinity, and the $2n-2$ resonant points of the circuit obtained by opening the branch in which the driving-point impedance is measured.

Upon setting $n=2$ in equations (43) and (42), formulas (1a) and (1b) are obtained, respectively.

From the fact that the coefficients L_{jk}, R_{jk}, and $1/C_{jk}$ satisfy the quadratic form conditions (39), it can be shown mathematically that the coefficients $a_0, a_1, \ldots, a_{2n}$ of (42) are all positive and that the roots $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ of (43) have negative real parts.[14] This can also be shown from the fact that the free oscillations of the circuit are of the forms $e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_{2n} t}$. Thus the roots occur in pairs each of which has negative real values or conjugate complex values with negative real parts.

The same restrictions hold for the coefficients $b_1, b_2, \ldots, b_{2n-1}$ and the poles $\beta_2, \beta_3, \ldots, \beta_{2n-1}$ since the denominator of S, with the exception of the factor λ^n, is also the discriminant of a circuit. Thus the general restrictions (2) are obtained.

In order to obtain the necessary and sufficient conditions that a function of the type (1b) represent a driving-point impedance realizable by a two-mesh circuit, set this function equal to the impedance of the most general two-mesh circuit and investigate the conditions which must hold upon the coefficients in order that the two forms may be equivalent.

[14] The mathematical work is identical with the mathematics of the corresponding dynamical problem. A detailed proof is given by A. G. Webster, *loc. cit.*

The discriminant of the most general two-mesh circuit is of the form

$$A = \begin{vmatrix} L_{11}\lambda+R_{11}+D_{11}\lambda^{-1}, & L_{12}\lambda+R_{12}+D_{12}\lambda^{-1} \\ L_{12}\lambda+R_{12}+D_{12}\lambda^{-1}, & L_{22}\lambda+R_{22}+D_{22}\lambda^{-1} \end{vmatrix}, \tag{44}$$

where the three sets of coefficients, using D_{jk} instead of $1/C_{jk}$, are subject to the restriction that the three determinants

$$\begin{vmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{vmatrix}, \quad \begin{vmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{vmatrix}, \text{ and } \begin{vmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{vmatrix} \tag{45}$$

are all positive or zero, as well as L_{11}, R_{11}, and D_{11}. This condition requires L_{22}, R_{22}, and D_{22} also to be positive or zero.

The most general driving-point impedance of a two-mesh circuit may be taken as the impedance in the first mesh of the circuit defined by the discriminant (44). Set A/A_{11} equal to the value of S given by (1b). Expanding into polynomials in λ, and equating coefficients of the numerators and denominators of the two expressions, the following relations are obtained:

$$L_{11}L_{22}-L_{12}^2=a_0k^2, \tag{46}$$
$$L_{11}R_{22}+L_{22}R_{11}-2L_{12}R_{12}=a_1k^2, \tag{47}$$
$$L_{11}D_{22}+L_{22}D_{11}+R_{11}R_{22}-2L_{12}D_{12}-R_{12}^2=a_2k^2, \tag{48}$$
$$R_{11}D_{22}+R_{22}D_{11}-2R_{12}D_{12}=a_3k^2, \tag{49}$$
$$D_{11}D_{22}-D_{12}^2=a_4k^2, \tag{50}$$
$$L_{22}=b_1k^2, \tag{51}$$
$$R_{22}=b_2k^2, \tag{52}$$
$$D_{22}=b_3k^2, \tag{53}$$

where k has any real value other than zero. Introduce the notation

$$R_{11}R_{22}-R_{12}^2=dk^2, \tag{54}$$

where d is positive or zero. Then, using (46), (54), and (50), eliminate L_{11}, R_{11}, and D_{11} from equations (47)–(49), obtaining

$$(L_{12}R_{22}-L_{22}R_{12})^2=k^2(-dL_{22}^2+a_1L_{22}R_{22}-a_0R_{22}^2), \tag{55}$$
$$(D_{12}L_{22}-D_{22}L_{12})^2=k^2[-a_0D_{22}^2+(a_2-d)D_{22}L_{22}-a_4L_{22}^2], \tag{56}$$
$$(R_{12}D_{22}-R_{22}D_{12})^2=k^2(-a_4R_{22}^2+a_3R_{22}D_{22}-dD_{22}^2). \tag{57}$$

Using (51)–(53), eliminate L_{22}, R_{22}, and D_{22} from the right-hand members of (55)–(57); extract the square root; rearrange the order of the equations, obtaining

$$R_{12}D_{22}-R_{22}D_{12}=\pm k^3(-a_4b_2^2+a_3b_2b_3-db_3^2)^{1/2}, \tag{58}$$
$$D_{12}L_{22}-D_{22}L_{12}=\pm k^3[-a_0b_3^2+(a_2-d)b_3b_1-a_4b_1^2]^{1/2}, \tag{59}$$
$$L_{12}R_{22}-L_{22}R_{12}=\pm k^3(-db_1^2+a_1b_1b_2-a_0b_2^2)^{1/2}. \tag{60}$$

Thus conditions (4)–(6) are obtained directly from (58)–(60). The left-hand members of (58)–(60) satisfy the identity

$$(R_{12}D_{22}-R_{22}D_{12})L_{22}+(D_{12}L_{22}-D_{22}L_{12})R_{22} \\ +(L_{12}R_{22}-L_{22}R_{12})D_{22}=0. \qquad (61)$$

Substituting (51)–(53) and (58)–(60) in this identity (61), and rationalizing, equation (3) and its equivalent (21) are obtained.

For the general network of Fig. 7,

$$\left.\begin{array}{lll} L_{11}=L_1'+L_2', & L_{12}=L_2', & L_{22}=L_2'+L_3', \\ R_{11}=R_1+R_2, & R_{12}=R_2, & R_{22}=R_2+R_3, \\ D_{11}=D_1+D_2, & D_{12}=D_2, & D_{22}=D_2+D_3, \end{array}\right\} \qquad (62)$$

where L_1', L_2', and L_3' are defined by (17)–(19). For this set of constants, branch 2 is made the branch common to the two meshes; the choice of branch 3 as the common branch would not affect the final formulas. Substituting these values (62) in (46), (54), (50)–(53), and (58)–(60), equations (7)–(15) are obtained directly.

Thus Theorems I and IV are completely proved. Theorems II and III are verified by the actual formulas for the elements given in Tables I and II, and by the census of networks presented in Table III.

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