The Theory of the Operation of the Howling Telephone with Experimental Confirmation

By HARVEY FLETCHER

SYNOPSIS: A general theory of the sustained oscillations of electromechanical systems is presented in the paper. The electrodynamical properties of the telephone transmitter and receiver are described and sufficient numerical data are given to enable one to calculate the intensity and frequency of howling for various types of systems. Detailed consideration is given to the following three systems, namely, one where the transmitter and receiver disphragms are coupled together mechanically by a lever system, one where they are coupled by a small box of air, and one where they are coupled by a long tube of air. The type of electrical circuit to use with each of these systems depends upon the type of performance desired.

WHEN the telephone receiver of a subscriber's set is held in front of the mouthpiece of the transmitter, a shrill note is emitted. A sustained oscillation is set up in the electro-mechanical system which is frequently called "howling" or "singing" or "humming."

This phenomenon was first observed by A. S. Hibbard of the United States in 1890. Frank Gill was the first to publish an account of the phenomenon. He first noted that the pitch of the howling note was changed by reversing the telephone receiver connection. In summarizing further his experimental results, he states "that the pitch of the note appears to be determined by the length of the column of air between the two diaphragms and the conditions of the circuit. As the periodic time of the circuit is increased, the time of the note rises. To some extent, the pitch is governed by the rate of the diaphragm, but I do not think this is so important a factor as the others. The main factors appear to be the angle of lag and the length of the column of air between the diaphragms. Although the vibration is a forced one, we could almost see that its rate is largely dependent on the free period of the circuit." [1]

In 1908 Kennelly and Upson extended Gill's work and made extensive experimental investigations of the case in which the transmitter and receiver are coupled together acoustically by means of a

[1] Taken from a paper on "Notes on the Humming Telephone" by F. Gill, read at a meeting of the Dublin Local Section of the Society of Telephone Engineers and published in the Journal of the Institution of Electrical Engineers, Vol. XXXI, 1901.

hollow circular tube of varying lengths and electrically by means of an induction coil. The summary of the conclusions is as follows:[2]

"(1) The mean frequency of the humming-telephone note is determined solely by the receiver diaphragm, and its natural free rate of vibration. (2) The ascending intersections of the frequency zig-zag with the mean frequency line will be formed approximately at tube lengths of $(3/4+m)\ v/n_o$ cm. for one connection, and of $(1/4+m)\ v/n_o$ cm. for the other connection, of the receiver; where v is the velocity of sound in air, n is the mean frequency in cycles per second, and m is any positive integer, within the working range of the tube. The constants 3/4 and 1/4 may be modified by the presence of condensers, and other circumstances. (3) The range of pitch variation. and the breaking positions, are determined by the transmitter, and by the reinforcing capability of the system. For systems that are weak, either electrically or acoustically, the range of pitch, above or below the mean, will be small. (4) The primary current, as measured by a DC instrument, is ordinarily a minimum at the mean frequency, and a maximum at a break. (5) Transmitters may be tested for effectiveness, by measuring their hum-extinguishing resistances in the primary or secondary circuit. The tube length should be such as to produce mean frequency if one connection of receiver only is used, but should favor both connections equally, if both connections of receiver are used."

They also give a first approximation theory to account for the changes in frequency as the length of the coupling tube is changed.

In 1917, H. W. Nichols gave the general equations for the special case where the two diaphragms act as pistons closing the ends of a tube of air. This case was given as an illustrative example of the "Theory of Variable Dynamical Electrical Systems."[3]

This paper gives a theoretical treatment of the behavior of a system containing a transmitter and a receiver coupled together acoustically and electrically, and with a source of electrical energy feeding the transmitter. Formulae are deduced which give the frequency and intensity of howling in terms of the physical constants of the system. Numerical calculations are given and sufficiently detailed solution of some special cases are given to enable one, who is interested in using the howling telephone as a source of alternating current or for other experimental work, to design the set for his particular purpose.

[2] "Humming Telephone" by A. E. Kennelly and Walter L. Upson, American Philosophical Society, July 20, 1908.

[3] Physical Review, Aug., 1917, p. 191.

General Solution of the Howling Circuit

The elements of a telephone system which is howling are the transmitter, the receiver, the mechanical coupler and the electrical coupler as indicated in Fig. 1. If there is a source of electrical power in the electrical coupler, which is released by movements of the transmitter diaphragm in the form of electrical vibrations, and also, if there is a proper relationship between these four elements, then a sustained

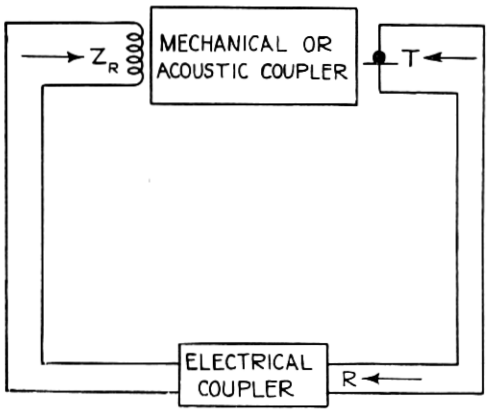

Fig. 1

howling will result. In other words, if the gain in the transmitter due to its amplifying action is just equal to the losses in the electrical and mechanical circuits, then a steady oscillatory state will be maintained. The problem is to determine the nature of these relationships.

Assume that the conditions are such that a steady oscillatory state has been set up. Under such conditions let T be the electrical impedance of the transmitter, R the impedance looking away from the transmitter terminals into the electrical coupler, and Z_R the impedance of the receiver. It is well known that the impedance Z_R is dependent upon the velocity of motion of the receiver diaphragm. Also, T is dependent upon the amplitude of motion of the transmitter diaphragm as well as upon the direct current supplied to it. Consequently, the impedances defined above are not only dependent upon frequency but also upon the mechanical coupling and magnitude of the current supplied to the transmitter.

If e is the electromotive force created in the transmitter, and i the current flowing through it both expressed in root mean square values[4], then

$$e=(T+R)i \tag{1}$$

It is convenient to define a quantity M which I shall call the unilateral mutual impedance by the qeuation

$$e_1=Mi_1 \tag{2}$$

where e_1 is the electromotive force created in the transmitter when a current i_1 flows in the receiver circuit. It is a quantity which is closely related to the effectiveness of the mechanical coupling and the efficiencies of the transmitter and receiver.

If the electrical coupler be considered part of the receiver, and the transmitter and receiver circuits are connected together as in Fig. 1, then $e=e_1$, and $i=i_1$. Consequently

$$M=T+R \tag{3}$$

is the condition for sustained oscillation. This condition is in effect a pair of conditions, as the two sides of the equation must be equal both in amplitude and in phase. These two conditions are sufficient to determine the frequency and intensity of howling.

In order to express M and R in more fundamental physical constants, it is necessary to examine more closely the mechanical and electrical connections. Before doing this for some important special cases, it will be necessary to discuss some of the electro-dynamical properties of transmitters and receivers.

Electrodynamical Properties of Transmitters and Receivers

For the sake of clarity the discussion will be confined to permanent magnet receivers and carbon transmitters. The modifications necessary for other types of instruments will, I think, be evident from the discussion. Representing by F_R and F_T the forces acting on the diaphragms of the receiver and transmitter respectively, and by y and z their displacements, we have the following equations defining the "stiffness factors" S_R and S_T

$$S_R=\frac{F_R}{y} \tag{4}$$

$$S_T=\frac{F_T}{z} \tag{5}$$

[4] In what follows all quantities involving periodic variations will be expressed as root mean square values unless otherwise specified, and the vector notation will be used for denoting phases.

These factors are usually complicated functions of the frequency while S_T likewise depends on the kind and amount of agitation. In the case of a system of a single degree of freedom which may be regarded as a first approximation to this case

$$S = m\omega^2 + j\omega r + s \tag{6}$$

where ω is 2π times the frequency. When referring to the movements of a diaphragm, the quantity m represents the mass, r the mechanical resistance, and s the elastic constant. The stiffness factor S divided by $j\omega$ is usually called the mechanical impedance.

Measurements have shown that for the transmitters and the

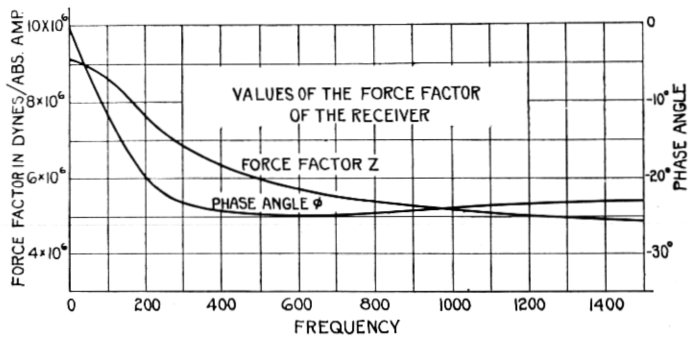

Fig. 2

receivers used in the experiments described below, the following constants represent approximately the two stiffness factors in the region of resonance

$$S_R = -.93\omega^2 + 230j\omega + 3\times10^7 \tag{6'}$$

$$S_T = -4.5\omega^2 + 2000j\omega + 2\times10^8 \tag{6''}$$

An important constant which enters into the determination of the unilateral mutual impedance M is the force factor of the receiver which will be designated by Z. It is defined as the force in dynes acting upon the diaphragm per unit of current. For the receivers used in this investigation, its values in magnitude and phase are shown for various frequencies in Fig. 2. These were determined by the method outlined by Wegel.[5] In the region of the resonant frequency its value in absolute units can be approximately represented by

$$Z = 5.3\times10^6 \,\underline{|24^\circ}. \tag{7}$$

[5] Theory of Telephone Receivers—Wegel, R. L., Jour. of A. I. E. E., Oct. 1921.

The impedance Z_R of the receiver varies with frequency and depends upon the load on the diaphragm. If S is the loaded stiffness of the diaphragm, that is, its resistance to force under actual working conditions, and Z_d is the impedance of the receiver when the diaphragm is prevented from moving, then it is well-known that

$$Z_R = Z_d + j\frac{\omega Z^2}{S}. \tag{8}$$

It was found that Z_d expressed in ohms could be represented in the frequency region near resonance by the formula

$$Z_d = 93 + .06f + j(43 + .15f) \tag{9}$$

where f denotes the frequency in cycles per second.

The electromotive force e created in the transmitter, the direct current I flowing through it, and the displacement of the diaphragm are related in a rather complicated way. For describing this relationship it is convenient to define a modulation factor h by the equation

$$e = Ihz \tag{10}$$

Combining this equation with (2) it is seen that

$$M = Ih\frac{z}{i} \tag{11}$$

which shows that the modulation factor is also an important one in determining the unilateral mutual impedance. For a sustained oscillation the factor Ih does not enter into the periodic variation and may be thought of as an electro-mechanical impedance between the electromotive force created in the button and the displacement of the diaphragm of the transmitter. However, for a different condition of sustained oscillation which results in giving z a different magnitude the value of h changes. In other words h is dependent upon the agitation of the carbon as represented by z, and also upon the direct current supplied to the transmitter. It is mainly this variable character of h that makes it possible to fulfill the conditions for sustained howling.

Simultaneous measurements of e, I and z were made upon several transmitters of the type used in this investigation. From the results obtained and from the defining equation (10) for h, it was found that the following empirical equation would represent approximately the relation between h, I and z, namely

$$h = \frac{32 + \frac{z}{2}}{\left(2.6 + 2z + \frac{1}{z}\right)(I + .03)} \tag{12}$$

where z is expressed in microns and I in amperes e in volts and h in ohms per micron. To facilitate solving for z when h and I are given, a set of curves showing this relation is given in Fig. 3. It is this modulation factor h which measures the efficiency of the transmitter button.

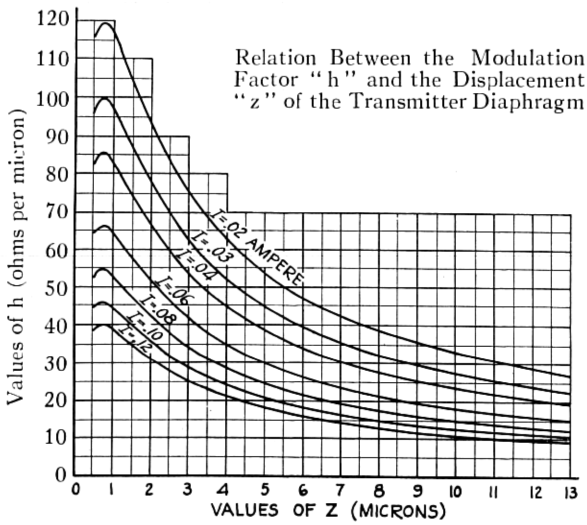

Fig. 3

It is also necessary to know the dependence of T upon z and I. To obtain this relation corresponding values of e and V, the DC drop across the transmitter as measured by direct current measuring instruments, were obtained for various degrees of agitation and amounts of direct current. Four transmitters were used in establishing the relation, the results being shown in Fig. 4. Then, for any value of the supply current I a value of T can be obtained from V. From the corresponding e a value of h and z can be obtained from equations (10) and (11). In this way the relations shown in Figs. 5

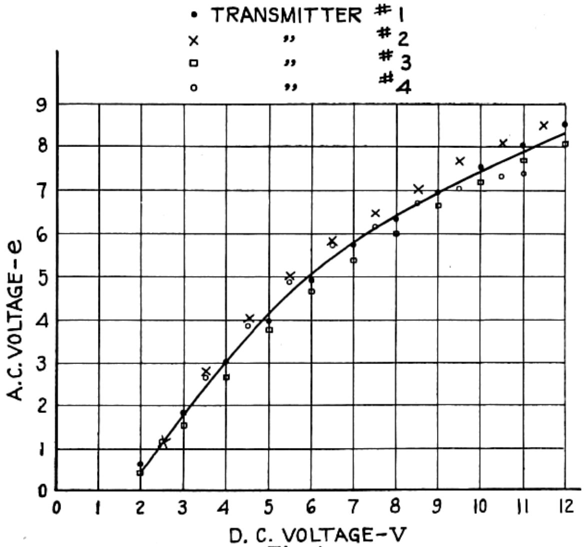
• TRANSMITTER #1
× " #2
□ " #3
○ " #4
A.C. VOLTAGE-e
D. C. VOLTAGE-V

Fig. 4

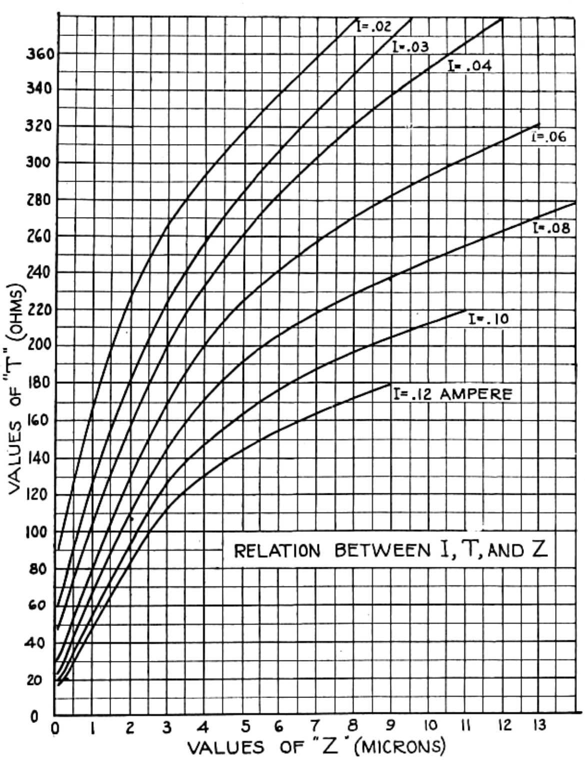
I=.02
I=.03
I=.04
I=.06
I=.08
I=.10
I=.12 AMPERE
RELATION BETWEEN I, T, AND Z
VALUES OF "T" (OHMS)
VALUES OF "Z" (MICRONS)

Fig. 5

and 6 were obtained. It is thus seen that for a given type of transmitter if the direct current and any one of the four quantities *e*, *h*, *z*, or *T* are known, the others are determined and may be obtained from suitable curves.

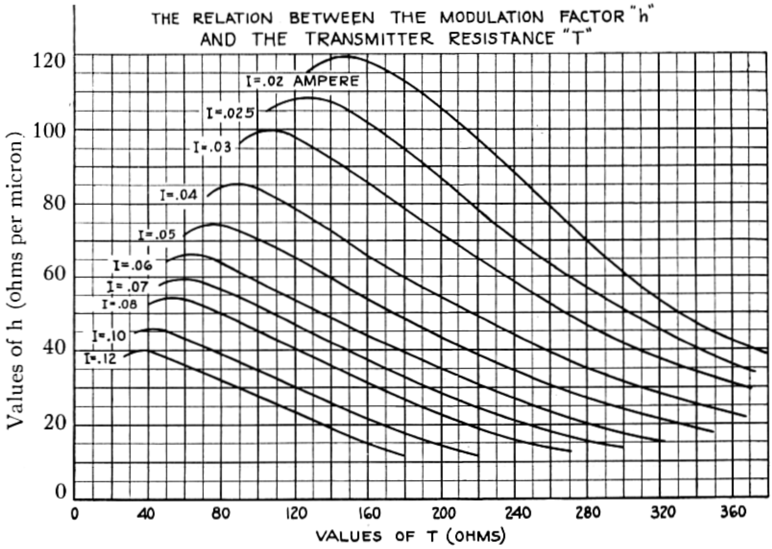

Fig. 6

Commercial receivers and transmitters have constants which vary largely from those given above. These values represent the general behavior of such instruments and are useful in understanding their operation in a howling circuit. Inasmuch as the performance of such instruments particularly the transmitter depends very largely upon the condition of operation the constants given cannot be applied with confidence to conditions greatly different from those mentioned in the paper. With these facts concerning telephone instruments in mind we are now in a position to treat some special cases.

Case 1—Diaphragms Connected Mechanically by a Rigid and Weightless Lever

To illustrate the method of solution this special case will be solved in some detail. A diagrammatic sketch illustrating the connections is shown in Fig. 7. Neglecting the reaction of the air, the vibration of the receiver diaphragm is controlled by the force *Zi* exerted by the receiver winding and the opposing force X exerted by the connecting rod.

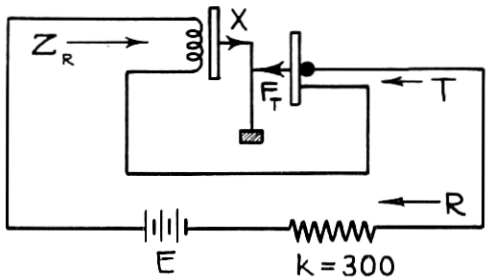

Fig. 7

The amplitude of motion of the receiver diaphragm is then given by

$$y=\frac{Zi-X}{S_R} \tag{13}$$

If the lever is rigid and weightless and has an arm ratio c, then

$$F_T=cF_R \tag{14}$$

and due to the restraint

$$y=cz=\frac{c^2X}{S_T} \tag{15}$$

Using these equations together with equation (11) it is seen that

$$M=\frac{IhZ}{cS_R+\frac{1}{c}S_T} \tag{16}$$

$$S=S_R+\frac{1}{c^2}S_T, \tag{17}$$

$$R=Z_d+j\frac{\omega Z^2}{S}+k. \tag{18}$$

The relation between I and T is given by

$$I=\frac{E}{T+R_{DC}+k} \tag{19}$$

where R_{DC} is the direct current resistance of the receiver winding and k is the line resistance. The condition (3) for howling then becomes

$$IhZ=(Z_d+k+T)\left(cS_R+\frac{1}{c}S_T\right)+j\omega Z^2c. \tag{20}$$

This is equivalent to two scalar equations and taken together with (19) and the curves of Fig. 6 gives the necessary four equations to solve for the unknowns f, h, T, and I.

The solution, however, is not straightforward since the relation between h, T, and I is only given empirically by a set of curves. By "cut and try" methods the solution for any numerical case can be obtained. The last term of (20) is usually negligible or at least it is of second order of magnitude. Consequently, the sum of the phase angles of the other factors must be approximately equal to the phase of Z. This completes the formal solution for this case.

The solution of a numerical case throws considerable light upon the physical phenomenon taking place, and also upon the method of calculation. Let the arm ratio be unity, a case corresponding to that when the diaphragms are connected directly together, and assume that the supply current is furnished by a battery of 24 volts through a line having a resistance of 300 ohms. Using the constants for the receivers and transmitters given above and expressing f in kilocycles, T in ohms, I in amperes and h in ohms per micron, equations (19) and (20) become

$$I=\frac{24}{384+T}, \tag{19'}$$

$$Ih\ 52\,|\overline{24^\circ}=[393+T+60f+j(43+150f)]\,[-2.14f^2+2.3+j.14f]+j\ 1.7f. \tag{20'}$$

If I is positive there is no solution for f, since the angle of the first factor is in the first quadrant, and that of the second factor either in the first or second; consequently, the phases cannot match at any frequency. If the supply current is reversed, then I is negative or 180° is added to the phase of the left hand member making it a positive 156°. The solution for this case is

$f=1072$ cycles	$i=8.2$ mils
$h=64$	$e=5.5$ volts
$T=150$ ohms	$y=z=1.9$ microns
$I=45$ mils	

If a value of c equal to 2.7, which is approximately equal to the square root of the ratio of mechanical impedances of the two diaphragms, then the solution for reversed DC supply becomes

$f=1001$ cycles	$i=10$ mils
$h=47.3$	$e=7.16$ volts
$T=236$ ohms	$z=3.9$ microns
$I=39$ mils	$y=10.5$ microns

It is thus seen that changing the ratio arm has increased the howling intensity, but the increase for the various elements is greatly different. The frequency is slightly lowered, the values of h and I have been

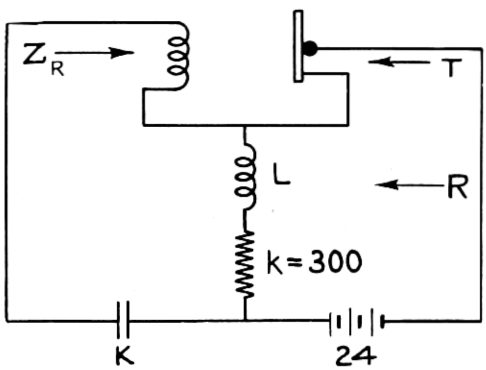

Fig. 8

reduced by 26% and 14% respectively, while the values of y, z, T, i and e have been increased 400%, 105%, 57%, 22% and 30% respectively.

If the circuit of Fig. 7 is modified as shown in Fig. 8, the inductance L being very large, then the condition for howling becomes

$$\frac{IhZ}{cS_R+\frac{1}{c}S_T}=T+Z_d+\frac{j}{K\omega}+j\frac{\omega Z^2}{S} \tag{21}$$

and

$$I=\frac{24}{300+T}. \tag{22}$$

Using the same constants as above the condition for howling becomes

$$Ih\ 52\,|\overline{24°}. = \left[93+T+60f+j(43+150f)-\frac{158}{Kf}\right][-2.14f^2+2.3+j.14f]+j1.7f \qquad (23)$$

The solution for values of $K=1$ *mf*, $K=1/2$ *mf*, and $K=1/5$ *mf* are given in Table I. When $K=1$ *mf* and the supply current is direct the solution which satisfies the phase equality is $f=506$. This corresponds to $h=220$ which is an impossible value. Therefore, no howling will be sustained for this condition. For $K=1/2$ *mf* the system will howl for both direct and reversed supply current, the frequency changing suddenly from 839 to 1119 cycles as the current is reversed while the other variables change only slightly.

TABLE I

	$K=1$		$K=1/2$		$K=1/5$	
	Direct	Reversed	Direct	Reversed	Direct	Reversed
f	......	1016	839	1119	935	......
h	220	33.5	53.5	57.4	44.2	......
T	......	275	160	140	220	......
I	......	42	52.2	54.5	46	......
i	......	20	18.3	17.6	10.9	......
e	......	8.2	6.15	5.6	7.5	......
z	......	5.9	2.2	1.8	3.7	......
y	......	16	5.95	4.9	10	......

It is interesting to note the change in the howling frequency as the value of K increases. When the supply current is negative, and for values larger than 1 *mf*, the frequency of howling is always close to 1000, as K goes from 1 to 1/2 the frequency increases to above 1100. For smaller values of K the frequency continues to slowly increase until, for values smaller than 1/3, the system ceases to sustain oscillations. For positive values of supply current no howling will result until K becomes smaller than 2/3 where the frequency is around 800. The frequency then increases reaching a howling frequency around 1000 for $K=1/7$. For smaller values of K no howling will be sustained.

Case II—Diaphragms Coupled Together by a Small Chamber of Air

It will be assumed that the air chamber is so small that the phase of the pressure variation is the same on both diaphragms. Let V be the volume of air between the diaphragms. Then

$$V = V_o + Q_R y + Q_T z \tag{24}$$

where V is the volume of air in the undisturbed state and Q_R and Q_T are the effective areas of the receiver and transmitter diaphragms respectively.

The pressure variation in the chamber (changes considered adiabatic) is given by

$$dp = -\gamma \frac{P}{V} dV = -(Q_R y + Q_T z)\gamma \frac{P}{V} \tag{25}$$

When the steady state is set up this may be considered a vector equation and the variables expressed in *rms* values.

The equations of motion for the diaphragms are

$$y = \frac{Zi - Q_R dp}{S_R} \tag{26}$$

and

$$z = \frac{Q_T dp}{S_T}. \tag{27}$$

Solving

$$y = \frac{Zi\left(S_T + \frac{\gamma P}{V} Q_T^2\right)}{S_R S_T + \frac{\gamma P}{V} Q_T^2 S_R + \frac{\gamma P}{V} Q_R^2 S_T}, \tag{28}$$

$$z = -\frac{\frac{\gamma P}{V} Q_R Q_T}{S_T + \frac{\gamma P}{V} Q_T^2} y, \tag{29}$$

$$M = \frac{IhZQ_R Q_T}{\frac{V}{\gamma P} S_R S_T + Q_T^2 S_R + Q_R^2 S_T}. \tag{30}$$

In this case the ratio between z and y is not fixed, but depends upon S_T which is a function of the frequency.

The loaded stiffness of the receiver diaphragm is

$$S=\frac{\frac{V}{\gamma P}S_R S_T+Q_T^2 S_R+Q_R^2 S_T}{\frac{V}{\gamma P}S_T+Q_T^2}. \quad (31)$$

For the transmitter and receiver used

$$Q_R=6.5,$$
$$Q_T=10.3.$$

Let the volume of entrapped air be taken as 10 cc., then

$$\frac{\gamma P}{V}=1.418\times 10^5.$$

Using these values and the values for S_R and S_T and the circuit of Fig. 8 with $K=\frac{1}{2}$ the condition for howling becomes

$$Ih\,3.48\underline{|24^\circ}=27.6f^5+(50.3+.459T)f^4-59.9f^3-(1.01T+85.7)f^2+31.9$$
$$+(.539T+33)+2\Big[68f^5+16.7f^4-(.0506T+11.1)f^3-40f^2$$
$$+(.0537T-233)f+23.2+\frac{172}{f}\Big] \quad (32)$$

where I is expressed in amperes, T in ohms, f in kilocycles and h in ohms per micron.

For reverse current or negative I the solution is

$f=970$ kilocycles	$i=24\,\underline{|17^\circ}$
$h=30.5$	$e=8.7$ volts
$T=290$ ohms	$z=7.0$ microns
$I=.0407$ mils	$y=1.9\,\underline{|158^\circ}$ microns

Comparing this to the case where the diaphragms are coupled by a lever having an arm ratio 2.7 it is seen that the air coupling produces a greater e.m.f. in the transmitter and only a slightly increased AC current. The receiver diaphragm in this case, however, has a smaller amplitude than the transmitter diaphragm. At this particular howling frequency the transmitter diaphragm stiffness is only about 1/4 that of the receiver diaphragm stiffness which explains this anomalous result. Also, it will be seen that the diaphragms vibrate almost oppositely in phase.

These cases are sufficient to illustrate the method of calculation, but there is one other important case for which I desire to give the results as this is the case handled experimentally by Kennelly and Upson.

Case III—Diaphragms Connected Acoustically by a Tube of Air of Uniform Cross-Section with an Air Chamber at Both Ends

In this case the two diaphragms are connected acoustically by the air, but since the tube has considerable length phase differences exist

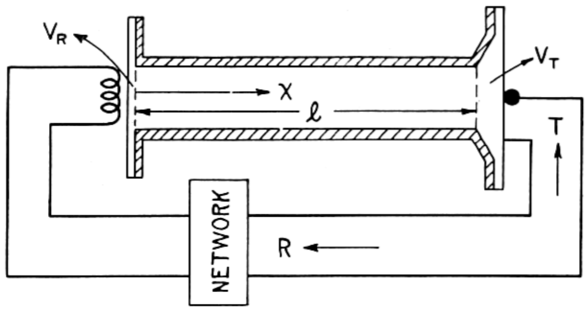

Fig. 9

at different points along it. The connections are shown schematically in Fig. 9.

The equation of motion for the receiver diaphragm is

$$y=\frac{Zi-Q_R dp_R}{S_R}=\frac{Zi}{S} \tag{33}$$

and for the transmitter diaphragm is

$$z=\frac{Q_T dp_T}{S_T} \tag{34}$$

where dp_R and dp_T are the pressure variations in the air chambers at the receiver and transmitter ends of the tube respectively.

The equations of motion for a gas in which the movements are small and in only one direction and in which the fluid friction is neglected are as follows: [6]

$$\frac{d^2\phi}{dt^2}=a^2\frac{d^2\phi}{dx^2}, \tag{35}$$

$$\frac{dp}{\rho}=-\frac{d\phi}{dt}, \tag{36}$$

[6] See Rayleigh "Theory of Sound," Vol. II, pp. 14 and 15, 49 and 50.

where ϕ is the velocity potential, t the time, a the velocity of sound in the air, x the distance along the tube, p the pressure and ρ the density of the air.

For the case in which we are interested, a sinusoidal oscillation is sustained, so that the special solution

$$\phi = e^{j\omega t}\left(A\cos\frac{\omega X}{a} + B\sin\frac{\omega x}{a}\right) \tag{37}$$

is suitable for our problem. Quantities A and B are arbitrary constants which are determined by the end conditions. Substituting this value of ϕ in equation (35), there results

$$dp = -\rho j\omega e^{j\omega t}\left(A\cos\frac{\omega x}{a} + B\sin\frac{\omega x}{a}\right) \tag{38}$$

It remains then to determine the arbitrary constants A and B.

At the receiver end of the tube, the displacement, ζ_R of the air diaphragm across the end of the tube is related to the displacement y of the receiver diaphragm. This relationship is established by the following consideration. If q is the cross-section of the tube, the increase in volume in the air chamber is given by

$$dV_R = (\zeta_R q - yQ_R). \tag{39}$$

Assuming that the air chamber is so small that the pressure change at any instant is the same throughout, and that it takes place adiabatically, we have:

$$dp_R = -\gamma\frac{p}{V_R}dV_R \tag{40}$$

Combining equations (33), (39), and (40), we obtain:

$$qS_R\zeta_R = Q_R Zi - \left(\frac{V_R}{\gamma p}S_R + Q_R{}^2\right)dp_R \tag{41}$$

Similarly,

$$qS_T\zeta_T = \left(\frac{V_T}{\gamma p}S_T + Q_T{}^2\right)dp_T \tag{42}$$

Then the following conditions must be fulfilled at the two ends of a tube of length l.

At $x=0$, $\quad dp = dp_R$ and $\dfrac{d\phi}{dx} = \dfrac{d\zeta_R}{dt}$;

at $x=l$, $\quad dp = dp_T$ and $\dfrac{d\phi}{dx} = \dfrac{d\zeta_T}{dt}$.

These conditions give the following equations:

$$a\omega\rho A + S'_R B = jaZ' i_O \tag{43}$$

$$\left(a\omega\rho \cos\frac{\omega l}{a} + S'_T \sin\frac{\omega l}{a}\right) A + \left(a\omega\rho \sin\frac{\omega l}{a} - S'_T \cos\frac{\omega l}{a}\right) B = 0 \tag{44}$$

where

$$S'_R = \frac{qS_R}{Q_R{}^2 + \frac{V_R}{\gamma p} S_R},\ S'_T = \frac{qS_T}{Q_T{}^2 + \frac{V_T}{\gamma p} S_T},\ Z' = \frac{ZQ_R}{Q_R{}^2 + \frac{V_T}{\gamma p} S_R}.$$

Solving for the constants A and B, we find their values to be:

$$A = jaZ' i_o \left(S'_T \cos\frac{\omega l}{a} - a\omega\rho \sin\frac{\omega l}{a}\right) \div D, \tag{45}$$

$$B = jaZ' i_o \left(S'_T \sin\frac{\omega l}{a} + a\omega\rho \cos\frac{\omega l}{a}\right) \div D, \tag{46}$$

where

$$D = [S'_R S'_T - (a\omega\rho)^2] \sin\frac{\omega l}{a} + a\omega\rho (S'_R + S'_T) \cos\frac{\omega l}{a}. \tag{47}$$

The two pressure values are then given by:

$$dp_R = Z'_I a\omega\rho \left(S_T \cos\frac{\omega l}{a} - a\omega\rho \sin\frac{\omega l}{a}\right) \div D, \tag{48}$$

$$dp_T = Z'_I a\omega\rho\, S'_T \div D, \tag{49}$$

and

$$y = \frac{Zi}{S_R D}\left[S'_R S'_T - (a\omega\rho)^2 \left(1 - Q_R \frac{Z'}{Z}\right) \sin\frac{\omega l}{a} + a\omega\rho \left(S'_R S'_T \left(1 - Q_R \frac{Z'}{Z}\right)\right) \cos\frac{\omega l}{a}\right], \tag{33'}$$

$$z = \frac{qZia\omega\rho}{D}\left[\frac{Q_T}{Q_T{}^2 + \frac{V_T}{\gamma p} S_T}\ \frac{Q_R}{Q_R{}^2 + \frac{V_R}{\gamma p} S_R}\right]. \tag{34'}$$

The loaded stiffness of the receiver diaphragm is given by

$$S = \frac{qQ_T Q_R a\omega\rho \left(N \sin\frac{\omega l}{a} + P \cos\frac{\omega l}{a}\right)}{S_T \frac{a\omega\rho}{\gamma p}\left[\left((q^2 - \frac{a\omega\rho}{\gamma p} V_R V_T\right) \sin\frac{\omega l}{a} + q(V_T + V_R) \cos\frac{\omega l}{a} - a\omega\rho \left(\frac{a\omega\rho}{\gamma p} V_R Q_R{}^2 \sin\frac{\omega l}{a} + qQ_T \cos\frac{\omega l}{a}\right)\right.}, \tag{50}$$

where

$$N=S_TS_R\frac{1}{a\omega\rho}\left[\frac{q^2}{Q_RQ_T}-\frac{(a\omega\rho)^2}{(\gamma p)^2}\frac{V_TV_R}{Q_RQ_T}\right]-S_R\left(\frac{a\omega\rho}{\gamma p}\right)\frac{Q_T}{Q_R}V_R-S_T\left(\frac{a\omega\rho}{\gamma p}\frac{Q_R}{Q_T}V_T,\right. \quad (51)$$

$$P=S_R\frac{Q_T}{Q_R}+S_T\frac{Q_R}{Q_T}+S_TS_R\frac{V_T+V_R}{Q_TQ_R}\frac{1}{\gamma p}. \quad (52)$$

The unilateral mutual impedance M is given by

$$M=\frac{IhZ}{N\sin\frac{\omega l}{a}+P\cos\frac{\omega l}{a}}. \quad (53)$$

The condition for sustained howling becomes

$$\frac{IZ}{T+R}h=N\sin\frac{\omega l}{a}+P\cos\frac{\omega l}{a}. \quad (54)$$

If the two diaphragms work directly into the connecting tube as pistons, then $Q_R=Q_T=q=Q$ and $V_R=V_T=O$ and the expressions for M and S become [7]

$$M=\frac{IhZ\,Q\,a\omega\rho}{[S_RS_T-(a\omega\rho)^2Q^2]\sin\frac{\omega l}{a}+(S_R+S_T)Q\,a\omega\rho\cos\frac{\omega l}{a}}, \quad (55)$$

$$S=\frac{[S_RS_T-(a\omega Q)^2]\sin\frac{\omega l}{a}+(a\omega\rho Q)\,(S_R+S_T)\cos\frac{\omega l}{a}}{S_T\sin\frac{\omega l}{a}+a\omega\rho\,Q\cos\frac{\omega l}{a}}. \quad (56)$$

The method of solution is the same as that given for the simpler cases, although it is evident that the actual work of calculation is more involved.

It is seen that in such a system the intensity and frequency depend upon a large number of quantities, namely: S_T and S_R, the diaphragm stiffness factors; Q_R and Q_T the effective areas of the two diaphragms; V_R and V_T the volumes of air entrapped between the diaphragm and the opening into connection tube; the length l and the cross section q of the connecting tube; the pressure a, the density s, and the velocity of sound a for the gas in the connecting tube; the resistance T, direct current I and modulation factor h of the transmitter; and

[7] These two equations were given by H. W. Nichols in essentially this form in the Physical Review, Vol. 10, p. 171; 1917.

the force factor and impedance of the receiving circuit. Modification of any of these may produce marked changes in the resulting howling.

The way the length l enters the formula (54) for sustained howling indicates that the curves representing the possible frequencies of howling, that is, frequencies which produce equality of phase on both sides of the equation, vary periodically with the length.

The intersection of the branches of these curves on any given frequency line will be separated by distances corresponding to $\frac{a}{f}$, that is, corresponding to a wave length at the pitch corresponding to f. Also, if the supply current is reversed, that is, the sign of I

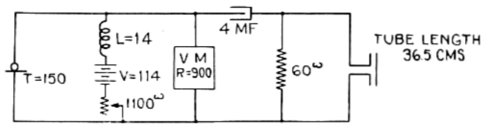

Fig. 10

changed, and the length of the tube varied until the frequency of howling is brought back to the original value, the change in length must be equal to $\frac{a}{2f}$. For since the frequency is unchanged all the quantities in equation (54) remain unchanged except the sine and cosine factors. Adding a half wave length is equivalent to adding π to the angle which makes the left hand member the negative of its first value, and consequently, restores the phase equality.

Using the circuit shown in Fig. 10 for the electrical coupling, the frequency of howling was computed for various tube lengths, the results being given in Fig. 11.

The instrument constants were those used before, the other values being $V_R=1.6$ cc., $V_T=6.4$ cc., and $q=.97$ cm.^{2}, $a=3.43\times10^4$ cm/sec. $\rho=.001203$ gm/cm^3. Using these values the formulae for N and P become

$$N=(-1.31f^5+7.5f^3-9.68f+3.26\frac{1}{f})\times10^8+j(.141f^4-.63f^2+.36)\times10^8,$$

$$P=(5.5f^4-12.35f^2+6.77)\times10^8+j(-.60f^3+.66f)\times10^8,$$

where f is the frequency in kilocycles.

The points on the calculated curves of Fig. 11 were obtained by direct experimental observation with the circuit shown, and with various lengths of brass tube coupling the transmitter and receiver together. The agreement between the calculated and observed values is well within the experimental error involved in determining the constants used in the calculation.

In Fig. 12 are shown similar calculated curves for a transmitter called "hollow," that is, for one having a lower natural period of vibration. It is coupled to the same receiver as used before. The dotted curves in each case represent the behavior for reversed current.

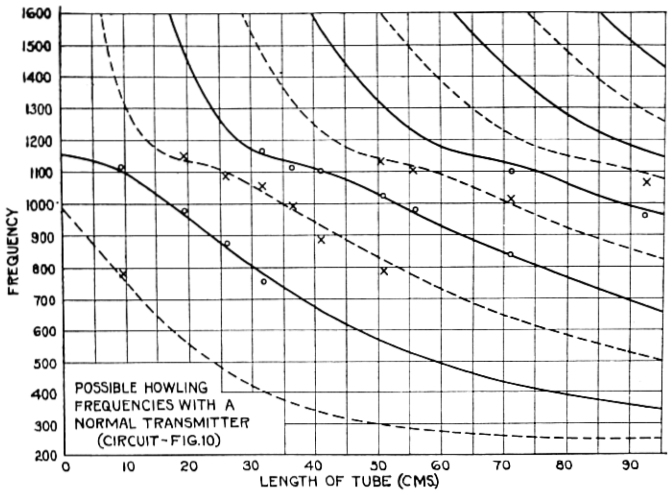

Fig. 11

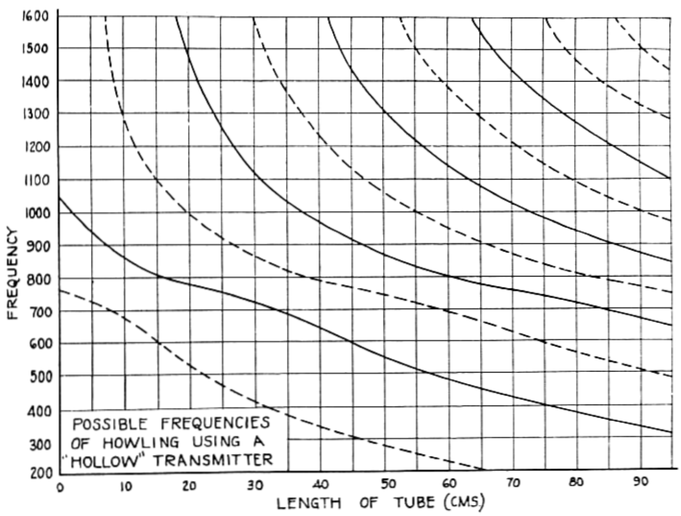

Fig. 12

In Figs. 13 and 14 are shown the probable frequencies of howling for these two transmitters as the tube length of the coupler is increased. The shaded areas are the so-called breaking points where the howling may be at either of the frequencies shown.

With these facts in mind let us review the conclusions reached by Kennelly and Upson given in the beginning of this paper. It is seen that conclusion (1) is not warranted. The transmitter and circuit conditions as well as the receiver diaphragm influence the mean frequency of humming. The second conclusion regarding the branches

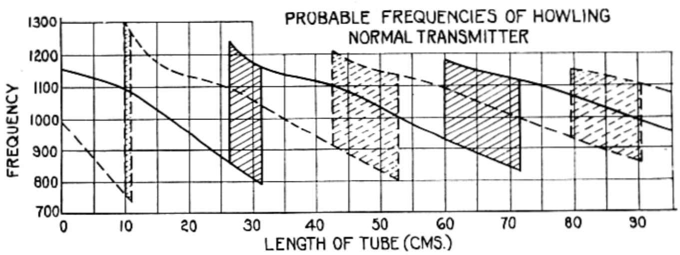

Fig. 13

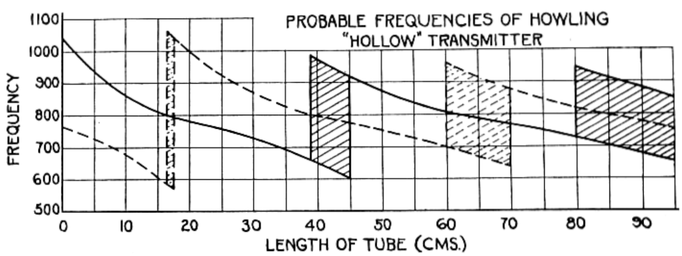

Fig. 14

of the curves representing the relation between frequency and tube length is correct and the explanation has just been given. This periodic relation is not only true of the mean frequency line but for every constant frequency line.

The terms corresponding to $\frac{1}{4}\frac{V}{n_o}$ and $\frac{3}{4}\frac{V}{n_o}$ depend upon a number of factors including the circuit and end conditions. Conclusion (3) is partially correct, the range of the howling frequencies depending upon the efficiencies of the transmitter, receiver, and circuit is evident from equation (54). Calculations show that conclusion (4) is generally correct although not necessarily so.

When the transmitter and receiver are coupled by the air in an open room the behavior is somewhat similar to the case just solved. The size and shape of the room as well as the disposition of articles of furniture will all influence the intensity and frequency of howling. In general when the two instruments are moved apart the frequency will go up and down similar to that when they are coupled by a tube.

NOMENCLATURE

T	Transmitter Resistance.
R	Impedance looking away from Transmitter Terminals.
Z_R	Impedance of Receiver.
Z_d	Damped Impedance of Receiver.
e	Electromotive Force Created in the Transmitter.
i	Alternating Current in the Transmitter Branch.
M	Unilateral Mutual Impedance between Receiver Current and Transmitter e.m.f.
F_R	Force on Receiver Diaphragm.
F_T	Force on Transmitter Diaphragm.
S_R	Stiffness Factor of Receiver Diaphragm.
S_T	Stiffness Factor of Transmitter Diaphragm.
y	Receiver Diaphragm Displacement.
z	Transmitter Diaphragm Displacement.
m	Mass of Diaphragm.
r	Mechanical Resistance of Diaphragm.
s	Elastic Constant of Diaphragm.
f	Frequency.
ω	2π times Frequency.
Z	Force Factor of Receiver.
S	Loaded Stiffness of Receiver Diaphragm.
I	Direct Current Supplied to Transmitter.
V	DC Voltage Drop across Transmitter Terminals.
h	Modulation Factor of the Transmitter.
X	Mechanical Force on Receiver Diaphragm for Case I.
E	Electromotive Force of Supply Battery.
k	Resistance in Line for Case I.
K	Capacity of Condenser.
V_R	Volume of Air in Front of Receiver Diaphragm.
V_T	Volume of Air in Front of Transmitter Diaphragm.
Q_R	Effective Area of Receiver Diaphragm.
Q_T	Effective Area of Transmitter Diaphragm.
p	Air Pressure.
γ	Adiabatic Constant.
ϕ	Velocity of Potential.
a	Velocity of Sound in Air.
x	Distance Along Connecting Tube.
ρ	Density of Air.
ζ_R	Displacement of Air Particle at Receiver End of Tube.
ζ_T	Displacement of Air Particle at Transmitter End of Tube.