Electric Circuit Theory and the Operational Calculus[1]

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CHAPTER VI

Propagation of Current and Voltage Along the Non-Inductive Cable

THE principal practical applications of the operational calculus in electrotechnics are to the theory of the propagation of current and voltage along transmission systems. Of such transmission systems the simplest is the non-inductive cable. The theory of the non-inductive cable is not only of great historic interest, relating as it does to Kelvin's early work on the possibility of transatlantic telegraphy, but is also of very considerable practical importance today, and serves as a basis for the theory of submarine telegraphy over long distances. We shall therefore consider the propagation phenomena in the non-inductive cable in some detail.

The propagation phenomena in any type of transmission system are isolated and exhibited in the clearest possible manner when we confine attention to the infinitely long line, with voltage applied directly to the line terminals. Furthermore, as we shall see later, the solution for the infinitely long line is fundamental and can be extended to the more practical case of the finite line with terminal impedances. We therefore, in this chapter, shall confine our attention to the case of the infinitely long cable with voltage applied directly to the cable terminals.

Consider a cable of distributed resistance R and capacity C per unit length, extending from $x=0$ along the positive x axis. From a previous chapter (see equations (64) and (65)), we are in possession of the operational equations of voltage and current; they are, for the infinitely long line,

$$V=e^{-\sqrt{\alpha p}}\ V_o, \tag{162}$$

$$I=\frac{1}{Rx}\sqrt{\alpha p}\ e^{-\sqrt{\alpha p}}\ V_o=\sqrt{\frac{Cp}{R}}e^{-\sqrt{\alpha p}}\ V_o, \tag{163}$$

where $\alpha=x^2RC$, and V_o is the terminal cable voltage at $x=0$. Let us now assume that the terminal voltage V_o is a "unit e.m.f."; then

$$V=e^{-\sqrt{\alpha p}}, \tag{164}$$

$$I=\frac{1}{Rx}\sqrt{\alpha p}\ e^{-\sqrt{\alpha p}}. \tag{165}$$

[1] Continued from the October, 1925, issue.

The solution of (164) for V was considered in some detail in the preceding chapter; it is, by (129)

$$V=\frac{1}{\sqrt{\pi}}\int_0^\tau \frac{e^{-1/\tau}}{\tau\sqrt{\tau}}d\tau \tag{166}$$

where $\tau=4t/\alpha=4t/x^2RC$. Series expansions of this solution were also given. Another equivalent form is, by (131)

$$V=1-\frac{2}{\sqrt{\pi}}\int_0^{1/\sqrt{\tau}} e^{-\tau^2}d\tau. \tag{167}$$

This last form, recognizable also from inspection of the series expansion (132), is useful because the integral term is what is called the error function and has been completely computed and tabulated.

Before discussing these formulas and the light they throw on propagation phenomena in the non-inductive cable, we shall derive the solution for the current. A very simple way of doing this is to make use of the differential equation (57)

$$I=-\frac{1}{R}\frac{\partial}{\partial x}V.$$

Now from (166) and the relation

$$\frac{\partial}{\partial x}=\frac{d\tau}{dx}\frac{d}{d\tau}$$

we get

$$\frac{\partial}{\partial x}V=\frac{1}{\sqrt{\pi}}\frac{e^{-1/\tau}}{\tau\sqrt{\tau}}\frac{d}{dx}\frac{4t}{x^2RC}$$

$$=-\frac{2}{x\sqrt{\pi}}\frac{e^{-1/\tau}}{\sqrt{\tau}},$$

whence

$$I=\frac{2}{xR\sqrt{\pi}}\frac{e^{-1/\tau}}{\sqrt{\tau}}=\sqrt{\frac{C}{\pi Rt}}e^{-1/\tau}. \tag{168}$$

It is worthwhile verifying the formula by direct solution from the operational equation (165). From formula (g) of the table of integrals, we have

$$h=e^{-2\sqrt{\lambda p}}\sqrt{p}\sqrt{\frac{C}{R}}$$

$$=\frac{e^{-\lambda/t}}{\sqrt{\pi t}}\sqrt{\frac{C}{R}}.$$

Comparison with the operational equation shows that they are identical, within a constant factor provided we put $\lambda=\alpha/4$. Consequently the solution of (165) is

$$I=\sqrt{\frac{C}{\pi Rt}}\,e^{-\alpha/4t}=\sqrt{\frac{C}{\pi Rt}}\,e^{-1/\tau}$$

which agrees with (168). This, it may be remarked, is an excellent example of the utility of the table of integrals in solving operational equations.

This formula is easily calculated for large values of t by expanding the exponential function; it is

$$\frac{2}{Rx}\frac{1}{\sqrt{\pi\tau}}\left[1-\left(\frac{1}{\tau}\right)+\frac{1}{2!}\left(\frac{1}{\tau}\right)^2-\ldots\right].$$

The propagation phenomena of the non-inductive cable are therefore determined by the pair of equations

$$V=\frac{1}{\sqrt{\pi}}\int_0^\tau\frac{e^{-1/\tau}}{\tau\sqrt{\tau}}\,d\tau=1-\frac{2}{\sqrt{\pi}}\int_0^{1/\sqrt{\tau}}e^{-\tau^2}\,d\tau \tag{169}$$

and

$$I=\frac{2}{\sqrt{\pi}xR}\frac{e^{-1/\tau}}{\sqrt{\tau}}=\sqrt{\frac{C}{\pi Rt}}\,e^{-1/\tau} \tag{170}$$

where $\tau=4t/\alpha=\dfrac{4t}{x^2RC}\cdot$

Now an important feature of these formulas is that the voltage at point x is a function only of $\dfrac{4}{x^2RC}t$; that is, of $4t$ *divided by the total resistance and capacity* of the cable from $x=0$ to $x=x$. The same statement holds for the *form* of the current wave: its magnitude, however, is inversely proportional to xR, or the total resistance of the cable up to point x. Consequently a single curve, with proper time scale serves to give the voltage wave at any point on the cable. Similarly a single curve, with proper time and amplitude scales, serves to depict the current wave at any distance from the cable terminals. These curves are given in Figs. 3 and 4.

Referring to the curve depicting the current wave, we observe that it is finite for all values of $t>0$; consequently, in the ideal cable, the velocity of propagation is infinite. This is a consequence, of course, of the fact that the distributed inductance of the cable is neglected. Actually, of course, the velocity of propagation cannot exceed the velocity of light. The error, however, in neglecting the inductance in the case of long cables is appreciable only near the head of the wave provided we confine attention to d.c. or low frequency voltages. This point will be discussed and explained more fully in connection with the transmission line.

The current, while finite, is negligibly small until τ reaches the

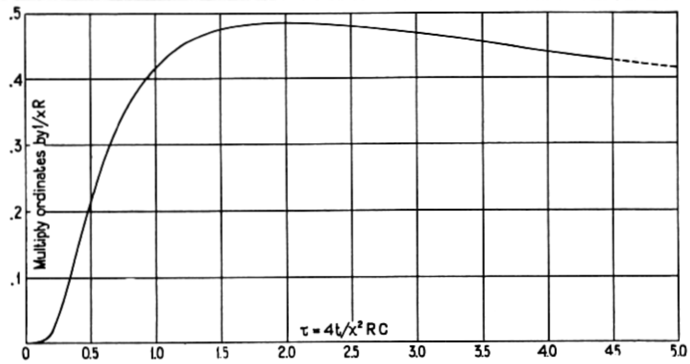

Fig. 3—Current in non-inductive cable ($G=0$) unit e.m.f. applied

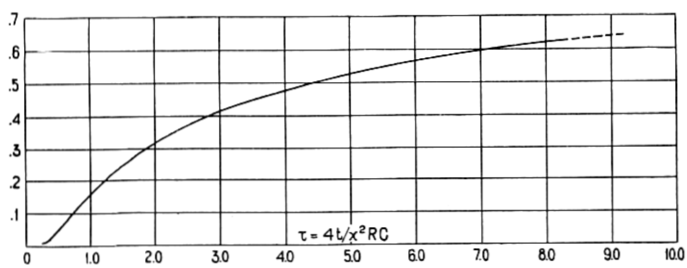

Fig. 4—Voltage in non-inductive cable ($G=0$) unit e.m.f. applied

value 0.2. In the neighborhood of this point it begins to build up rapidly; reaches at $\tau=2$ its maximum value

$$\frac{2}{\sqrt{\pi}\,xR}\,\frac{e^{-0.5}}{\sqrt{2}}=\frac{2}{\sqrt{\pi}\,xR}(0.429)$$

and then begins to decrease, ultimately dying away in accordance with the formula

$$\frac{2}{\sqrt{\pi}\,xR}\,\frac{1}{\sqrt{\tau}}\left\{1-\frac{1}{\tau}+\frac{1}{2!}\left(\frac{1}{\tau}\right)^2-\ldots\right\}.$$

Its subsidence to its final zero value is very slow; for example, when $\tau=100$ its value is still

$$\frac{2}{\sqrt{\pi}\,xR}(0.10).$$

Turning to the voltage curve, Fig. 4, we see that it is negligibly small until τ reaches the value 0.25, at which point it begins to build up. Its maximum rate of building up occurs when $\tau=2/3$, after

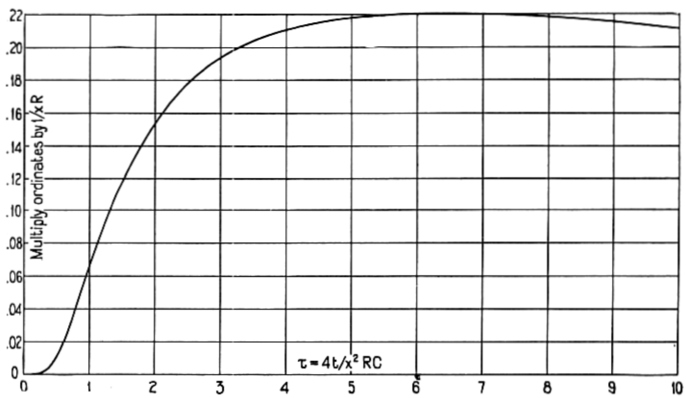

Fig. 5—Power transmitted in non-inductive cable ($G=0$)

which it builds up more and more slowly. Its approach to its final steady value is in accordance with the formula

$$V=1-\frac{2}{\sqrt{\pi\tau}}\left(1-\frac{1}{3\tau}+\frac{1}{2!}\frac{1}{5\tau^2}-\ \cdot\ \cdot\ \cdot\right).$$

Even, therefore, when τ is as great as 100, V differs sensibly from its ultimate value, unity, its value being 0.8876.

Since the actual time is $\dfrac{x^2RC}{4}\tau$, it follows that the speed of building up is inversely proportional to the square of the length of the cable.

The power curve VI is given in Fig. 5. V.I is the rate at which energy is being transmitted past the point x of the cable.

The fact that the form of the current and voltage waves depends only on $4t/x^2RC$ is at the basis of Kelvin's famous "KR" law, long applied to cable telegraphy and sometimes incorrectly applied to telephony. When the first transatlantic telegraph cable was under consideration, Kelvin attacked the problem of propagation along the non-inductive cable and arrived at formulas equivalent to (169) and

(170). From these formulas he announced the law that the "speed" of the cable, i.e., the number of signals transmissible per unit time, is inversely proportional to the product of the total capacity and total resistance of the cable (KR in the English notation). To see just what this means requires a little digression into the elementary theory of telegraph transmission.

Telegraph signals are transmitted in code by means of "dots" and "dashes." The "dot" is the signal which results when a battery is impressed on the cable for a definite interval of time, after which the cable is short circuited. A "dash" is the same except that the time interval during which the battery is connected to the cable is increased. The "dots" and "dashes" are separated by intervals, called "spaces", during which the cable is short circuited. Now when the cable is short-circuited we may imagine a negative battery impressed on the cable in series with the original battery. Consequently the current in the cable, corresponding to a signal composed of a series of dots, dashes and spaces, will be represented by a series of the form

$$I(t)-I(t-t_1)+I(t-t_2)-I(t-t_3)+I(t-t_4)-\ \ldots \tag{171}$$

where, in the cable under consideration, $I(t)$ is given by (168). t_1 is the duration of the first impulse, t_2-t_1 of the first space, t_3-t_2 of the second impulse, etc.

Now by (168)

$$I(t)=\frac{2}{xR\sqrt{\pi}}\frac{e^{-1/\tau}}{\sqrt{\tau}}=\frac{2}{xR\sqrt{\pi}}\phi(\tau).$$

τ is, of course, $4t/x^2CR=4t/KR$ (in the English notation). Now suppose that

$$\tau_1=\frac{4t_1}{x^2CR},$$

$$\tau_2=\frac{4t_2}{x^2CR},\ \text{etc.}$$

Then the signal can be written as

$$\frac{2}{xR\sqrt{\pi}}\{\phi(\tau)-\phi(\tau-\tau_1)+\phi(\tau-\tau_2)-\ \ldots\} \tag{172}$$

Now if the relative time intervals τ_1, τ_2 . . . are kept constant (as the length of the cable is varied), the actual time intervals t_1, t_2 . . . are proportional to x^2CR or to KR, and the wave form of the total signal is independent of KR, when referred to the relative time scale τ.

Hence, if T is the total time of the signal, T is proportional to x^2CR (or to KR). That is to say, if the duration of the component dots, dashes, spaces of the signal are proportional to the "KR" of the cable, the wave form of the received signal, referred to the τ time scale, is invariable, and the total time required to transmit the signal is proportional to the "KR" of the cable. Now the maximum theoretical speed of transmission on the cable is limited by the requirement that the received signal shall bear a recognizable likeness to the original system of dots and dashes: in other words there is a

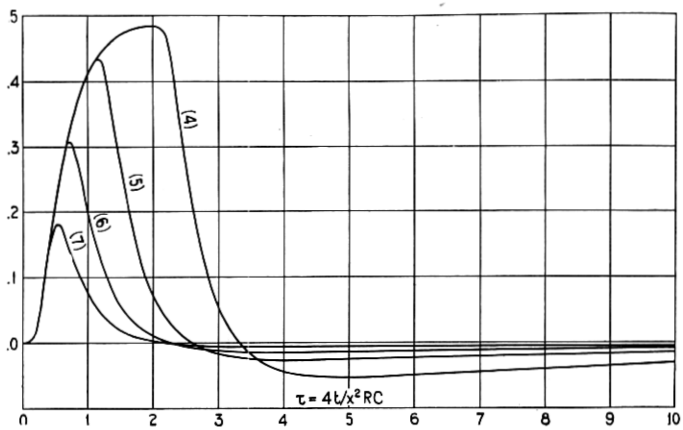

Fig. 6—Elementary telegraph signals in non-inductive cable

maximum allowable departure in wave form between received and transmitted signals. If, therefore, the actual speeds of two cables are inversely proportional to their "KRs," the wave form will be the same. This establishes Kelvin's "KR" law. As a corollary, if the length of the cable is doubled the speed of signaling is reduced to one-quarter, assuming the same definition of signals.

The foregoing will be somewhat clearer, perhaps, if we refer to curves, 4, 5, 6, 7 of Fig. 6 which illustrate the distortion suffered by elementary dot signals in cable transmission. Curve 4 shows the dot signal produced by a unit battery applied to the cable terminals for a time interval $t=2\,\frac{x^2RC}{4}$, while curves 5, 6 and 7 are the corresponding dot signals when the battery is applied for the time intervals $\frac{x^2RC}{4}$, $\frac{1}{2}\,\frac{x^2RC}{4}$ and $\frac{1}{4}\,\frac{x^2RC}{4}$. Any further decrease in the duration of the impressed dot, beyond that shown in curve 7, does not affect the *shape* of the transmitted dot, which means that the cable speed has reached its theoretical maximum. These curves, it should be observed, can be interpreted in two ways. First, we can regard the length x of the cable as fixed and the duration of the impressed dot as varied. On the other hand, we can regard the actual duration of the impressed dot as constant and the length of the cable as varied. From the latter standpoint the curves illustrate the progressive distortion of the signal as it is transmitted along the cable.

The dot signal of relative duration T can be written as

$$D = I(\tau), \qquad \tau < T$$
$$= I(\tau) - I(\tau - T), \qquad \tau < T$$

and the second expression can be expanded in a Taylor's series, giving

$$D = T\frac{d}{d\tau} I(\tau) - \frac{T^2}{2!}\frac{d^2}{d\tau^2} I(\tau) + \dots$$

If T is sufficiently short this becomes

$$D = T\ I'(\tau). \tag{173}$$

Hence when the dot signal is of sufficiently short relative duration T, the wave shape of the received signal is constant, $I'(\tau)$, and its amplitude is proportional to the relative duration of the dot.

This can be generalized for any type of transmission system: Let the dot signal be produced by an e.m.f. $f(t)$ of actual duration T. Then the received dot signal, by formula (31), is

$$D = \frac{d}{dt}\int_0^t f(\tau) I(t-\tau) d\tau, \qquad t < T$$
$$= \frac{d}{dt}\int_0^T f(\tau) I(t-\tau) d\tau, \qquad t > T.$$

For $t > T$ this becomes

$$D = I'(t)\int_0^T f(\tau) d\tau - I''(t)\int_0^T \tau f(\tau) d\tau + \dots$$

and for sufficiently short duration T, we have approximately,

$$D = I'(t)\int_0^T f(\tau) d\tau. \tag{174}$$

Hence for a sufficiently short duration of the impressed e.m.f. the received dot signal is of constant wave form, independent of the shape of the impressed e.m.f., and its amplitude is proportional to the time integral of the impressed e.m.f. These principles are of considerable practical importance in telegraphy.

The *leaky cable*, that is, a cable with distributed leakage conductance G in addition to resistance R and capacity C, is of some interest. The differential equations of the problem are given in equations (70); the operational formulas for the case of voltage directly impressed on the terminals of the infinitely long line are

$$V=e^{-x\sqrt{CRp+RG}}\ V_o,$$

$$I=\sqrt{\frac{pC}{R}+\frac{G}{R}}\ e^{-x\sqrt{CRp+RG}}\ V_o.$$

Writing $CRx^2=\alpha$ and $RGx^2=\beta$, $G/C=\lambda$, and assuming a "unit e.m.f." impressed on the cable, this becomes

$$V=e^{-\sqrt{\alpha p+\beta}}, \tag{175}$$

$$I=\sqrt{\frac{C}{R}}\sqrt{p+\lambda}\ e^{-\sqrt{\alpha p+\beta}}. \tag{176}$$

These equations are readily solved by means of the table and formulas given in a preceding chapter.

But first let us attempt to solve the operational equation (175) for the voltage by Heaviside methods, guided by the solution of the operational equation

$$V=e^{-\sqrt{\alpha p}} \tag{124}$$

of the preceding chapter. Expand the exponential function in (175) in the usual power series; it is

$$V=1-\sqrt{\alpha p+\beta}+\frac{(\alpha p+\beta)}{2!}-\frac{(\alpha p+\beta)\sqrt{\alpha p+\beta}}{3!}+\ldots \tag{177}$$

Now discard the integral terms and write

$$V=1-\left\{1+\frac{\alpha p+\beta}{3!}+\frac{(\alpha p+\beta)^2}{5!}+\ldots\right\}\sqrt{\alpha p+\beta}. \tag{178}$$

We have now to interpret the expression $\sqrt{\alpha p+\beta}$. We have by ordinary algebra

$$\begin{aligned}\sqrt{\alpha p+\beta}&=\left(1+\frac{\beta}{\alpha p}\right)^{1/2}\sqrt{\alpha p}=\left(1+\frac{\lambda}{p}\right)^{1/2}\sqrt{\alpha p}\\&=\left[1+\frac{\lambda}{2p}-\frac{1}{2!}\left(\frac{\lambda}{2p}\right)^2+\frac{1.3}{3!}\left(\frac{\lambda}{2p}\right)^3+\ldots\right]\sqrt{\alpha p}.\end{aligned} \tag{179}$$

Now identify $\sqrt{p}$ with $1/\sqrt{\pi t}$ in accordance with the Heaviside rule, and $1/p$ with $\int dt$. We get

$$\sqrt{\alpha p+\beta}=\sqrt{\frac{\alpha}{\pi t}}\left\{1+\frac{\lambda t}{1!}-\frac{(\lambda t)^2}{3!}+\frac{1.4}{5!}(\lambda t)^3-\dots\right\}. \tag{180}$$

Now in the terms of the expansion (178) identify p^n with d^n/dt^n and substitute (180); we get

$$\begin{aligned} V=1-\left\{1+\frac{1}{3!}\left(\alpha\frac{d}{dt}+\beta\right)+\frac{1}{5!}\left(\alpha^2\frac{d^2}{dt^2}+2\alpha\beta\frac{d}{dt}+\beta^2\right)+\dots\right\} \\ \times\sqrt{\frac{\alpha}{\pi t}}\left\{1+\frac{\lambda t}{1!}-\frac{(\lambda t)^2}{3!}+1.4\frac{(\lambda t)^3}{5!}-\dots\right\}. \end{aligned} \tag{181}$$

This series is hopelessly complicated to either interpret or compute. It is, in fact, an excellent illustration of the grave disadvantages under which many of Heaviside's series solutions labor. We shall therefore attack the solution by aid of the theorems and formulas of a preceding section. The simplicity of the solutions which result is remarkable.

The operational formula for the voltage is

$$V=e^{-\sqrt{\alpha p+\beta}}. \tag{175}$$

Now the operational formula for the voltage in the non-leaky cable is (see equation (164))

$$V=e^{-\sqrt{\alpha p}}.$$

In order to distinguish between the two cases, let us denote the voltage in the latter case by V^o; thus

$$V^o=e^{-\sqrt{\alpha p}}. \tag{182}$$

Now by theorem (VII) and equation (182) we have

$$\begin{aligned} V^o e^{-\lambda t}&=\frac{p}{p+\lambda}\,e^{-\sqrt{\alpha(p+\lambda)}}, \\ &=\frac{p}{p+\lambda}\,e^{-\sqrt{\alpha p+\beta}}. \end{aligned} \tag{183}$$

Now write (175) as

$$\begin{aligned} V&=\frac{p+\lambda}{p}\cdot\frac{p}{p+\lambda}\,e^{-\sqrt{\alpha p+\beta}}, \\ &=\left(1+\frac{\lambda}{p}\right)\cdot\frac{p}{p+\lambda}\,e^{-\sqrt{\alpha p+\beta}}. \end{aligned} \tag{184}$$

It follows at once by comparison with (183) and the rule that $1/p$ is to be replaced by $\int dt$, that

$$V=\left(1+\lambda\int_0^t dt\right)V^o e^{-\lambda t}. \tag{185}$$

By a precisely similar procedure with the operational formula (176) for the current, we get

$$I=\left(1+\lambda\int_0^t dt\right)I^o e^{-\lambda t} \tag{186}$$

where I^o is the current in the non-leaky cable. Now by formulas (169) and (170)

$$V^o=\frac{1}{\sqrt{\pi}}\int_0^{4t/\alpha}\frac{e^{-1/t}}{t\sqrt{t}}\,dt, \tag{169}$$

$$I^o=\sqrt{\frac{C}{\pi R t}}\,e^{-\alpha/4t}, \tag{170}$$

which completes the formal solution of the problem.

Formulas (185) and (186) are extremely interesting, first as showing the superiority of the definite integral to the series expansion—compare (185) with the series expansions (181)—and secondly as exhibiting clearly the effect of leakage on the propagated waves of current and voltage. We see that in both the current and voltage the effect of leakage is two-fold: first it attenuates the wave by the factor $e^{-\lambda t}$, $(\lambda=G/C)$, and secondly it adds a component consisting of the progressive integral of the attenuated wave. This, it may be remarked, is the general effect of leakage in all types of transmission systems. Its effect is, therefore, easily computed and interpreted.

Formulas (185) and (186) are very easy to compute with the aid of a planimeter or integraph; or, failing these devices, by numerical integration. However, for large values of t, the character of the waves is more clearly exhibited if we make use of the identity

$$\int_0^t dt=\int_0^\infty dt-\int_t^\infty dt$$

whence

$$V=\left(1+\lambda\int_0^\infty dt\right)V^o e^{-\lambda t}-\lambda\int_t^\infty V^o e^{-\lambda t}dt \tag{187}$$

and

$$I=\left(1+\lambda\int_0^\infty dt\right)I^o e^{-\lambda t}-\lambda\int_t^\infty I^o e^{-\lambda t}dt. \tag{188}$$

The first two terms of these formulas are clearly the ultimate steady state values of the voltage and current waves, and can be determined by evaluating the infinite integrals. A far simpler and more direct way, however, is to make use of the fact that the ultimate steady values of V and I are gotten from the operational formulas by setting $p=0$. That this statement is true is easily seen if we reflect that the steady d.c. voltage and current are gotten from the original differential equations of the problem by assuming a steady state and setting $d/dt=0$.

From the operational formulas we get, therefore,

$$\left(1+\lambda\int_0^\infty dt\right) V^o e^{-\lambda t} = e^{-\sqrt{\beta}} = e^{-x\sqrt{RG}}, \tag{189}$$

$$\left(1+\lambda\int_0^\infty dt\right) I^o e^{-\lambda t} = \sqrt{\frac{C\lambda}{R}}\, e^{-\sqrt{\beta}} = \sqrt{\frac{G}{R}}\, e^{-x\sqrt{RG}}. \tag{190}$$

Introducing these expressions into (187) and (188) respectively, we get

$$V = e^{-x\sqrt{RG}} - \lambda\int_t^\infty V^o e^{-\lambda t} dt, \tag{191}$$

$$I = \sqrt{\frac{G}{R}}\, e^{-x\sqrt{RG}} - \lambda\int_t^\infty I^o e^{-\lambda t} dt. \tag{192}$$

The definite integrals can be expanded by partial integration; thus

$$\begin{aligned} -\lambda\int_t^\infty V^o e^{-\lambda t} dt &= \int_t^\infty V^o de^{-\lambda t} \\ &= -V^o e^{-\lambda t} - \int_t^\infty e^{-\lambda t} \frac{d}{dt} V^o dt. \end{aligned}$$

Continuing this process we get

$$V = e^{-x\sqrt{RG}} - e^{-\lambda t}\left(1+\frac{d}{\lambda dt}+\frac{d^2}{\lambda^2 dt^2}+ \,.\,.\right) V^o, \tag{193}$$

$$I = \sqrt{\frac{G}{R}}\, e^{-x\sqrt{RG}} - e^{-\lambda t}\left(1+\frac{d}{\lambda dt}+\frac{d^2}{\lambda^2 dt^2}+ \,.\,.\right) I^o. \tag{194}$$

Using the values of V^o and I^o, as given by (169) and (170), it is extremely easy to compute V and I, for large values of t, from (193) and (194).

So far we have considered the current and voltage waves in response to a "unit e.m.f.," impressed on the cable at $x=0$. It is of interest and importance to examine the waves due to sinusoidal e.m.fs., suddenly impressed on the cable, particularly in view of proposals to employ alternating currents in cable telegraphy.

We start with the fundamental formula

$$x(t)=\frac{d}{dt}\int_0^t f(t-\tau)h(\tau)d\tau$$
$$=\int_0^t f(t-\tau)h'(\tau)d\tau$$

provided $h(o)=0$, which is the case in the cable.

If $f(t)=\sin\,\omega t$, we write

$$x_s(t)=\sin\,\omega t\int_0^t \cos\,\omega t.h'(t)dt$$
$$-\cos\,\omega t\int_0^t \sin\,\omega t.h'(t)dt. \tag{194-a}$$

Similarly, if the impressed e.m.f. is cos ωt,

$$x_c(t)=\cos\,\omega t\int_0^t \cos\,\omega t.h'(t)dt$$
$$+\sin\,\omega t\int_0^t \sin\,\omega t.h'(t)dt. \tag{194-b}$$

The investigation of the building-up of alternating currents and voltages, therefore, depends on the progressive integrals

$$C=\int_0^t \cos\,\omega t.h'(t)dt,$$
$$S=\int_0^t \sin\,\omega t.h'(t)dt. \tag{194-c}$$

For the case of the *voltage* waves on the non-inductive, non-leaky cable these integrals, by aid of equations (169), become, if we write $\omega'=\alpha\omega/4$,

$$C=\frac{1}{\sqrt{\pi}}\int_0^\tau \frac{e^{-1/\tau}\cos\,\omega'\tau}{\tau\sqrt{\tau}}\,d\tau,$$
$$S=\frac{1}{\sqrt{\pi}}\int_0^\tau \frac{e^{-1/\tau}\sin\,\omega'\tau}{\tau\sqrt{\tau}}\,d\tau, \tag{194-d}$$

where, as before, $\tau=4t/\alpha$.

For the current wave we have, by (170),

$$C=\frac{2}{\sqrt{\pi}\,xR}\int_0^\tau\left(\frac{1}{\tau}-\frac{1}{2}\right)\frac{e^{-1/\tau}\cos\,\omega'\tau}{\tau\sqrt{\tau}}\,d\tau,$$
$$S=\frac{2}{\sqrt{\pi}\,xR}\int_0^\tau\left(\frac{1}{\tau}-\frac{1}{2}\right)\frac{e^{-1/\tau}\sin\,\omega'\tau}{\tau\sqrt{\tau}}\,d\tau. \tag{194-e}$$

For small values of τ and ω' these integrals can be numerically evaluated without great labor. Mechanical devices, such as the Coradi Harmonic Analyzer, are here of great assistance. In fact the Coradi Analyzer gives these progressive integrals automatically. It may be said, therefore, that a complete mathematical investigation of the building-up of alternating current and voltage waves on the non-inductive cable presents no serious difficulties, although the labor of computation is necessarily considerable. One fact makes the complete investigations much less laborious than might be sup-

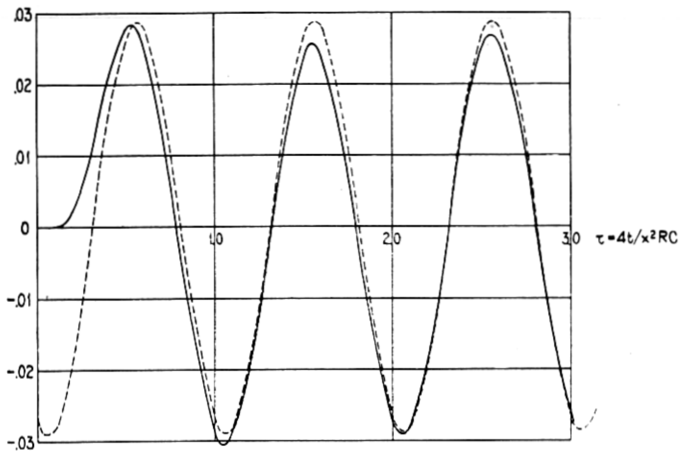

Fig. 7—Non-inductive cable ($G=0$), building-up of alternating current.
Applied e.m.f. $\cos \omega t$; $\omega = 2\pi \dfrac{4}{x^2RC}$

posed. This is, if the foregoing integrals are calculated for a given value of ω', the results apply to all lengths of cable and all actual frequencies $\omega/2\pi$, such that $\alpha\omega$ is a constant. Then if we double the length of the cable and quarter the frequency, the integrals are unaffected.

The solid curve of Fig. 7 shows the building-up of the cable voltage in response to an e.m.f. $\cos \omega t$, impressed at time $t=0$. The frequency $\omega/2\pi$ is so chosen that $\omega' = \alpha\omega/4 = 2\pi$, and the curve is calculated from equations (194-b) and (194-e). The dotted curve shows the corresponding *steady-state* voltage on the cable; that is, the voltage which would exist if the e.m.f. $\cos \omega t$ had been applied at a long time preceding $t=0$. We observe that, for this frequency, the building-up is effectually accomplished in about one cycle, and that the transient distortion is only appreciable during the first half-cycle.

The case is very much different when a higher frequency is applied. Fig. 8 shows the building-up of the alternating current in the cable when an e.m.f. sin ωt is applied at time $t=0$. The frequency is so chosen that $\omega' = \alpha\omega/4 = 10\pi$. The outstanding features of this curve are that the initial current surge is very large compared with the final steady-state, and that the transient distortion is relatively very large. It is evident that the frequency here shown could not be

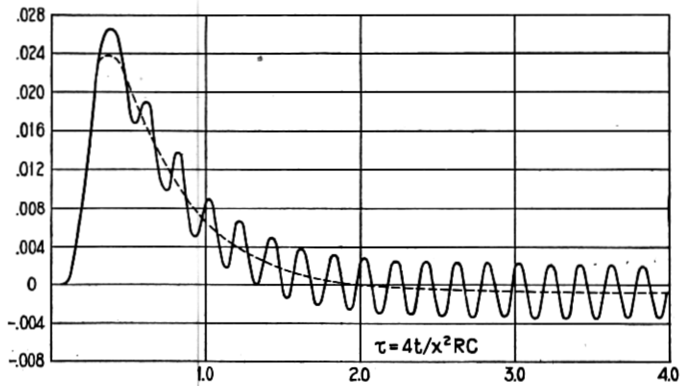

Fig. 8—Non-inductive cable ($G=0$). Building-up of alternating current. Applied e.m.f. sin ωt; $\omega = 10\pi \dfrac{4}{x^2RC}$

employed for signaling purposes. This curve has been computed from the steady-state formulas, and equations (160) and (161) for the transient distortion.

If the applied frequency $\omega/2\pi$ is very high, the steady-state becomes negligibly small, and the complete current is obtained to a good approximation by taking the leading terms of (160) and (161). Thus if the applied e.m.f. is sin ωt, and ω is sufficiently large, the cable current is

$$\frac{2}{\sqrt{\pi x R}} \frac{1}{\omega'} \frac{d}{d\tau} \frac{e^{-1/\tau}}{\sqrt{\tau}}$$

by (160) and (170) while, if the impressed e.m.f. is cos ωt, it is

$$\frac{2}{\sqrt{\pi x R}} \left(\frac{1}{\omega'}\right)^2 \frac{d^2}{d\tau^2} \frac{e^{-1/\tau}}{\sqrt{\tau}}$$

by (161) and (170). Here $\omega' = \alpha\omega/4$ and $\tau = 4t/\alpha$.

CHAPTER VII

The Propagation of Current and Voltage Along the Transmission Line

We now take up the more important and difficult problem of investigating the propagation phenomena in the transmission line. The transmission line has distributed series resistance R and inductance L, and distributed shunt capacity C and leakage conductance G. It is the addition of the series inductance L which makes our problem more difficult and at the same time introduces the phenomena of true propagation with finite velocity, as distinguished from the diffusion phenomena of the cable problem. The cable theory serves very well for the problems of trans-oceanic telegraphy [8] but is quite inadequate in the problems of telephonic transmission.

If I denotes the current and V the voltage at point x on the line, the well known differential equations of the problem are:—

$$\left(L\frac{d}{dt}+R\right) I= -\frac{\partial}{\partial x}V,$$
$$\left(C\frac{d}{dt}+G\right) V= -\frac{\partial}{\partial x}I. \tag{195}$$

Replacing d/dt by p, these become

$$(Lp+R)\, I= -\frac{\partial}{\partial x}V,$$
$$(Cp+G)\, V= -\frac{\partial}{\partial x}I. \tag{196}$$

From the second of these equations

$$\frac{\partial V}{\partial x} = -\frac{1}{Cp+G}\frac{\partial^2}{\partial x^2}I$$

and substitution in the first gives

$$(Lp+R)(Cp+G)I=\frac{\partial^2}{\partial x^2}I. \tag{197}$$

Similarly if we eliminate I, we get

$$(Lp+R)(Cp+G)\, V=\frac{\partial^2}{\partial x^2}V. \tag{198}$$

[8] With the installation of the new submarine cable, continuously loaded with permalloy, this statement must be modified. In this cable, the inductance plays a very important part, and is responsible for the greatly increased speed of signaling obtainable.

If we assume a solution of the form

$$V=Ae^{-\gamma x}+Be^{\gamma x}$$

where A and B are arbitrary constants, substitution shows that the solution satisfies the differential equation for V provided

$$\gamma^2=(Lp+R)(Cp+G). \tag{199}$$

From equation (196) it then follows that

$$\begin{aligned} I&=\frac{\gamma}{Lp+R}\,(Ae^{-\gamma x}-Be^{\gamma x}) \\ &=\frac{Cp+G}{\gamma}\,(Ae^{-\gamma x}-Be^{\gamma x}). \end{aligned} \tag{200}$$

Now restricting attention to the infinitely long line extending along the positive x axis, with voltage V_o impressed directly on the line at $x=0$, the reflected wave vanishes and we get

$$\begin{aligned} V&=V_o e^{-\gamma x}, \\ I&=\frac{Cp+G}{\gamma}V_o e^{-\gamma x}, \\ \gamma^2&=(Lp+R)(Cp+G). \end{aligned} \tag{201}$$

Now let us write

$$\gamma^2=\frac{1}{v^2}[(p+\rho)^2-\sigma^2] \tag{202}$$

where

$$v=1/\sqrt{LC},$$

$$\rho=\frac{R}{2L}+\frac{G}{2C},$$

$$\sigma=\frac{R}{2L}-\frac{G}{2C}.$$

Then setting $V_o=1$, the *operational equations* of the problem become

$$V=e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}, \tag{203}$$

$$I=v\left(C+\frac{G}{p}\right)p\,\frac{e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}}{\sqrt{(p+\rho)^2-\sigma^2}}. \tag{204}$$

Now consider the operational equation, defining a new variable F:

$$F=p\,\frac{e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}}{\sqrt{(p+\rho)^2-\sigma^2}}. \tag{205}$$

It follows at once from our operational rules, and (203) and (204), that

$$I=v\left(C+G\int_0^t dt\right)F, \tag{206}$$

$$V=-v\int_0^t \frac{\partial F}{\partial x}dt. \tag{207}$$

Our problem is thus reduced to evaluating the function F, from the operational equation (205). This equation can be solved by aid of the operational rules and formulas already given. The process is rather complicated, and there is less chance of error if we deal instead with the integral equation of the problem

$$\frac{e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}}{\sqrt{(p+\rho)^2-\sigma^2}}=\int_0^\infty F(t)e^{-pt}dt. \tag{208}$$

Now let us search through our table of definite integrals. We do not find this integral equation as it stands, but we do observe that formula (m) resembles it, and this resemblance suggests that formula (m) can be suitably transformed to give the solution of (208). We therefore start with the formula

$$\frac{e^{-\lambda\sqrt{p^2+1}}}{\sqrt{p^2+1}}=\int_\lambda^\infty e^{-pt}J_o(\sqrt{t^2-\lambda^2})\,dt. \tag{m}$$

This, regarded as an integral equation, defines a function which is zero for $t<\lambda$ and has the value $J_o(\sqrt{t^2-\lambda^2})$ for $t\geqq\lambda$, J_o being the Bessel function of order zero. We now transform (m) as follows:

(1) Let $\lambda p=q$ and $t/\lambda=t_1$. Substituting in (m) we get

$$\frac{e^{-\sqrt{q^2+\lambda^2}}}{\sqrt{q^2+\lambda^2}}=\int_1^\infty e^{-qt_1}J_o(\lambda\sqrt{t_1^2-1})\,dt_1.$$

Now, in order to keep our original notation in p and t, replace q by p and t_1 by t; we get

$$\frac{e^{-\sqrt{p^2+\lambda^2}}}{\sqrt{p^2+\lambda^2}}=\int_1^\infty e^{-pt}J_o(\lambda\sqrt{t^2-1})\,dt. \tag{m.1}$$

(2) In (m.1) make the substitution $p=q+\mu$ and then in the final expression replace q by p; we get

$$\int_1^\infty e^{-pt}.e^{-\mu t}J_o(\lambda\sqrt{t^2-1})dt=\frac{e^{-\sqrt{(p+\mu)^2+\lambda^2}}}{\sqrt{(p+\mu)^2+\lambda^2}}. \tag{m.2}$$

(3) In (m.2) make the substitution $p=\frac{x}{v}q$ and $t_2=\frac{x}{v}t$, and ultimately replace q by p and t_2 by t; we get

$$\int_{x/v}^{\infty} e^{-pt}.e^{-\mu_1 t} J_o\left(\lambda_1\sqrt{t^2-\frac{x^2}{v^2}}\right)dt=\frac{e^{-\frac{x}{v}\sqrt{(p+\mu_1)^2+\lambda_1^2}}}{\sqrt{(p+\mu_1)^2+\lambda_1^2}} \qquad \text{(m.3)}$$

where $\lambda_1=\frac{v}{x}\lambda$ and $\mu_1=\frac{v}{x}\mu$. (They are, of course, as yet, arbitrary parameters, except that they are restricted to positive values).

(4) Now if we compare (m.3) with the integral equation (208) for F, we see that they are identical provided we get

$$\mu_1=\rho,$$

$$\lambda_1=i\sigma=\sigma\sqrt{-1},$$

which is possible, since $\rho>\sigma$.

Introducing these relations, we have

$$\int_{x/v}^{\infty} e^{-pt}.e^{-\rho t}\, I_o(\sigma\sqrt{t^2-x^2/v^2})dt=\frac{e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}}{\sqrt{(p+\rho)^2-\sigma^2}}. \qquad \text{(m.4)}$$

Here I_o denotes the Bessel function of imaginary argument; thus $J_o(iz)=I_o(z)$.

It follows from (m.4) and the integral equation (208) that

$$F(t)=0 \text{ for } t<x/v, \qquad (209)$$
$$=e^{-\rho t}I_o(\sigma\sqrt{t^2-x^2/v^2}) \text{ for } t\geqq x/v.$$

Having now solved for $F=F(t)$, the current and voltage are gotten from equations (206) and (207). Thus

$$I=0 \text{ for } t<x/v,$$
$$=\sqrt{\frac{C}{L}}F(t)+vG\int_{x/v}^{t}F(t)dt \text{ for } t\geqq x/v. \qquad (210)$$

The corresponding voltage formula is

$$V=0 \text{ for } t<x/v,$$
$$=e^{-\rho x/v}+\frac{\sigma x}{v}\int_{x/v}^{t}\frac{e^{-\rho\tau}I_1(\sigma\sqrt{\tau^2-x^2/v^2})}{\sqrt{\tau^2-x^2/v^2}}d\tau \text{ for } t\geqq x/v. \qquad (211)$$

Here $I_1(\sigma\sqrt{\tau^2-x^2/v^2})$ is the Bessel function of order 1: thus $-iJ_1(iz)=I_1(z)$. The function is entirely real. The derivation of formula (211) is a little troublesome, owing to the discontinuous character of the function F: the detailed steps are given in an appendix.

The preceding solution depends for its outstanding directness and simplicity on the recognition of the infinite integral identity (m), into which the integral equation of the problem can be transformed. When such identities are known their value in connection with the solution of operational equations requires no emphasis. On the other hand, we cannot always expect to find such an identity in the case of every operational equation; and, particularly in the case of such an important case as the transmission equation it would be unfortunate to have no alternative mode of solution. Fortunately a quite direct series expansion solution is obtainable from the operational equation, and this will now be derived. As a matter of convenience we shall restrict the derivation to the voltage formula

$$V=e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}. \tag{203}$$

As a further matter of mere convenience we shall assume that $G=0$, so that $\sigma=\rho$ and (203) becomes

$$V=e^{-\tau\sqrt{p^2+2\rho p}} \tag{203-a}$$

where $\tau=x/v$.

The method holds equally well for the current equation (204) and for the general case $\sigma\neq\rho$.

Write (203-a) as

$$V=e^{-\tau p(1+2\rho/p)^{1/2}}$$

and expand the exponential factor $(1+2\rho/p)^{1/2}$ by the binomial theorem; thus

$$(1+2\rho/p)^{1/2}=1+\frac{\rho}{p}+\alpha_2\left(\frac{\rho}{p}\right)^2+\alpha_3\left(\frac{\rho}{p}\right)^3+\ldots$$

so that

$$V=e^{-\tau p}.e^{-\rho\tau}.\exp\left(-\frac{\alpha_2\tau\rho^2}{p}-\frac{\alpha_3\tau\rho^3}{p^2}-\frac{\alpha_4\tau\rho^4}{p^3}-\ldots\right).$$

Now the operational equation

$$v=\exp\left(-\frac{\alpha_2\tau\rho^2}{p}-\frac{\alpha_3\tau\rho^3}{p^2}-\frac{\alpha_4\tau\rho^4}{p^3}-\ldots\right)$$

can be expanded in inverse powers of p; thus

$$v=1+\frac{\beta_1}{p}+\frac{\beta_2}{p^2}+\frac{\beta_3}{p^3}+\ldots\ldots$$

the power series solution of which is

$$v(t)=1+\frac{\beta_1 t}{1!}+\frac{\beta_2 t^2}{2!}+\frac{\beta_3 t^3}{3!}+\ldots\ldots$$

It follows at once from the preceding and Theorem VII that

$$V(t) = 0 \text{ for } t < \tau$$
$$= e^{-\rho\tau}\left(1 + \beta_1 \frac{(t-\tau)}{1!} + \beta_2 \frac{(t-\tau)^2}{2!} + \ldots\right) \text{ for } t > \tau$$

If the coefficients β_1, β_2 . . . are evaluated, a simple matter of elementary algebra, the foregoing expansion in the retarded time $t-\tau$ will be found to agree with the solution (211) when σ is put equal to ρ.

We shall now discuss the outstanding features of the propagation phenomena in the light of equations (210) and (211) for the current and voltage. We observe, first, that we have a true finite velocity of propagation $v = 1/\sqrt{LC}$. No matter what the form of impressed e.m.f. at the beginning of the line ($x=0$), its effect does not reach the point x of the line until a time $t = x/v$ has elapsed. Consequently $v = x/t$ is the velocity with which the wave is propagated. This is a strict consequence of the distributed inductance and capacity of the line and depends only on them, since $v = 1/\sqrt{LC}$. It will be recalled that in the case of the cable, where the inductance is ignored, no finite velocity of propagation exists.

The question of velocity of propagation of the wave has been the subject of considerable confusion and misinterpretation when dealing with the steady-state phenomena. It seems worth while to briefly touch on this in passing.

As has been pointed out in preceding chapters, the symbolic or complex steady-state formula is gotten from the operational equation by replacing the symbol p by $i\omega$ where $i = \sqrt{-1}$ and $\omega/2\pi$ is the frequency. If this is done in the operational equation (203) for the voltage, the symbolic formula is

$$V = e^{-\frac{x}{v}\sqrt{(i\omega+\rho)^2 - \sigma^2}} e^{i\omega t}.$$

If the expression $\sqrt{(i\omega+\rho)^2 - \sigma^2}$ is separated into its real and imaginary parts we get an expression of the form

$$V = e^{-\alpha x} e^{i\omega\left(t - \beta\frac{x}{v}\right)},$$

where

$$\beta = \sqrt{\frac{\omega^2 + \sigma^2 - \rho^2 + \sqrt{(\omega^2 + \sigma^2 - \rho^2)^2 + 4\omega^2\rho^2}}{2\omega^2}}$$

and

$$\alpha = \rho/\beta v.$$

Now if we keep the expression $t - \beta\frac{x}{v}$ constant, that is, if we move along the line with velocity $dx/dt = v/\beta$, the phase of the wave will remain constant. This is interpreted often as meaning that the velocity of propagation of the wave is v/β. Now since β is greater than unity and only approaches unity as the frequency becomes indefinitely great, the inference is frequently made that the velocity of propagation depends upon and increases to a limiting value v, with the frequency. This velocity, however, is not the true velocity of propagation, which is always v, but is the *velocity of phase propagation in the steady-state.* This distinction is quite important and failure to bear it in mind has led to serious mistakes.

Returning to equation (211) and (210) we see that after a time interval $t=x/v$ has elapsed since the unit e.m.f. was impressed on the cable, the voltage at point x suddenly jumps from zero to the value $e^{-\rho x/v}$ while the current correspondingly jumps to the value $\sqrt{\frac{C}{L}}\,e^{-\rho x/v}$. The exponential factor $\rho x/v$ is

$$x\left(\frac{R}{2L}+\frac{G}{2C}\right)\sqrt{LC}=x\left(\frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}}\right)=ax$$

which will be recognized as the *steady-state attenuation factor* for high frequencies. Similarly $\sqrt{C/L}$ is the steady-state admittance of the line for high frequencies. The sudden jumps in the current and voltage at time $t=x/v$ are called the heads of the current and voltage waves. If, instead of a unit e.m.f., a voltage $f(t)$ is impressed on the line at time $t=0$, the corresponding heads of the waves are $f(o)e^{-ax}$ and $\sqrt{C/L}\,f(o)e^{-ax}$ for voltage and current respectively. These expressions follow at once from the integral formula

$$\begin{aligned} x(t) &= \frac{d}{dt}\int_0^t f(t-\tau)h(\tau)d\tau \\ &= f(o)h(t)+\int_0^t f'(t-\tau)h(\tau)d\tau. \end{aligned}$$

The tails of the waves, that is, the parts of the waves subsequent to the time $t=x/v$, are more complicated and will depend on the distance x along the line and on the line parameters ρ and σ. The two simplest cases are the *non-dissipative* line, and the *distortionless* line.

The ideal non-dissipative line, quite unrealizable in practice, is one in which both R and G are zero. In this case $\rho=\sigma=0$, and formulas (210) and (211) become

$$\begin{aligned} I &= 0 \text{ for } t<x/v, \\ &= \sqrt{\frac{C}{L}} \text{ for } t\geq x/v, \\ V &= 0 \text{ for } t<x/v, \\ &= 1 \text{ for } t\geq x/v. \end{aligned}$$

Both current and voltage jump, at time $t=x/v$, to their steady values. If an e.m.f. $f(t)$ is impressed on the line at time $t=0$, the corresponding current and voltage waves are

$$I=0 \text{ for } t<x/v,$$
$$=\sqrt{\frac{C}{L}}f(t-x/v) \text{ for } t\geq x/v,$$
$$V=0 \text{ for } t<x/v,$$
$$=f(t-x/v) \text{ for } t\geq x/v.$$

Consequently the ideal non-dissipative line transmits the waves with finite velocity v, without attenuation or distortion. Such a line is, of course, the ideal transmission system.

The non-dissipative line is, of course, purely theoretical and unrealizable in practice; the *distortionless* line is, however, approximately realizable, and as the name implies, transmits without distortion of wave form. The distortionless line is one in which the line constants are so related that

$$\sigma=\frac{R}{2L}-\frac{G}{2C}=0.$$

If this condition is satisfied, formulas (210) and (211) become

$$I=0 \text{ for } t<x/v,$$
$$=\sqrt{\frac{C}{L}}e^{-ax} \text{ for } t\geq x/v,$$
$$V=0 \text{ for } t<x/v,$$
$$=e^{-ax} \text{ for } t\geq x/v.$$

Furthermore, if the impressed e.m.f. is $f(t)$, the corresponding current and voltage waves are:—

$$I=0 \text{ for } t<x/v,$$
$$=\sqrt{\frac{C}{L}}e^{-ax}f(t-x/v) \text{ for } t\geq x/v,$$
$$V=0 \text{ for } t<x/v,$$
$$=e^{-ax}f(t-x/v) \text{ for } t\geq x/v.$$

The distortionless line, therefore, transmits the waves without distortion of wave form, but attenuates the waves by the factor e^{-ax}. Such a line is an ideal transmission system as regards preservation of wave form, but introduces serious attenuation losses. For example, if a line has normally negligible leakage, and leakage is introduced to secure the condition $R/L=G/C$, the line is thereby rendered distortionless but the attenuation is doubled.

One of Heaviside's most important contributions to wire transmission theory was to point out the properties of the distortionless line, its approximately realizable character, and to base on it a correct theory of telephonic transmission.

The character of the wave propagation when the parameters ρ and σ are not restricted to special values, can only be roughly inferred from inspection of the formulas, and then only when the properties of the Bessel function I_o and I_1 have been studied. Fortunately these functions have been computed and tabulated for small values of the argument, and have simple asymptotic expansions for large values. It is therefore a simple matter to compute and graph a representative set of curves which show the current and voltage waves for various values of ρ, σ and x. For this purpose it is convenient to introduce a change of variables and write:

$$\tau=vt$$

$$a=\rho/v$$

$$b=\sigma/v$$

whence the formulas for current and voltage become:

$$I=\sqrt{\frac{C}{L}}\,e^{-a\tau}I_o(b\sqrt{\tau^2-x^2}) + (a-b)\sqrt{\frac{C}{L}}\int_x^\tau e^{-a\tau}I_o(b\sqrt{\tau^2-x^2})d\tau, \tag{210a}$$

$$V=e^{-ax}+bx\int_x^\tau \frac{e^{-a\tau}I_1(b\sqrt{\tau^2-x^2})}{\sqrt{\tau^2-x^2}}\,d\tau. \tag{211a}$$

Figs. (9) to (18) give a representative set of curves illustrating the form of the propagated current and voltage waves for different lengths of line, and different values of the line parameters a and b, or ρ and σ.

The curves of Figs. (9) and (10) show the current entering the line in response to a unit e.m.f. applied at time $t=0$. The line is assumed to be non-leaky ($b=0$) and is computed for two different values of the parameter a. We see that the current instantly jumps to the value $\sqrt{C/L}$ and then begins to die away, the rate at which it dies away depending on and increasing with the parameter $a=\frac{R}{2}\sqrt{\frac{C}{L}}$.

If we now consider a point x out on the line, the current is zero until $\tau=x$, at which time it jumps to the value $\sqrt{C/L}\,e^{-ax}$. It then begins to die away provided x and a are such that $ax<2$. If, however, we are considering a point at which $ax>2$, the current begins to rise instead of fall after the initial jump, and may attain a maximum value very large compared with the head before it starts to die away. This is shown in the curves of Figs. (11), (12) and (13), also computed

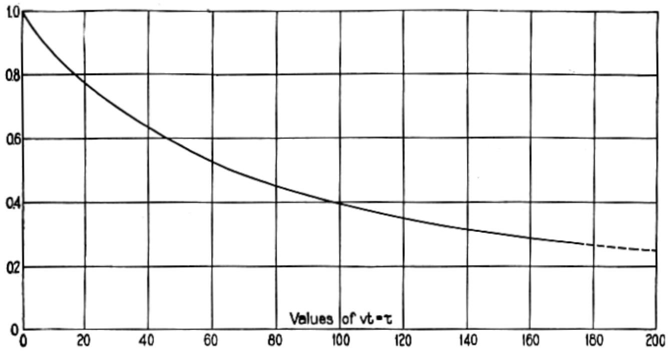

Fig. 9—Current entering line; $\frac{R}{2}\sqrt{\frac{C}{L}}=a=0.0132$; $G=0$.

Multiply ordinates by $\sqrt{C/L}$

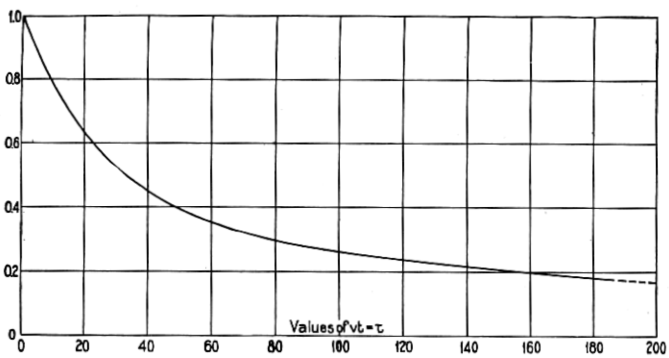

Fig. 10—Current entering line; $\frac{R}{2}\sqrt{\frac{C}{L}}=a=0.2645$; $G=0$.

Multiply ordinates by $\sqrt{C/L}$

for the non-leaky line ($b=0$). From these curves we see that, as the length of the line and the parameter a increase, the relative magnitude of the tail, as compared with the head of the wave, increases. Finally when the line becomes very long, the head of the wave becomes negligibly small, and the wave, except in the neighborhood of its head, becomes very close to that of the corresponding non-inductive cable. This is shown in curves (13) and (14), for the line and the corresponding cable, which are plotted to the same time scale and ordinate scale to facilitate comparison. Curve (15) shows the effect of leakage in eliminating the tail. This line is not quite distortionless but nearly so.

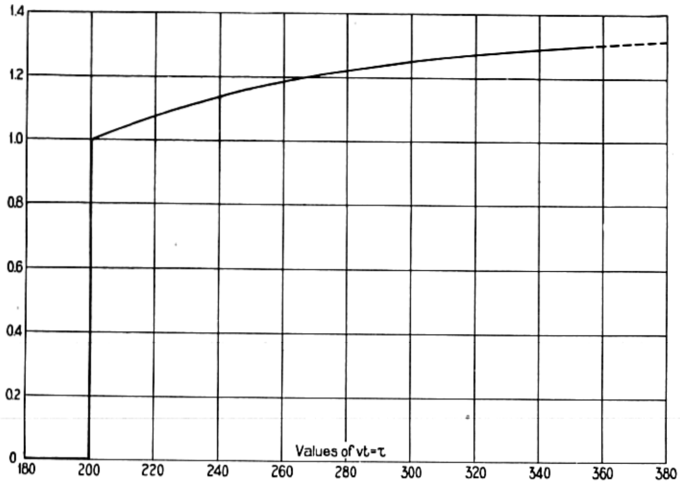

Fig. 11—Propagated current in line; $x=200$; $\frac{R}{2}\sqrt{\frac{C}{L}}=a=0.0132$; $G=0$.

Multiply ordinates by $\sqrt{C/L}.e^{-2.64}$

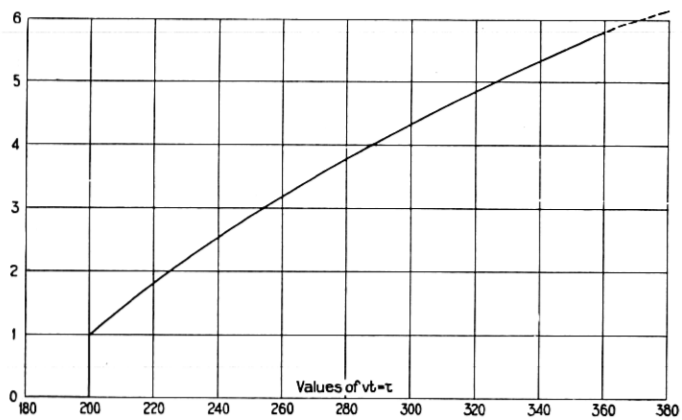

Fig. 12—Propagated current in line; $x=200$; $\frac{R}{2}\sqrt{\frac{C}{L}}=0.02645$: $G=0$.

Multiply ordinates by $\sqrt{C/L}.e^{-5.29}$

An interesting feature of both current and voltage waves is that when a sufficient time has elapsed after the arrival of the head of the wave, the waves become closer and closer to the wave of the corresponding non-inductive cable; that is, to the cable having the same R,C and G. Consequently the inductance plays no part in the subsidence of the waves to their final values.

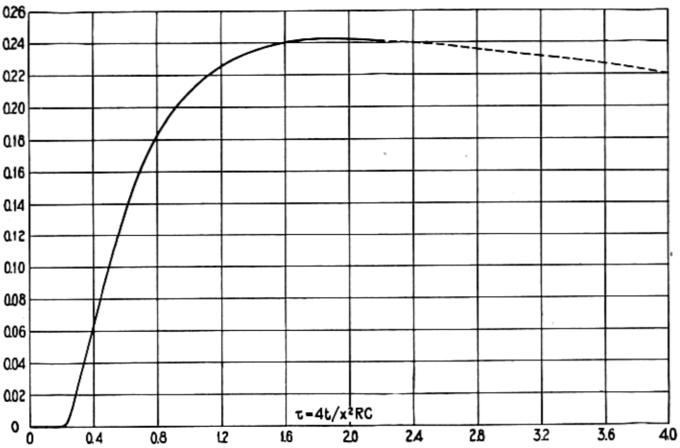

Fig. 13—Propagated current in line; $\frac{R}{2}\sqrt{\frac{C}{L}}x=10$; $G=0$.
Multiply ordinates by $2/Rx$.

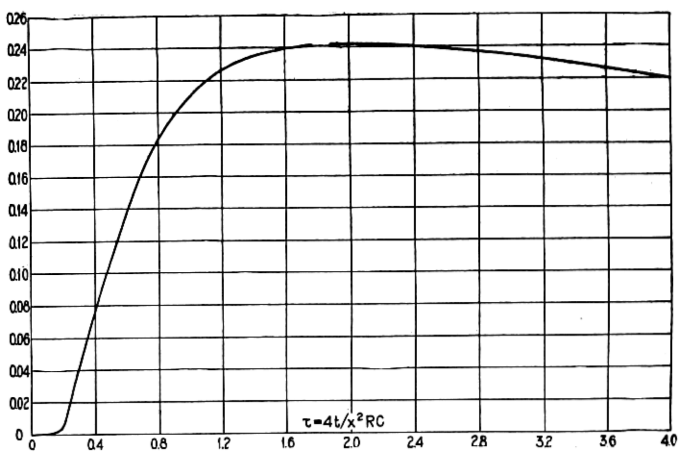

Fig. 14—Propagated current in cable. Multiply ordinates by $2/Rx$.

Curves (16), (17) and (18) illustrate the voltage wave for several conditions. After the arrival of the head, the wave slowly builds up to its final value. Curve (18) represents the case where the line is very nearly distortionless, showing how completely the distorting tail of the wave is eliminated.

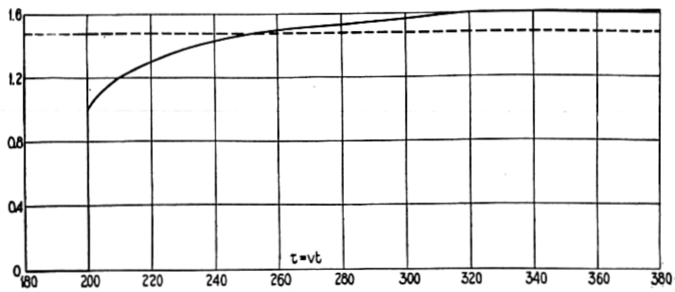

Fig. 15—Propagated current in line; $x=200$

$$a=\frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}}=0.0353$$

$$b=\frac{R}{2}\sqrt{\frac{C}{L}}-\frac{G}{2}\sqrt{\frac{L}{C}}=0.01765$$

Multiply ordinates by $\sqrt{C/L}.e^{-7.06}$

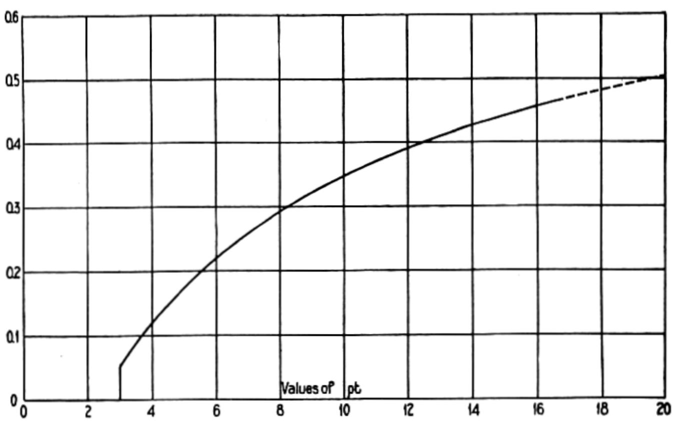

Fig. 16—Propagated voltage in line; $\frac{R}{2}\sqrt{\frac{C}{L}}\,x=ax=3$; $G=0$.

So far we have confined attention to the current and voltage waves in response to a unit e.m.f. applied at time $t=0$ to the line terminals. Of much greater technical importance is the question of the waves in response to a sinusoidal e.m.f. suddenly applied to the line terminals. In order to investigate this important problem it is convenient to divide the expressions for the current and voltage waves as given by equations (210-a) and (211-a) into two components. We write for $\tau \geq x$,

$$I = \sqrt{\frac{C}{L}} e^{-ax} + J(t), \tag{210-b}$$

$$V = e^{-ax} + W(t), \tag{211-b}$$

where, by definition, $J(t)$ and $W(t)$ are the differences between the total waves and their heads. The advantage of analyzing the waves into these components is that the distortion of the waves is due to

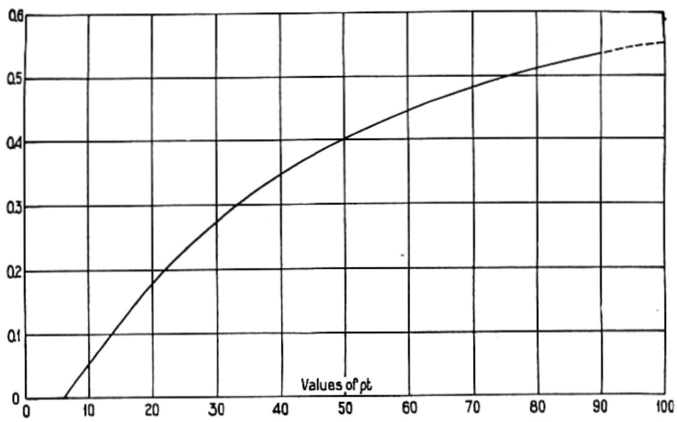

Fig. 17—Propagated voltage in line; $\frac{R}{2}\sqrt{\frac{C}{L}}\, x = ax = 6$; $G = 0$.

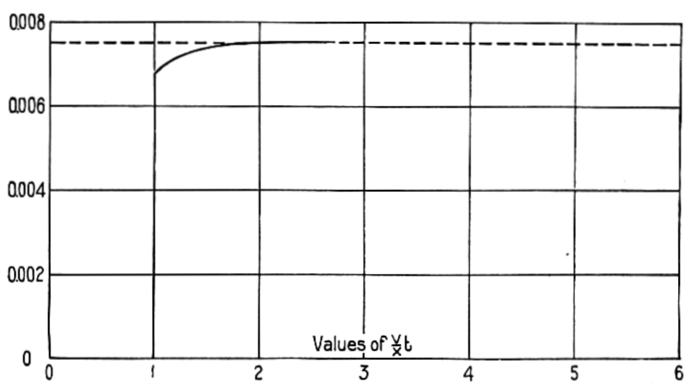

Fig. 18—Propagated voltage in line; $\frac{R}{2}\sqrt{\frac{C}{L}}\, x = ax = 3$; $\frac{G}{2}\sqrt{\frac{L}{C}}\, x = bx = 2$.

$J(t)$ and $W(t)$ respectively, while the first component of (210-b) and (211-b) introduce merely a delay. Thus, if the e.m.f. impressed at time $t=0$ is $f(t)$, the corresponding waves for $t\geq x/v$ or $\tau\geq x$, are

$$I=\sqrt{\frac{C}{L}}e^{-ax}f(t-x/v) + \int_{x/v}^{t} f(t-t_1)J'(t_1)dt_1, \tag{212}$$

$$V=e^{-ax}f(t-x/v) + \int_{x/v} f(t-t_1)W'(t_1)dt_1, \tag{213}$$

where $J'(t)=\frac{d}{dt}J(t)$ and $W'(t)=\frac{d}{dt}W(t)$.

The integrals of (212) and (213) can be computed and analyzed in precisely the same way as discussed in connection with the non-inductive cable problem, and are of very much the same character as the alternating current waves of the cable. In the total waves, however, as given by (212) and (213), a very essential difference is introduced by the absence of the first terms, which represent undistorted waves propagated with velocity v. Thus, if the impressed e.m.f. is $\sin \omega t$, (212) and (213) become

$$I=\sqrt{\frac{C}{L}}e^{-ax}\sin \omega(t-x/v) + \int_{x/v}^{t} \sin \omega(t-t_1)J'(t_1)dt_1, \text{ for } t\geq x/v \tag{214}$$

$$V=e^{-ax}\sin \omega(t-x/v) + \int_{x/v}^{t} \sin \omega(t-t_1)W'(t_1)dt_1, \text{ for } t\geq x/v. \tag{215}$$

Now the first terms of (214) and (215) are simply the usual steady-state expressions for the current and voltage waves when the frequency is sufficiently high to make the steady-state attenuation constant equal to a and the phase velocity equal to v. Furthermore the integral terms become smaller and smaller as the applied frequency $\omega/2\pi$ is increased. It follows, therefore, that for high frequencies the waves assume substantially their final steady value at time $t=x/v$, and that the tails of the waves, or the transient distortion, becomes negligible. This is a consequence entirely of the presence of inductance in the line, and shows its extreme importance in the propagation of alternating waves and the reduction of transient distortion.

It should be pointed out, however, that if the line is very long and the attenuation is very high, the integral terms of (214) and (215) are not negligible unless the applied frequency is correspondingly very high. For example, on a long submarine cable, the a.c. attenuation is so large that the first terms of (214) and (215) are very small, and $J(t)$ is very large compared with $\sqrt{C/L}\ e^{-ax}$. Consequently here there is very serious transient distortion and alternating currents are therefore not adapted for submarine telegraph signalling.

This discussion may possibly be made a little clearer, without detailed analysis, if we recall the discussion of alternating current propagation in the non-inductive cable of the preceding chapter. From that analysis it follows that, when the applied frequency $\omega/2\pi$ is sufficiently high, the integral term of (214) becomes approximately

$$\frac{1}{\omega}J'(t)$$

and the complete current wave is

$$\sqrt{\frac{C}{L}}e^{-ax}\sin\omega(t-x/v)+\frac{1}{\omega}J'(t) \tag{216}$$

and similarly the voltage wave is

$$e^{-ax}\sin\omega(t-x/v)+\frac{1}{\omega}W'(t). \tag{217}$$

Now if the total attenuation ax is large the last terms of (216) and (217), before they ultimately die away, may become very large compared with the first terms, which represent the ultimate steady-state.

Appendix to Chapter VII. Derivation of Formula (211)

The only troublesome question involved in deriving (211) from (207) and (209) is that we have to differentiate with respect to x, in accordance with (207), the discontinuous function $F(t)$. To accomplish this we write (209) in the form

$$F(t)=\phi(t-x/v)e^{-\rho t}I_o(\sigma\sqrt{t^2-x^2/v^2}) \tag{209-a}$$

where $\phi(t)$ is defined as a function which is zero for $t<x/v$ and unity for $t\geq x/v$. Clearly this is equivalent to (209) and permits us to deal

with $F(t)$ as a *continuous* function. Now, in accordance with (207), perform the operation of differentiation upon (209-a): we get

$$-v\frac{\partial F}{\partial x}=\frac{\partial}{\partial t}\phi(t-x/v)e^{-\rho t}I_o(\sigma\sqrt{t^2-x^2/v^2})$$
$$-v\phi(t-x/v)\frac{\partial}{\partial x}e^{-\rho t}I_o(\sigma\sqrt{t^2-x^2/v^2}).$$

The first expression follows from the fact that

$$\frac{\partial}{\partial x}\phi(t-x/v)=-\frac{1}{v}\frac{\partial}{\partial t}\phi(t-x/v).$$

We observe also that $\frac{\partial}{\partial t}\phi(t-x/v)=0$ except at $t=x/v$, when it is infinite. We also observe that, for $t\geq x/v$,

$$\int_0^t\frac{\partial}{\partial t}\phi(t-x/v)dt=1$$

and that the whole contribution to the integral occurs at $t=x/v$. With these points clearly in mind, the expression

$$V=-v\int_0^t\frac{\partial F}{\partial x}dt$$

reduces to (211) without difficulty.

CHAPTER VIII

Propagation of Current and Voltage in Artificial Lines and Wave Filters

The artificial line here considered is a periodic structure, composed of a series of sections connected in tandem, each section consisting of a lumped impedance z_1 in series with the line, and a lumped

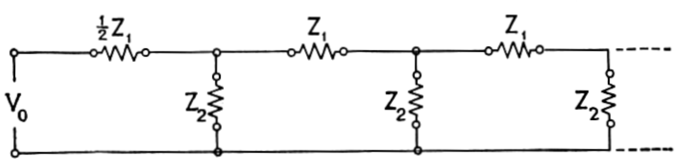

Fig. 19

impedance z_2 in shunt across the line. In the artificial line which we shall consider it will be assumed that the voltage is applied at the middle of the initial or zeroth section, as shown in Fig. 19. This termination is chosen because of its practical importance, and because also of the fact that the mathematical analysis is simplified thereby. Furthermore any other termination can be regarded and dealt with as an additional terminal impedance, so there is no essential loss of generality involved.

A study of the properties of the artificial line is of practical importance for several reasons:

1. The artificial line is often used as a model of an actual transmission line and it is therefore of importance to determine theoretically the degree of correspondence between the two.

2. The solution for the corresponding transmission line with continuously distributed constants is derivable from the solution for the artificial line by keeping the total inductance, resistance, capacity and leakage constant or finite, and letting the number of sections approach infinity.

3. The artificial line is very closely related, in its properties and performance, to the periodically loaded line, and its solution is, to a first approximation, a working solution for the loaded line.

4. The structure is of great importance in its own right, and when the impedance elements are properly chosen, constitutes a "wave filter."

We shall now derive the operational and symbolic equations which formulate the propagation phenomena in the artificial line. Let I_n denote the mesh current in the nth section of the line; I_{n-1} the mesh current in the $(n-1)^{\text{th}}$ section, etc. Now write down the expression for the voltage drop in the n^{th} section; in accordance with Kirchhoff's law we get:

$$(z_1+2z_2)I_n - z_2(I_{n-1}+I_{n+1}) = 0 \tag{218}$$

where, of course, the impedances have the usual significance.

Now this is a difference equation, as distinguished from a differential equation, but the method of solution is essentially the same. We *assume* a solution of the form

$$I_n = Ae^{-n\Gamma} + Be^{n\Gamma} \tag{219}$$

where A, B and Γ are independent of n, and substitute in (218). After some simple rearrangements we get

$$\{(z_1+2z_2) - 2z_2 \cosh \Gamma\} \cdot \{Ae^{-n\Gamma} + Be^{n\Gamma}\} = 0. \tag{220}$$

Equation (218) is clearly satisfied by the assumed form of solution, and furthermore leaves the constants A and B arbitrary and at our disposal to satisfy any boundary conditions, provided Γ is so chosen that

$$\cosh \Gamma = \frac{z_1+2z_2}{2z_2} = 1+2\rho \tag{221}$$

where $\rho = z_1/4z_2$.

Now by reference to equation (219) it is easily seen that Γ is the *propagation constant* of the artificial line, precisely analogous to the propagation constant γ of the smooth line. In terms of the impedances z_1 and z_2, the propagation constant of the artificial line is determined by (221). This equation may either be regarded as an operational equation or a symbolic equation, depending on whether the impedances are expressed in terms of the operator p or in terms of $i\omega$, where ω is 2π times the frequency.

Now suppose in (221) we write $e^{\Gamma} = x$; the equation becomes

$$x+1/x = 2(1+2\rho)$$

and solving for x we get

$$x = e^{\Gamma} = (1+2\rho)+\sqrt{(1+2\rho)^2-1} = (\sqrt{1+\rho}+\sqrt{\rho})^2 = (\sqrt{1+\rho}-\sqrt{\rho})^{-2} \tag{222}$$

which is an explicit formula for Γ.

Now return to equation (219) and let us assume that the line is either infinitely long, or, what amounts to the same thing, that it is closed by an impedance which suppresses the reflected wave. We assume also that a voltage V_o is impressed at mid-series position of the zeroth section $(n=0)$. Equation (219) becomes

$$I_n = Ae^{-n\Gamma}$$

and the currents in the zeroth and 1^{st} sections are

$$I_o = A, \qquad I_1 = Ae^{-\Gamma}.$$

Now, by direct application of Kirchhoff's law to the zeroth section, we have

$$V_o = (\tfrac{1}{2}z_1+z_2)I_o - z_2 I_1,$$

whence

$$A\left\{\tfrac{1}{2}z_1+z_2(1-e^{-\Gamma})\right\} = V_o. \tag{223}$$

But

$$I_o = A = \frac{1}{K}V_o,$$

$$I_n = \frac{V_o}{K}e^{-n\Gamma},$$

where K is the *characteristic impedance* of the artificial line (at mid-series position). Hence by (223) and (222)

$$\begin{aligned}\frac{1}{K}&=\frac{1}{z_2}\frac{1}{(1-e^{-\Gamma})+2\rho}\\&=\frac{1}{2z_2}\frac{1}{\sqrt{\rho+\rho^2}}=\frac{1}{\sqrt{z_1z_2}}\frac{1}{\sqrt{1+\rho}}.\end{aligned}\tag{224}$$

By aid of the preceding the direct current wave can be written as

$$I_n=\frac{V_o}{\sqrt{z_1z_2}}\frac{[\sqrt{1+\rho}-\sqrt{\rho}\,]^{2n}}{\sqrt{1+\rho}}.\tag{225}$$

This formula is not so physically suggestive as its equivalent

$$I_n=\frac{V_o}{K}e^{-n\Gamma}$$

but is useful when we come to the solution of the operational equation.

Before proceeding with the operational equation, and the investigation of transient phenomena in artificial lines, it will be of interest to deduce from the foregoing the unique and remarkable properties of wave filters in the steady state. For this purpose we return to equation (221)

$$\cosh\Gamma=1+2\rho.$$

Now suppose that the series impedance z_1 is an inductance L and the shunt impedance z_2 a capacity C, so that, symbolically,

$$z_1=i\omega L,\ z_2=\frac{1}{i\omega C},\ \rho=-\frac{\omega^2LC}{4},$$

and

$$\cosh\Gamma=1-\tfrac{1}{2}\,\omega^2LC.\tag{226}$$

Now let us write $\Gamma=i\theta$, where $i=\sqrt{-1}$; the preceding equation becomes

$$\cos\theta=1-\tfrac{1}{2}\,\omega^2LC\tag{227}$$

and the *ratio of currents* in adjacent sections is $e^{-i\theta}$. *Consequently if θ is a real quantity the ratio of the absolute values of the currents in adjacent sections is unity, and the current is propagated without attenuation.*

Inspection of equation (227) shows that θ is real provided the right hand side lines between $+1$ and -1: or that ω lies between 0 and $2/\sqrt{LC}$. Consequently this type of artificial line transmits, in the steady state, sinusoidal currents of all frequencies from zero to $1/\pi\sqrt{LC}$ without attenuation. It is known as the low-pass filter.

If we invert the structure, that is, make the series impedance z_1 a capacity C and the shunt impedance z_2 an inductance L, so that

$$z_1=\frac{1}{i\omega C},\ z_2=i\omega L,\ \rho=-\frac{1}{4\omega^2 LC},$$

we get, corresponding to (226) and (227),

$$\cosh\Gamma=1-\frac{1}{2\omega^2 LC}, \tag{228}$$

$$\cos\theta=1-\frac{1}{2\omega^2 LC}. \tag{228a}$$

This type of artificial line transmits without attenuation currents of all frequencies for which the right hand side of (228-a) lies between $+1$ and -1; that is, all frequencies from infinity to a lower limiting frequency $1/4\pi\sqrt{LC}$, while it attenuates all frequencies below this range. It is known, on this account, as the high-pass filter.

It is possible by using more complicated impedances to design filters which transmit a series of bands of frequencies. We cannot, however, go into the complicated theory of wave filters here, which has been covered in a series of important papers. One point should be noted, however: transmission without attenuation implies that the impedance elements are non-dissipative. Actually, of course, all the elements introduce some loss, so that in practice the filter attenuates all frequencies. Careful design, however, keeps the attenuation very low in the transmission bands.

We shall now derive the indicial admittance formulas for some representative types of artificial lines and wave filters from the operational formula

$$A_n=\frac{1}{\sqrt{(1+\rho)z_1z_2}}[\sqrt{1+\rho}+\sqrt{\rho}\,]^{-2n}. \tag{229}$$

This equation follows directly from (225) on putting $V_o=1$.

We start with the so-called low-pass filter on account of its simplicity and also its great importance in technical applications. This type of filter consists of series inductance L and shunt capacity C. The general case which includes series resistance R and shunt leakage G has been worked out (see Transient Oscillations, Trans. A. I. E. E., 1919). The solution is, however, extremely complicated and will not be dealt with here. We shall, instead, consider the important and illuminating case where the series and shunt losses are so related as to make the circuit quasi-distortionless. We therefore take, operationally,

$$\begin{aligned} z_1 &= pL+R = L(p+\lambda) \\ 1/z_2 &= pC+G = C(p+\lambda) \end{aligned} \tag{230}$$

where $\lambda = R/L = G/C$.

We then have

$$\begin{aligned} z_1 z_2 &= L/C, \\ z_1/z_2 &= LC(p+\lambda)^2, \\ \rho &= \frac{LC}{4}(p+\lambda)^2. \end{aligned} \tag{231}$$

Now by reference to formula (229) we see that A_n is a function of $(p+\lambda)$; thus

$$A_n = \frac{1}{Z_n(p+\lambda)} = \left(1+\frac{\lambda}{\rho}\right)\frac{\rho}{(\rho+\lambda)Z_n(\rho+\lambda)}.$$

Now write

$$A_n^o = \frac{1}{Z_n(p)}.$$

It follows at once from reference to theorem VII that

$$A_n = \left(1+\lambda\int_0^t dt\right)A_n^o e^{-\lambda t} \tag{232}$$

so that the problem is reduced to the solution of the operational equation for A_n^o. Writing $\omega_c = 2/\sqrt{LC}$, we have

$$\begin{aligned} A_n^o &= \sqrt{\frac{C}{L}}\frac{1}{\sqrt{1+(p/\omega_c)^2}}\left[\sqrt{1+(p/\omega_c)^2}+p/\omega_c\right]^{-2n} \\ &= \sqrt{\frac{C}{L}}\frac{\omega_c}{\sqrt{p^2+\omega_c^2}}\left[\frac{\sqrt{p^2+\omega_c^2}-p}{\omega_c}\right]^{2n}. \end{aligned} \tag{233}$$

Now refer to formula (n) of the table of integrals; writing $\sqrt{L/C}=k$, we see by Theorem V that

$$A_n^o = \frac{1}{k}\int_0^{\omega_c t} J_{2n}(\tau)d\tau \tag{234}$$

where $J_{2n}(\tau)$ is the Bessel function of order $2n$ and argument τ. We note also that this is the indicial admittance of the non-dissipative low-pass wave filter; that is, the current in the n^{th} section in response to a unit e.m.f. applied to the initial section ($n=0$). From (232) and (234) it follows at once that

$$A_n = e^{-\lambda t}\frac{1}{k}\int_0^{\omega_c t} J_{2n}(\tau)d\tau$$
$$+\frac{\lambda}{k}\int_0^t d\tau e^{-\lambda\tau}\int_0^{\omega_c\tau} J_{2n}(\tau_1)d\tau_1.$$

Integrating the second member by parts and noting that $A_n^o(o)=0$, this reduces to

$$A_n = \frac{1}{k}\int_0^{\omega_c t} e^{-\frac{\lambda}{\omega_c}\tau} J_{2n}(\tau)d\tau \qquad (235)$$

which is the indicial admittance formula for the quasi-distortionless low-pass filter, or artificial line.

Before discussing these formulas, it is of interest to derive the formula for A_n^o by power series expansion. Formula (233) can be written

$$A_n^o = \frac{1}{k}\left(\frac{\omega_c}{p}\right)^{2n+1}\frac{1}{\sqrt{1+(\omega_c/p)^2}}\,\frac{1}{[1+\sqrt{1+(\omega_c/p)^2}]^{2n}}.$$

This can be expanded in a series in inverse powers of p; thus

$$A_n^o = \frac{1}{k2^{2n}}\left\{\left(\frac{\omega_c}{p}\right)^{2n+1} - \frac{2n+2}{2^2 1!}\left(\frac{\omega_c}{p}\right)^{2n+3}\right.$$
$$\left.+\frac{(2n+3)\,(2n+4)}{2^4 2!}\left(\frac{\omega_c}{p}\right)^{2n+5} - \ldots\ldots\right\}.$$

Replacing $1/p^n$ by $t^n/n!$ in accordance with the Heaviside Rule we get

$$A_n^o = \frac{2}{k}\left\{\frac{1}{(2n+1)!}\left(\frac{\omega_c t}{2}\right)^{2n+1} - \frac{2n+2}{1!(2n+3)!}\left(\frac{\omega_c t}{2}\right)^{2n+3}\right.$$
$$\left.+\frac{(2n+3)\,(2n+4)}{2!(2n+5)!}\left(\frac{\omega_c t}{2}\right)^{2n+5} - \ldots\ldots\right\}. \qquad (235\text{-a})$$

This can be recognized as the power series expansion of (234).

The *artificial cable* is also of interest and practical importance. In this structure the series impedance is a resistance R and the shunt impedance is a capacity C, so that

$$z_1 = R,\ 1/z_2 = pC,$$
$$z_1 z_2 = R/pC,\quad z_1/z_2 = pRC, \qquad (236)$$
$$\rho = pRC/4.$$

Now let us return to formula (229), and expand in inverse powers of ρ: we get

$$A_n=\frac{1}{2^{2n}\sqrt{\rho z_1 z_2}}\left\{\frac{1}{\rho^n}-\frac{2n+2}{2^2 1!}\frac{1}{\rho^{n+1}}+\frac{(2n+3)(2n+4)}{2^4 2!}\frac{1}{\rho^{n+2}}-\ldots\right\} \quad (237)$$

Now since $\sqrt{\rho z_1 z_2}=\frac{R}{2}$, we have

$$A_n=\frac{2}{2^n R}\left\{\left(\frac{2}{RCp}\right)^n-\frac{2n+2}{2.1!}\left(\frac{2}{RCp}\right)^{n+1}+\frac{(2n+3)(2n+4)}{2^2 2!}\left(\frac{2}{RCp}\right)^{n+2}-\ldots\right\}.$$

Replacing $1/p^n$ by $t^n/n!$ we get finally

$$A_n=\frac{2}{2^n R}\left\{\frac{1}{n!}\left(\frac{2t}{RC}\right)^n-\frac{(2n+2)}{2.1!(n+1)!}\left(\frac{2t}{RC}\right)^{n+1}+\frac{(2n+3)(2n+4)}{2^2.2!(n+2)!}\left(\frac{2t}{RC}\right)^{n+2}-\ldots\right\}. \quad (238)$$

For large values of n and t this series is difficult to compute or interpret. It can, however, be recognized as the series expansion of the function

$$A_n=\frac{2}{R}e^{-\frac{2t}{RC}}I_n\left(\frac{2t}{RC}\right) \quad (239)$$

where $I_n(2t/RC)$ is the Bessel function I_n of order n and argument $(2t/RC)$. This solution, it may be remarked, can be derived directly by a modification of the integral formula (n).

It is beyond the scope of this paper to consider other types of artificial lines and wave filters; for a fairly extensive discussion the reader is referred to "Transient Oscillations in Electric Wave-Filters," B. S. T. J., July, 1923. The low-pass wave filter, however, both in its own right and on account of its close relation to the periodically loaded line, deserves further discussion.

For the non-dissipative low-pass wave filter, we have

$$A_n^{\circ}=\frac{1}{k}\int_0^{\omega_c t}J_{2n}(\tau)\,d\tau \quad (234)$$

while for the quasi-distortionless low-pass wave filter

$$A_n=\frac{1}{k}\int_0^{\omega_c t}e^{-\mu\tau}J_{2n}(\tau)\,d\tau \quad (235)$$

where $\mu=\lambda/\omega_c=R/L\omega_c=R/2vL$.

Computation and analysis of these formulas involve an elementary knowledge of Bessel functions. The properties necessary for our purposes are briefly discussed in an appendix to this chapter.

The indicial admittances for the non-dissipative low-pass filter,

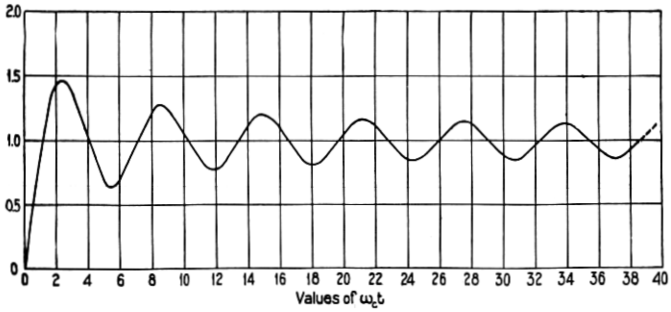

Fig. 20—Low pass wave filter. Indicial admittance of initial section ($n=0$). Multiply ordinates by $\sqrt{C/L}$

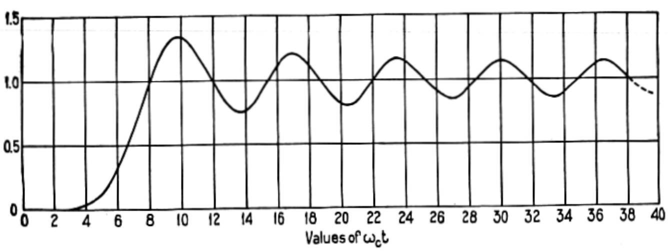

Fig. 21—Low pass wave filter. Indicial admittance of third section ($n=2$). Multiply ordinates by $\sqrt{C/L}$

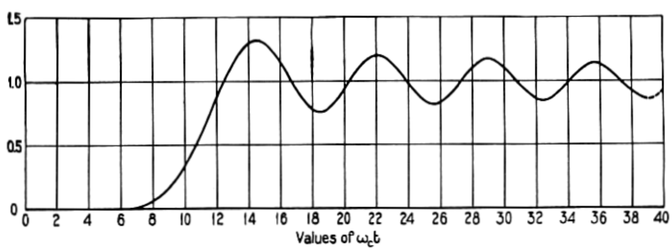

Fig. 22—Low pass wave filter. Indicial admittance of fifth section ($n=4$). Multiply ordinates by $\sqrt{C/L}$

that is, the current in response to a steady unit e.m.f. applied at time $t=0$, are shown in the curves of Figs. 20, 21 and 22, for the initial or zeroth, the 3rd and the 5th sections, respectively. These curves together with the exact and approximate formulas given above are sufficient to give a reasonably comprehensive idea of the general character of these oscillations and their dependence on the number of sections and the constants of the filter.

It will be observed that the current is small until a time approximately equal to $2n/\omega_c = n\sqrt{L_1C_2}$ has elapsed after the voltage is applied. Consequently the low-pass filter behaves as though currents were transmitted with a finite velocity of propagation $\omega_c/2 = 1/\sqrt{L_1C_2}$ sections per second. This velocity is, however, only apparent or virtual since in every section the currents are actually finite for all values of time >0.

After time $t = n\sqrt{L_1C_2}$ has elapsed the current oscillates about the value $1/k$ with increasing frequency and diminishing amplitude. The amplitude of these oscillations is approximately

$$\frac{1/k}{\sqrt{1-(2n/\omega_c t)^2}}\sqrt{\frac{2}{\pi\omega_c t}}$$

and their instantaneous frequency (measured by intervals between zeros)

$$\frac{\omega_c}{2\pi}\sqrt{1-(2n/\omega_c t)^2}.$$

The oscillations are therefore ultimately of cut-off or critical frequency $\omega_c/2\pi$ in all sections, but this frequency is approached more and more slowly as the number of filter sections is increased.

Figs. 23, 24, 25, give the indicial admittance in the 100th, 500th and 1000th section of the low-pass wave filter. The filter itself seldom

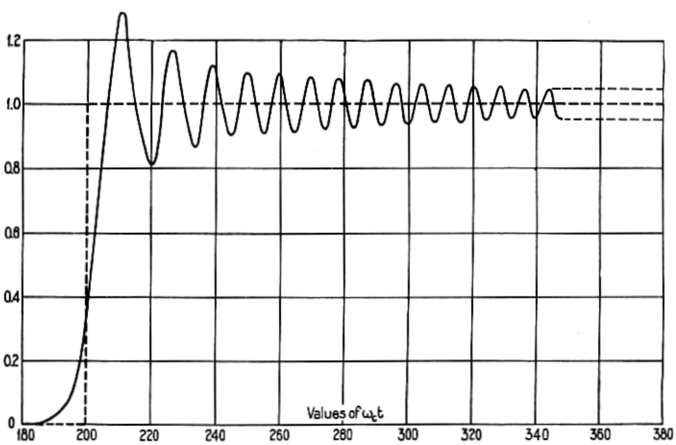

Fig. 23—Low pass wave filter. Indicial admittance of 100th section ($n=99$). Multiply ordinates by $\sqrt{C/L}$

embodies more than 5 sections. The case of a large number of sections is of interest, however, because it represents a first approximation to the periodically loaded line. While the non-dissipative

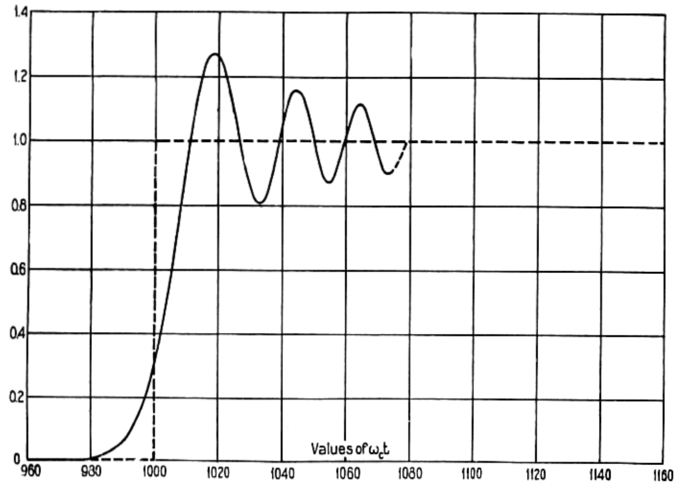

Fig. 24—Low pass wave filter. Indicial admittance of 500th section ($n=499$). Multiply ordinates by $\sqrt{C/L}$

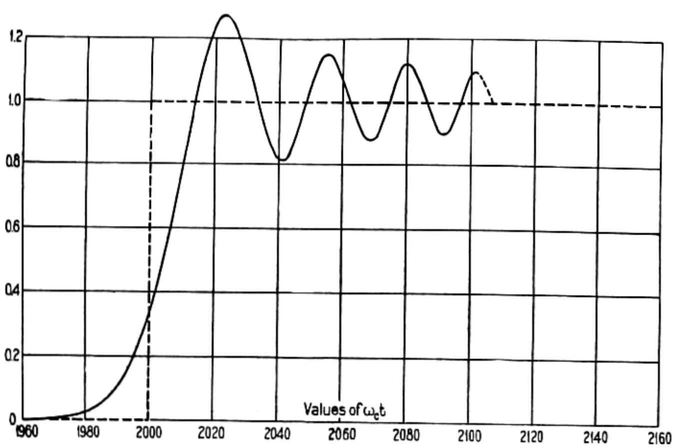

Fig. 25—Low pass wave filter. Indicial admittance of 1000th section ($n=999$). Multiply ordinates by $\sqrt{C/L}$

line is ideal and unrealizable, its study is of practical importance because in this type of line the effect of the discontinuous character of the loading of the periodically loaded line is isolated and exhibited in the clearest possible manner.

The dotted curves represent the current in the corresponding smooth line. For the smooth line, the current, as we have seen, is discontinuous, being identically zero for a time $vt=n$ and having an instantaneous jump to its final value $\sqrt{C/L}$ at $vt=n$. The current in the artificial or periodically loaded line differs from that in the corresponding smooth line in three important respects: (1) the absence of the abrupt discontinuous wave front, (2) the presence of superposed oscillations, and (3) *the absence of a true finite velocity of propagation*. It will be observed, however, that the current in any section is negligibly small or even sensibly zero until $vt=n$, so that the current is propagated with a *virtual* velocity $1/\sqrt{LC}$ per section. The presence of a well marked wave front is also evident although this is not abrupt, as in the smooth line. The effective slope of the wave front becomes smaller as the current wave travels out on the line, decreasing noticeably as the number of sections is increased. When the number of sections becomes large, however, the decrease in the slope is not rapid, being in the 500$^{\text{th}}$ section about 60 per cent. of that in the 100$^{\text{th}}$ section.

The superposed oscillations are of interest. These are initially of a frequency depending upon and decreasing with the number of sections, n, but in all sections ultimately attaining the frequency

$$\frac{1}{\pi\sqrt{LC}}=\frac{v}{\pi}$$

which is the critical or cut-off frequency of the line, above which steady-state currents are attenuated during transmission and below which they are unattenuated. When vt is large compared with n the amplitude of these oscillations becomes $\sqrt{1/\pi vt}$ so that they ultimately die away and the current approaches the value $\sqrt{C/L}$ for all sections. The current in the loaded line is thus asymptotic to the current in the corresponding smooth line and oscillates about it with diminishing amplitude and increasing frequency.

Since the abscissas of these curves represent values of $2vt=2t/\sqrt{LC}$, and the ordinates are to be multiplied by $\sqrt{C/L}$ to translate into actual values, the curves are of universal application for all values of the constants L and C.

The investigation of the building-up of alternating currents in wave filters and loaded lines is very important. It depends for the non-dissipative case on the properties of the definite integrals

$$\int_0^{\omega_c t} \sin w\tau \; J_n(\tau)d\tau,$$

$$\int_0^{\omega_c t} \cos w\tau \; J_n(\tau)d\tau,$$

where $w=\omega/\omega_c$ and $\omega=2\pi$ times the applied frequency. The mathematical discussion is, however, quite complicated and will not be entered into here. The reader, who wishes to follow this further, is referred to Transient Oscillations, Trans. A. I. E. E., 1919 and Transient Oscillations in Electric Wave Filters, B. S. T. J., July, 1923.

Appendix to Chapter VIII. Note on Besel Functions

The Bessel Functions of the first kind, $J_n(x)$ and $I_n(x)$, are defined, when n is zero or a positive integer, by the absolutely convergent series

$$J_n(x)=\frac{x^n}{2^n.n!}\left\{1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2.4(2n+2)(2n+4)}-\frac{x^6}{2.4.6.(2n+2)(2n+4)(2n+6)}+\ldots\right\},$$

$$I_n(x)=\frac{x^n}{2^n.n!}\left\{1+\frac{x^2}{2(2n+2)}+\frac{x^4}{2.4(2n+2)(2n+4)}+\frac{x^6}{2.4.6.(2n+2)(2n+4)(2n+6)}+\ldots\right\}.$$

In the following discussion of the properties of these functions it will be assumed that the argument x is a pure real quantity.

For large values of the argument (x large compared with n), the behavior of the functions is shown by the asymptotic expansions:—

$$I_n(x)=\frac{e^x}{\sqrt{2\pi x}}\left\{1-\frac{4n^2-1}{1!\,(8x)}+\frac{(4n^2-1)(4n^2-9)}{2!\,(8x)^2}-\frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!\,(8x)^3}+\ldots\right\},$$

$$J_n(x)=\sqrt{\frac{2}{\pi x}}\left\{P_n\cos\left(x-\frac{2n+1}{4}\pi\right)-Q_n\sin\left(x-\frac{2n+1}{4}\pi\right)\right\},$$

where

$$P_n=1-\frac{(4n^2-1)(4n^2-9)}{2!\,(8x)^2}+\frac{(4n^2-1)(4n^2-9)(4n^2-25)(4n^2-49)}{4!\,(8x)^4}-\ldots,$$

$$Q_n=\frac{4n^2-1}{8x}-\frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!(8x)^3}+\ldots.$$

We thus see that I_n increases indefinitely and behaves ultimately as

$$\frac{e^x}{\sqrt{2\pi x}}.$$

The function $J_n(x)$, however, is oscillatory and ultimately behaves as

$$\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{2n+1}{4}\pi\right).$$

For all orders of n

$$\int_0^\infty J_n(x)dx=1.$$

The properties of $J_n(x)$ may be described qualitatively as follows:—

When the argument is less than the order ($0\leqq x<n$) the function is very small and positive, and is initially zero (except when $n=0$). In the neighborhood of $x=n$, the function begins to build up and reaches a maximum a little beyond the point $x=n$. Thereafter the function oscillates with increasing frequency and diminishing amplitude, and ultimately behaves as

$$\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{2n+1}{4}\pi\right).$$

When $n=0$, the initial value is unity, but the subsequent behavior of the function is as described above.

A more precise description of the function is gotten from the following approximate formulas.

$$J_n(x)\doteq B_n(x)\cos\Omega_n(x),\qquad \text{for } x>n$$

where

$$B_n(x)=\sqrt{\frac{2}{\pi x}}\frac{1}{\left(1-\frac{m^2}{x^2}+\frac{3}{2}\frac{m^2}{x^4}\frac{1}{(1-m^2/x^2)^2}\right)^{1/4}},$$

$$\Omega_n(x)=x\left[\sqrt{1-\frac{m^2}{x^2}}+\frac{m}{x}\sin^{-1}\left(\frac{m}{x}\right)-\frac{m^2}{4x^4}\frac{1}{(1-m^2/x^2)^{3/2}}\right]-\frac{2n+1}{4}\pi,$$

$$\Omega_n'(x)=\frac{d}{dx}\Omega_n(x),$$

$$=\sqrt{1-\frac{m^2}{x^2}+\frac{3}{2}\frac{m^2}{x^4}\frac{1}{(1-m^2/x^2)^2}},$$

and

$$m^2=n^2-1/4.$$

This approximate formula is valid only where $x>n$, its accuracy increasing with x and with n. For all orders of n it is quite accurate beyond the first zero of the function.

The "instantaneous frequency" of oscillation is approximately

$$\frac{1}{2\pi}\Omega_n'(x)=\frac{1}{2\pi}\sqrt{1-\frac{m^2}{x^2}+\frac{3}{2}\,\frac{m^2}{x^4}\,\frac{1}{(1-m^2/x^2)^2}}.$$

By this it is meant that at any point $x(x>n)$ the interval between successive zeros is approximately $\pi/\Omega'(x)$. Otherwise stated, in the neighborhood of any point x, the function behaves like a sinusoid of amplitude $B_n(x)$ and frequency $\omega/2\pi$ where $\omega=\Omega_n'(x)$.

The following approximate formulas, while not sufficiently precise for the purposes of accurate computation except for quite large values of x, clearly exhibit the character of the functions for values of the argument $x>n$, and of the order $n>2$.

$$J_n(x)\doteq h_n\sqrt{\frac{2}{\pi x}}\cos\,(q_n x-\theta_n),$$

$$J_n'(x)=-q_n h_n\sqrt{\frac{2}{\pi x}}\sin\,(q_n x-\theta_n),$$

$$\int_0^x J_n(x)dx=1+\frac{h_n}{q}\sqrt{\frac{2}{\pi x}}\sin\,(q_n x-\theta_n),$$

where

$$h_n=\left(\frac{1}{1-n^2/x^2}\right)^{1/4}=1+\frac{n^2}{4x^2},$$

$$q_n=\sqrt{1-n^2/x^2},$$

$$\theta_n=\frac{2n+1}{4}\pi-n\sin^{-1}(n/x).$$