Notes on the Heaviside Operational Calculus

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This paper briefly discusses the following topics: (1) the asymptotic solution of operational equations; (2) Bromwich's formulation of the Heaviside problem, and its relation to the classical Fourier integral; and (3) the existence of solutions of the operational equation. The paper closes with some general remarks on the interpretation of the operator and the operational equation, emphasizing the purely symbolic character of the latter.

THE large amount of work done in the past thirteen years, starting with important papers by Bromwich [1] and K. W. Wagner,[2] has served to remove whatever mystery may have surrounded the Heaviside operator, and has placed his operational calculus on a quite secure and logical foundation. However, certain phases of the problem still do not appear to the writer to have as clear or adequate treatment as perhaps might be desired; these it is the object of the present paper to discuss. The topics dealt with are (1) the asymptotic solution of operational equations; (2) Bromwich's very important formula and its relation to the classical Fourier integral; and (3) the existence of solutions of the operational equation.

In the following it will be assumed that the reader has a general acquaintance with the Heaviside operational calculus as well as the Fourier integral, but a brief sketch of the former may not be out of place. It will be recalled that the Heaviside processes were originally developed in connection with the solution of electrical problems:[3] more precisely, the determination of the oscillations of a linearly connected system specified by a set of linear differential equations with constant coefficients or a partial differential equation of the type of the wave equation. This system is supposed to be in a state of equilibrium at reference time $t = 0$, when it is suddenly acted upon by a 'unit' force (zero before, unity after time $t = 0$); the subsequent behavior of the system is required. In the solution of this problem, Heaviside's first step was the purely formal and symbolic one of replacing the differential operator $\partial/\partial t$ by the symbol p, thereby

[1] "Normal Coordinates in Dynamical Systems," *Proc. Lond. Math. Soc.* (2), 15, 1916.

[2] "Über eine Formel von Heaviside zur Berechnung von Einschaltvorgänge," *Archiv. Elektrotechnik*, Vol. 4, 1916.

[3] Since this paper is addressed largely to physicists and engineers, we shall employ to some extent the language of circuit theory rather than pure mathematics; no loss of essential generality is involved.

reducing the differential equations to an algebraic form, the formal solution of which we shall write

$$h = \frac{1}{H(p)}, \qquad t \geqq 0. \tag{1}$$

Here $h = h(t)$ is the variable with whose determination we are concerned and $H(p)$ is the Heaviside function, derived as stated from the differential equations of the problem. This equation is as yet purely symbolic, and its conversion into an explicit solution for h, as a function of t, constitutes the Heaviside problem.

Bromwich [1] formulates the problem as the infinite integral

$$h(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{pt}}{pH(p)}\,dp. \tag{2}$$

The writer's formulation of the problem is, that h is uniquely determined by the integral equation [4]

$$\int_0^\infty h(t)e^{-pt}dt = \frac{1}{pH(p)} \tag{3}$$

This equation is valid for all values of p, for which its *real part* is greater than some finite constant c; c must be at least large enough to make the infinite integral converge. In the majority of physical problems this constant may be taken as 0; in some, however, the equation is valid only when c is greater than some finite constant.

The equivalence of (2) and (3) is very easily established in a number of ways; perhaps the simplest is to show, following March,[5] that (2) is the formal solution of (3). Either can be deduced from the other. The Bromwich solution can, of course, be derived directly from the Heaviside problem, as shown below.

I

One of the most interesting and perhaps the least generally understood of Heaviside's methods of solving the operational equation is the process whereby he derives a series solution, usually divergent and asymptotic, in inverse fractional powers of t. What I have termed the Heaviside Rule [6] for deriving this type of solution may be formulated as follows:

[4] "The Heaviside Operational Calculus," *B. S. T. J.*, 1922; *Bulletin Amer. Math. Soc.*, 1926.

[5] "The Heaviside Operational Calculus," *Bulletin Amer. Math. Soc.*, 1927.

[6] In terming this process the Heaviside Rule I do not in any sense imply that Heaviside himself would have applied it incorrectly. In fact in one case he adds an extra term which contributes to numerical accuracy although the series itself is

If the operational equation $h = 1/H(p)$ admits of formal series expansion in the form

$$h = a_0 + a_1\sqrt{p} + a_2 p + a_3 p\sqrt{p} + a_4 p^2 + \ldots, \tag{4}$$

a solution, usually divergent and asymptotic, results from discarding the terms in integral powers of p, and replacing $p^n\sqrt{p}$ by $\frac{d^n}{dt^n}\frac{1}{\sqrt{\pi t}}$, whence

$$h \sim a_0 + \left\{ a_1 + a_3\frac{d}{dt} + a_5\frac{d^2}{dt^2} + \cdots \right\} \frac{1}{\sqrt{\pi t}}\cdot \tag{5}$$

As stated in a forthcoming paper, this divergent series is a true asymptotic expansion, as defined by Poincare, if and only if, the singularities in $1/H(p)$ all lie to the left of the imaginary axis in the complex plane. Otherwise the series may require the addition of an extra term or factor, or even be quite meaningless.

An excellent illustration of the preceding principle is furnished by the operational equation,

$$h = \frac{\sqrt{p}}{\sqrt{p + \lambda}}\cdot \tag{6}$$

For convenience and without loss of essential generality we take $|\lambda| = 1$ and $\lambda = e^{i\theta}$; that is, the parameter λ may lie anywhere on a circle of unit radius in the complex plane.

Now the solution of (6) is easily derived by well known processes of the operational calculus: it is

$$h(t) = \frac{1}{\pi}\int_0^t \frac{e^{-\lambda\tau}}{\sqrt{\tau}\sqrt{t-\tau}}\,d\tau \tag{7}$$

$$= \frac{e^{-\lambda t}}{\pi}\int_0^t \frac{e^{\lambda\tau}}{\sqrt{\tau}\sqrt{t-\tau}}\,d\tau. \tag{8}$$

The solution is also known to be [7]

$$h(t) = e^{-(\lambda t/2)} I_0\left(\frac{\lambda t}{2}\right), \tag{9}$$

where $I_0(\lambda)$ is the Bessel function $J_0(ix)$.

a true asymptotic expansion. On the other hand Heaviside in his frequent applications of the Rule gives no hint or indication of the restrictions imposed on its applicability. Fortunately in most applications of the operational calculus to physical problems, the Rule leads to correct results.

[7] See formula (p) of the table of integrals in Chap. IV, "Electric Circuit Theory and Operational Calculus."

Now return to the operational equation (6), and expand as follows, without reference to convergence,

$$h = \frac{1}{\sqrt{\lambda}}\left(1+\frac{p}{\lambda}\right)^{-1/2}\sqrt{p}$$

$$= \frac{1}{\sqrt{\lambda}}\left(1-\frac{1}{2}\left(\frac{p}{\lambda}\right)+\frac{1}{2!}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{p}{\lambda}\right)^2-\cdots\right)\sqrt{p}.$$

Application of the Heaviside Rule now gives the divergent solution

$$h(t) \sim \left\{1+\frac{1}{1!}\left(\frac{1}{4\lambda t}\right)+\frac{1^2\cdot 3^2}{2!}\left(\frac{1}{4\lambda t}\right)^2+\cdots\right\}\frac{\sqrt{\pi\lambda t}}{1},$$

$$\sim S(\lambda t). \tag{10}$$

We have now to distinguish three cases:

1. $\lambda_R > 0$. (Real part of $\lambda > 0$.)

In this case it can be shown from (7) that [8]

$$h(t) \sim S(\lambda t) \tag{11}$$

and that the Heaviside Rule leads to a true asymptotic expansion, as defined by Poincare. When $\lambda = 1$, by the known expansion of the right hand function in equation (9) we find that the error committed by stopping with any term in the divergent series is less than that term. This property, however, does not characterize the series for all complex values of λ for which the real part is positive.

2. $\lambda_R < 0$, $\lambda = -\mu$, $\mu_R > 0$.

In this case, comparison of (8) with (7), gives by aid of (11),

$$h(t) \sim e^{\mu t}S(\mu t), \tag{12}$$

which again is a true asymptotic expansion. The expansion differs, however, from that given by the Heaviside Rule, by the factor $e^{\mu t}$, and the alternation in sign of the odd terms of the series.

3. $\lambda_R = 0$, $\lambda = i\omega$.

In this case it is easily shown that [9]

$$h(t) = e^{-(i\omega t/2)}J_0\left(\frac{\omega t}{2}\right), \tag{13}$$

where J_0 is the Bessel function of order zero. From the known asymptotic expansion of this function, we find that

$$h(t) \sim e^{-(i\omega t/2)}\left[e^{(i\omega t/2)}S(i\omega t)\right]_{\text{Real Part}} \tag{14}$$

with an error less than the last term included.

[8] L.c. by the process described in Chap. V.
[9] L.c. formula (n) of table of integrals, Chap. IV.

Perhaps the simplest way of establishing the Heaviside Rule for the asymptotic solution of the operational equation $h = 1/H(p)$ and the conditions under which it is valid, is as follows: We start with the integral equation

$$\int_0^\infty h(t)e^{-pt}dt = 1/pH(p) \tag{15}$$

and specify that the singularities of $1/pH(p)$ and its derivatives are all confined to the left hand side of the complex plane, except at the point $p = 0$, in the neighborhood of which

$$\frac{1}{pH(p)} = \frac{a_0}{\sqrt{p}} + a_1 + a_2\sqrt{p} + a_3 p + a_4 p\sqrt{p} + \cdots. \tag{16}$$

In other words, $1/pH(p)$ admits of expansion in powers of $\sqrt{p}$.

Now since

$$\int_0^\infty \frac{e^{-pt}}{\sqrt{\pi t}}dt = \frac{1}{\sqrt{p}} \tag{17}$$

we have from (15)

$$\int_0^\infty \left(h - \frac{a_0}{\sqrt{\pi t}}\right) e^{-pt}dt = \frac{1}{pH(p)} - \frac{a_0}{\sqrt{p}}. \tag{18}$$

By virtue of the restrictions imposed on $1/pH(p)$, equation (18) is valid at $p = 0$, whence by (16)

$$\int_0^\infty \left(h - \frac{a_0}{\sqrt{\pi t}}\right) dt = a_1. \tag{19}$$

Now differentiate (18) with respect to p; we get

$$\int_0^\infty \left(h - \frac{a_0}{\sqrt{\pi t}}\right) te^{-pt}dt = -\frac{d}{dp}\left(\frac{1}{pH(p)} - \frac{a_0}{\sqrt{p}}\right). \tag{20}$$

Now add $\int_0^\infty \frac{a_2}{2}\frac{e^{-pt}}{\sqrt{\pi t}}dt$ to the left of (20) and its value $a_2/2\sqrt{p}$ to the right hand side; we have

$$\int_0^\infty \left(h - \frac{a_0}{\sqrt{\pi t}} + \frac{a_2}{2t}\frac{1}{\sqrt{\pi t}}\right) te^{-pt}dt = -\frac{d}{dp}\left(\frac{1}{pH(p)} - \frac{a_0}{\sqrt{p}}\right) + \frac{a_2}{2\sqrt{p}}. \tag{21}$$

Now set $p = 0$; from (16) we have

$$\int_0^\infty \left(h - \frac{a_0}{\sqrt{\pi t}} + \frac{a_2}{2t} \frac{1}{\sqrt{\pi t}} \right) t dt = - a_3, \tag{22}$$

a formula which again is valid by reason of the restrictions imposed on $1/pH(p)$.

Proceeding in this manner we get the formula

$$\int_0^\infty (h - S_n) \cdot t^n dt = (-1)^n n! a_{2n+1}, \tag{23}$$

where

$$S_n = \frac{1}{\sqrt{\pi t}} \left(a_0 - \frac{a_2}{2t} + 1.3 \frac{a_4}{(2t)^2} + \cdots + (-1)^n 1.3 \cdots (2n-1) \frac{a_{2n}}{(2t)^n} \right)$$

$=$ first $(n + 1)$ terms of the divergent Heaviside series. (24)

Also since

$$S_{n+1} = S_n + (-1)^{n+1} \frac{1.3 \cdots (2n+1)}{(2t)^{n+1}} \frac{a_{2n+2}}{\sqrt{\pi t}} \tag{25}$$

we have from (23) by changing n to $(n + 1)$,

$$\int_0^\infty \left(h - S_n - (-1)^{n+1} \frac{1.3 \cdots (2n+1)}{(2t)^{n+1}} \frac{a_{2n+2}}{\sqrt{\pi t}} \right) t^{n+1} dt = (-1)^{n+1} (n+1)! a_{2n+3}. \tag{26}$$

Equations (23) and (26) establish the fact that $(h - S_n)$ converges, for indefinitely great values of t, at least as rapidly as $1/t^{n+1}\sqrt{t}$, since otherwise the integrand of (26) would diverge; stated in mathematical notation

$$h - S_n = 0(1/t^{n+3/2}). \tag{27}$$

Consequently the series S when divergent is a true asymptotic expansion, as defined by Poincare, of the function h.

The foregoing says nothing, it will be noted, regarding the error committed when S_n is employed to compute the function h. Nothing, in general, can be said about this question, which requires an independent investigation in every specific problem. In some cases the error will be less than the magnitude of the last term of S_n, but this is the exception rather than the rule. In other exceptional cases the series may even be absolutely convergent.

The foregoing results can undoubtedly be derived by integration of the Bromwich integral (2) along the contour suggested by March (*l.c.*). Wiener in his paper on "The Operational Calculus" (*Math. Annalen*, Bd. 95, 1925) gives an entirely different treatment of the problem. The operational calculus he deals with, however, differs under some circumstances from that of Heaviside, as Wiener himself remarks. A paper by Tibor v. Stacho on "Operatoren Kakül von Heaviside und Laplaceshe Transformation" (publication 1927 VI 15 by the Hungarian University, Francis Joseph) may also be consulted.

II

Subject to certain well known restrictions a function $f(t)$ can be expressed as the Fourier integral

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F(p)e^{pt}dp. \tag{28}$$

the path of integration being along the imaginary axis. We assume for the moment that this equation is valid.

Now suppose that $f(t)$ represents a force applied to an electrical or dynamic system whose "steady state" or forced response to an applied force $F(p)e^{pt}$ is

$$\frac{F(p)}{H(p)}e^{pt}.$$

Then the *forced* response $g(t)$ of the system to the applied force $f(t)$ is given by

$$g(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{F(p)}{H(p)}e^{tp}dp. \tag{29}$$

However, in applying the foregoing to the Heaviside problem we encounter an initial difficulty. This is that if $f(t)$ is taken as the unit function (zero before unity after, $t = 0$) it does not admit of formulation as the Fourier integral (28). The unit function, however, when multiplied by e^{-ct} when c is a positive real constant, does admit of such formulation, and it is easy to show that the unit function itself is given by

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{pt}}{p}dp \qquad c > 0. \tag{30}$$

Consequently, if the unit function is the force impressed on the system, the *forced* response is

$$k(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{pt}}{pH(p)}dp \qquad c > 0. \tag{31}$$

If now all the singularities of the integrand lie to the left of the imaginary axis, then $k(t) = h(t)$ and (31) is the formulation of the Heaviside problem. Suppose, however, that the electrical or dynamic system specified by $H(p)$ is "unstable"; that is, it contains some internal source of energy which makes its transient oscillations increase with time t instead of dying away. In such a case $H(p)$ will have zeros to the right of the imaginary axis, and in order that (31) shall be the solution of the Heaviside problem, c must be taken so large that all the singularities of the integrand lie to the left of the path of integration. Consequently

$$h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{pt}}{pH(p)} dp, \qquad (2)$$

provided c is so chosen that all the singularities lie to the left of the path of integration in the complex plane. This is Bromwich's formulation of the Heaviside problem.[10]

From the foregoing it follows that the Fourier integral

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{pt}}{pH(p)} dp \qquad (2a)$$

is, in general, the formulation of the Heaviside problem if and only if, all the singularities of the integrand lie to the left of the imaginary axis. If there are singularities on the imaginary axis, the integral is ambiguous, while if there are singularities to the right of the imaginary axis, the integral gives an incorrect solution of the Heaviside problem.[11]

As a simple example consider the operational equation

$$h = 1/H(p) = \frac{p}{p - \beta},$$

where the real part β_r of β is positive. The correct solution as given by either (2) or (3) is

$$\begin{aligned} h &= 0 \qquad & t < 0 \\ &= e^{\beta t} \qquad & t > 0, \end{aligned}$$

[10] The appropriate mathematical methods of solving the infinite integral (2) are dealt with in great detail by Jeffreys in his "Operational Methods in Mathematical Physics" (Cambridge University Tracts).

[11] To prevent misunderstanding it should be stated that the application, when permissible, of the classical Fourier integral (2a) to the Heaviside problem, was known long prior to the work of Bromwich. Bromwich's essential and important contribution lay in showing that the path of integration must be shifted to the right of all the singularities, together with a verification of an important form of solution, first given by Heaviside, of the operational equation.

whereas the Fourier integral (2a) gives

$$\begin{aligned} h &= -e^{\beta t} & t &< 0. \\ &= 0 & t &> 0. \end{aligned}$$

There is another reason why care must be exercised in applying the classical Fourier integral to the Heaviside problem. This is that in solving the operational equation, $h = 1/H(p)$, the appropriate expansion of $1/H(p)$ may introduce singularities on or to the right of the imaginary axis in the component terms. This offers no difficulty if either (2) or (3) is employed, but renders the Fourier integral (2a) inapplicable. As an example consider the equation

$$h = \frac{1}{\sqrt{p} + 1}.$$

One form of solution is gotten by multiplying numerator and denominator by $\sqrt{p} - 1$, whence

$$h = \frac{\sqrt{p}}{p - 1} - \frac{1}{p - 1}$$

and each term has a singularity at $p = 1$.

A physical interpretation of the foregoing may not be without interest. Suppose that an elementary force $F(p)e^{pt}dp$, where $p = c + i\omega$, is applied at an indefinitely remote past (negative) time to a system specified by $H(p)$. The response of the system is then

$$\frac{F(p)}{H(p)} e^{pt} dp + T_p(t) dp,$$

where $T_p(t)dp$ is the concomitant transient or characteristic oscillation of the system. If c is chosen sufficiently large then at least for $t \geq 0$ the transient term can be made as small as we please compared with the first term. Finally if the impressed force is the unit function (zero before, unity after, time $t = 0$) and it is written as

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{tp}}{p} dp,$$

the total response and therefore $h(t)$ is given by

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{tp}}{pH(p)} dp,$$

provided c is sufficiently large to make the transient term

$$\int_{c-i\infty}^{c+i\infty} T_p(t)dp$$

negligibly small. Analytically this requires that c be so large that the zeros of $pH(p)$ shall all lie to the left of the axis $p_R = c$.

III

The foregoing discussion tacitly assumes the existence of an unique solution of the operational equation. On the part of the physicist this assumption is entirely proper because if the operational equation is the symbolic formulation of a correctly set physical problem an unique solution must and does exist. When approached from the purely mathematical standpoint, however, the case is different and there is no assurance of the existence of a solution. As an example consider the operational equation

$$h = e^p$$

The corresponding integral equation

$$\frac{e^p}{p} = \int_0^\infty h(t)e^{-pt}dt \qquad p_R > 0$$

has no solution, while Bromwich's formula

$$h(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^p}{p} e^{tp}dp$$

gives

$$\begin{aligned} h &= 0 \qquad t < -1 \\ &= 1 \qquad t > -1 \end{aligned}$$

which is obviously incorrect. As a matter of fact the operational equation itself has no solution.

To formulate the necessary and sufficient conditions for the existence of a solution we may proceed as follows: If a solution exists it is given by either of the equations

$$h(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(p)e^{tp}dp, \tag{2}$$

$$f(p) = \int_0^\infty h(t)e^{-pt}dt \qquad p_R \geq c, \tag{3}$$

where $f(p)$ denotes $1/pH(p)$. Substitution of the value of $h(t)$, as given by (2), in (3), gives the transform

$$f(p) = \frac{1}{2\pi i}\int_0^\infty e^{-pt}dt \int_{c-i\infty}^{c+i\infty} f(z)e^{tz}dz. \tag{32}$$

In addition, since $h(t) = 0$ for $t < 0$, we must have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(p)e^{tp}dp = 0 \text{ when } t < 0. \tag{33}$$

Equations (32) and (33) formulate the necessary and sufficient restrictions on $f(p)$ for the existence of a solution of the operational equation

$$h = pf(p) = 1/H(p).$$

To correlate the transform (32) more closely with the classical Fourier transform, write $p = u + i\omega$ and

$$f(u + i\omega) = \phi(\omega) \qquad u \text{ and } \omega \text{ real.}$$

Then the transform (32) becomes

$$\phi(\omega) = \frac{1}{2\pi}\int_0^\infty e^{-i\omega t}dt \int_{-\infty}^{\infty} \phi(x)e^{itx}dx \tag{34}$$

for all values of $u \geq c$. Also since $h(t) = 0$, for $t < 0$, the lower limit of integration with respect to t in (33) may be replaced by $-\infty$, whence

$$\phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\omega t}dt \int_{-\infty}^{\infty} \phi(x)e^{itx}dx, \tag{35}$$

which is the classical Fourier transform.

The foregoing naturally suggests a few remarks regarding the mode of approach to the operational calculus. If we regard, as Heaviside certainly did, the operational equation as the symbolic formulation of a definite physical problem, it is not permissible to define the significance of the operator p *a priori*. The meaning of the operator p and methods of solution of the equation must be so determined as to give the correct solution of the original physical problem. Heaviside's procedure here was purely heuristic and "experimental"; equations (2) and (3), however, provide a sound logical basis for the development of the operational calculus. On the other hand, from the purely mathematical standpoint it is possible to develop an operational calculus on the basis of certain mutually consistent definitions and conventions adopted at the outset, just as it is possible to develop different geometries and algebras. An operational calculus so developed, however, may or may not agree with that of Heaviside and may or may not give the correct solution of the Heaviside problem. In a number of recent papers on the Heaviside operator this procedure has been adopted. To the writer this appears both illogical and doubtful, and is certainly not the method of Heaviside himself, as is sometimes implied.

In the interpretation of the operational equation $h = 1/H(p)$ it is, in the writer's opinion, extremely important to recognize the fact that it is not a true equation and has no literal significance of itself, but is simply and solely the symbolic or shorthand way of writing down equation (2) or its equivalent (3). If this fact is kept clearly in mind the 'operator' p loses the mysterious character it seems to possess for so many students and all real danger of misinterpretation and incorrect solution is eliminated. In the writer's opinion, Heaviside's achievement in the development of his operational calculus does not consist in inventing a novel and mysterious kind of mathematics, but in formulating a body of rules and processes whereby recourse to the actual equations of the problem is rendered unnecessary.

There is another fact which it is also important to clearly recognize. In the original differential equations from which the operational equation is derived, the symbol p^n denotes d^n/dt^n and its reciprocal p^{-n}, corresponding multiple integration, and the index n is always integral. If, as in the case in important electrotechnical problems, non-integral or fractional powers of the symbol p occur in the operational equation, it is due to algebraic manipulations and operations, which in essence rob p of its original significance. That is to say, in such cases it is not permissible nor indeed possible to assign to the operator p its original significance. For example the operational equation

$$h = \sqrt{p}$$

does not mean

$$h(t) = \left(\frac{d}{dt}\right)^{1/2} \cdot 1 \qquad (1 = \text{unit function})$$

which is itself meaningless, but simply

$$h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{tp}}{\sqrt{p}} dt \qquad c > 0$$

or

$$\frac{1}{\sqrt{p}} = \int_0^\infty h(t)e^{-pt}dt \qquad p_R > 0$$

More broadly stated, the operational equation is the shorthand statement of true equations in which p has lost its original significance and is simply the complex argument of functions which obey all the laws of algebra and analysis.

Failure to recognize these simple principles is responsible for a large amount of confusion, loose reasoning and profitless discussion of so called 'fractional differentiation,' a term which, to the writer at least, is quite meaningless. On the other hand, their recognition should go far towards removing whatever mystery may have surrounded the Heaviside operator and the Heaviside processes.

CORRECTION SLIP FOR ISSUE OF JANUARY, 1930

Page 153: Equation (10) should read

$$h(t) \sim \left\{ 1 + \frac{1}{1!}\left(\frac{1}{4\lambda t}\right) + \frac{1^2 \cdot 3^2}{2!}\left(\frac{1}{4\lambda t}\right)^2 + \cdots \right\} \frac{1}{\sqrt{\pi \lambda t}},$$

$$h(t) \sim S(\lambda t) \qquad (10)$$