Wave Propagation Over Continuously Loaded Fine Wires

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The paper contains the results of a theoretical investigation of wave propagation along a pair of wires that are "loaded" by enclosing each wire in a continuous sheath of magnetic material. The results of greatest practical interest are certain approximate formulas that are sufficiently simple to be adapted to engineering design studies, while having a high degree of precision for all practical dimensions and frequencies.

THE purpose of this investigation is to define the character of wave transmission along a pair of wires each of which is loaded with a continuous sheath of magnetic material. Exact expressions for the propagation constants are developed from the general theory that applies to such a system. Also, simple approximate formulas are given for the sizes of wires that are generally used in paper-insulated cables.

Wave Propagation Along a Pair of Wires with Magnetic Sheaths

For the benefit of those who are not interested in following the theoretical work in detail, a general sketch of the method and a summary of the mathematical results will be given first, together with a discussion of some numerical examples. Details of the theoretical work have been placed in the Appendices.

The analysis here given follows closely the methods developed by John R. Carson [1] in a solution of the transmission of periodic currents along a system of coaxial cylinders. The analysis for the case where the outgoing and return conductors are coaxial is applied, with only small modifications, to the case where the two conductors are parallel and not coaxial. This application of the theory ignores the "proximity effect." [2] That is to say, it assumes that the electric and magnetic forces within each conductor are functions only of the distance from its axis and of the coordinate in the direction of propagation, which is strictly true where the cylindrical conductors are coaxial.

[1] "Transmission Characteristics of the Submarine Cable," John R. Carson and J. J. Gilbert, *Journal of the Franklin Institute,* December 1921.

[2] This is the usual method of dealing with problems involving balanced parallel conductors. The alternating-current resistance of the system may be expressed as the product of the a.c. resistance, assuming a concentric return, and a "proximity effect correction factor," which takes into account the influence of the parallel return conductor. The "proximity effect" is in general negligible at voice frequencies for conductors of sufficiently small cross-section, such as those of paper insulated cables. References: "Wave Propagation over Parallel Wires: The Proximity Effect," John R. Carson, *Phil. Mag.,* April 1921, and "Wave Propagation over Parallel Tubular Conductors," Sallie Pero Mead, *Bell System Technical Journal,* April 1925.

The physical system contemplated is shown in Fig. 1. The outgoing and return systems of conductors, each comprising a cylindrical wire with insulated cylindrical sheath, are assumed to be identical in all respects. For the sake of generality, it is assumed that the magnetic sheaths may be insulated from the wires, as shown. The interesting practical case where wire and sheath are contiguous, forming a bi-metallic conductor, then appears as the limiting case of infinitesimally thin insulation.

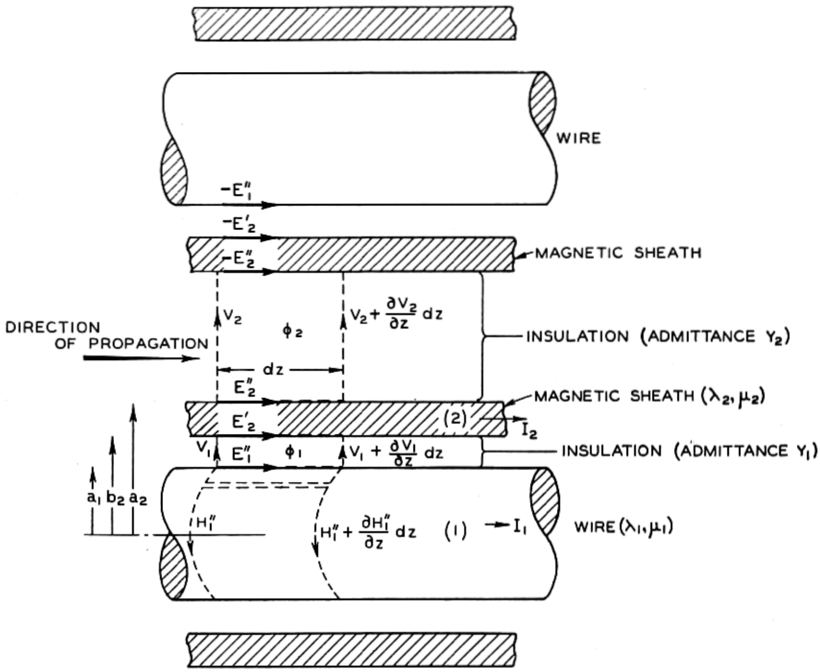

Fig. 1—Illustrating various quantities involved in the analysis.

The problem consists in finding a solution for the propagation constant of the system from Maxwell's equations. If the magnetic sheaths are in contact with the wires, the propagation constant is given in the usual form, $\Gamma = \sqrt{Y_2 Z}$, where Y_2 is the admittance across the insulation between the sheaths and Z is the series impedance of the system. The admittance is, in general, either a known, or an experimentally determined, quantity; so that for this case the theoretical problem resolves itself into that of finding the series impedance.

An important part of the investigation is, however, to determine what the effect would be of introducing insulation between the wire and its sheath. In this more general system, shown by the sketch, the solution for the propagation constant has two values, because two layers of insulation are involved, and cannot be expressed in the usual form. It is found, however, that it can be expressed in terms of the propagation constant for the elementary case where wire and sheath are in contact by introducing two other known propagation constants that determine transmission along the separate pairs of conductors in the system. The expression for the propagation constant, when given in this form, shows directly the effect of insulating the wires from their sheaths.

It is necessary first to define certain impedances. Let I_1 be the total current in one of the wires and I_2 the total current in its sheath. The tangential electric forces in the surfaces of wire and sheath are denoted by E_1'', E_2' and E_2'', as shown in Fig. 1. These electric forces are linear functions of the currents, as follows:

$$\begin{aligned} E_2'' &= Z_{21}''I_1 + Z_{22}''I_2, \\ E_2' &= Z_{21}'I_1 + Z_{22}'I_2, \\ E_1'' &= Z_{11}''I_1. \end{aligned} \qquad (1)$$

The impedances which appear in these equations as the coefficients of the currents are functions of the electrical constants and dimensions of the wires and sheaths. Their values are given in Appendix A.

Now let

γ = propagation constant determining transmission along the loaded wires if the wires and their sheaths were in contact $= \sqrt{Y_2Z}$.

γ_{12} = propagation constant determining transmission along one wire with its sheath as the return, when the sheath is insulated from the wire $= \sqrt{Y_1Z_{12}}$.

γ_{22} = propagation constant determining transmission along the two sheaths if the wires were removed $= \sqrt{Y_2Z_{22}}$.

Then, from (1)

$$\left.\begin{aligned} Z_{12} &= \frac{E_2' - E_1''}{I_2} + X_1 = Z_{11}'' - Z_{21}' + Z_{22}' + X_1, \quad I_2 = -I_1 \\ Z_{22} &= \frac{2E_2''}{I_2} + X_2 = 2Z_{22}'' + X_2 \qquad\qquad I_1 = 0 \\ Z &= \frac{2E_2''}{I_1 + I_2} + X_2 = 2\left[Z_{22}'' - \frac{Z_{22}'^2}{Z_{11}'' - Z_{21}' + Z_{22}'}\right] + X_2 \end{aligned}\right\} \qquad (2)$$

In these equations,

$X_1 = i\omega L_{12}$ = reactance arising from the magnetic field between the outer surface of the wire and the inner surface of the sheath.

$X_2 = i\omega L_{22}$ = reactance arising from the magnetic field between the two sheaths.

The terms in brackets in the equation for Z give the "internal impedance" of one of the loaded wires for the elementary case where wire and sheath are in contact, and X_2 is the additional reactance that arises from the magnetic field outside the wires.

With the elementary propagation constants, γ, γ_{12} and γ_{22} so defined, it is found that the propagation constant, Γ, of the general system can be expressed as follows:

$$2\Gamma^2 = \gamma_{12}^2 + \gamma_{22}^2 \pm \sqrt{(\gamma_{12}^2 + \gamma_{22}^2)^2 - 4\gamma^2\gamma_{12}^2}. \qquad (3)$$

It is convenient also to express the two solutions for Γ in the form of series:

$$\Gamma_1^2 = \gamma^2 \frac{\gamma_{12}^2}{\gamma_{12}^2 + \gamma_{22}^2} + \gamma^4 \frac{\gamma_{12}^4}{(\gamma_{12}^2 + \gamma_{22}^2)^3} + 2\gamma^6 \frac{\gamma_{12}^6}{(\gamma_{12}^2 + \gamma_{22}^2)^5} + \cdots,$$
$$\Gamma_2^2 = \gamma_{12}^2 + \gamma_{22}^2 - \Gamma_1^2. \qquad (4)$$

The solutions in the series form show the effect of introducing insulation between the wire and sheath. For, if $\left|\frac{4\gamma^2\gamma_{12}^2}{(\gamma_{12}^2 + \gamma_{22}^2)^2}\right|$ is small compared to unity, as it would be in a continuously loaded wire with a thin magnetic sheath of high resistance, then, to a first order of approximation, the principal propagation constant Γ_1 is less than γ, the propagation constant that determines transmission when wire and sheath are in contact, by the factor

$$\sqrt{\frac{1}{1 + \frac{\gamma_{22}^2}{\gamma_{12}^2}}}.$$

The other propagation constant, Γ_2, is, in this case, very large compared to Γ_1 and plays no appreciable part in defining the character of transmission except at points very near to the terminals of the system. For practical purposes, the system may be considered to have only one significant mode of propagation.

Case of a Wire with Contiguous Sheath

The Internal Impedance

The practical case where the magnetic sheath and the wire are contiguous, forming a bi-metallic conductor, is of special interest.

In this case, the propagation constant is uniquely determined from a knowledge of the admittance between the loaded wires and of their series impedance. The "internal impedance" of the loaded wires comprises the larger part of this impedance. For the purpose of engineering design work, it is convenient to have at hand approximate formulas for the "internal impedance."

The exact expression for the impedance is given by the last of equations (2). When the magnetic sheath is thin, as compared to the radius of the copper wire, certain approximations can be made. These are explained in Appendix A. The result is the following formula for the "internal impedance":

$$\frac{Z_i}{2} = \frac{1 - \omega^2 F + i\omega G}{\dfrac{1}{R} + i\omega H}, \tag{5}$$

where $F = \pi\mu_2 b\left[\frac{4}{3}\pi\lambda_2\lambda_1\mu_2 t^3 + \mu_1 b\left(\frac{\lambda_1}{2R_2} - \frac{\lambda_2}{R_1}\log\frac{a}{b}\right)\right]$,

$$G = \frac{\mu_2}{R_2} + \frac{\mu_1}{R_1} + \pi b^2 L_2(\lambda_1 - \lambda_2),$$

$$H = 2\pi^2\lambda_2\mu_2 a t^2\left[\frac{2}{3}t(\lambda_2 - \lambda_1) + \lambda_1 b\right] + \frac{\mu_1}{2R_1R_2},$$

$R = \dfrac{R_1R_2}{R_1 + R_2}$ = d.-c. resistance of one of the pair of bi-metallic conductors,

$R_1 = \dfrac{1}{\pi\lambda_1 b^2}$ = d.-c. resistance of the inner part of the conductor (the wire),

$R_2 = \dfrac{1}{\pi\lambda_2(a^2 - b^2)}$ = d.-c. resistance of the outer part of the conductor (the sheath),

$L_2 = 2\mu_2\log\dfrac{a}{b}$ = low-frequency inductance contributed by the sheath,

b = radius of the wire,
a = outside radius of the sheath,
$t = a - b$ = thickness of the sheath,
λ_1, μ_1 = conductivity and permeability of the wire,
λ_2, μ_2 = conductivity and permeability of the sheath,
$\omega = 2\pi$ times the frequency,
$i = \sqrt{-1}$.

The total series "loop" impedance of the pair of loaded conductors per centimeter is $Z = Z_i + X_2$.[3]

For the purpose of indicating the degree of precision of the approximate formula, data are given in Fig. 2 on the internal resistance and inductance of various copper wires coated with loading material to

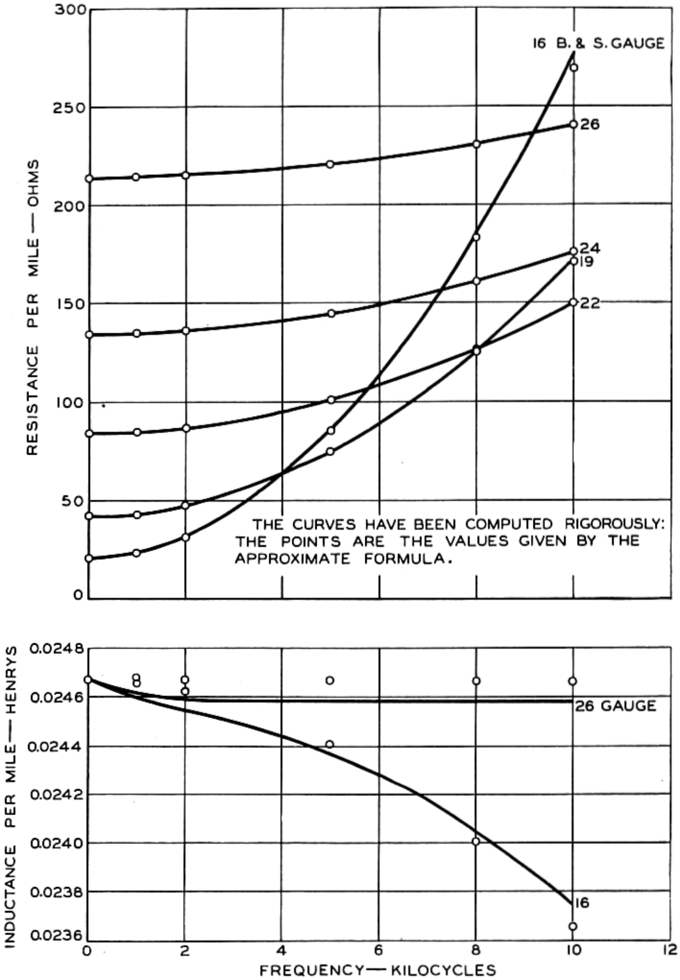

Fig. 2—Internal impedance of wires of various sizes with continuous loading of approximately 25 millihenrys per wire mile (for very small currents, i.e., hysteresis losses not included).

[3] All quantities are expressed in the electromagnetic c.g.s. system of units. To obtain the result in ohms per loop mile, multiply by 160,934 (10^{-9}). In the case of cable circuits, X_2 ($= i\omega L_{22}$) is an experimentally determined quantity, L_{22} having a value of about .001 henry per mile.

such a depth as to give an internal inductance of about .025 henry per wire mile. The magnetic material in the sheath has been assumed to have a permeability of 3,000 and a conductivity of $.77 \times 10^{-4}$ in e.m.u. (resistivity 13 microhm-centimeters, in practical units). The data shown by the solid lines are exact while the points give the results obtained by means of the approximate formula (5). A comparison of results is tabulated below for the largest wire (16 B. & S. gauge), where the errors of the approximate formula are greatest.

Frequency—Kilocycles	Internal Resistance and Inductance (of One Wire) Ohms and Millihenrys per Mile					
	Exact		Approximate		Errors	
	Res.	Ind.	Res.	Ind.	Res.	Ind.
0	21.065	24.77	21.065	24.77	—	—
2	31.674	24.56	31.63	24.63	− .14%	+.29%
5	86.795	24.37	86.18	24.41	− .71	+.16
8	186.65	24.05	183.6	24.01	−1.63	−.17
10	276.04	23.75	269.4	23.66	−2.41	−.39

The errors are roughly proportional to the quantity, $t\sqrt{\omega\mu_2\lambda_2}$. For a loading material having, say, one-quarter the permeability and the same conductivity, the errors would be about twice as large, therefore, if the inductance and the wire size remain the same.

Hysteresis Loss

The real part of the internal impedance given by (2) or (5) is the effective internal resistance of the bi-metallic wire, taking into account the heat losses that arise from the electric current, namely, d.-c. resistance, eddy current loss and "skin effect loss." The formulas do not take into account hysteresis loss, which is a magnetic phenomenon as distinguished from these electric phenomena. The determination of hysteresis loss rests upon experimental data. If the energy loss due to hysteresis in the magnetic material per unit volume per cycle is h (ergs), then the resistance increment due to hysteresis is

$$R_h = \frac{\omega}{I^2}\int_b^a hrdr. \tag{6}$$

For the low values of magnetic force that obtain in telephony, it is found that $h \doteq \eta B^3$, where η is the hysteresis coefficient and B the induction density. Therefore

$$R_h = \frac{\eta\omega\mu^3}{I^2}\int_b^a H^3rdr. \tag{7}$$

Since the magnetic coating is thin, and the "demagnetizing," or "screening," effect of eddy currents small, it may be assumed that $H = 2I/r$. (It will not exceed that value, at least.) Using this approximate value for H, the resistance increment due to hysteresis is

$$R_h \doteq 8\eta\omega\mu^3 I \frac{a-b}{ab} \doteq 2\eta\mu\omega B_a L_2, \tag{8}$$

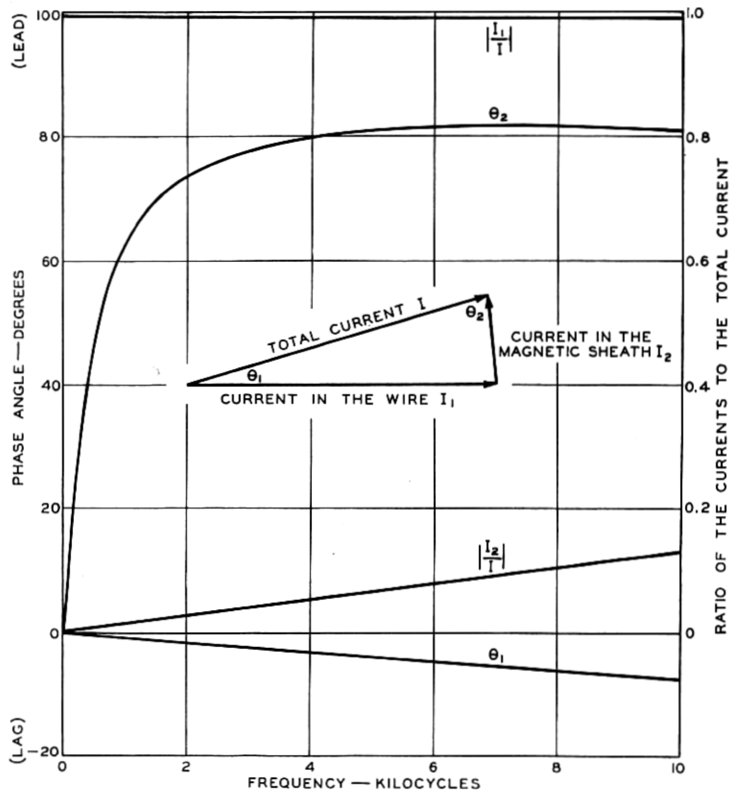

Fig. 3—Illustrating the fractions of the total current that are carried by the copper wire and by the magnetic sheath (19 gauge (B. & S.) wire with continuous loading of 25 millihenrys per mile).

where B_a is the induction density at the outside boundary of the sheath.

The Distribution of the Current in Wire and Sheath

It is a matter of interest to know how much of the current is carried by the magnetic sheath and how the current is distributed over the cross-section of the wire and sheath at various frequencies. The solution of this problem is not an essential part of the investigation, but it helps in understanding what takes place in the bi-metallic conductor.

The ratio of the currents in wire and sheath to the total current, as computed from (1), is plotted in Fig. 3. It will be noted that the fraction of the total current carried by the sheath becomes greater

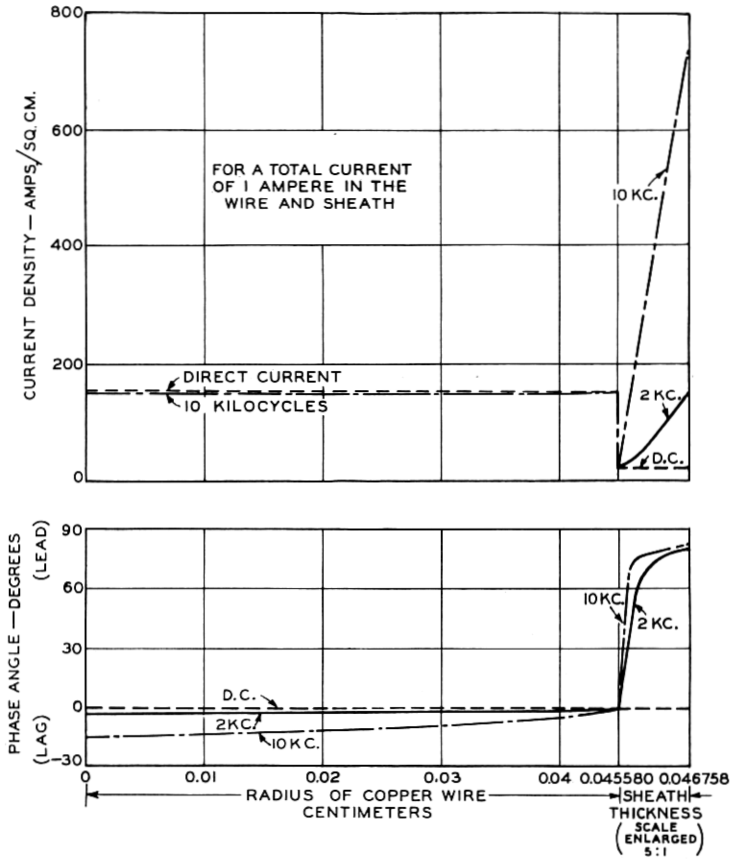

Fig. 4—Illustrating the current density throughout the cross-section of a wire loaded with a continuous magnetic sheath—for direct current and 2 and 10 kilocycle alternating currents. (Same 19 B. & S. gauge wire as that of Fig. 3.)

as the frequency increases. But the fraction carried by the copper nevertheless remains very nearly unity at all frequencies. This behavior is explained by the curves representing the phase angles involved. These show, of course, that at very low frequencies the copper current and the sheath current are nearly in phase, but with increasing frequency, the copper current lags behind the sheath current, until at high frequencies the two currents approach a quadrature phase relation.

It may be said that at high frequencies the current in the loading material is practically all "wattless" current, in the sense that it contributes very little to the energy delivered to any receiving device connected to the line, but it dissipates energy, of course. At 10 kilocycles, for the 19-gauge loaded wire, the current carried by the magnetic sheath contributes only 2 per cent of the useful current (see Fig. 3); yet 75 per cent of the energy loss occurs in the sheath (see Fig. 2).

The difference in phase between the component currents in wire and sheath is explained by the consideration that the reactance of a given filament of current is proportional to the magnetic flux external to it. In the copper, therefore, the elementary current paths have a small resistance, but a large reactance, due to the fact that nearly all the magnetic flux is in the loading material. Near the outer surface of the loading material, on the other hand, the current paths have less internal reactance, but the resistance is large.

This brings the discussion to Fig. 4, which shows how the amplitude and phase of the current varies over the cross-section of the bi-metallic conductor for direct current and for 2 and 10 kilocycle alternating currents. For the 19-gauge loaded wire, illustrated, the "skin effect" in the copper is seen to be very small, the alternating current distribution being practically uniform, as for direct current. At the boundary between the copper and the magnetic material, the current amplitude suffers a discontinuity, but the phase is continuous. The discontinuity in the current amplitude conforms to the law that the component of electric force along the conductor must be continuous at a boundary, which requires that the ratio between the current amplitudes on the two sides of the boundary must equal the ratio of the conductivities of the two materials. The current density distribution over the cross-section of the magnetic sheath is uniform for direct current, of course, but for alternating currents, the density increases and the phase advances abruptly toward the outer surface of the sheath.

APPENDIX A

The impedances [4] which appear in equation (1) in the body of the paper as the coefficients of the currents are given by:

[4] See above noted paper (reference 1) for the development of these formulas.

$$Z_{21}'' = \frac{2i\omega\mu_2}{x_2}\frac{U_2 - 1}{U_2'},$$

$$Z_{22}'' = \frac{2i\omega\mu_2}{x_2}\frac{U_2}{U_2'},$$

$$Z_{21}' = \frac{2i\omega\mu_2}{x_2}\frac{1 - \frac{a_2}{b_2}V_2'}{U_2'}, \tag{9}$$

$$Z_{22}' = \frac{2i\omega\mu_2}{x_2}\frac{1}{U_2'},$$

$$Z_{11}'' = \frac{2i\omega\mu_1}{x_1}\frac{1}{U_1'} = \frac{2i\omega\mu_1}{x_1}\frac{J_0(x_1)}{J_0'(x_1)}$$

(Note that $Z_{22}' = Z_{22}'' - Z_{21}''$),

where

$$\begin{aligned} U_j &= -\, y_j[J_0(x_j)K_0'(y_j) - J_0'(y_j)K_0(x_j)], \\ V_j &= -\, y_j[J_0(y_j)K_0(x_j) - J_0(x_j)K_0(y_j)], \\ U_j' &= -\, y_j[J_0'(x_j)K_0'(y_j) - J_0'(y_j)K_0'(x_j)], \\ V_j' &= -\, y_j[J_0(y_j)K_0'(x_j) - J_0'(x_j)K_0(y_j)]. \end{aligned} \tag{10}$$

J_0 and K_0 are Bessel functions of zero order of the first and second kind, respectively, and J_0' and K_0' are their derivatives with respect to the arguments, which are given by

$$\begin{aligned} x_j &= a_j i\sqrt{4\pi i\omega\mu_j\lambda_j}, \\ y_j &= b_j i\sqrt{4\pi i\omega\mu_j\lambda_j}, \end{aligned} \tag{11}$$

where $\omega = 2\pi$ times the frequency, $i = \sqrt{-1}$, a_j and b_j are the outer and inner radii respectively of conductor j, and μ_j and λ_j are its permeability and conductivity. Quantities with the subscript 1 refer to the wire and those with the subscript 2 refer to the sheath. All quantities are expressed in the electromagnetic c.g.s. system of units.

Writing Maxwell's Law, curl $E = -\frac{d\Phi}{dt}$, around the contours indicated by the dotted rectangles in Fig. 1 gives

$$E_2' - E_1'' - \frac{\partial V_1}{\partial z} = \frac{d\Phi_1}{dt}, \tag{12}$$

$$-\, 2E_2'' - \frac{\partial V_2}{\partial z} = \frac{d\Phi_2}{dt}, \tag{13}$$

where V_1, V_2 are the potential differences between the surfaces of the conductors, as shown, and Φ_1, Φ_2 are the normal values of the magnetic flux that cuts the surfaces bounded by the contours. The term $-2E_2''$ results from the symmetry of the system, which imposes the condition that the electric and magnetic forces at corresponding points in the outgoing and return conductors are equal and oppositely directed. Also, it is unnecessary to write a third equation for the field between the other wire and its sheath, because this equation would be the same as (12). Therefore, the transmission is characterized by only two modes of propagation.

Since all the variables are propagated at the same rate, and since sinusoidal currents are being considered, $\partial/\partial z$ may be replaced by $-\Gamma$ and d/dt by $i\omega$. Then

$$E_2' - E_1'' + \Gamma V_1 = X_1 I_1, \tag{14}$$

$$-2E_2'' + \Gamma V_2 = X_2(I_1 + I_2), \tag{15}$$

where Γ is the propagation constant and

$X_1 = i\omega L_{12} =$ reactance arising from the magnetic field between the outer surface of the wire and the inner surface of the sheath.

$X_2 = i\omega L_{22} =$ reactance arising from the magnetic field between the two sheaths.

The potential differences V_1, V_2 can be expressed in terms of the currents by writing Maxwell's Law, curl $H = 4\pi I$, around contours in the outside surfaces of wire and sheath. (Such a contour for the wire is indicated by dotted lines in the sketch.) This gives

$$2\pi a_1 \frac{\partial H_1''}{\partial z} = -4\pi V_1 Y_1, \tag{16}$$

$$2\pi a_2 \frac{\partial H_2''}{\partial z} = -4\pi V_2 Y_2. \tag{17}$$

where

$Y_1 =$ admittance across the insulation between wire and sheath.

$Y_2 =$ admittance across the insulation between the two sheaths.

Since $H_1'' = \dfrac{2I_1}{a_1}$ and $H_2'' = \dfrac{2(I_1 + I_2)}{a_2}$,

$$\Gamma I_1 = V_1 Y_1, \tag{18}$$

$$\Gamma(I_1 + I_2) = V_2 Y_2. \tag{19}$$

Substituting (18) and (19) in (14) and (15), respectively, gives

$$\left(X_1 - \frac{\Gamma^2}{Y_1}\right) I_1 = E_2' - E_1'', \tag{20}$$

$$\left(X_2 - \frac{\Gamma^2}{Y_2}\right) (I_1 + I_2) = -2E_2'', \tag{21}$$

and substituting (1) in (20) and (21) gives the two equations of the currents. In order that they shall be consistent, the determinant of the coefficients must vanish. Therefore

$$\begin{vmatrix} X_1 - \frac{\Gamma^2}{Y_1} - Z_{21}' + Z_{11}'' & -Z_{22}' \\ X_2 - \frac{\Gamma^2}{Y_2} + 2Z_{21}'' & X_2 - \frac{\Gamma^2}{Y_2} + 2Z_{22}'' \end{vmatrix} = 0. \tag{22}$$

The roots of this equation give the required solutions for the propagation constant. First, however, it is convenient to introduce two known propagation constants. Let

γ_{12} = propagation constant determining transmission along one wire with its sheath as the return $= \sqrt{Y_1 Z_{12}}$.

γ_{22} = propagation constant determining transmission along the two sheaths if the wires were removed $= \sqrt{Y_2 Z_{22}}$.

Then, from (1), (20) and (21),

$$\begin{aligned} Z_{12} &= Z_{11}'' - Z_{21}' + Z_{22}' + X_1, \\ Z_{22} &= 2Z_{22}'' + X_2, \end{aligned} \tag{23}$$

substituting (23) in (22) and rearranging,

$$\begin{vmatrix} \frac{\gamma_{12}^2 - \Gamma^2}{Y_1} & -Z_{22}' \\ -2Z_{22}' & \frac{\gamma_{22}^2 - \Gamma^2}{Y_2} \end{vmatrix} = 0. \tag{24}$$

Expanding

$$\Gamma^4 - \Gamma^2(\gamma_{22}^2 + \gamma_{12}^2) + \gamma_{12}^2\gamma_{22}^2 - 2Z_{22}'^2 Y_1 Y_2 = 0. \tag{25}$$

The remaining impedance can be eliminated by introducing γ, the propagation constant that would characterize transmission if the wires were in contact with the sheaths. (In order not to disturb the dimensions, it may be imagined that the insulation between wire and sheath be replaced by an infinitely conducting material, which, however, is assumed to conduct no current axially. Then $E_2' - E_1'' = X_1 I_1$.) To find γ, make Y_1 infinite and solve (25). Then

$$\gamma^2 = Y_2 Z,$$

and

$$Z = Z_{22} - \frac{2Z_{22}'^2}{Z_{12}}. \tag{26}$$

Therefore

$$2Z_{22}'^2 Y_1 Y_2 = \gamma_{22}^2 \gamma_{12}^2 - \gamma^2 \gamma_{12}^2. \tag{27}$$

Finally, substituting (27) in (25) and solving the resulting equation gives the two solutions for the propagation constant,

$$2\Gamma^2 = \gamma_{12}^2 + \gamma_{22}^2 \pm \sqrt{(\gamma_{12}^2 + \gamma_{22}^2)^2 - 4\gamma^2\gamma_{12}^2}. \tag{28}$$

The arbitrary constants remain to be determined. The currents are, in general,

$$\begin{aligned} I_1 &= A_{11}\epsilon^{-\Gamma_1 z} + A_{12}\epsilon^{-\Gamma_2 z} + B_{11}\epsilon^{\Gamma_1 z} + B_{12}\epsilon^{\Gamma_2 z}, \\ I_2 &= A_{21}\epsilon^{-\Gamma_1 z} + A_{22}\epsilon^{-\Gamma_2 z} + B_{21}\epsilon^{\Gamma_1 z} + B_{22}\epsilon^{\Gamma_2 z}. \end{aligned} \tag{29}$$

The condition of principal practical interest is that of a long cable with connection made to the two wires and with the sheaths left free at the sending end. For this case, the conditions are

(1) At $z = 0$, $I_1 = I_0$ and $I_2 = 0$,

(2) At $z = \infty$, $I_1 = 0$ and $I_2 = 0$,

where I_0 is the current delivered to the cable pair at the sending end. From the second condition,

$$B_{11} = B_{12} = B_{21} = B_{22} = 0. \tag{30}$$

From the first condition,

$$\begin{aligned} A_{11} + A_{12} &= I_0, \\ A_{21} + A_{22} &= 0. \end{aligned} \tag{31}$$

But these constants must satisfy, for each of the two values of Γ, the equations of the currents, whose coefficients are given in the determinant (22). Therefore

$$A_{21} = K_1 A_{11},$$
$$A_{22} = K_2 A_{12}, \tag{32}$$

where

$$K_1 = \frac{Z_{12} - Z_{22}' - \dfrac{\Gamma_1^2}{Y_1}}{Z_{22}'} = \frac{Z_{22} - 2Z_{22}' - \dfrac{\Gamma_1^2}{Y_2}}{-Z_{22} + \dfrac{\Gamma_1^2}{Y_2}},$$

$$K_2 = \frac{Z_{12} - Z_{22}' - \dfrac{\Gamma_2^2}{Y_1}}{Z_{22}'} = \frac{Z_{22} - 2Z_{22}' - \dfrac{\Gamma_2^2}{Y_2}}{-Z_{22} + \dfrac{\Gamma_2^2}{Y_2}}. \tag{33}$$

Substituting (32) in (31) and solving

$$A_{11} = I_0 \frac{K_2}{K_2 - K_1}, \qquad A_{12} = -I_0 \frac{K_1}{K_2 - K_1},$$
$$A_{21} = I_0 \frac{K_1 K_2}{K_2 - K_1} = -A_{22}. \tag{34}$$

Finally, the currents are given by

$$I_1 = \frac{I_0}{K_2 - K_1}\left[K_2 \epsilon^{-\Gamma_1 z} - K_1 \epsilon^{-\Gamma_2 z}\right],$$
$$I_2 = \frac{I_0 K_1 K_2}{K_2 - K_1}\left[\epsilon^{-\Gamma_1 z} - \epsilon^{-\Gamma_2 z}\right]. \tag{35}$$

This completes the analysis for the more general system where the magnetic sheaths are insulated from the wires. For the special case where wire and sheath are contiguous, γ_{12}^2 is infinite and (28) shows that $\Gamma_1 = \gamma$ and $\Gamma_2 = \infty$. The transmission is, therefore, defined by only one mode of propagation. The series impedance of the system is, from (23) and (26),

$$Z = 2\left[Z_{22}'' - \frac{Z_{22}'^2}{Z_{11}'' - Z_{21}' + Z_{22}'}\right] + X_2, \tag{36}$$

where the terms in brackets give the internal impedance of one of the loaded wires, and X_2 is the reactance that arises from the magnetic field between them. The internal impedance can be obtained also by finding $\dfrac{2E_2''}{I_1 + I_2}$ directly from the last two of equations (1), of course.

The constant K_2 becomes -1 and the total current, $I = I_1 + I_2$, is propagated in accordance with $I = I_0\epsilon^{-\gamma z}$, where $\gamma = \sqrt{ZY_2}$.

The constant K_1, which is the ratio of the current in the sheath to that in the wire, is of interest. It becomes

$$\frac{I_2}{I_1} = K_1 = \frac{Z_{12} - Z_{22}'}{Z_{22}'} = \frac{Z_{11}'' - Z_{21}'}{Z_{22}'}. \tag{37}$$

The approximate formulas for the case where wire and sheath are contiguous are derived as follows: The arguments, x_2 and y_2, of the Bessel functions differ by only a small amount when the magnetic sheath is thin. This situation is favorable to an advantageous use of Taylor's series. $J_0(x_2)$, for example, can be expressed in terms of $J_0(y_2)$, its derivatives and the difference of the arguments in a Taylor series as follows:

$$J_0(x) = J_0(y) + \tau J_0'(y) + \frac{\tau^2}{2!}J_0''(y) + \frac{\tau^3}{6!}J_0'''(y) + \cdots, \tag{38}$$

where $\tau = x - y$ (x_2, y_2 being written simply, x, y, here, for convenience). Furthermore, Bessel functions are subject to recurrence formulas,[5] which enable us to express each of the derivatives occurring in the series in terms of the function of zero order, its first derivative and the argument. Therefore, by applying the recurrence formulas to the Taylor series, we find functions U and V (see Appendix B) such that

$$J_0(x) = UJ_0(y) + VJ_0'(y), \tag{39}$$

$$K_0(x) = UK_0(y) + VK_0'(y) \tag{40}$$

(U_2, V_2 being also written now, U, V). Differentiating (39) and (40) with respect to τ,

$$J_0'(x) = U'J_0(y) + V'J_0'(y), \tag{41}$$

$$K_0'(x) = U'K_0(y) + V'K_0'(y), \tag{42}$$

where $U' = \dfrac{\partial U}{\partial \tau}$, $V' = \dfrac{\partial V}{\partial \tau}$.

[5] The two recurrence formulas required are:

$$J_n'(z) = \frac{n}{z}J(z) - J_{n+1}(z),$$

$$J_n'(z) = J_{n-1}(z) - \frac{n}{z}J_n(z).$$

The Bessel Functions of the second kind satisfy the same formulas.

If (39) to (42) be solved for U, V, U', V', it can be verified that the solutions are the definitions of these functions already given in equations (10).[6]

The exact formula for the internal impedance of a wire with contiguous sheath has been given in (36). In terms of the functions U and V, this formula becomes

$$\frac{Z_i}{2} = \frac{2i\omega\mu_2}{x_2} \cdot \frac{\sqrt{\dfrac{\lambda_2\mu_1}{\lambda_1\mu_2}} \dfrac{U_2}{U_1'} + V_2}{\sqrt{\dfrac{\lambda_2\mu_1}{\lambda_1\mu_2}} \dfrac{U_2'}{U_1'} + V_2'}. \tag{43}$$

By using the series for these functions and discarding all terms of degree higher than ω^2, the approximation given in the body of the paper (equation 5) may be obtained.

APPENDIX *B*

When the recurrence formulas are applied to the Taylor series, it is found that

$$U = 1 + \frac{\tau^2}{2} + \frac{\tau^3}{6y} + \frac{\tau^4}{24}\left(1 - \frac{3}{y^2}\right) - \frac{\tau^5}{120}\left(\frac{2}{y} - \frac{12}{y^3}\right) - \frac{\tau^6}{720}\left(1 + \frac{10}{y^2} + \frac{60}{y^4}\right) + \cdots, \tag{44}$$

$$V = \tau - \frac{\tau^2}{2y} - \frac{\tau^3}{6}\left(1 - \frac{2}{y^2}\right) + \frac{\tau^4}{24}\left(\frac{2}{y} - \frac{6}{y^3}\right) + \frac{\tau^5}{120}\left(1 - \frac{7}{y^2} + \frac{24}{y^4}\right) - \frac{\tau^6}{720}\left(\frac{3}{y} - \frac{33}{y^3} + \frac{120}{y^5}\right) + \cdots. \tag{45}$$

These series converge for $\left|\frac{\tau}{y}\right| < 1$, which condition is satisfied by the sheath dimensions of any practical continuously loaded conductor.

A considerable number of the terms in the series for U and V are

[6] A relation that can be used to advantage at times is

$$U'V - UV' = -\frac{y}{x} = -\frac{b}{a}.$$

This relation corresponds to the similar one for the Bessel functions themselves, namely:

$$J_n'(z)K_n(z) - J_n(z)K_n'(z) = \frac{1}{z}.$$

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parts of well-known series that define certain elementary functions. It can be verified readily that

$$U = \cos\tau + \frac{y^2}{2}\left[\log\left(1+\frac{\tau}{y}\right) - \frac{\tau}{y} + \frac{\tau^2}{2y^2}\right] - \frac{y^4}{12}\left[\log\left(1+\frac{\tau}{y}\right) - \frac{\tau}{y} + \frac{\tau^2}{2y^2} - \frac{\tau^3}{3y^3} + \frac{\tau^4}{4y^4}\right] + \cdots, \quad (46)$$

$$V = \sin\tau + y\left[\log\left(1+\frac{\tau}{y}\right) - \frac{\tau}{y}\right] + \frac{\tau^4}{12y} - \frac{7\tau^5}{120y^2} - \frac{\tau^6}{240y} + \frac{11\tau^6}{240y^3} + \cdots, \quad (47)$$

$$U' = -\sin\tau + \frac{y}{2}\frac{\left(\frac{\tau}{y}\right)^2}{1+\frac{\tau}{y}} + \frac{\partial}{\partial\tau}\text{(above remainder of (3))}, \quad (48)$$

$$V' = \cos\tau - \frac{\frac{\tau}{y}}{1+\frac{\tau}{y}} + \frac{\partial}{\partial\tau}\text{(above remainder of (4))}. \quad (49)$$

The series (46) to (49) possess a certain advantage for computing in that the quantities in brackets are real numbers. $\left(\text{Note that } \frac{\tau}{y} = \frac{a_2 - b_2}{b_2}.\right)$ They have been used also in obtaining the approximate formulas given in the body of the paper.

The quantities discussed above all pertain to the sheath. For finding U_1', involved in the last of the formulas (9), the series are not valid, of course. For this we have the well-known series,

$$\frac{1}{U_1'} = \frac{J_0(x_1)}{J_0'(x_1)} = \frac{1}{x_1}\left[-2 + \frac{x_1^2}{4} + \frac{x_1^4}{96} + \frac{x_1^6}{1536} + \frac{x_1^8}{23040} + \cdots\right] \quad (50)$$

—see, e.g., Gray, Mathews and McRoberts, "Bessel Functions," 2d edition, p. 170.