The Reciprocal Energy Theorem

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This paper gives a simple theorem determining relative transmission efficiencies in a two-way transducer, and showing that the conditions for equal efficiencies of transmission in the two directions are simply those for maximum output and maximum reception of energy. The theorem is then applied to radio communication and a second theorem stated and proved by which the ratio of the transmitting efficiences of any two antenna systems is expressed in terms of their receiving efficiences. The paper closes with a mathematical note on a generalization of Rayleigh's Reciprocal Theorem.

THE Reciprocal Theorem, originally enunciated by Rayleigh, which has proved so useful to communication engineers, may be stated, with sufficient generality for engineering purposes, as follows:

Let an e.m.f. E_1', *inserted in any branch, designated as No. 1, of a transducer,*[1] *produce a current* I_2' *in any other branch No. 2; correspondingly let an e.m.f.* E_2'' *inserted in branch No. 2 produce a current* I_1'' *in branch No. 1; then*

$$I_1''E_1' = I_2'E_2''$$

and when $E_1' = E_2''$ the currents in the two branches are equal.

The engineer, however, is primarily interested in energy rather than current relations, whereas the theorem says nothing explicitly regarding energy relations and relative efficiencies in two-way transmission. It is, however, a simple matter to deduce from it quite general and useful formulas relating to relative transmission efficiencies. In the present paper there will be formulated and proved a reciprocal energy theorem for the general transducer, after which it will be applied to the question of antenna transmission efficiency in radio communication.

Consider a transducer having two sets of accessible terminals 1,1 and 2,2. With terminals 2,2 closed by an impedance $z_2 = r_2 + ix_2$, let the driving point impedance, as measured from terminals 1,1 be denoted by $Z_{11} = R_{11} + iX_{11}$; similarly with terminals 1,1 closed by an impedance $z_1 = r_1 + ix_1$, let the driving point impedance, as measured from terminals 2,2 be denoted by $Z_{22} = R_{22} + iX_{22}$. Now with the terminals closed by the impedances z_1 and z_2, let an e.m.f. E_1 be inserted in series with the terminal impedance z_1; then the current I_{11}, delivered to the transducer at the sending terminals 1,1 is

[1] A transducer is defined as a complete transmission system which may or may not include a radio link, which has two accessible branches, either of which may act as the transmitting branch while the other acts as the receiving branch. These branches may be designated as operating branches.

$$I_{11} = \frac{E_1}{Z_{11} + z_1} \tag{1}$$

and the current I_{12}, received by the terminal or load impedance, z_2, is given by

$$I_{12} = \frac{E_1}{Z_{12}}. \tag{2}$$

Here Z_{12} is the transfer impedance of the transducer for the specified terminations.

The power P_{11}^0 developed by the generator of e.m.f. E_1 is

$$P_{11}^0 = (R_{11} + r_1) \mid I_{11} \mid^2 = \frac{R_{11} + r_1}{\mid Z_{11} + z_1 \mid^2} E_1^2. \tag{3}$$

The power P_{11} delivered to the transducer is

$$P_{11} = R_{11} \mid I_{11} \mid^2 = \frac{R_{11}}{\mid Z_{11} + z_1 \mid^2} E_1^2 \tag{4}$$

and the power P_{12} delivered to the load impedance z_2 is

$$P_{12} = r_2 \mid I_{12} \mid^2 = \frac{r_2}{\mid Z_{12} \mid^2} E_1^2. \tag{5}$$

Now reverse the direction of transmission; that is insert an e.m.f. E_2 in series with the terminal impedance z_2; corresponding to equations (3)–(5) we have then

$$P_{22}^0 = \frac{R_{22} + r_2}{\mid Z_{22} + z_2 \mid^2} E_2^2, \tag{6}$$

$$P_{22} = \frac{R_{22}}{\mid Z_{22} + z_2 \mid^2} E_2^2, \tag{7}$$

$$P_{21} = \frac{r_1}{\mid Z_{21} \mid^2} E_2^2. \tag{8}$$

As a consequence of the Reciprocal Theorem the transfer impedances are equal; that is

$$Z_{21} = Z_{12}. \tag{9}$$

From the preceding we get at once the following expressions for the ratios of the powers delivered to the load impedances;

$$\frac{P_{12}}{P_{21}} = \frac{r_2}{r_1} \left(\frac{E_1}{E_2} \right)^2$$

$$= \left(\frac{r_2}{r_1}\right)\left(\frac{R_{22}+r_2}{R_{11}+r_1}\right)\left|\frac{Z_{11}+z_1}{Z_{22}+z_2}\right|^2 \frac{P_{11}^0}{P_{22}^0} \tag{10}$$

$$= \left(\frac{r_2}{r_1}\right)\left(\frac{R_{22}}{R_{11}}\right)\left|\frac{Z_{11}+z_1}{Z_{22}+z_2}\right|^2 \frac{P_{11}}{P_{22}}. \tag{11}$$

From (10) it follows that for equal total generated powers, the relative transmission efficiency in the two directions is given by

$$\eta^0 = \frac{P_{12}}{P_{21}} = \left(\frac{r_2}{r_1}\right)\left(\frac{R_{22}+r_2}{R_{11}+r_1}\right)\left|\frac{Z_{11}+z_1}{Z_{22}+z_2}\right|^2, \tag{12}$$

while on the basis of equal powers delivered to the transducer, the relative transmission efficiency is, by (11)

$$\eta = \frac{P_{12}}{P_{21}} = \left(\frac{r_2}{r_1}\right)\left(\frac{R_{22}}{R_{11}}\right)\left|\frac{Z_{11}+z_1}{Z_{22}+z_2}\right|^2. \tag{13}$$

Now in correctly designed communication transmission systems, the terminal impedances are so proportioned with reference to the characteristics of the transducer itself as to secure maximum output and maximum transfer of power from generator to load; the required condition is that the terminal impedances z_1 and z_2 be the 'conjugate image impedances' of the transducer; analytically stated

$$z_1 = R_{11} - iX_{11} \qquad \text{and} \qquad z_2 = R_{22} - iX_{22}.$$

Introducing these relations into (12) and (13), we have

$$\eta^0 = \eta = 1 \tag{14}$$

and the relative transmission efficiencies are the same in the two directions. We thus have the following propositions:—

If a transducer is terminated in its conjugate image impedances—the condition for maximum output and maximum transfer of power—the efficiency of transmission is the same in the two directions.

We shall now apply the preceding to the derivation of a simple formula which enables us to determine the relative transmission efficiencies of any two long wave radio antennas.[2]

Consider any antenna, designated as No. 1, and let it be acting as

[2] As pointed out in the paper on "Reciprocal Theorems in Radio Transmission," *Proc. I. R. E.*, the Reciprocal Theorem does not hold rigorously in radio transmission if the earth's magnetic field plays an appreciable part in the transmission phenomena. Consequently the formula and proposition which follow apply rigorously only to long wave transmission; they are probably, however, approximately correct for short wave transmission except in the neighborhood of the critical wave-length 214 meters. See a paper by Nichols and Shelling, "Propagation of Electric Waves over the Earth," *B. S. T. J.*, April 1925.

a transmitter to a reference antenna, designated as No. 3, which is located at any desired point 3. Let E_{13} denote the intensity of the (vertical) electric field produced at point 3 by antenna No. 1. Then the current induced in the receiving branch of No. 3 will be $\alpha_3 E_{13}$, the parameter α_3 being the receiving sensitivity of antenna No. 3. The power P_{13} transferred from 1 to 3 is then

$$P_{13} = r_3 \alpha_3^2 E_{13}^2,$$

where r_3 is the equivalent resistance of the receiving branch of antenna No. 3.

Now reverse the direction of transmission; we have

$$P_{31} = r_1 \alpha_1^2 E_{31}^2.$$

We now suppose that the terminal impedances are adjusted for maximum output and maximum transfer of power and that the power P_{11} developed by No. 1 when transmitting is equal to the power P_{33} developed by No. 3 when transmitting. Then it follows at once from the reciprocal energy theorem, that $P_{13} = P_{31}$, and

$$\left(\frac{E_{13}}{E_{31}}\right)^2 = \frac{r_1 \alpha_1^2}{r_3 \alpha_3^2}.$$

Now replace antenna No. 1 by any other antenna, designated as No. 2; we then have from the foregoing

$$\left(\frac{E_{23}}{E_{32}}\right)^2 = \frac{r_2 \alpha_2^2}{r_3 \alpha_3^2}.$$

By virtue of the terminal impedances specified, $r_1 = R_1$ and $r_2 = R_2$ where R_1 and R_2 are the resistances of the two antennas as measured from their operating terminals. Consequently, since $E_{32} = E_{31}$, we have

$$\left(\frac{E_{13}}{E_{23}}\right)^2 = \eta_{12} = \frac{R_1 \alpha_1^2}{R_2 \alpha_2^2} = \frac{R_1 h_1^2}{R_2 h_2^2},$$

where h_1 and h_2 are the equivalent heights of the two antennas.

The ratio η_{12} will be termed the 'relative transmission figure of merit' of the two antennas No. 1 and No. 2 with respect to transmission between any two specified points. For directional antennas, the parameters α_1 and α_2 will depend on the direction of transmission; that is, the location of the receiving with respect to the transmitting point.

The foregoing may be summed up in the following proposition.

The relative transmission figure of merit of any two antennas with respect to transmission from a given transmitting point to a given receiving point is equal to the ratio of their resistances as measured from their operating branches, multiplied by the square of the ratio of their receiving sensitivities with respect to transmission from the receiving point to the transmitting point.

This theorem has a considerable field of practical utility. For example it enables us to deduce the relative transmitting properties and efficiency of any antenna system from its receiving efficiency. It has already been so applied in one actual case of large importance.

Note on the Reciprocal Theorem

The proof of the Reciprocal Theorem, as given originally by Lord Rayleigh, was applicable only to 'quasi-stationary' transducers, that is transducers which obey the simple laws of electric circuit theory. In the July 1924 issue of the *Bell System Technical Journal* the writer stated and proved a generalized theorem subject, however, to the restriction that the permeability μ of the medium shall be everywhere unity. The theorem referred to is

Let a distribution of impressed periodic electric intensity $\boldsymbol{F}' = \boldsymbol{F}'(x, y, z)$ *produce a corresponding distribution of current intensity* $\mathbf{u}' = \mathbf{u}'(x, y, z)$, *and let a second distribution of equi-periodic impressed electric intensity* $\boldsymbol{F}'' = \boldsymbol{F}''(x, y, z)$ *produce a second distribution of current intensity* $\boldsymbol{u}'' = \boldsymbol{u}''(x, y, z)$, *then*

$$\int (\boldsymbol{F}' \cdot \boldsymbol{u}'')dv = \int (\boldsymbol{F}'' \cdot \boldsymbol{u}')dv,$$

the volume integration being extended over all conducting and dielectric media. $\boldsymbol{F}$ *and* $\boldsymbol{u}$ *are vectors and the expression* $(\boldsymbol{F} \cdot \boldsymbol{u})$ *denotes the scalar product of the two vectors.*

Later Pleijel [3] stated the theorem for unrestricted values of μ. In discussing reciprocal theorems in the June 1929 issue of the *Proc. I. R. E.* the writer expressed some doubt as to the validity of Pleijel's proof (which is entirely different from my own). Subsequent study, however, has convinced me that except for minor and easily remedied errors, the proof is entirely sound. Later the writer discovered that the restriction $\mu = 1$ can easily be removed from his own original proof as will now be shown.[4]

[3] "Two Reciprocal Theorems in Electricity," Ingeniors Vetenskaps Akademien Nr. 68, 1927.

[4] Another and somewhat different extension of the proof has been derived by my associate Dr. W. H. Wise.

If $\mu \neq 1$ everywhere and if we write

$$\boldsymbol{w} = \boldsymbol{u} + \text{curl}\ \boldsymbol{M} = \lambda \boldsymbol{E} + \text{curl}\ \boldsymbol{M} \tag{1'}$$

equation (8) of my paper becomes [5]

$$\frac{1}{\lambda}\boldsymbol{w} + \frac{i\omega}{c}\int \frac{\boldsymbol{w}}{r}\exp\left(-\frac{i\omega r}{c}\right)dv = \boldsymbol{G} + \frac{1}{\lambda}\text{curl}\ \boldsymbol{M} \tag{2'}$$

and correspondingly equation (9) becomes

$$\int \{\boldsymbol{w}'\cdot\boldsymbol{G}'') - (\boldsymbol{w}''\cdot\boldsymbol{G}')\}dv + \int \frac{1}{\lambda}\{\boldsymbol{w}'\cdot\text{curl}\ \boldsymbol{M}'') - (\boldsymbol{w}''\cdot\text{curl}\ \boldsymbol{M}')\}dv = 0. \tag{3'}$$

If now in (3′) we replace $\boldsymbol{w}$ by $\boldsymbol{u} + \text{curl}\ \boldsymbol{M}$ and note that $\boldsymbol{u}/\lambda = \boldsymbol{E}$, (3′) reduces to

$$\int \{(\boldsymbol{u}'\cdot\boldsymbol{G}'') - (\boldsymbol{u}''\cdot\boldsymbol{G}')\}dv - \int\{(\boldsymbol{G}'\cdot\text{curl}\ \boldsymbol{M}'') - (\boldsymbol{G}''\cdot\text{curl}\ \boldsymbol{M}')\}dv + \int\{\boldsymbol{E}'\cdot\text{curl}\ \boldsymbol{M}'') - (\boldsymbol{E}''\cdot\text{curl}\ \boldsymbol{M}')\}dv = 0. \tag{4'}$$

Finally since $\boldsymbol{E} - \boldsymbol{G} = -\frac{i\omega}{c}\boldsymbol{A}$, (4′) reduces to

$$\int \{(\boldsymbol{u}'\cdot\boldsymbol{G}'') - (\boldsymbol{u}''\cdot\boldsymbol{G}')\}dv - \frac{i\omega}{c}\int\{(\boldsymbol{A}'\cdot\text{curl}\ \boldsymbol{M}'') - (\boldsymbol{A}''\cdot\text{curl}\ \boldsymbol{M}')\}dv = 0. \tag{5'}$$

But

$$\begin{aligned}\int(\boldsymbol{A}'\cdot\text{curl}\ \boldsymbol{M}'')dv &= \int(\boldsymbol{M}''\cdot\text{curl}\ \boldsymbol{A}')dv \\ &= \frac{1}{4\pi}\int\frac{\mu - 1}{\mu}(\boldsymbol{B}''\cdot\text{curl}\ \boldsymbol{A}')dv \\ &= \frac{1}{4\pi}\int\frac{\mu - 1}{\mu}(\text{curl}\ \boldsymbol{A}''\cdot\text{curl}\ \boldsymbol{A}')dv,\end{aligned}$$

so that the second integral of (5′) vanishes and

$$\int\{(\boldsymbol{u}'\cdot\boldsymbol{G}'') - (\boldsymbol{u}''\cdot\boldsymbol{G}')\}dv = 0, \tag{6'}$$

which is equation (9) of the original paper. The rest of the proof of the theorem is now simply that of the original paper.

It will be observed the theorem is stated for the current $\boldsymbol{u} = \lambda\boldsymbol{E}$; that is the conduction (plus polarization) current. Ballantine [6] in

[5] The paper itself must be consulted for the significance of the symbols and the method of attack and proof.

[6] June 1929 issue of *Proc. I. R. E.*

discussing this subject states that the theorem holds for the current $\boldsymbol{w} = \lambda \boldsymbol{E} + \text{curl } \boldsymbol{M}$. This cannot be true in general, however, because from the foregoing in order that the theorem should hold for the current $\boldsymbol{w}$, it is clearly necessary that

$$\int \{(\boldsymbol{F}' \cdot \text{curl } \boldsymbol{M}'') - (\boldsymbol{F}'' \cdot \text{curl } \boldsymbol{M}')\} dv = 0.$$

This is only true in the exceptional cases where the impressed force is derivable from a potential; that is, curl $\boldsymbol{F} = 0$, or else $\boldsymbol{F} = 0$ where $\boldsymbol{M} \neq 0$.