Transients in Parallel Grounded Circuits, One of Which is of Infinite Length

By LISS C. PETERSON

This paper deals with a mathematical discussion of induction due to transient currents of the forms $I = \sin \omega t$ and $I = e^{-\beta t}$. Formulas and curves are developed for the calculation of the induced voltage in exposed telephone lines due to currents of the above types.

Part I

THE problem of mutual impedance between grounded circuits of infinite length for steady state sinusoidal currents has been treated by a number of authors, and the solution of this problem is now well established.[1, 2, 3] In addition to the steady state voltages induced the transient voltages are also of importance. Rüdenberg [4] and Ollendorf [5] have given approximate solutions for transient voltages due either to d.-c. switching or the sudden flow of a sinusoidal current on the assumption of circular symmetry and for circuits one of which is of infinite length. Since the assumption of circular symmetry holds only for a limited set of conditions it is desirable to develop formulas for the transient induced voltages based on the exact solution for steady state conditions referred to above.

The discussion in this paper will be limited to the case of parallel wires, one of which is of infinite length, and both located on the surface of the earth but insulated from it except at their ends. Disturbing currents of the forms $I = \sin \omega t$ and $I = e^{-\beta t}$ will be assumed.

A more general case with both wires above the earth's surface leads to complicated expressions for the induced voltage not well adapted for engineering use. The restriction to wires on the earth's surface results in appreciable simplification and does not introduce a serious departure from actual conditions.

With these assumptions, the following formulas holding for small and large values of time, determine the induced voltage per unit length on a secondary wire 2 due to the sudden flow of a current $I(t) = \sin \omega t$ in a primary wire 1 infinite in length, separated from wire 2 by a distance x centimeters.

[1] Pollaczek, F., *E. N. T.*, Vol. 3, 1926.
[2] Carson, J. R., *Bell System Technical Journal*, Vol. 5, 1926.
[3] Haberland G., *Z. ang. Math. U. Mech.*, Vol. 6, No. 5, 1926.
[4] Wiss. Veroff. a. d. Siemens-Konzern, Vol. 5, No. 3, 1927.
[5] E. N. T., Vol. 5, No. 3, 1928.

$$V_{12}(t) = \frac{\sin \omega t}{\pi \lambda x^2} - \frac{\omega}{\pi \lambda x^2} e^{-\pi\lambda x^2/t} \left[\frac{t^2}{\pi\lambda x^2} - \frac{2t^3}{(\pi\lambda x^2)^2} + \frac{6t^4 - \omega^2 t^6}{(\pi\lambda x^2)^3} - \frac{24t^5 - 12\omega^2 t^7}{(\pi\lambda x^2)^4} + \cdots \right]$$

and

$$V_{12}(t) = \frac{\sin \omega t}{\pi \lambda x^2} + \frac{2\sqrt{\omega}}{x\sqrt{\pi\lambda}} [\cos \omega t \ \mathrm{kei}'(2x\sqrt{\pi\lambda\omega}) + \sin \omega t \ \mathrm{ker}'(2x\sqrt{\pi\lambda\omega})] - e^{-\pi\lambda x^2/t} \left[\frac{1}{\omega t^2} - \frac{1}{\omega^2} \left(\frac{6}{t^4} - \frac{6\pi\lambda x^2}{t^5} + \frac{(\pi\lambda x^2)^2}{t^6} \right) + \cdots \right]$$

and for such values of time where neither of these series expansions would give very accurate results the following formulas may be used

$$V_{12}(t) = \frac{\sin \omega t}{\pi\lambda x^2} - \frac{\sqrt{A^2 + B^2}}{\pi\lambda x^2} \cos (\omega t - \varphi),$$

$$\tan \varphi = \frac{B}{A} \cdot$$

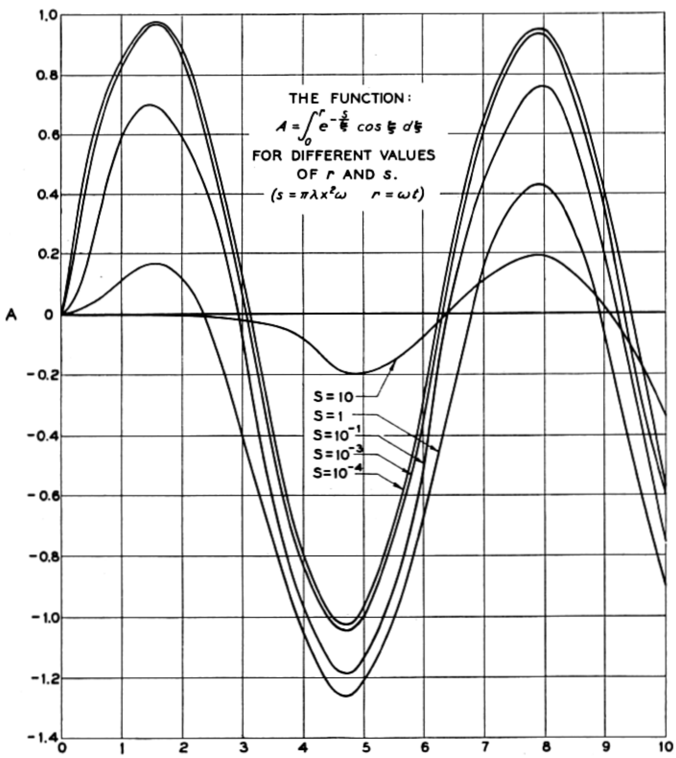

Fig. 1—Plot of the integral A as a function of r for different values of s.

A and B are given by

$$A = \int_0^{\omega t} e^{-\pi\lambda x^2\omega/\xi} \cos \xi d\xi,$$

$$B = \int_0^{\omega t} e^{-\pi\lambda x^2\omega/\xi} \sin \xi d\xi.$$

With a disturbing current $I = e^{-\beta t}$ in wire 1, the induced voltage

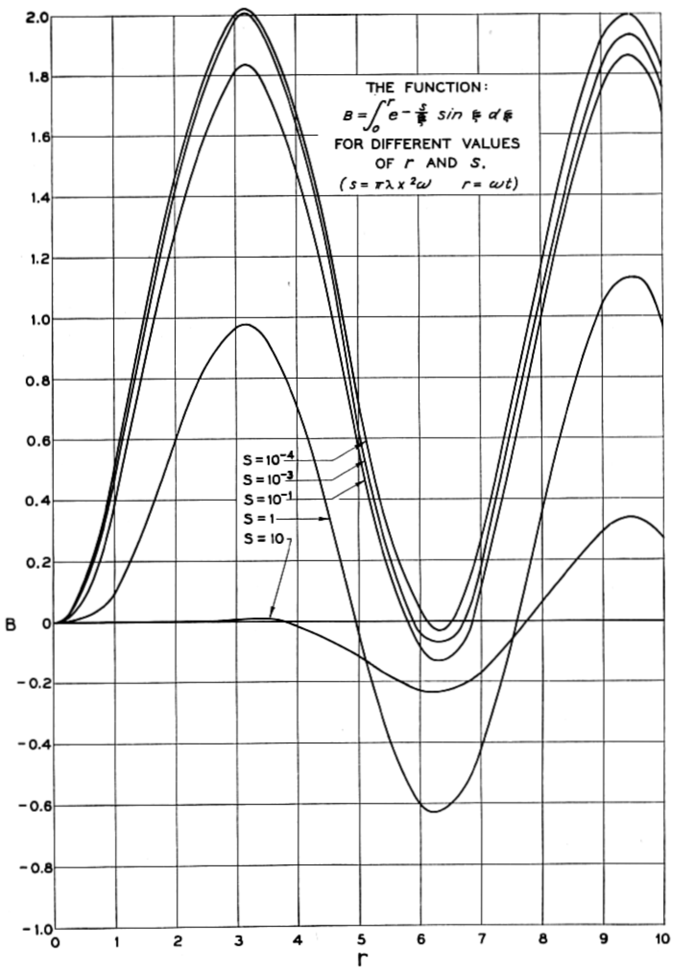

Fig. 2—Plot of the integral B as a function of r for different values of s.

per unit length in wire 2 for small values of time is given by

$$
\begin{aligned}
V_{12}(t) = {} & \frac{1}{\pi\lambda x^2}\left(e^{-\beta t} - e^{-\pi\lambda x^2/t}\right) \\
& + \frac{\beta}{\pi\lambda x^2} e^{-\pi\lambda x^2/t}\left[\frac{t^2}{\pi\lambda x^2} - \frac{2t^3 + \beta t^4}{(\pi\lambda x^2)^2} + \frac{6t^4 + 6\beta t^5 + \beta^2 t^6}{(\pi\lambda x^2)^3}\right. \\
& \left. - \frac{24t^5 + 36\beta t^6 + 12\beta^2 t^7 + \beta^3 t^8}{(\pi\lambda x^2)^4} + \cdots\right]
\end{aligned}
$$

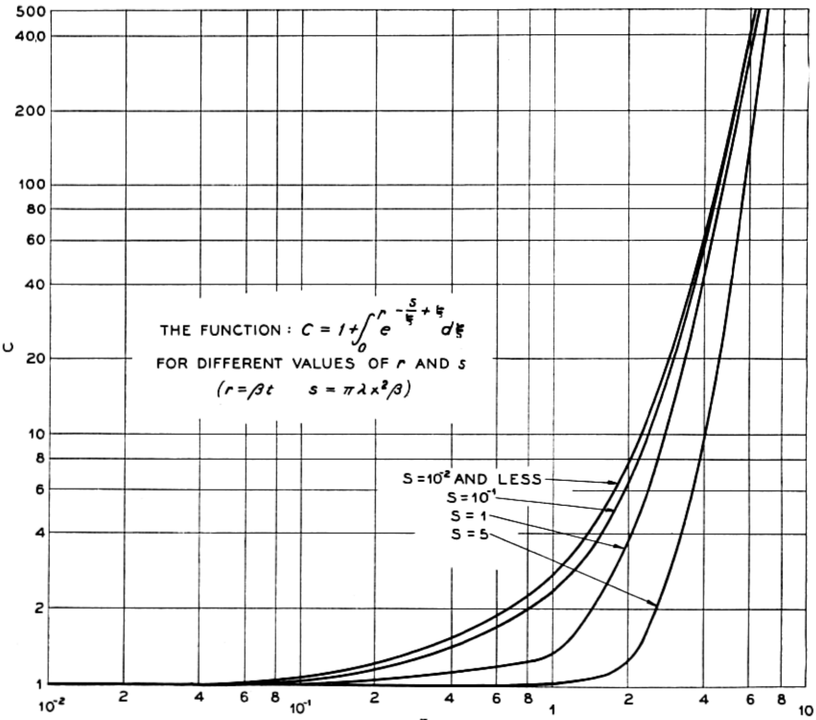

Fig. 3—Plot of the quantity C as a function of r for different values of s.

and for values of time such that the above series can not be used by

$$V_{12}(t) = \frac{1}{\pi\lambda x^2}\left(Ce^{-\beta t} - e^{-\pi\lambda x^2/t}\right),$$

where

$$C = 1 + \int_0^{\beta t} e^{-(\pi\lambda x^2\beta/\xi)+\xi} d\xi.$$

Finally, the induced voltage $Z_{12}(t)$ due to a unit step current in wire

1 is determined by

$$Z_{12}(t) = \frac{1}{\pi\lambda x^2}(1 - e^{-\pi\lambda x^2/t}).$$

The functions A, B, and C are plotted in Figs. 1, 2, and 3 for some values of the parameters often to be found in practice.

In these formulas λ is the ground conductivity in electromagnetic c.g.s. units, x the separation between wires in centimeters, t the time in seconds, and $j = \sqrt{-1}$. The functions ker′ and kei′ are related to the Bessel function of the second kind for imaginary arguments defined by G. N. Watson, " Bessel Functions " as follows

$$\mathrm{ker}'(z) \pm j\, \mathrm{kei}'(z) = -j^{\pm 1/2} K_1(z j^{\pm 1/2})$$

Values of these functions are tabulated in Table I of " Bessel Functions for A-C Problems " by H. B. Dwight A. I. E. E. Trans. 1929 pp. 812–820.

The induced voltage is in units of abvolts per cm. which is transformed to volts per mile by the factor 1.61×10^{-4}.

Part II

The second part of this paper will be devoted to a discussion of the theory leading to the above results.

Consider a system of two wires, 1 and 2, wire 1 being of infinite length, parallel with each other, with the heights h_1 and h_2 above earth and separated by a distance x. The general problem is to calculate the voltage on wire 2 as a function of time due to the sudden flow of a current in wire 1, this current being zero before $t = 0$ and $I(t)$ thereafter. Let the voltage on wire 2 due to a unit current step, that is, a current equal to zero before $t = 0$ and unity after $t = 0$, be denoted by $Z_{12}(t)$, then the voltage due to a current $I(t)$ is given by

$$V_{12}(t) = \frac{d}{dt}\int_0^t Z_{12}(\tau) I(t - \tau) d\tau.\text{[6]} \qquad (1)$$

The fundamental quantity thus necessary in the solution of the problem is $Z_{12}(t)$. This quantity completely determines the voltage $V_{12}(t)$ for all types of disturbing currents. $Z_{12}(t)$ may be written as a Fourier integral:

$$Z_{12}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{j\omega t}}{j\omega} Z_{12}(\omega) d\omega, \qquad (2)$$

where $Z_{12}(\omega)$ is the mutual impedance for periodic earth currents and

[6] Carson, Electric Circuit Theory and the Operational Calculus, page 16.

is given by [7]

$$Z_{12}(\omega) = 2j\omega \log \frac{\rho''}{\rho'} + 4\omega \int_0^\infty [\sqrt{\mu^2 + j} - \mu] e^{-(h_1+h_2)\mu\sqrt{\alpha}} \cos \mu x\sqrt{\alpha}\, d\mu. \quad (3)$$

where

$$\alpha = 4\pi\lambda\omega$$
$$\rho'' = \sqrt{(h_1 + h_2)^2 + x^2}$$
$$\rho' = \sqrt{(h_1 - h_2)^2 + x^2}$$

$Z_{12}(\omega)$ is limited by the neglect of displacement currents to frequencies such that the propagation constant is a small quantity in c.g.s. units. To obtain $Z_{12}(t)$ it is necessary to integrate over all frequencies as shown by (2); this introduces an approximation in all results for small values of the time.

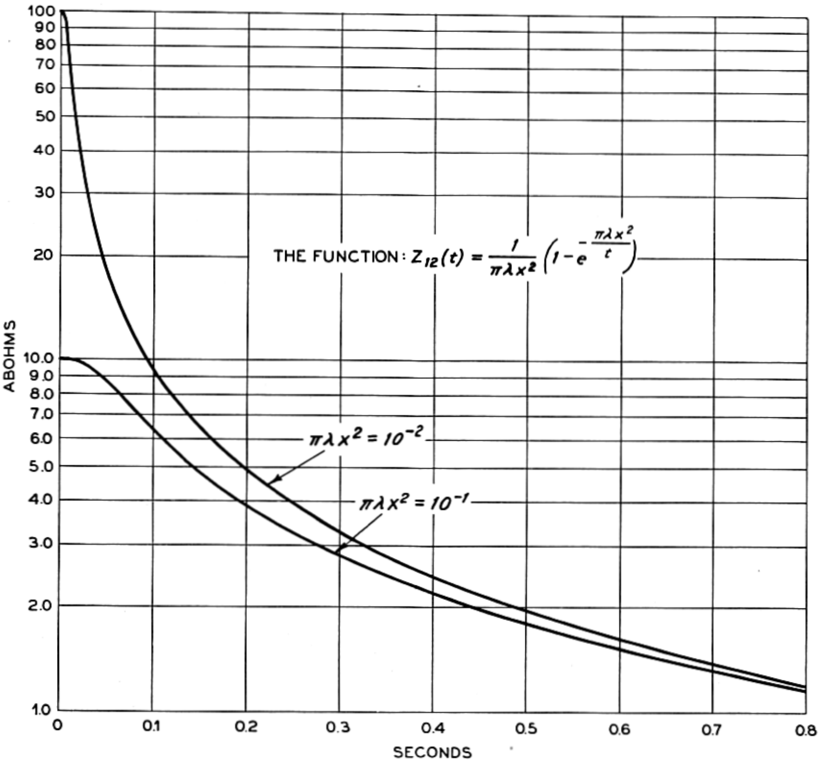

Fig. 4—Plot of the voltage $Z_{12}(t)$ on wire 2 as a function of t for two different values of separation or conductivity as given by the product $\pi\lambda x^2$.

[7] *Bell System Technical Journal*, Vol. V, page 544, October, 1926.

I am indebted to Dr. F. H. Murray of the American Telephone and Telegraph Company for the following solution of $Z_{12}(t)$ as given by (2):

$$Z_{12}(t) = \frac{2h}{|M|^2\sqrt{\lambda}\sqrt{t}} - \left[\frac{1}{2\lambda}\left(\frac{1}{M_1^2}+\frac{1}{M_2^2}\right)\right] + \frac{1}{2\lambda}\left[\frac{e^{\frac{M_1^2\lambda}{t}}}{M_1^2}\operatorname{erfc}\frac{M_1\sqrt{\lambda}}{\sqrt{t}} + \frac{e^{\frac{M_2^2\lambda}{t}}}{M_2^2}\operatorname{erfc}\frac{M_2\sqrt{\lambda}}{\sqrt{t}}\right], \quad (4)$$

where

$$|M| = |M_1| = |M_2|,$$
$$M_1 = (h - jx)\sqrt{\pi},$$
$$M_2 = (h + jx)\sqrt{\pi},$$
$$h = h_1 + h_2$$

and

$$\operatorname{erfc} Z = 1 - \operatorname{erf} Z = 1 - \frac{2}{\sqrt{\pi}}\int_0^Z e^{-x^2}dx.$$

Taking the limit of equation (4) as h approaches zero there results

$$Z_{12}(t) = \frac{1}{\pi\lambda x^2}(1 - e^{-\pi\lambda x^2/t}), \quad (5)$$

which formula is of fundamental importance in the present analysis.[8] This equation is also plotted on Fig. 4 for two different values of $\pi\lambda x^2$.

Assuming now $I(t) = \sin \omega t$ formula (1) gives

$$V_{12}(t) = \frac{\sin \omega t}{\alpha} - \frac{\omega}{2\alpha}\left[e^{\beta t}\int_0^t e^{-(\alpha/\tau)-\beta\tau}d\tau + e^{-\beta t}\int_0^t e^{-(\alpha/\tau)+\beta\tau}d\tau\right], \quad (6)$$

where

$$\alpha = \pi\lambda x^2,$$
$$\beta = j\omega.$$

[8] This formula can readily be checked in the following manner. The mutual impedance between wires on the surface of the ground is

$$Z_{12}(\omega) = \frac{1}{\pi\lambda x^2} - \frac{\gamma}{\pi\lambda x}K_1(\gamma x),$$

where $\gamma = \sqrt{4\pi\lambda j\omega}$ and K_1 is the Bessel function of the second kind with imaginary argument defined by Watson, "Bessel Functions."

Replace $j\omega$ by p, and interpret the function of p so obtained according to operational methods. The first term is independent of p and therefore of t. The second term is transformed according to the equivalent

$$\alpha\sqrt{p}K_1(\alpha\sqrt{p}) = e^{-\alpha^2/4t},$$

given as pair 922 in G. A. Campbell's paper "The Practical Application of the Fourier Integral," *Bell System Technical Journal*, Oct. 1928.

Equation (5) is then immediately obtained.

The integrals appearing in (6) are apparently not known in closed form. Series expansions holding for small and large values of time may be derived however.

By successive integration by parts we obtain:

$$\int_0^t e^{-(\alpha/\tau)+\beta\tau}d\tau = e^{-(\alpha/t)+\beta t}\left[\frac{t^2}{\alpha}-\frac{2t^3+\beta t^4}{\alpha^2}+\frac{6t^4+6\beta t^5+\beta^2t^6}{\alpha^3}\right.$$
$$\left.-\frac{24t^5+36\beta t^6+12\beta^2t^7+\beta^3t^8}{\alpha^4}+\cdots\right]. \quad (7)$$

$e^{\beta t}$ appearing on the right hand side of equation (7) is cancelled by $e^{-\beta t}$ appearing before the integral, and similarly for the first term in brackets in equation (6). In the complete expression for the voltage odd powers of β cancel and we have:

$$V_{12}(t) = \frac{\sin \omega t}{\pi\lambda x^2} - \frac{\omega}{\pi\lambda x^2}e^{-\pi\lambda x^2/t}\left[\frac{t^2}{\pi\lambda x^2}-\frac{2t^3}{(\pi\lambda x^2)^2}+\frac{6t^4-\omega^2t^6}{(\pi\lambda x^2)^3}\right.$$
$$\left.-\frac{24t^5-12\omega^2t^7}{(\pi\lambda x^2)^4}+\cdots\right]. \quad (8)$$

For large values of time equation (6) is written as

$$V_{12}(t) = \frac{\sin \omega t}{\alpha}+\frac{\omega}{2\beta}\left[e^{-\beta t}\int_0^\infty \frac{e^{-(\alpha/\tau)+\beta\tau}}{\tau^2}d\tau - e^{\beta t}\int_0^\infty \frac{e^{-(\alpha/\tau)-\beta\tau}}{\tau^2}d\tau\right]$$
$$+\frac{\omega}{2\beta}\left[e^{\beta t}\int_t^\infty \frac{e^{-(\alpha/\tau)-\beta\tau}}{\tau^2}d\tau - e^{-\beta t}\int_t^\infty \frac{e^{-(\alpha/\tau)+\beta\tau}}{\tau^2}d\tau\right], \quad (9)$$

where the integrals between zero and infinity correspond to the steady state condition while the integrals between t and infinity give the transient distortion. The integral between t and infinity may be evaluated in a manner quite similar to that used above. The result with plus sign for β is

$$\int_t^\infty \frac{e^{-(\alpha/\tau)+\beta\tau}}{\tau^2}d\tau = e^{-(\alpha/t)+\beta t}\left[\frac{1}{\beta t^2}+\frac{1}{\beta^2}\left(\frac{2}{t^3}-\frac{\alpha}{t^4}\right)+\frac{1}{\beta^3}\left(\frac{6}{t^4}-\frac{6\alpha}{t^5}+\frac{\alpha^2}{t^6}\right)\right.$$
$$\left.+\frac{1}{\beta^4}\left(\frac{24}{t^5}-\frac{36\alpha}{t^6}+\frac{12\alpha^2}{t^7}-\frac{\alpha^3}{t^8}\right)+\cdots\right]. \quad (10)$$

The integrals between 0 and infinity are evaluated by:

$$\int_0^\infty \frac{e^{-(\alpha/\tau)\pm\beta t}}{\tau^2}d\tau = \frac{2}{\sqrt{\alpha}}(\mp\beta)^{1/2}K_1(2\sqrt{\mp\alpha\beta}), \quad (11)$$

in which the real and imaginary parts of the right hand side may be expressed by the ker′ and kei′ functions by the relation already given.

The complete expression for the voltage is:

$$V_{12}(t) = \frac{\sin \omega t}{\pi\lambda x^2} + \frac{2\sqrt{\omega}}{x\sqrt{\pi\lambda}}[\cos \omega t \ \mathrm{kei}'(2x\sqrt{\pi\lambda\omega}) + \sin \omega t \ \mathrm{ker}'(2x\sqrt{\pi\lambda\omega})]$$
$$- e^{-\pi\lambda x^2/t}\left[\frac{1}{\omega t^2} - \frac{1}{\omega^2}\left(\frac{6}{t^4} - \frac{6\pi\lambda x^2}{t^5} + \frac{(\pi\lambda x^2)^2}{t^6}\right) + \cdots\right]. \quad (12)$$

For such values of time where neither of the formulas (8) or (12) give very accurate results it is necessary to perform mechanical integration.

In so doing it is convenient to introduce a new variable ξ of integration. Let $\xi = \omega\tau$, and the integrals become

$$\int_0^t e^{-(\alpha/\tau)\pm j\omega\tau}d\tau = \frac{1}{\omega}\int_0^r e^{-(s/\xi)\pm j\xi}d\xi$$
$$= \frac{1}{\omega}\left[\int_0^r e^{-s/\xi}\cos \xi d\xi \pm j\int_0^r e^{-s/\xi}\sin \xi d\xi\right],$$

where

$$\left.\begin{aligned} s &= \alpha\omega = \pi\lambda\omega x^2, \\ r &= \omega t. \end{aligned}\right\} \quad (13)$$

Now let

$$A(s, r) = \int_0^r e^{-s/\xi}\cos \xi d\xi, \quad (14)$$

$$B(s, r) = \int_0^r e^{-s/\xi}\sin \xi d\xi, \quad (15)$$

and formula (6) becomes

$$V_{12}(t) = \frac{\sin \omega t}{\pi\lambda x^2} - \frac{\sqrt{A^2 + B^2}}{\pi\lambda x^2}\cos(\omega t - \varphi), \quad (16)$$

where

$$\tan\varphi = \frac{B}{A}. \quad (17)$$

The values of the functions A and B are given in Figs. 1 and 2 for some values of s and r which are frequently met in practice.

Assuming finally $I(t) = e^{-\beta t}$ equation (1) gives after simplifications

$$V_{12}(t) = \frac{1}{\pi\lambda x^2}(e^{-\beta t} - e^{-\pi\lambda x^2/t}) + \frac{\beta e^{-\beta t}}{\pi\lambda x^2}\int_0^t e^{-(\pi\lambda x^2/\tau)+\beta\tau}d\tau. \quad (18)$$

For small values of t the series expansion (7) may be used. The result for this case then becomes

$$V_{12}(t) = \frac{1}{\pi\lambda x^2}(e^{-\beta t} - e^{-\pi\lambda x^2/t}) + \frac{\beta}{\pi\lambda x^2}e^{-\pi\lambda x^2/t}\left[\frac{t^2}{\pi\lambda x^2} - \frac{2t^3+\beta t^4}{(\pi\lambda x^2)^2} + \frac{6t^4+6\beta t^5+\beta^2 t^6}{(\pi\lambda x^2)^3} - \frac{24t^5+36\beta^2 t^6+12\beta^2 t^7+\beta^3 t^8}{(\pi\lambda x^2)^4} + \cdots\right]. \quad (19)$$

For large values of time, introduce a new variable $\xi = \beta\tau$ of integration in the integral in (18). Then

$$V_{12}(t) = \frac{1}{\pi\lambda x^2}(Ce^{-\beta t} - e^{-\pi\lambda x^2/t}) \quad (20)$$

where

$$\left.\begin{aligned} C &= 1 + \int_0^r e^{-(s/\xi)+\xi}d\xi, \\ r &= \beta t, \\ s &= \pi\lambda x^2\beta. \end{aligned}\right\} \quad (21)$$

The values of C are given on Fig. 3 for important ranges of r and s. For s equal to and less than 10^{-2}, C is for practical purposes independent of s.

I am indebted to Dr. F. H. Murray, and Mr. R. M. Foster of the American Telephone and Telegraph Company for valuable suggestions during the course of this work, and to Miss R. Pedersen who carried out all the numerical calculations.