A Method of Impedance Correction

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This paper gives a theoretical treatment of some recently developed wave filter terminating sections whose application is discussed in the accompanying paper on "Impedance Correction of Wave Filters." The sections consist primarily of non-recurrent ladder networks which operate, over the transmission bands of the associated filters, as transformers whose ratio varies with frequency. The transformation ratio of the network is specified, as a function of frequency, by a power series containing a limited number of terms and the design procedure therefore depends upon the construction of power series approximations to the ratio between the resistance of the filter proper and the desired resistance. A separate network is added to secure control of the reactance component. An increased number of terms in the power series, and therefore an improved approximation to the desired transformation ratio, can be obtained by increasing the number of branches in the network. The method thus leads to a series of sections of progressively increasing complexity and with progressively improving impedance characteristics. By an inversion of the analysis a second series of sections can also be obtained. The paper is chiefly devoted to a discussion of these two series of filter sections, but other possible applications of the method are also described briefly.

THE analysis of transmission circuits with which telephone engineers are familiar is an outgrowth of the general physical theory of the propagation of wave disturbances in continuous media. Problems analogous to the analysis of a smooth transmission line are found, for example, in optical and acoustical theory and in the theory of the vibrations of a taut string. The situations of most importance from the standpoint of general physics are those in which the continuous medium extends indefinitely in at least one direction. Since, moreover, this is also the simplest case, it has been customary to base our transmission analysis upon the analogous concept of an infinite line with distributed constants. The analysis of such a structure, since it depends upon only two quantities, the characteristic impedance and the propagation constant, is of course very simple.

An actual telephone transmission circuit, however, is by no means an infinite structure containing distributed constants. Many lines, for example, are loaded. Whether loaded or unloaded, they do not extend indefinitely, but are interrupted by terminal apparatus and intermediate repeaters. Each of these, moreover, contains a miscellany of apparatus, such as modulators, transformers, amplifiers, filters, equalizers, by-pass circuits, and the like, having little physical resemblance to a continuous medium.

This physical contrast between an ideal continuous medium and an actual physical telephone circuit does not necessarily mean that the application of the wave theory to circuit analysis is a difficult matter. To a first approximation we can determine the response of a circuit merely by adding together the propagation constants of its various constituents. Unfortunately, however, the diverse components of a typical circuit usually have characteristic impedances which are widely different functions of frequency. Thus, for example the impedances of the amplifiers and modulators in most telephone systems are nearly constant pure resistances. Non-loaded lines approach such a characteristic at high frequencies but at low frequencies their impedance is usually large and may have a considerable reactive component. Loaded lines depart from a constant resistance at high frequencies as well. An even more complicated characteristic, consisting of a varying resistance in the transmitted band, changing abruptly to a pure reactance as we pass the cutoff, is exhibited by a wave filter. In addition to the normal propagation constants of the circuit, therefore, we must take account of reflection effects at all of the junctions between these various types of characteristic impedance. In a long circuit containing impedance irregularities at many junctions, moreover, we must give consideration to an enormous variety of waves which suffer multiple reflections from a number of junctions. This complicated system of factors may make life burdensome to the man who must evaluate them, but since they are seldom large enough to grossly affect the transmission characteristic of a circuit, they usually play otherwise a secondary role in practical transmission analyses. They do, however, blur the original clarity of the wave picture and from the standpoint of theoretical simplicity at least, therefore, they should be eliminated. For this purpose we should have at our disposal a network whose impedances at its two ends could be assigned arbitrarily to match the impedances actually present at any junction.

The networks which form the subject of this paper were developed to eliminate reflection effects which, in addition to being a nuisance from the theoretical standpoint, were attended by serious practical consequences as well. The engineering problem involved is described in the paper on "Impedance Correction of Wave Filters" by E. B. Payne appearing simultaneously in this *Journal*. Briefly, it appears from the discussion in that paper that impedance mismatches at the junctions between terminal or repeater equipment of carrier systems and the line give rise to reflected waves which may produce cross-talk in neighboring systems. This cross-talk can be reduced as much as we like by means of line transpositions but the required transposition scheme is so extremely complicated and expensive that the reduction in the amplitude of the reflected waves by improvement of the reflection coefficient at these junctions is of considerable economic importance. The impedances of the terminals and repeaters at the junctions at which reflections occur are chiefly determined by their filters, which are the apparatus immediately facing the line. A detailed study of the relationship between the actual input impedance of a filter and mismatches of characteristic impedance which may occur at further junction points in the circuit shows that by far the simplest method of obtaining a low reflection coefficient at the line terminals is to produce a match of characteristic impedances at all junction points of the filter system. Fortunately speech currents beyond the transmitted band of the filters carry so little energy that the reflection coefficient of the structure in these ranges is of no importance. The technical problem therefore reduces to the construction of a new type of filter section for use at terminations, the new filter section having an image impedance within the transmitted band which at one end matches that of the standard sections forming the main body of the structure and at the other approximates a constant resistance, matching the terminating impedances. Of course the new filter sections must also be so chosen that they will not impair the transmission properties of the system.

This immediate problem has been solved. It still leaves unsettled, however, the question as to whether we can devise a type of network capable of correcting for reflection effects not only at these particular junctions but also at any other impedance irregularity in the circuit. Such a structure would transform one arbitrary impedance characteristic into another preassigned characteristic without decreasing the transmission efficiency of the circuit, much as the familiar attenuation equalizer changes the attenuation characteristic of a circuit by a preassigned amount without changing its impedance and without greatly affecting its phase characteristic. The mathematical analysis underlying the sections which have been developed for filter impedance correction is easily extended to a much broader class of terminating impedances. Judged from a purely formal standpoint, therefore, the networks appear to be a long step forward in the development of such a general impedance equalizing device. Unfortunately, it seems certain from other considerations that much of the promise thus inherent in the formal mathematical analysis may not be realized in practical applications, but since the network has been thoroughly studied only in its application to filters, its precise limitations are still uncertain. In the discussion which follows the general method of impedance correction is first sketched briefly, and is succeeded by a detailed treatment of its application to filters. Some of the probable limitations of the method in other applications are suggested near the end of this paper.

The analysis used in impedance correction can also be applied to the construction of networks having transmission properties somewhat like those of the familiar wave filter. In contrast to the usual filter theory, developed, after the analogy of wave propagation in continuous media, from the conception of an infinite recurrent structure, however, it leads to networks which are not recurrent and are not divisible into separate sections with matched image impedances. In its present state of development the analysis is unquestionably much less powerful than the established theory. Since it may be of interest as an example showing at least the possibility of an alternative approach to filter design, however, it is discussed briefly at the conclusion of the paper.

General Impedance Correcting Process

If no transmission requirements were imposed upon electrical structures, a wide variety of networks might be used for impedance correction. For example, we might make up deficiencies of impedance or admittance by a simple two-terminal network in series or in shunt with the circuit. In almost all circuits, however, we are interested in securing minimum transmission loss, that is to say, maximum energy in the receiving impedance, throughout the frequency bands containing the transmitted signals. The energy which goes into a system terminated by a correcting network depends only upon generator and the corrected impedance, both of which are specified by the conditions of the problem. We can increase the energy delivered to the receiving device, therefore, only by reducing the amount absorbed in the correcting network. Obviously the best possible condition is found when the correcting network is composed of pure reactances. Unless either the resistance or the conductance of the circuit happens to be ideal, however, impedance correction cannot be obtained by a simple two-terminal reactive network. For this reason, the impedance correcting structures which have been developed are four-terminal networks of pure reactances. Control of the resistance or conductance component is gained, not by the direct addition of resistance, but rather through the use of the network as a sort of variable transformer, whose impedance ratio changes as we go over the frequency range. In such a circuit the insertion loss of the network is determined entirely by the ratio of the energy drawn from the generator by the original and the corrected impedance. Ideal dissipationless network elements are, of course, not available in practice. Except for the possible influence of this factor, impedance correction, since it normally means an improvement in the match between generator and load impedances, should result in a slight increase of transmission efficiency.

Reciprocal Impedance Relations at Terminals of a Reactive Network

Our restriction to networks of pure reactances allows us to make use of a principle by means of which the impedances measured at the two ends of the network under certain terminal conditions can be reciprocally related to one another. The theorem will be given here since it is of frequent application in further discussion. Referring to Fig. 1, let us assume that the impedance measured at terminals *cd*, with an impedance $\overline{Z}_1$ connected to terminals *ab*, is equal to $\overline{Z}_2$, as is shown in the diagram. The theorem is concerned with the impedance Z looking into terminals *ab* when Z_2, the conjugate of $\overline{Z}_2$, is connected across *cd*. Let us suppose that the generator e in Z_2 produces a current i in $\overline{Z}_1$. Then, by the usual principle of reciprocity, the generator e when inserted in $\overline{Z}_1$ will produce the current i in Z_2. In the first case the power entering the network is obviously $\frac{e^2}{4R_2}$ and the power flowing from it into $\overline{Z}_1$ is i^2R_1. In the second case these powers are $\frac{e^2R}{(R_1 + R)^2 + (X_1 + X)^2}$ and i^2R_2. Since the network is non-dissipative the power entering the network equals the power leaving it in both cases.

$$\frac{e^2}{4R_2} = i^2R_1$$

$$\frac{e^2R}{(R_1 + R)^2 + (X_1 + X)^2} = i^2R_2$$

Upon dividing the two equations and simplifying we find:

$$(R_1 - R)^2 + (X_1 + X)^2 = 0$$

which can be true only if:

$$R = R_1 \text{ and } X = -X_1$$

In other words, Z is the conjugate of $\overline{Z}_1$. We can state this result in the following words:

A network composed of pure reactances will have a given impedance, Z_1 at one pair of terminals when an impedance Z_2 is connected to a second set of terminals if, when the conjugate of Z_1 is connected to the first pair of terminals, the impedance measured at the second pair of terminals is the conjugate of Z_2.

This theorem can be applied immediately to the problem of filter impedance correction discussed in the introduction. The networks required for this problem were defined there as sections which within the transmitting band would have image impedances matching the line at one end and matching the image impedance of the main body of the filter at the other. If we represent the filter proper and the line by $\overline{Z}_1$ and Z_2 in Fig. 1, these image impedance requirements reduce to the

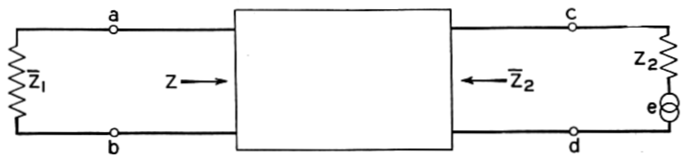

Fig. 1—Diagram to illustrate the reciprocal properties of impedance correcting networks.

statement that the network must be so chosen that an impedance match exists both at *ab* and at *cd*. $\overline{Z}_1$ and Z_2 for this particular circuit are however, pure resistances, and therefore equal to their conjugates, within the required frequency range. The theorem shows that an impedance match will be obtained at *cd* provided an impedance match exists at *ab*, and vice versa.[1] If we please, therefore, we can consider that our problem is that of obtaining a network which, when terminated by a filter, has an actual impedance equal to a constant resistance. On the other hand we can start with the resistance and attempt to build up a network whose impedance matches that of the filter. Both the first or "direct" and the second or "reverse" methods of constructing terminating networks for filters are considered in the next section. With either procedure the resulting networks have both required image impedances and can be used, when properly connected, either at the line or the receiving end of the filter. The "correction" of one impedance to match another and the construction of a network having given image impedance characteristics are therefore interchangeable conceptions.

Separate Correction of Real and Imaginary Components of Impedance or Admittance

The image impedance method of defining the properties of the terminating network is a convenient one when we are concerned with the operation of the structure in the transmission system as a whole. The methods used in designing the network can, however, be described

[1] See also Feldtkeller's paper, "Über einige Endnetzwerke von Kettenleitern" in the *Elektrische Nachrichten-Technik*, June 1927, for a very similar use of this property of reactive networks.

more simply if we reject the image impedance statement of the problem in favor of its alternative. For the time being, therefore, we will assume that we are attempting to design a reactive network having a preassigned input impedance when terminated by a given load impedance.

In accordance with a principle originally stated by O. J. Zobel,[2] this problem of impedance correction can be simplified if we consider separately the resistance and reactance of the corrected circuit. To be more explicit, since a reactance in series with the circuit will change its reactance without changing its resistance, it is simplest to consider first the construction of a network which will produce the required resistance characteristic. Of course the reactance characteristic furnished by such a structure will not in general be ideal, but we may be able to correct it to the proper value by the later addition of a series reactive network. Quite obviously, it is equally easy to base the analysis upon admittances and construct first a network which will give the required conductance characteristic and make up any faults in its susceptance characteristic by a final shunting branch.

This division of the network into two separate structures is, of course, not a necessary one and in view of the extremely limited range of reactance or susceptance characteristics which can be compensated for by a final, physically realizable, two-terminal reactive network may seem scarcely desirable. An alternative procedure in which this division is not attempted is mentioned in the concluding section. The reason for assuming separate correction of the real and imaginary components of impedance and admittance in the present discussion is simply one of convenience. The difficulties which might be anticipated in the design of the final reactive compensator do not appear in filter impedance correcting problems, at least. On the other hand, the division has the advantage that it makes each step simple and allows us to meet fairly severe impedance requirements with a small number of variables. As we shall see later the method has the further advantage in its application to filters that it lends itself readily to the modifications necessary when a number of filters must operate together.

The Resistance or Conductance Controlling Network

Since the characteristics of two-terminal reactance networks are well understood, the construction of the final reactive branch demands no

[2] See, for example, U. S. Patents No. 1,557,229 and 1,557,230 where he applies it to "x-terminated" filters. The method of this paper is in some respects merely a generalization of that analysis. The relation of "m-derived" sections and "x-terminations" to the filter terminations developed in this paper is indicated in the following section. In this connection, the previous work of R. S. Hoyt on loaded lines should also be mentioned. See this *Journal*, Vol. 3, p. 414, 1924.

extensive discussion. The problem of designing a four-terminal reactive network which will transform one arbitrary resistance or conductance characteristic into another arbitrary characteristic must, however, be treated with more respect. The configuration which has been adopted for this purpose is shown in Fig. 2. The quantities of

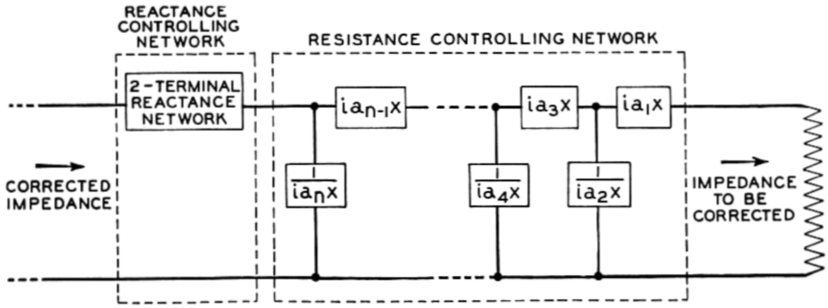

Fig. 2—Generalized schematic of impedance correcting network.

the general form ia_jx are analytic representations of the impedances of the series branches and admittances of the shunt branches. The a's are constants whose choice determines the particular resistance or conductance controlling properties of the structure, and x is a function of frequency. Since the series impedances and shunt admittances are all proportional to x all of the series branches will have a given physical configuration and all of the shunt branches will have the inverse configuration. For example if the series branches are inductances the shunt branches will be capacities, while x, of course, will be proportional to frequency. By using other series arm configurations we can obtain a considerable variety of networks. Each such network, it will be noticed, is similar to a "constant-k" filter in physical configuration. The appropriate network in any particular situation is that one which resembles a constant-k filter transmitting the frequency range of interest.

The property of this network configuration which makes it particularly suitable as a resistance or conductance controlling device is the fact that in most instances the modification it produces in the resistance or conductance of the load can be expressed as a single polynomial. To be more explicit, when the load impedance is of a certain mathematical type, which includes the impedances in which we are most interested, the resistance or conductance of the corrected structure is given by a formula of the following sort.

$$R(\text{or } G) = \frac{F(x)}{A_0 + A_1x + A_2x^2 + \cdots + A_nx^n}$$

in which the A's are constants involving the arbitrary quantities a_1, a_2, etc. which specify the network elements.

The quantity $F(x)$ is usually either the resistance or conductance component of the load impedance, and in any case is a quantity entirely determined by that impedance. In order to secure the proper resistance or conductance from the corrected structure, therefore, it is merely necessary to choose such values of the constants $A_0 \cdots A_n$ that the polynomial satisfies the equation

$$A_0 + A_1 x + \cdots A_n x^n = \frac{F(x)}{R(\text{or } G)}$$

with sufficient accuracy when R is given the desired value of the corrected resistance or conductance. The problem of approximating a given curve by a polynomial of given degree is a well known one in mathematics and such general methods as expansions in power series or Legendrian harmonics exist for its solution. We can, therefore, consider that the choice of these constants presents no particular difficulty. Even without the help of these general methods, however, the problem is so simple that suitable approximations can be obtained by cut-and-try methods.

These polynomial coefficients $A_0 \cdots A_n$ are merely intermediate parameters which specify the values of the elements in the network implicitly but do not give them directly. In order to determine the relation between these coefficients and the actual element values it is necessary to make a direct computation of the impedance of the network in terms of the a's and sort out the various powers of x in the resulting expression. Each of the quantities $A_1 \cdots A_n$ is thereby expressed as a function of the a's. Our next step must then be to determine values of the network elements by solving the set of simultaneous equations relating them to the numerical values of the polynomial coefficients. In accordance with the procedure we have adopted, the design is completed by the computation of the reactance or the susceptance of the network, and its adjustment to the desired value by the addition of a suitable final branch. The discussion of the application of the method to filter impedances given in the next section will illustrate the process in detail.

Proof of Properties of Ladder Type Resistance Correctors

As we observed in a previous paragraph the ratio of the load resistance or conductance to the corrected resistance or conductance can be expressed in this simple fashion as a polynomial in x only when the load impedance belongs to a certain mathematical class. Appropriate load impedances are those whose real components can be written as the square roots of rational functions[3] of x and whose imaginary components are rational functions of x. We can make this conclusion plausible by direct inspection. It is obvious that the general nature of the mathematical expression for the impedance of the network cannot change radically as we add successive branches. When we add a series branch, however, the reactance is increased by $a_j x$, while the resistance is not altered. The functional form of the impedance then will be unchanged if the reactance was originally an algebraic function of x. But, since we must add shunt as well as series arms to the network the functional forms must be symmetrical whether taken on an impedance or admittance basis. By analogy, therefore, the susceptance also must be a rational algebraic function. The susceptance B is expressed in terms of R and X, the resistance and reactance, by $B = X/(R^2 + X^2)$, but X (and therefore X^2) has already been fixed as a rational algebraic function and R^2 must have a similar form if the whole susceptance expression is to be such a function. This conclusion, since it applies equally at any part of the network, must, of course, be valid for the load impedance also.

This argument is sufficient to indicate what sort of a load impedance *might* have the property for which we are looking—that of allowing the change in resistance or conductance produced by the insertion of the ladder network to be expressible as a simple polynomial. In order to show definitely that this type of load impedance *will* have that property it is simplest to begin by finding out whether the relation holds when the network consists of a single branch. In accordance with the previous discussion, the load impedance will be taken as

$$\sqrt{\frac{F_1(x)}{F_2(x)}} + i\frac{G_1(x)}{G_2(x)},$$

where $F_1(x)$, $F_2(x)$, $G_1(x)$, and $G_2(x)$ are polynomials in x. Upon multiplying and dividing the resistance expression by $\sqrt{F_2(x)}C(x)$, where $C(x)$ is a new polynomial so chosen that when the product $F_2(x)C(x)$ is divided by $G^2(x)$ the quotient is a polynomial, the load impedance is transformed into

$$\frac{\sqrt{F_1(x)F_2(x)C^2(x)}}{F_2(x)C(x)} + i\frac{G_1(x)}{G_2(x)} = \frac{F(x)}{F_2(x)C(x)} + i\frac{G_1(x)}{G_2(x)}.$$

$F(x)$ is a new symbol, written for $\sqrt{F_1(x)F_2(x)C^2(x)}$, and, as we shall

[3] Including as special cases real components which are simply rational functions, without the square root.

proceed to prove, it is the common numerator of all of the resistance and conductance expressions throughout the network.

Let us suppose now that the first branch, ia_1x, of the network is added in series. The admittance after its addition is

$$\frac{1}{\dfrac{F(x)}{F_2(x)C(x)} + i\dfrac{G_1(x)}{G_2(x)} + ia_1x} = \frac{F(x)}{F_2(x)C(x)\left[\dfrac{F_1(x)}{F_2(x)} + \left(\dfrac{G_1(x)}{G_2(x)} + a_1x\right)^2\right]}$$

$$- i\frac{F_2(x)C(x)\left(\dfrac{G_1(x)}{G_2(x)} + a_1x\right)}{F_2(x)C(x)\left[\dfrac{F_1(x)}{F_2(x)} + \left(\dfrac{G_1(x)}{G_2(x)} + a_1x\right)^2\right]}.$$

Upon remembering the way in which $C(x)$ was chosen we observe that the expressions in the denominators of the conductance and susceptance fraction and in the numerator of the susceptance fraction reduce to polynomials.

So far we have been able to show that the impedance of the load and the admittance of the network after one branch is added can be so expressed that (1) their imaginary components are rational functions, (2) the numerators of their real components are equal to $F(x)$, and (3) the denominators of their real components are polynomials. It is also possible, however, to show that if these statements hold for the impedance and admittance at *any* two consecutive junctions they will hold also at the next following junction. Referring to Fig. 3, let us

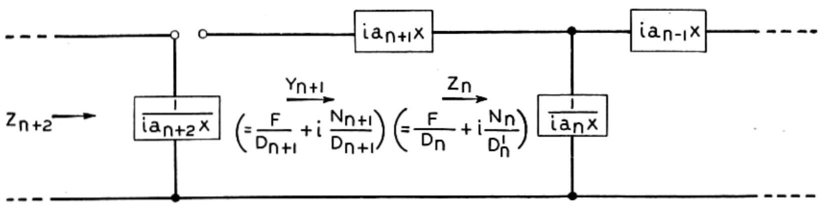

Fig. 3—Impedance and admittance relations at $n + 1$st branch of network.

suppose that the impedance after n branches of the network have been added is

$$Z_n = \frac{F(x)}{D_n(x)} + i\frac{N_n(x)}{D_n'(x)}$$

and that the admittance after $n + 1$ branches have been added is

$$Y_{n+1} = \frac{F(x)}{D_{n+1}(x)} + i\frac{N_{n+1}(x)}{D_{n+1}(x)}.$$

We wish to show that the impedance after the addition of the $n + 2nd$ branch is

$$Z_{n+2} = \frac{F(x)}{D_{n+2}(x)} + i\frac{N_{n+2}(x)}{D_{n+2}(x)} \cdot$$

The various N's and D's, of course, represent polynomials. The denominator of the imaginary component of Z_n is accented, to indicate that it is not necessarily equal to the denominator of the real component. The denominators in the Y_{n+1} expression, however, have been given the same designation, since they are equal in the expression we have set up for the admittance at the terminals of the first network branch. This fact is not essential in the proof which follows, but its use somewhat simplifies the procedure. Direct mesh computation gives

$$Z_{n+2} = \frac{F(x)}{D_{n+1}\left[\frac{F^2}{D_{n+1}^2} + \frac{N_{n+1}^2}{D_{n+1}^2}\right] + a_{n+2}^2x^2D_{n+1} + 2a_{n+2}x}$$

$$- i\frac{[N_{n+1} + a_{n+2}xD_{n+1}]}{D_{n+1}\left[\frac{F^2}{D_{n+1}^2} + \frac{N_{n+1}^2}{D_{n+1}^2}\right] + a_{n+2}^2x^2D_{n+1} + 2a_{n+2}x}.$$

Since a_{n+2} is arbitrary, the resistance component will have the specified form only if

$$D_{n+1}\left[\frac{F^2}{D_{n+1}^2} + \frac{N_{n+1}^2}{D_{n+1}^2}\right]$$

is a polynomial in x. If this condition is satisfied the reactance expression can obviously be put in the required form.

In order to examine the denominator of the resistance expression more closely we state N_{n+1}, and D_{n+1} in terms of N_n, D_n', and D_n. Direct mesh computation, again, gives

$$Y_{n+1} = \frac{F(x)}{D_n\left[\frac{F^2}{D_n^2} + \frac{N_n^2}{D_n'^2}\right] + a_{n+1}^2x^2D_n + 2a_{n+1}xD_n\frac{N_n}{D_n'}}$$

$$- i\frac{\left[a_{n+1}x + \frac{N_n}{D_n'}\right]D_n}{D_n\left[\frac{F^2}{D_n^2} + \frac{N_n^2}{D_n'^2}\right] + a_{n+1}^2x^2D_n + 2a_{n+1}xD_n\frac{N_n}{D_n'}};$$

$$\therefore D_{n+1} = D_n\left[\frac{F^2}{D_n^2} + \frac{N_n^2}{D_n'^2}\right] + a_{n+1}^2x^2D_n + 2a_{n+1}xD_n\frac{N_n}{D_n'}$$

and

$$\frac{N_{n+1}}{D_{n+1}} = -\frac{\left[a_{n+1}x + \dfrac{N_n}{D'_n}\right] D_n}{D_n\left[\dfrac{F^2}{D_n^2} + \dfrac{N_n^2}{D_n'^2}\right] + a_{n+1}^2 x^2 D_n + 2a_{n+1}xD_n\dfrac{N_n}{D'_n}}$$

Substitution of these values for D_{n+1} and N_{n+1} reduces the expression for Z_{n+2} to

$$Z_{n+2} = \frac{F(x)}{D_n + a_{n+2}^2 x^2 D_{n+1} + 2a_{n+2}xN_{n+1}}$$
$$- i\frac{N_{n+1} + a_{n+2}xD_{n+1}}{D_n + a_{n+2}^2 x^2 D_{n+1} + 2a_{n+2}xN_{n+1}}.$$

We have, however, assumed that D_n, D_{n+1}, and N_{n+1} were polynomials. The sums of the quantities constituting the numerator of the imaginary component of Z_{n+2} and the denominators of both components are therefore also polynomials, and, consequently, Z_{n+2} is written in the specified form.

The rest of the proof follows the usual argument from mathematical induction. In brief, we have established directly the fact that the formula holds when the network has no branches, or only one branch. Knowing that it holds for these two cases, we conclude from the above reasoning that it holds when there are two branches. If it is valid for one branch and two branches it must also be valid for three branches, and so on. Therefore the formula holds generally.

It will be observed that we have considered the admittance, rather than the impedance, when a series branch is added, and the impedance, rather than the admittance, when a shunt branch is added. Quite obviously the cases not considered are of little interest. If the analysis is stated in terms of impedance a final series branch contributes nothing to the resistance and can be considered as part of the reactance correcting network, while an analysis based upon admittances would similarly have no use for a final shunt branch except as a constituent of the susceptance correcting network. The general formula does hold, however, for these cases also. For example the addition of a series branch simply changes one rational function, representing the reactance at the terminals of the previous shunt branch into another rational function. The fact that the impedance at the terminals of the shunt branch falls into our general form is therefore sufficient to prove that the impedance after the series branch has been added can be written in this form also. This indicates, incidentally, that an alternative form of the proof we have been considering, based upon the impedance and admittance relation at a single junction, can be developed. Using the previous notation, the impedance Z_n will be in the required form if Z_{n+1} is in that form, and not otherwise. Instead of assuming that the impedance at one junction and the admittance at an adjacent junction can be fitted into the formula, therefore, it is sufficient to assume that both the impedance and admittance at a single junction satisfy the formula in order to show that the impedance and admittance at the next succeeding junction satisfy this formula also.

Application of General Analysis to Filter Impedance Correction

The reciprocal property of the impedances at the terminals of a reactive network indicates two possible methods of applying a ladder network of the sort we have been describing to the correction of wave filter impedances. We can either terminate the network by the filter impedance and adjust its parameters to match the line impedance, or we can consider that the load impedance of the network is a constant pure resistance, representing the line impedance, and attempt to produce a match at the filter terminals. These two methods of procedure lead to distinct results, since in one case the reactance or susceptance correcting branch adjoins the line, while in the other it adjoins the filter. Both are, however, admissible under the general mathematical specifications we have set up for the load impedance of the resistance or conductance controlling network and both lead to reasonably satisfactory impedance correction.

The fact that a constant pure resistance is an admissible load impedance for the ladder network is easily established by inspection. The rational function $G_1(x)/G_2(x)$, representing the imaginary component, reduces to zero, of course, while the rational function $F_1(x)/F_2(x)$, whose square root represents the real component becomes a constant. A filter image impedance within transmission bands is similarly a pure resistance. As a function of frequency it may be defined as the geometric mean of the open and short-circuit impedances. An open or short-circuit filter, whatever its configuration is, however, simply a network of pure reactances. The open and short-circuit impedances are therefore rational functions of frequency and the image impedance they define falls within the scope of the mathematical specification we have set up for the load impedance of the correcting network.

Terminating Networks of the First Type

While both of these methods of approaching the problem lead to satisfactory impedance correction, other considerations to be discussed later recommend that one in which the filter is taken to be the load impedance for most designs. This approach will therefore be considered first and in greatest detail. We will, moreover, limit ourselves to image impedances of the "constant-k" type. Practical filter designs of course are usually composite structures containing several types of sections. The image impedances at the junctions between the sections are however, nearly always of the "constant-k" type and our restriction to image impedances belonging to this class does not, therefore, seriously limit the field of application for the network.

Notation

The image impedance of a mid-series terminated "constant-k" filter is usually written as

$$Z_0\sqrt{1+\frac{Z_{1k}}{4Z_{2k}}};$$

that of a mid-shunt terminated filter as

$$\frac{Z_0}{\sqrt{1+\dfrac{Z_{1k}}{4Z_{2k}}}},$$

where Z_{1k} and Z_{2k} represent in each case the series and shunt impedances of the "constant-k" filter, and $Z_0(=\sqrt{Z_{1k}Z_{2k}})$ is a constant which can be chosen arbitrarily to fix the impedance level of the circuit.

We will find it convenient to represent the way in which the various branches vary with frequency by a new quantity x, defined by the relation

$$\frac{Z_{1k}}{2}=iZ_0x.$$

In a low pass filter, for example, $x=f/f_c$, in a high-pass filter $x=f_c/f$, and in a band-pass filter

$$x=\frac{\dfrac{f}{f_m}-\dfrac{f_m}{f}}{\sqrt{\dfrac{f_2}{f_1}}-\sqrt{\dfrac{f_1}{f_2}}}.$$

Upon making use of the relation $Z_{1k}Z_{2k}=Z_0^2$ the formulæ for mid-series and mid-shunt "constant-k" image impedances can be written as $Z_0\sqrt{1-x^2}$ and $Z_0/\sqrt{1-x^2}$.[4]

This method of representing the image impedances suggests that

[4] In terms of the usual filter notation this $x=\sqrt{-U_k}$.

the configuration of the resistance or conductance controlling network be so chosen that the impedances of its series branches and the admittances of its shunt branches are proportional to x. In other words the series and shunt branches of the correcting network should be similar physically to those of the "constant-k" filter. The complete network is then that shown in Fig. 4. It is similar to that of Fig. 2

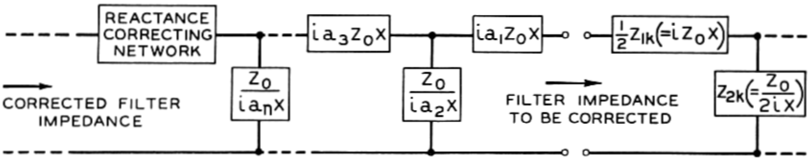

Fig. 4—Generalized schematic of first or "direct" type of filter terminations.

except that the explicit introduction of the factor Z_0 into the expressions for the series and shunt branches reduces the a's to constants of proportionality which can be fixed, once for all, for all "constant-k" filters. Following the analogy of ordinary filter structures it will be assumed that the first branch of the network is in series when the filter proper is mid-series terminated, and vice versa. It is then easily shown that the preceding general formula for the resistance[5] of the system reduces, both for mid-series and mid-shunt terminated filters, to

$$R = \frac{Z_0\sqrt{1 - x^2}}{1 + A_1x^2 + A_2x^4 + \ . \ . \ . \ A_nx^{2n}},$$

when n is the number of branches in the network. It will be observed that odd powers of x are missing.

The possibilities of manipulating this expression to secure desirable resistance characteristics are obviously determined by the number, n, of variable terms in the denominator of the expression. Since n is, however, also equal to the number of branches of the resistance or conductance controlling network, and therefore determines both the cost of this network and the extent to which the resistance or conductance can be made to approximate a given curve, it offers a convenient basis for differentiating between the various structures. The simplest cases, and the only ones of practical importance in contemporary filter design, are those for which $n = 1$, 2, or 3. They are illustrated in Fig. 5 and will be taken up in order. Our first step will be the establishment of the algebraic relations between the element values $a_1 \ . \ . \ . \ a_n$ and the parameters $A_1 \ . \ . \ . \ A_n$ for each of these three cases.

[5] Assuming that the final branch, $1/ia_nx$ is in shunt as in Fig. 4. When the analysis is stated in terms admittances the results are precisely similar, except for an obvious change from Z_0 to $1/Z_0$.

The analyses are stated in terms of conductance and susceptance, since in this form they are most conveniently applied to the impedance correction of systems of parallel filters, which constitute a large proportion of practical cases. The formulæ and curves can, however, be used directly in analyses stated in terms of impedance if we merely replace conductance and susceptance by resistance and reactance and write Z_0 in the numerator rather than in the denominator whenever it

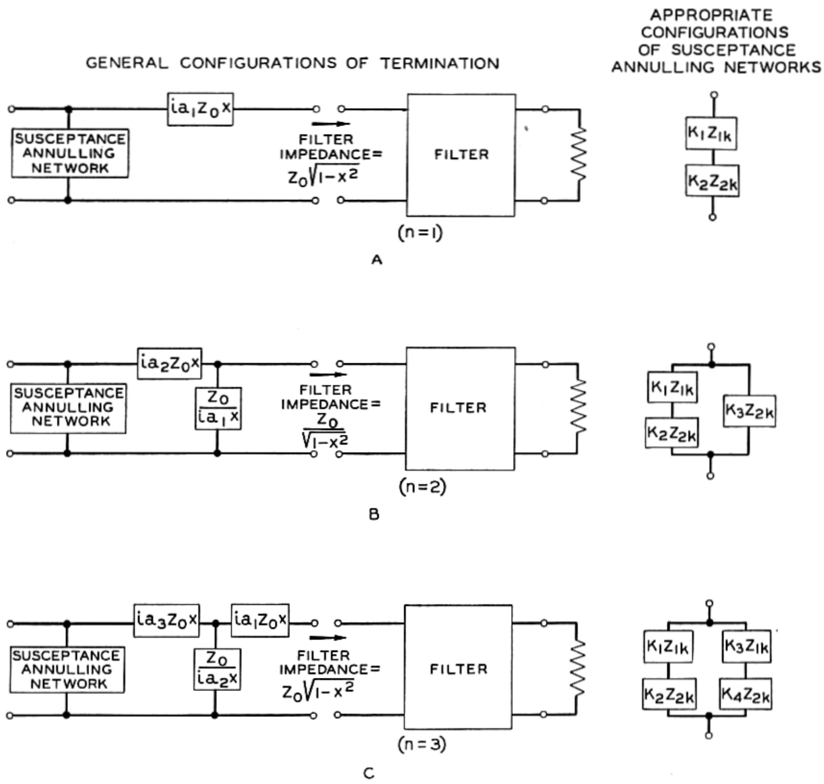

Fig. 5—Configurations of 1, 2 and 3 branch terminations.

appears. After the relations between $a_1 \ldots a_n$ and $A_1 \ldots A_n$ have been determined we shall proceed to a discussion of methods of choosing values of $A_1 \ldots A_n$ giving a suitable resistance or conductance characteristic. The final steps are the computation of the element values of the network from these values of the polynomial coefficients, the calculation of the resulting reactance or susceptance characteristic and the design of a final branch giving the complete structure the desired reactance or susceptance characteristic.

Analytical Relations between Polynomial Coefficients and Element Values

Case I—$n = 1$.

The general analysis shows that the conductance of the system must be expressible in the form

$$G = \frac{1}{Z_0} \frac{\sqrt{1 - x^2}}{1 + A_1 x^2}.$$

A direct mesh computation of the network of Fig. 5-*a* gives

$$G = \frac{1}{Z_0} \frac{\sqrt{1 - x^2}}{1 - (1 - a_1^2)x^2}.$$

From which, by comparison of coefficients,

$$A_1 = -(1 - a_1^2)$$

or

$$a_1 = \sqrt{1 + A_1}.$$

The susceptance characteristic is given by

$$B = -\frac{1}{Z_0} \frac{a_1 x}{1 + A_1 x^2}.$$

It can be annulled exactly by the reactance

$$iX = i\,\frac{1 - a_1^2}{a_1} Z_0 x + \frac{Z_0}{i a_1 x} = \left(\frac{1 - a_1^2}{2a_1}\right) Z_{1k} + \frac{2}{a_1} Z_{2k}$$

where Z_{1k} and Z_{2k} are, as before, the series and shunt impedances of the "constant-*k*" filter.

If the conductance and susceptance controlling portions of the network are combined the resulting structure is identical with a half section of the conventional "m-derived" type. We have merely to replace a_1 by m. Single branch conductance controlling networks therefore contribute nothing new to filter impedance correction. Multiple branch networks, which can be considered, if one pleases, as natural extensions of the "m-derived" scheme, must be looked to for the solution of impedance problems for which standard sections are inadequate.

Case II—$n = 2$.

A direct computation of the network shown in Fig. 5-*b* gives

$$G = \frac{1}{Z_0} \frac{\sqrt{1 - x^2}}{1 + (a_2^2 - 2a_1a_2)x^2 + a_2^2(a_1^2 - 1)x^4},$$

whence

$$A_1 = a_2^2 - 2a_1a_2,$$
$$A_2 = a_2^2(a_1^2 - 1),$$

$$a_1 = \frac{1 \pm \sqrt{1 + A_1 + A_2}}{\sqrt{(1 \pm \sqrt{1 + A_1 + A_2})^2 - A_2}},$$

$$a_2 = \sqrt{(1 \pm \sqrt{1 + A_1 + A_2})^2 - A_2}.$$

The upper of the alternative signs usually gives the better reactance characteristic.

The susceptance of this network is

$$B = -a_2x \frac{\left(1 - \frac{a_1}{a_2}\right) - (1 - a_1^2)\,x^2}{1 + A_1x^2 + A_2x^4}.$$

Case III—$n = 3$

The general conductance expression is

$$G = \frac{1}{Z_0} \frac{\sqrt{1 - x^2}}{1 + A_1x^2 + A_2x^4 + A_3x^6},$$

where

$$A_1 = a_1^2 + 2a_1a_3 + a_3^2 - 2a_2a_3 - 1,$$
$$A_2 = a_2^2a_3^2 + 2a_2a_3 - 2a_1^2a_2a_3 - 2a_1a_2a_3^2,$$
$$A_3 = a_1^2a_2^2a_3^2 - a_2^2a_3^2.$$

These equations can be reduced to

$$a_1 + a_3 - a_1a_2a_3 = \pm\sqrt{1 + A_1 + A_2 + A_3},$$
$$a_1 + a_3 = \sqrt{1 + A_1 + 2a_2a_3},$$
$$a_1a_2a_3 = \sqrt{A_3 + a_2^2a_3^2},$$

from which

$$\sqrt{1 + A_1 + 2a_2a_3} - \sqrt{A_3 + a_2^2a_3^2} = \pm\sqrt{1 + A_1 + A_2 + A_3}.$$

Upon examining the form of the radicals on the left we see that a_2a_3 is determined by the intersection of a parabola and a hyperbola.

Once a_2a_3 are known the individual values of a_1, a_2, and a_3 can be found directly from the previous equations. The two radicals on the left side of the equation must be taken as positive in order to secure positive elements, which is the same as saying that the two conic sections must intersect in the first quadrant. The square root on the right hand side may be taken either as positive or negative, the susceptance characteristic obtained with the negative sign being usually preferable.

It is also possible to eliminate two of the a's directly, obtaining the equation

$$[A_2^2 - 4A_2A_3 - 4A_3]a_1^4 + 8A_3\sqrt{1 + A_1 + A_2 + A_3}\,a_1^3 - [2A_1^2 + 2A_2A_3 - 4A_1A_3]a_1^2 - 8A_3\sqrt{1 + A_1 + A_2 + A_3}\,a_1 + [(A_2 + A_3)^2 + 4A_3] = 0,$$

which can be solved by standard methods. The former method is shorter, however.

The susceptance is given by

$$B = -x\,\frac{B_0 + B_1x^2 + B_2x^4}{1 + A_1x^2 + A_2x^4 + A_3x^6},$$

where

$$B_0 = a_1 + a_3 - a_2,$$

$$B_1 = a_2 + a_3a_2^2 - a_1^2a_2 - 2a_1a_2a_3,$$

$$B_2 = a_1^2a_2^2a_3 - a_2^2a_3.$$

Methods of Choosing Power Series Coefficients

Having developed the relations between the power series coefficients and the network elements we are now ready to consider methods of choosing the parameters to fit given impedance requirements. Upon rewriting our equation for the real component of the network admittance in the form

$$1 + A_1x^2 + A_2x^4 + \cdots A_nx^{2n} = \frac{\sqrt{1 - x^2}}{Z_0G}$$

we see that the problem reduces to the approximation of the ratio of $\frac{1}{Z_0}\sqrt{1 - x^2}$ to the desired conductance G, both of which are known, by means of the polynomial $1 + A_1x^2 + \ldots + A_nx^{2n}$. In most practical designs the desired filter impedance will be a constant resistance. It is then convenient to rewrite the equation as

$$\frac{Z_0}{R_0}(1 + A_1x^2 + \cdots A_nx^{2n}) = \sqrt{1 - x^2},$$

where R_0 denotes the desired constant resistance. The problem thus becomes that of simulating $\sqrt{1 - x^2}$ in the range $0 < x < 1$ by means of a polynomial in x^2 of degree n, and if we assume that the parameter Z_0 can be chosen arbitrarily the polynomial is completely unrestricted, since the constant term as well as the coefficients of the various powers of x can be taken at pleasure.

There are several ways of proceeding from this point. The simplest makes use of the binomial theorem. Upon expanding $\sqrt{1 - x^2}$ with the help of this theorem we reach the relation

$$\frac{Z_0}{R_0}(1 + A_1 x^2 + \cdots A_nx^{2n}) = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 \cdots$$

Equating corresponding powers of x gives

$$\begin{aligned} Z_0 &= R_0, \\ A_1 &= -1/2, \\ A_2 &= -1/8, \\ A_3 &= -1/16, \\ &\cdots\cdots \\ &\cdots\cdots \end{aligned}$$

Using n branches in the conductance controlling network it is possible to take the first n terms of the binomial expansion into account. The elements corresponding to these values of A_1, A_2, etc. can of course be found by the equations derived previously. The results are summarized in the following table.

TABLE I

Number of Branches	A_1	A_2	A_3	a_1	a_2	a_3
1	− 0.5000	0	0	0.7071	0	0
2	− 0.5000	− 0.1250	0	0.97679	1.6507	0
3	− 0.5000	− 0.1250	− 0.0625	1.00308	1.96227	1.62715

The conductance characteristics corresponding to these choices of parameters are shown on Fig. 6. The curve $n = 0$, which corresponds to the "constant-k" type image impedance, has also been added for comparison. It will be seen from the curves that these values of the coefficients $A_1 \ldots A_n$ give very good approximations for small values of x, but inferior ones for values near unity. It is preferable in most designs to sacrifice something at the lower end of the characteristic in order to secure better performance in the higher range.

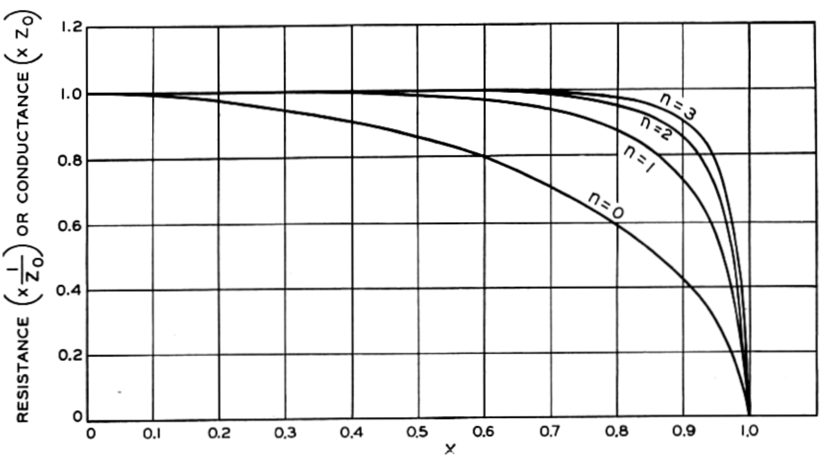

Fig. 6—Resistance and conductance characteristics secured from the binomial expansion.

The advantage of an approximation distributed over the band is gained by an expansion in terms of Legendrian harmonics. These functions are discussed in standard reference books, such as Byerly "Fourier Series and Spherical Harmonics" or Whittaker and Watson "Modern Analysis." It is important to mention here, however, that they are simply polynomials. Any polynomial such as $\frac{Z_0}{R_0}(1 + A_1x^2 + \ldots A_nx^{2n})$ can be broken up into a linear combination of even ordered harmonics, and, conversely, any linear combination of even ordered harmonics can be reduced to the form $\frac{Z_0}{R_0}(1 + A_1x^2 + \ldots A_nx^{2n})$. It is therefore easy to convert an expansion in terms of even harmonics into a power series of the sort with which we are directly concerned. The property of these functions of most interest here is the fact that, for an expansion of any given degree, they give the best "least squares" approximation to the desired function. In the range between $x = 0$ and $x = 1$, therefore, the approximation they furnish is much better for most purposes than that given by the binomial theorem. The expansion of $\sqrt{1 - x^2}$ in terms of Legendrian harmonics is given on p. 184 of Byerly as

$$\sqrt{1-x^2} = \frac{\pi}{2}\left[\frac{1}{2}P_0(x) - 5\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^2 P_2(x) - 9\left(\frac{3}{6}\right)\left(\frac{1}{2.4}\right)^2 P_4(x) - 13\left(\frac{5}{8}\right)\left(\frac{1.3}{2.4.6}\right)^2 P_6(x) + \ldots\right].$$

Upon replacing the harmonics by their values in terms of x,—

$$P_0(x) = 1,$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),$$

$$P_6(x) = \frac{1}{16}(231x^4 - 315x^4 + 105x^2 - 3),$$

and sorting out the various powers of x, values of the coefficients $A_1 \ldots A_n$ are secured, and from these the actual element values are found by means of formulæ developed previously. The following table summarizes the results

TABLE II

Number of Branches	$\frac{R_0}{Z_0}$	K_0	K_1	K_2	K_3	
0	1.273	0.7855	0	0	0	
1	0.9699	0.7855	− 0.4909	0	0	
2	1.011	0.7855	− 0.4909	− 0.1105	0	
3	0.9948	0.7855	− 0.4909	− 0.1105	− 0.04986	
Number of Branches	A_1	A_2	A_3	a_1	a_2	a_3
0	0	0	0	0	0	0
1	− 0.7142	0	0	0.5546	0	0
2	− 0.3236	− 0.4884	0	0.8986	1.593	0
3	− 0.0461	+ 0.4958	− .7162	0.9597	1.924	1.565

The quantities $K_0 \ldots K_3$ are the numerical coefficients of the corresponding harmonics. It will be observed that with this method of determining the network parameters Z_0 is not quite equal to R_0. When the analysis is based upon impedances instead of admittances the ratio R_0/Z_0 should be replaced by Z_0/R_0. The conductance characteristics secured by this process are shown in Fig. 7.

It is, of course, always possible to dispense with these general methods entirely and make an empirical determination of the design parameters. The particular requirements of specific design projects

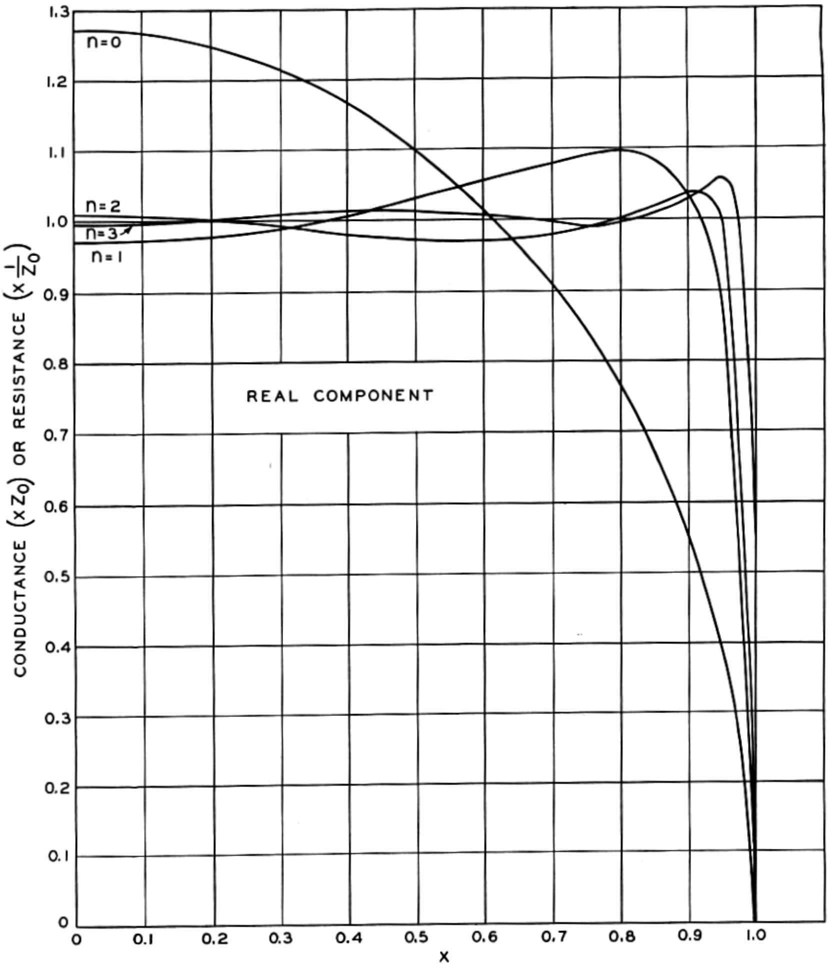

Fig. 7—Resistance and conductance characteristics secured from expansion in terms of Legendrian harmonics.

are thereby given the fullest recognition. This method was used in constructing the sections described in the accompanying paper. Even when the empirical method is adopted, however, the networks determined by the general expansions, particularly that in terms of Legendrian harmonics, should be valuable as starting points.

In most designs it is desirable to make the maximum departure from the ideal characteristic within the operating range as small as possible. A method of doing this for the 2-branch networks has been developed. The method assumes that the Z_0 of the filter has been taken equal to the terminating impedance, which assures a correct conductance at the point $x = 0$. The manipulation of the parameters A_1 and A_2 allows us to secure the desired value of conductance at two additional points. The result is a two looped characteristic, similar, if we make allowance for the difference in the assumptions regarding Z_0, to that already determined for this network by means of the Legendrian expansion. The requirement that the maximum departure from the ideal within the operating range be a minimum is equivalent to saying that the amplitudes of the downward and upward loops must be equal. It can be shown that a 2-branch conductance network will satisfy this condition if

$$\frac{-\frac{27}{16}A_2}{1 + A_1 + A_2} = (A_1{}^2 - 4A_2)^2.$$

In view of the relations which have been developed between A_1, A_2,

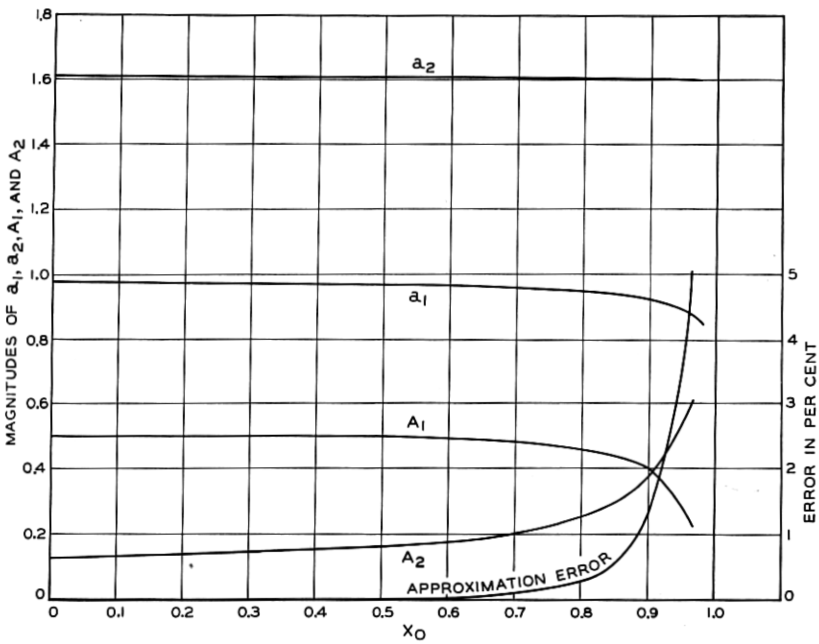

Fig. 8—Design chart for 2-branch termination.

and a_1, a_2 this condition can also be written in the form

$$\frac{27}{16}(1 - a_1^2) = a_2^2[4(1 - a_1a_2)^2 + a_2^2(1 - a_1a_2)]^2.$$

A second condition upon these quantities is found by specifying the range within which the impedance is to remain as flat as possible. The results of computations to determine this relationship are given in Fig. 8. x_0 in this diagram signifies the highest value of x in the operating range. Fig. 8 also gives the maximum departure of the conductance characteristic from its ideal value as a function of x_0. Numerical data taken from these curves should of course be confirmed by the equations given herewith before they are used to specify element values.

Susceptance Correcting Networks [6]

Once the conductance controlling portion of the network has been determined by one or another of these methods our general procedure calls for the computation of the susceptance characteristic it furnishes and the design of a final shunting reactance network which will annul

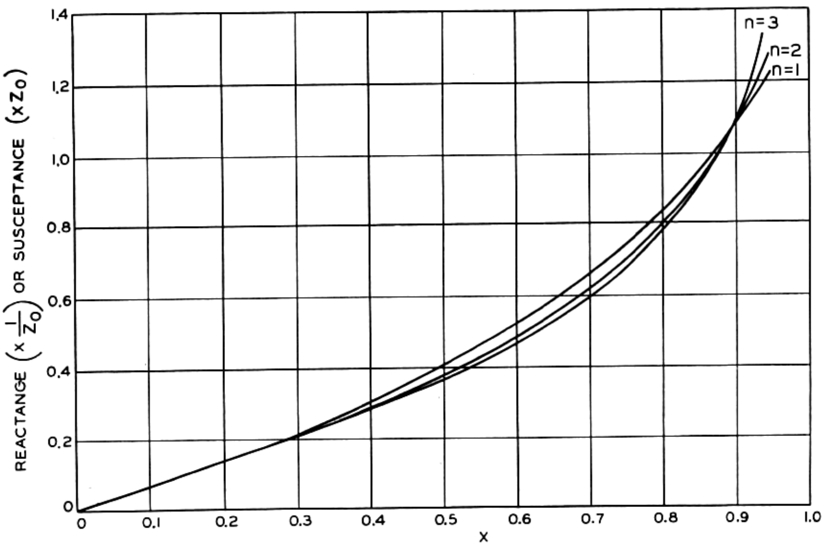

Fig. 9—Reactance and susceptance characteristics secured from binomial expansion.

[6] This section gives only a general description of the characteristics required of the susceptance correcting networks and the configurations which have been found appropriate for them. The design of these networks may be conveniently approached by means of the formulæ contained in R. M. Foster's article "A Reactance Theorem," in the Oct. 1924 issue of this *Journal*.

this susceptance to a suitable approximation. Fortunately the characteristics required of this network are of a type which can readily be obtained with physically realizable elements. The curves of Figs. 9 and 10 represent the susceptance characteristics required for the

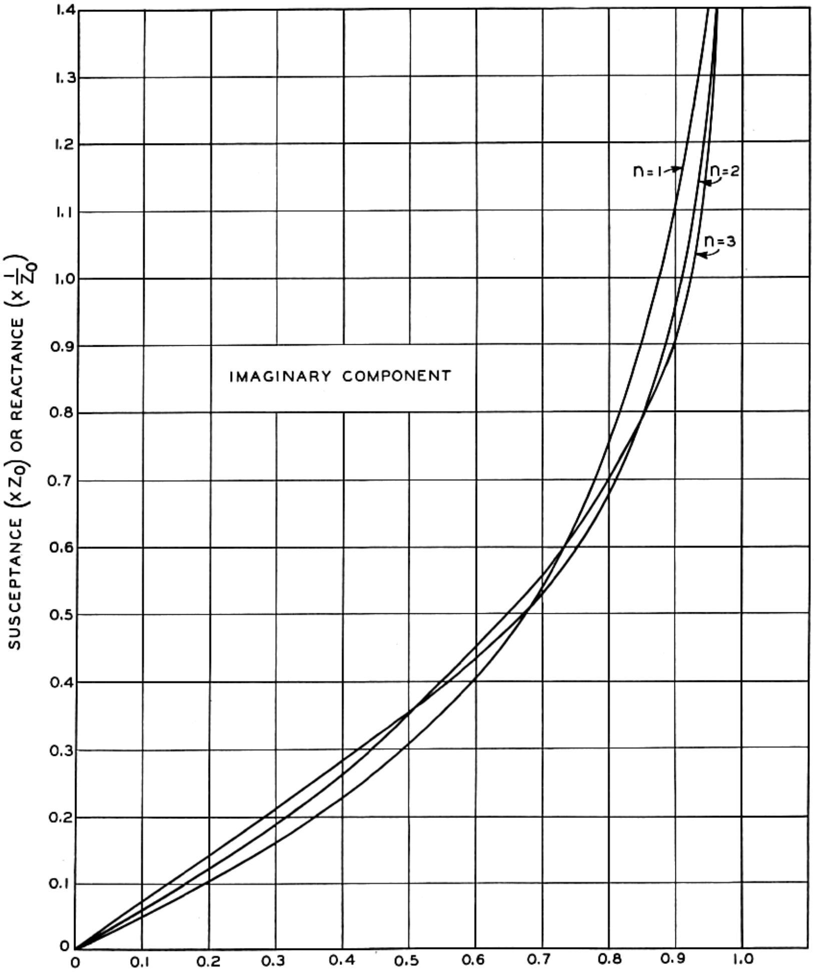

Fig. 10—Reactance and susceptance characteristics secured from expansion in terms of Legendrian harmonics.

Legendrian and binomial expansion networks. Empirically determined networks give very similar results. The general configuration of appropriate susceptance correcting networks can be determined from an inspection of these curves. For example, if we assume that a low pass filter is in question, which means that the variable "x"

is proportional to frequency, the desired susceptance curves will be recognized as being approximately those which would be obtained from tuned circuits resonating slightly beyond the cutoff. Since a tuned circuit can be considered as being a series combination of the series and shunt impedance of the "constant-k" filter, any such correcting network designed for a low-pass filter can be adapted to

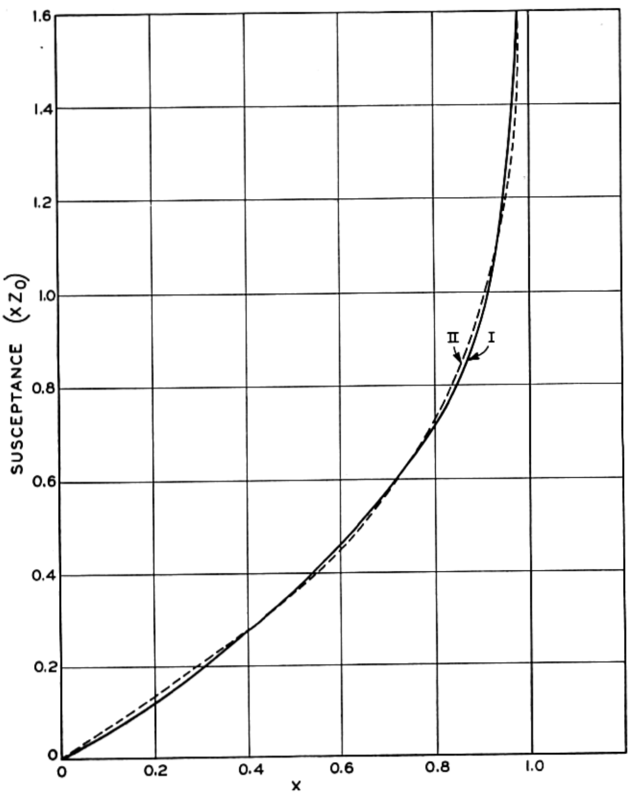

Fig. 11—Susceptance correction of a 3-branch termination.
I—Desired susceptance.
II—Susceptance actually obtained.

another type of "constant-k" structure by replacing inductances and capacities by the homologous impedances of the other filter.

This simple combination of series and shunt impedances is, as we have previously seen, capable of giving exact susceptance correction when the conductance controlling network contains only one branch, but it is not, in general, sufficient for 2 and 3 branch networks. Indeed, no physically realizable reactive network will cancel the susceptance furnished by these more complicated structures exactly. Close approximations however can be obtained by modifying the "tuned circuit" characteristic slightly through the introduction of extra elements. Suitable configurations for 2 and 3 branch networks have already been given in Fig. 5. They should furnish susceptance characteristics at least as good as the corresponding conductance characteristics. An example of the susceptance correction of a three branch network, using the configuration of Fig. 5-c, is shown in Fig. 11. Curve I represents the ideal susceptance characteristic, Curve II that actually obtained.

Impedance Correction of Paralleled Filters

An interesting modification of the process of susceptance correction occurs when a number of filters are to be connected in parallel. Since the impedance of an attenuating filter is almost a pure reactance the conductance component of a system of parallel filters at a given frequency is furnished almost entirely by the filter in whose transmission band that frequency lies. If the system as a whole is to have the correct conductance throughout each transmission band, therefore, every filter must be given the conductance controlling network which would be appropriate if it were operating alone. While the process of conductance correction is thus exactly the same for multipled and individual filters, the process of susceptance correction of paralleled filters must be modified somewhat to take account of the susceptance component furnished by the attenuating filters. A single susceptance network will serve for the whole system. We have merely to compute the susceptance characteristics furnished by the various filters terminated in their conductance controlling networks and annul them throughout every transmission band by a two terminal network in parallel with the system as a whole. An example of the application of the method to a pair of parallel complementary filters having 2 branch conductance controlling networks is given by Fig. 12. Curve I in this diagram represents the susceptance of the transmitting filter, Curves II the susceptance of the attenuating filter for several different choices of its cutoff frequency, Curves III the susceptances of the corresponding auxiliary networks, and Curves IV the net result. A series combination of the series and shunt impedances of either filter[7] resonating at the geometric mean of the cutoff frequencies was chosen for the

[7] Since the filters are complementary the series impedance of one is similar to the shunt impedance of the other, and vice versa. By choosing the resonance frequency of the auxiliary network symmetrically with respect to the two filters, as we have done, all of the susceptance relations become symmetrical, and the network functions as well for one filter as it does for the other.

auxiliary network. By using two resonant arms with closely adjacent resonance frequencies still better susceptance correction could have been secured.

Filters which must operate in parallel are usually given x-terminations. Since an x-termination can be thought of as being a one element

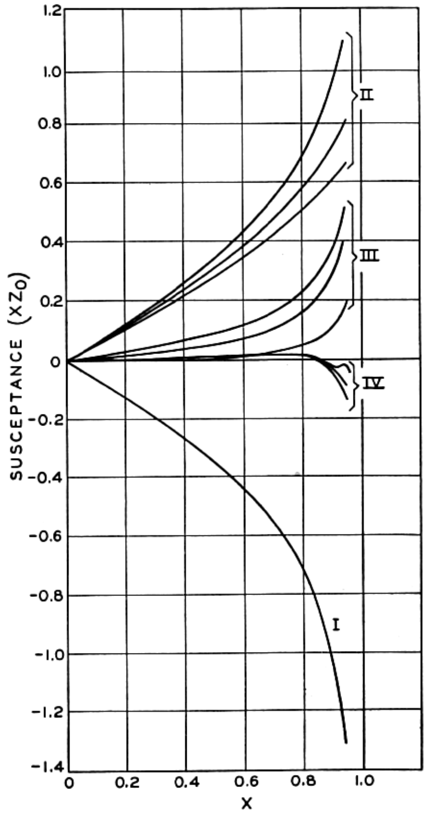

Fig. 12—Susceptance relations at the line terminals of a pair of parallel complementary filters having 2-branch conductance controlling networks.

conductance controlling network the method we are discussing can be applied here also. It is interesting to note that the introduction of an auxiliary susceptance controlling network considerably improves the performance even of this well known circuit. The susceptance relations at the line terminals of a pair of parallel complementary x-terminated filters are shown in Fig. 13, the arrangement of the curves being similar to that of Fig. 12. The improvement can be estimated from the magnitude of the auxiliary susceptance.

The auxiliary network improves the susceptance of parallel band pass filters even more than it does that of complementary filters. Curve I of Fig. 14 represents the susceptance of a typical uncorrected set of band pass filters. The first step in the improvement of this characteristic is due to Mr. R. H. Mills, who suggested that networks whose impedances resemble that of filters above and below the actual set of bands be added to the system. This reduces the susceptance

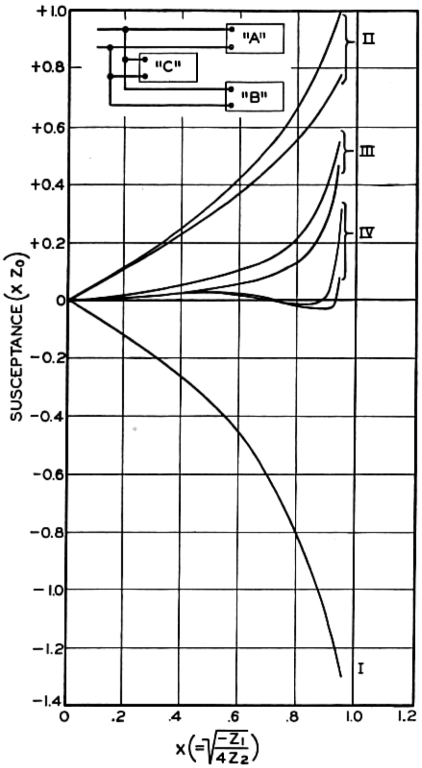

Fig. 13—Susceptance relations at the line terminals of a pair of parallel complementary x-terminated filters.

to the level shown by Curve II. Curve III gives the completely corrected characteristic. The auxiliary susceptance correcting network consists of a number of tuned circuits in parallel, one resonating between each pair of successive bands, together with one resonating above the topmost band and one resonating below the lowest band. The insertion of the auxiliary network has the further advantage that it produces peaks of attenuation near the cutoffs of the filters, thus enhancing their selectivity.

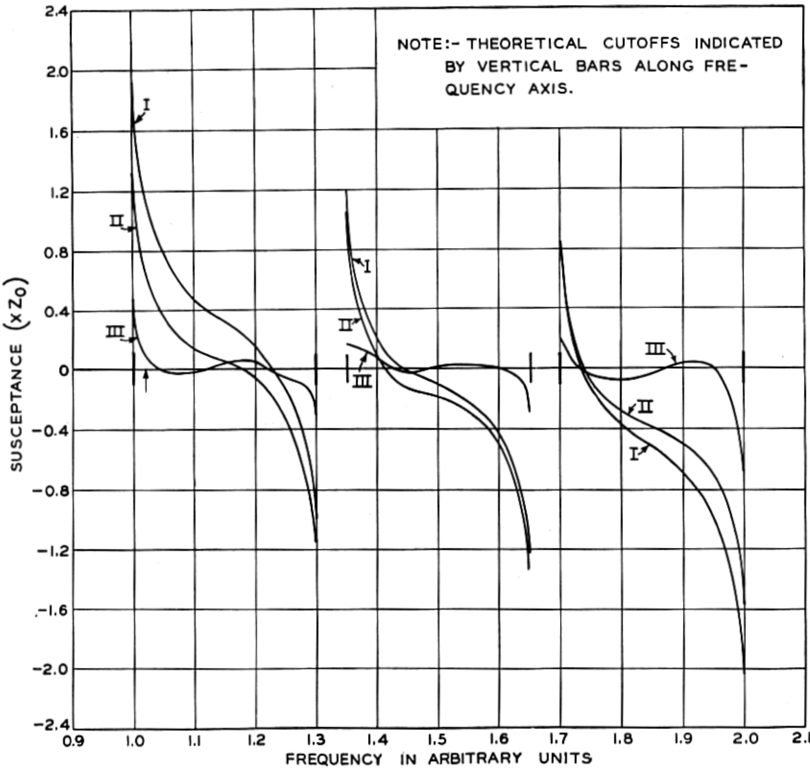

Fig. 14—Susceptance correction of a set of parallel x-terminated band-pass filters.

I—Uncorrected susceptance.
II—Susceptance after the addition of a simple auxiliary network.
III—Susceptance after the addition of a more elaborate auxiliary network.

Reverse Method of Designing Terminating Sections

Hitherto we have assumed that the load impedance of the terminating network was the filter image impedance, and our procedure has consisted essentially in determining an adjustment of the network parameters which would make its input impedance a constant pure resistance. As we have already seen, however, it is equally legitimate to assume that the network is terminated in the line resistance, and determine parameter values which will produce a match between its impedance and that of the filter. This assumption leads to the circuit arrangement shown in Fig. 15.

Upon examining what happens to the general expression for the resistance of the network when the load impedance reduces to the constant pure resistance, R_0, we easily find that it turns out to be

$$R = \frac{R_0}{1 + A_1x^2 + A_2x^4 + \ldots + A_nx^{2n}},$$

where n is the number of branches in the network. Odd powers of x are missing, just as they were when the network was terminated in a filter impedance.

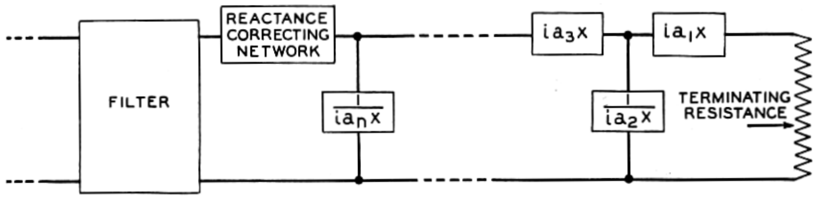

Fig. 15—Generalized schematic of second or "reverse" type of filter terminations.

Our problem consists in matching this expression to the filter impedance, $Z_0\sqrt{1 - x^2}$. Upon assuming that $R_0 = Z_0$, for simplicity, we see that it reduces to the selection of values of $A_1 \ldots A_n$ which will secure approximate satisfaction of the equation

$$1 + A_1x^2 + \ldots + A_nx^{2n} = \frac{1}{\sqrt{1 - x^2}}$$

Two empirical [8] choices of these parameters have been made, one

[8] Our previous methods of approximation, in terms of Taylor's series and Legendrian harmonics, are of course available here also. In addition, if we rewrite the expression as

$$\sqrt{1 - x^2}\,\frac{Z_0}{R_0}\,(1 + A_1x^2 + \ldots + A_nx^{2n}) = 1$$

the left hand side appears as a linear combination of the associated Legendrian functions $P_1'(x)$, $P_3'(x)$, ..., defined by the general formula

$$P_n'(x) = \sqrt{1 - x^2}\,\frac{d}{dx}\,P_n(x),$$

where $P_n(x)$ is the usual Legendrian function. The problem can therefore be considered as that of approximating unity by a series of the associated functions. These methods of approach differ chiefly in the relative weights which they ascribe to various portions of the frequency band. Judged by this criterion neither of the first two methods is very satisfactory for practical applications. The Taylor's series expansion, of course, is best in the neighborhood of $x = 0$. The "least squares" property of the ordinary Legendrian functions, on the other hand, tends to produce rough equality in the numerical values of the departures from the desired function in various portions of the frequency band. From the engineering standpoint, however, it is the percentage departure from the desired impedance, and not the numerical departure, which is of interest. This type of approximation therefore leads to a relative over-emphasis of the region near $x = 1$, where the desired function $1/\sqrt{1 - x^2}$ is large. The approach by means of the associated functions, however, avoids this objection, since the approximated function is in this case a constant, and leads to characteristics substantially as good as those obtained by means of the empirically determined parameters discussed in the text.

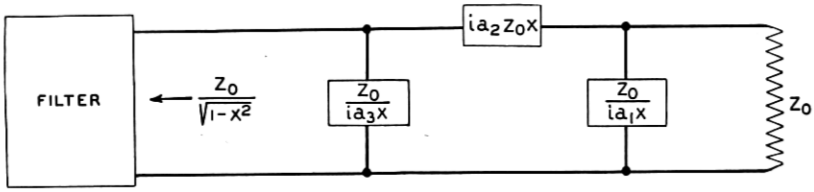

Fig. 16—A 2-branch termination of the "reverse" type.

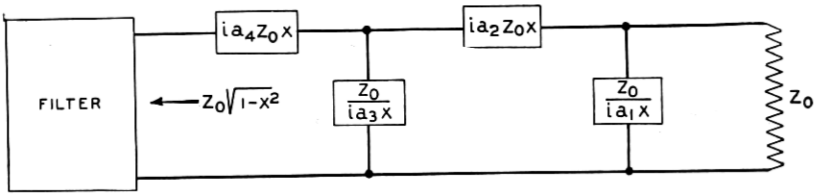

Fig. 17—A 3-branch termination of the "reverse" type.

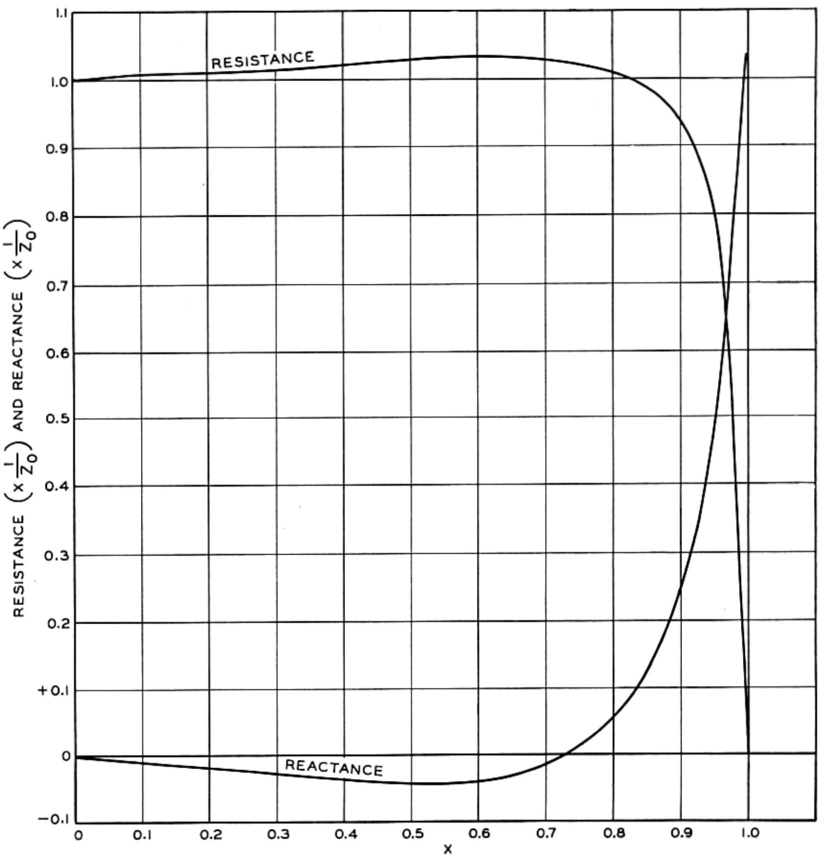

Fig. 18—Impedance characteristic secured from the network of Fig. 16.

when the network contained two branches, and the other when it contained three. In both instances the appropriate reactance or susceptance annulling networks were found to be simple arms, similar to the series or shunt branches of the remainder of the termination in physical configuration. The complete networks are shown in Figs. 16

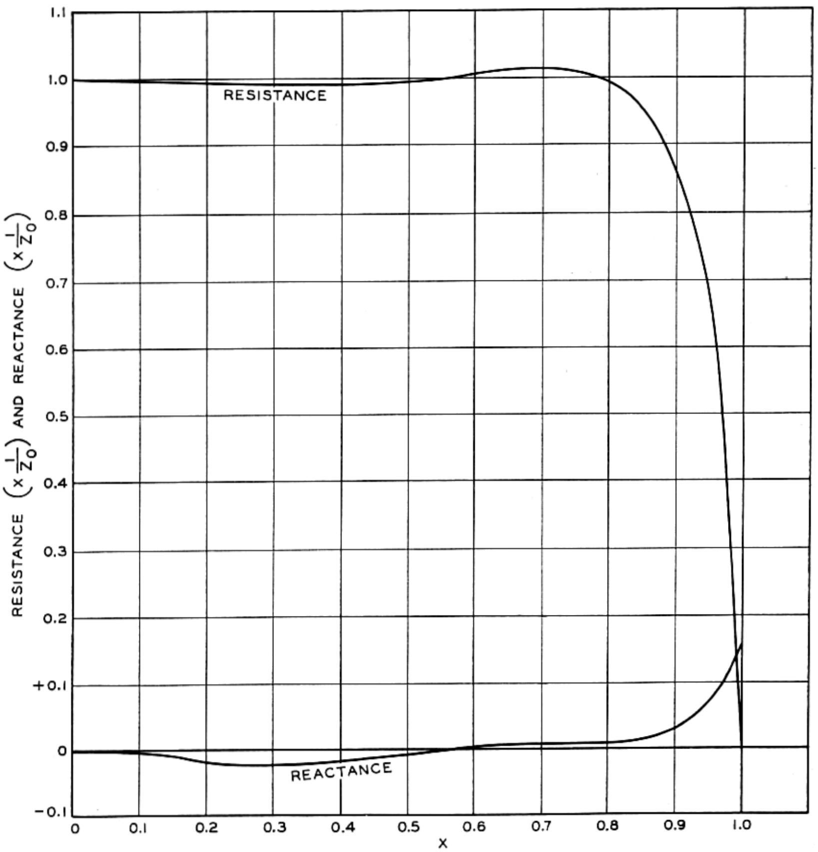

Fig. 19—Impedance characteristic secured from the network of Fig. 17.

and 17, where the final branches, Z_0/ia_3x in Fig. 16, and ia_4Z_0x in Fig. 17 are the susceptance or reactance annulling networks. The values of the various parameters are given in the following table.

TABLE III

n	A_1	A_2	A_3	a_1	a_2	a_3	a_4
2	+ 0.0505	+ 1.6508	0	0.7973	1.6186	0.904	0
3	+ 0.9114	− 1.8488	+ 3.2823	0.6733	1.466	1.835	0.925

If a perfect match were secured at the filter terminals then, by the reciprocity principle, a perfect match should be secured at the line terminals also. In order to evaluate the performance of the networks, therefore, the impedances they present to the line were computed. The results are shown in Figs. 18 and 19.

Comparison of Direct and Reverse Networks

At first glance the curves of Figs. 18 and 19 seem to show that while networks of the reverse type produce a good impedance match over a moderate fraction of band they will be much less successful than the structures previously described at frequencies very near the cutoff. This apparent advantage in favor of the networks first described is discounted considerably however by the economy of elements resulting from the relative simplicity of the reactance or susceptance controlling networks used with terminations of the second type. If we adopt as our standard in comparing the two types of networks the total number of elements each requires, rather than the number of branches they contain, the advantage of networks of the first type becomes much less impressive, if it does not actually disappear. More important considerations recommending the first type of terminations in preference to the second for most practical designs appear to be the greater ease with which they can be designed to meet a given reflection coefficient requirement, resulting from the relatively smaller number of branches they contain, the greater ease with which they can be adapted to filters which must operate in parallel, and the fact that the attenuation they contribute to the total filter suppression is usually more useful than that furnished by terminations of the second type.

Under certain circumstances, however, the second type of terminating sections have a definite advantage over the others. When a filter operates in conjunction with a modulating device a high modulator efficiency with low distortion demands that the impedance of the filter to the untransmitted side band be low (or high) and nearly constant. In spite of their poor characteristics within the transmitting band it has hitherto been necessary to use mid-shunt image impedance terminations of the "constant-k" type in these circuits. Impedance correcting sections of the first type are not suitable for this service because the complicated susceptance and reactance annulling networks at their line terminals produce sharp changes in reactance in the attenuating region. The outermost branch of terminations of the second type, however, is of simple configuration and if we choose it to resemble the final branch of a mid-shunt terminated "constant-k" type filter, as has been done in the sections shown in Figs. 16 and 17, we will secure an impedance characteristic beyond the band almost as good as that of the "constant-k" filter. Within the transmitting band, of course, its impedance is much better than that of the normal filter section.

Attenuation Characteristics of Terminating Sections

In the practical application of either type of terminating section some others of their characteristics, such as their transmitting efficiency and the effect produced upon them by parasitic dissipation of energy in the network elements, are also of importance. The transmission characteristics of the networks can be determined roughly by comparing them with standard filter sections. Let us consider, for example, the two branch termination shown in Fig. 20. If we neglect for

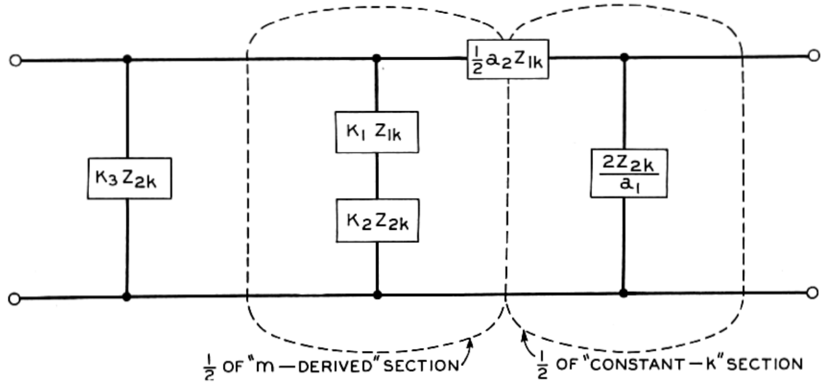

Fig. 20—Figure illustrating approximate transmission characteristics of 2-branch terminations.

the moment the third element of the susceptance correcting network, the remainder of the structure can be divided, in the manner indicated by the broken lines, into two portions, one of which resembles half of a "constant-k" section and the other half of an "m-derived" section in physical configuration. The transmission characteristic of the actual network is substantially similar to that which would be furnished by standard filter sections of these types. The mere fact that the network functions as an impedance corrector is, of course, sufficient to show that it will transmit efficiently frequencies within the nominal transmission band of the filter. Beyond the transmission band the attenuation characteristic would be almost exactly coincident with that of the suggested filter equivalent if it were not for the extra element in the final shunt branch. The extra element produces an anti-resonance in this arm somewhat beyond the resonance and near the anti-resonance point the attenuation is somewhat less than that which would be secured from ordinary filter sections. On the other hand the extra element considerably increases the admittance of the final shunt arm, and therefore the attenuation of the network, at frequencies remote from the cutoff. In spite of these modifications the analogy to standard sections is a fairly trustworthy guide to the attenuation of the networks. Several examples are given in the accompanying paper.

Since the ideal pure reactances contemplated by the theory are not physically available these conclusions must be modified somewhat in practical designs. As we might expect, however, unavoidable dissipation of energy in the network elements will alter the transmission characteristic of the correcting device about as it would that of an ordinary filter. In the attenuating range the effect can be neglected. In the nominal transmission band absorption of energy in the termination will reduce the transmitting efficiency of the circuit somewhat, but the loss in efficiency is no more serious than it would be in standard filter sections having the same general configuration.

Parasitic resistances in the network elements may of course affect the impedance as well as the transmission properties of the circuit. Since the structure is used primarily because of the impedance characteristic it furnishes, possible changes in impedance, caused by variations in the phase angles of the network elements, are of particular interest. Changes in impedance produced by dissipation of energy in the correcting networks, are easily estimated when the complete circuit with whose impedance we are concerned can be considered as a network of ordinary resistances, inductances and capacities and when dissipation affects the phase angles of all reactive elements equally. It can be shown that in such a network the change produced by dissipation in the resistance of the structure is proportional, to a first approximation, to the derivative of its reactance characteristic with respect to frequency, and that conversely the change in the reactance characteristic is proportional to the frequency derivative of the resistance characteristic. The explicit formulæ are:

$$\Delta R = \quad fd\frac{dX}{df},$$

$$\Delta X = -fd\frac{dR}{df},$$

where f is frequency and d the dissipation constant (defined as ratio of resistance to reactance) for each reactive element.

A filter, with its terminating sections and load resistance, is a network of resistances inductances and capacities to which the theorem applies. It seldom happens of course, that all of the reactive elements of the structure actually have the same dissipation constant. It is usually sufficient, however, to assume that "d" in the above formulæ is the average of the dissipation constants for coils and condensers. When well designed impedance correcting networks are used the reactance and resistance characteristics of the structure will be approximately constant over the operating range. The derivatives occurring in the above formulæ will consequently reflect only the presence of slight ripples in these characteristics about their mean values. The slopes of these ripples will usually be quite small. We can therefore conclude that *moderate amounts of dissipation will have no appreciable effect upon the impedance of a properly terminated filter.* The chief exceptions to this rule occur in low pass filters, where, at low frequencies the assumption that the dissipation constant is small is no longer satisfied.

In attempting to extend this principle to broader problems in impedance correction it is, of course, necessary to bear in mind that the analysis holds only for networks of resistances, inductances and capacities. We cannot expect the same results when the load impedance of the circuit has some arbitrary variation with frequency. For example, if we take the load impedance as the image impedance of a dissipationless "constant-k" filter and assume that parasitic resistances occur only in the termination, we will find that dissipation does change the impedance of the circuit. The circuit impedance will be insensitive to dissipation only when we include the complete structure, and not merely the terminations, in our analysis.

More General Problems of Impedance Correction

This general method of impedance correction having worked with reasonable success in its application to "constant-k" wave filter impedances, it is natural to inquire whether it can be applied to other problems with equal ease. Further possibilities for example might include the correction of other types of filter impedances, or the correction over extremely wide frequency bands for the effects of leakage inductance and finite mutual inductance in transformers, or the reduction of actual transmission line impedances to constant resistances. All of these possible applications assume that the impedance correcting device is a 4-terminal network, transmitting useful signal energy to its load impedance. When terminated by such an element as a simple resistance, however, it might also be used as a 2-terminal network, forming one branch of a complete system. By appropriate adjustment of the impedance controlling parameters the network could, theoretically at least, be given a wide range of impedance characteristics.

We might, for example, use it to approximate a pure resistance varying in an arbitrary manner with frequency, which would be a valuable impedance element in certain circumstances.

None of these possibilities has been investigated in detail, and naturally the measure of success which can be achieved with any one of them will depend largely upon the precise conditions of the problem. The mathematical form we have specified for the load impedance of the network is so broad however that if we were to consider only this aspect of the situation we might conclude that the scope of the structure is well nigh universal. For example, the impedance of any finite network of resistances, inductances, and capacities can be written in the appropriate mathematical form. Even when the load impedance is not described in the required manner, either because it is empirically determined or because it has the wrong theoretical formula, the type of algebraic expression we have been considering is so general that it can always be matched approximately.

Unfortunately, the range of application promised by this rather superficial mathematical discussion may be severely restricted by other considerations. In the general case, for instance, the number of terms in the denominator of the resistance expression will be greater than the number of branches in the correcting network and it will not be possible to choose them all arbitrarily. Moreover, even when the correct number of conditions have been imposed upon the power series coefficients we have no assurance that the resulting system of simultaneous equations between coefficients and element values can be solved, or that the solutions, if obtained, will always correspond to physically realizable elements. Finally, we may observe that although no difficulty was experienced in the reactance or susceptance correction of filters, it seems probable that, in view of the limited range of characteristics which can be simulated by physically realizable reactive structures, a straightforward application of the general method of resistance correction will often leave us with a reactive characteristic which cannot be corrected.

These difficulties may occasionally be overcome by slight modifications in the design process. Among other possibilities for example, we can adjust the lowest powers both in the denominator of the resistance expression and numerator of the reactance expression [9] to desirable values, obtain an approximate value for the effect of higher powers in both expressions by a trial computation and readjust the coefficients of the lower powers to take account of these previously neglected terms.

[9] Since the denominator of the reactance is equal to that of the resistance, whose value is prescribed by the requirements, the reactance expression can be determined completely from its numerator alone.

Difficulties appearing in a direct application of the impedance correcting process may also be avoided if we adopt the reverse method of impedance correction suggested by the theorem on reciprocal impedance relationships. The method has already been applied to the construction of alternative filter impedance correcting sections. Similar alternative configurations can be built up in any impedance correcting problem if we consider that the structure is terminated by the conjugate of the desired impedance and adjust its parameters to produce the conjugate of the given impedance. Since the desired impedance will in general be a relatively simple function of frequency, this alternative procedure at least avoids analytical complexities. In spite of these possibilities however it seems probable that the method will fail in many situations. It seems best adapted to such problems as that of filter impedance correction, where a transformation must be made from one fairly simple characteristic to another simple characteristic. An attempt to apply it to more difficult problems should result, at best, in very complicated networks.

Transmission Properties of Impedance Correcting Networks

The close relationship between the impedance correcting properties of our networks and their transmission characteristics has been manifest from time to time in the previous discussion. The networks used at filter terminations, for example, transmitted freely within the range in which they functioned satisfactorily as impedance correctors but attenuated other frequencies. That this will be true in general is easily seen by inspection. Within the range in which a desired impedance characteristic is obtained, of course, our previous argument from the principle of conservation of energy is alone sufficient to show that the networks transmit with the maximum efficiency compatible with the impedance requirements imposed upon the circuit. On the other hand, it is evident from the filter-like configuration of the networks that at frequencies remote from the operating range of the networks, where the parameter "x" becomes large, the structures will ordinarily introduce attenuation. From the impedance standpoint this means merely that for sufficiently large values of x the polynomial approximations upon which the analysis is based no longer hold, and the resulting mismatch between the generator and network impedances diminishes the amount of power which can enter the structure.

When the impedance correcting analysis is stated in a slightly modified form, whose possibilities have not as yet been completely investigated, the impedance and transmission characteristics of the circuit are still more firmly related. Thus for example the attenuation of the structure beyond its operating range results chiefly from the readily computed departure of the resistance or conductance characteristic of the network from that of the generator. It is also produced, in part, however, by the failure of the reactance or susceptance correcting network to annul in this range the imaginary component furnished by the resistance or inductance controlling network and the effect of this factor is less easy to determine. In the modified analysis it is often possible to do away with the distinction between the two types of networks. The complete insertion loss characteristic is then embodied in a single polynomial expression. In the modified form, moreover, the analysis may often be used to determine the phase as well as the attenuation of the circuit.

Granted these results, it is but a short step to the conclusion that the impedance correcting analysis offers a possible approach to the design of filters. While it is usually true that the networks will attenuate frequencies beyond the region in which impedance requirements have been set, the amount of the mismatch which produces this attenuation, since it depends upon the impedance correcting parameters, is still more or less under our control. By suitable adjustments of the correcting network, therefore, we can design a structure to meet attenuation as well as impedance requirements. A particularly interesting situation occurs when the load impedance is a constant pure resistance.[10] As we have already seen, a load impedance of this type satisfies our mathematical specification and it can therefore be used with a ladder network. Since a perfect impedance match already exists in the circuit an inserted network can be called an impedance correcting device only by courtesy. Unless the network contains so many branches that the mathematical complexity of the problem is overwhelming, however, it is possible to so manipulate the impedance correcting parameters that the network impedance matches the generator impedance approximately over a certain frequency band but is very poor outside this range. It follows from our previous discussion that the network will transmit frequencies lying within this band efficiently, but will attenuate other frequencies. Networks designed in accordance with this method therefore function as filters. They differ from conventional filters in several respects, however. For example they are non-recurrent, they cannot be divided into discrete sections with matched image impedances, and they do not possess definite cutoffs.

[10] This circuit arrangement was first investigated by E. L. Norton and W. R. Bennett, who developed a complete analysis for a number of particular cases.