Magnet Steels and Permanent Magnets—Relationships Among Their Magnetic Properties *

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Introduction

A GENERAL study of the magnetic properties of magnet steels and permanent magnets embraces a number of related problems. Of chief interest to the designer of magnets is the problem of determining the relationship between the open-circuit remanence of a permanent magnet and the various factors which determine its value. These factors include the magnetic characteristics or properties of the steel, as displayed by its hysteresis loop, and the shape and dimensions of the magnet. Of interest to the manufacturer of magnet steel are the questions of how chemical composition, melting practise, and rolling practise affect the magnetic and mechanical properties of the steel. The manufacturer of magnets is interested in the effect of the necessary manufacturing operations, such as heating for hot forming and annealing to increase machinability, upon the flux obtainable in the finished magnet; on the tendency of the steel to warp and crack upon quenching; and on the proper hardening treatment to use.

These interests overlap, of course, and include many special subjects, among which may be listed the manner in which various influences may affect the state of magnetization of a magnet, the correlation of the microstructure and mechanical properties of a magnet steel with its magnetic properties, the determination of a suitable criterion of magnetic quality, the development of accurate and convenient testing equipment, and various other matters of greater or less importance.

It is the purpose of this paper to present data relating to some of the above topics, which have been collected by the writer during the course of several years connection with the manufacture of permanent magnets, both in the laboratory and in the shop.

Symbols and Notations

The following symbols and notations will be used. See Fig. 1. All values of B are intrinsic or ferric induction.

B_{max} = The value of magnetic induction corresponding to the tip of a given hysteresis loop, in gauss.

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H_{max} = The value of H corresponding to B_{max}, in oersteds.
B_s = The saturation value of flux density corresponding to an infinite magnetizing force.
B_r = Residual induction, the magnetic induction in a ring or infinitely long straight bar after the value of H has been reduced from H_{max} to O. The value of B at the intersection of the hysteresis loop with the B axis, in gauss.
H_c = Coercive force, the value of H required to reduce B from B_r to O in a ring or infinitely long bar. The value of H at the intersection of the hysteresis loop with the H axis, in oersteds.
B_{rem} = Remanence, the magnetic induction at the magnetic equator of a permanent magnet with no external magnetizing or demagnetizing force. Values of B_{rem} given in this paper are without pole pieces on the magnets, in gauss.
L = The actual developed length of a magnet.
A = The area of cross section of a magnet.
D = The equivalent diameter of a magnet
$= 2\sqrt{A}/\sqrt{\pi}$
L/D = The dimension ratio of a permanent magnet.
$(BH)_{max}$ = The maximum value of the product of the coordinates of the demagnetization curve for a given steel. The demagnetization curve is the portion of the hysteresis loop between B_r and H_c.
T = Hardening temperature, absolute scale.
T_0 = Optimum hardening temperature, absolute scale.

Factors that Determine the Remanence of a Permanent Magnet

Of the subjects enumerated in the introduction, the first is the one that has been of greatest technical interest to the writer. During the course of several laboratory investigations connected with shop problems, advantage was taken of the opportunity to secure data which might be used for the purpose of making generalizations regarding relations among magnetic properties of magnet steels and permanent magnets.

The specific problem was to find out exactly the way in which the remanence of a magnet is related to its shape and dimensions as well as to the magnetic properties of the material of which it is made. Although it requires the complete family of hysteresis loops for a given piece of magnet steel to portray all of its various magnetic characteristics, it was felt that possibly the combined influence upon the remanence of a magnet of the sum of these characteristics could be expressed in terms of the values of B_r and H_c for the material. Such an assumption, if valid, would simplify the problem considerably.

It is true that in the literature on the subject there are several papers [1,5,6] describing more or less accurate methods by means of which a designer may predict the flux of a given magnet provided the demagnetization curve for the steel is known. But these methods are laborious, and unless a complete redesign is worked out for each case,

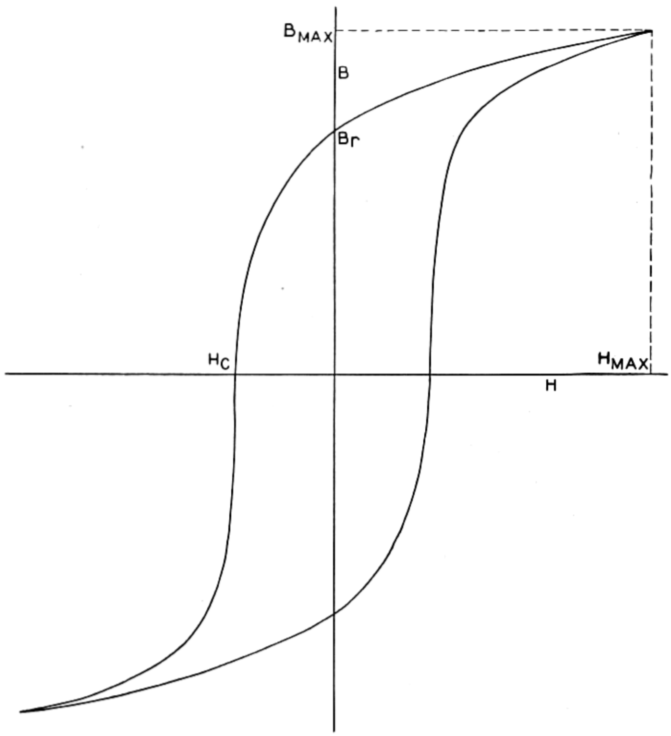

Fig. 1—Hysteresis loop of magnet steel.

they do not answer such questions, for example, as how much a certain change in B_r or H_c will affect the remanence of a given magnet, other factors remaining constant.

Because the flux value of a magnet is affected by shape as well as other factors, and because magnets are made in a bewildering variety of shapes, it was decided to confine the investigations to straight bar magnets and leave the matter of the effect of shape on flux for a later study.

[1] For references see end of paper.

Test Specimens. Accordingly, a large number of straight bars of magnet steel were used for test specimens. Each of the bars was 12 inches in length. In the lot were bars of 1/8 by 1 1/2 in., 1/4 by 1/4 in., 3/16 by 3/4 in., 1/4 by 1 1/4 in., 3/4 by 5/8 in., and 3/8 by 1/2 in. cross sections. All of the types of steel available were used, including 0.9 to 3.5 per cent chromium steel, 0.85 per cent manganese steel, 5 per cent tungsten steel, and 20, 25 and 36 per cent cobalt steels. Along with this range in composition a range in coercive force of from 40 oersteds to 260 oersteds was available, accompanied by a variation in residual induction of from 6,000 gauss to 11,000 gauss.

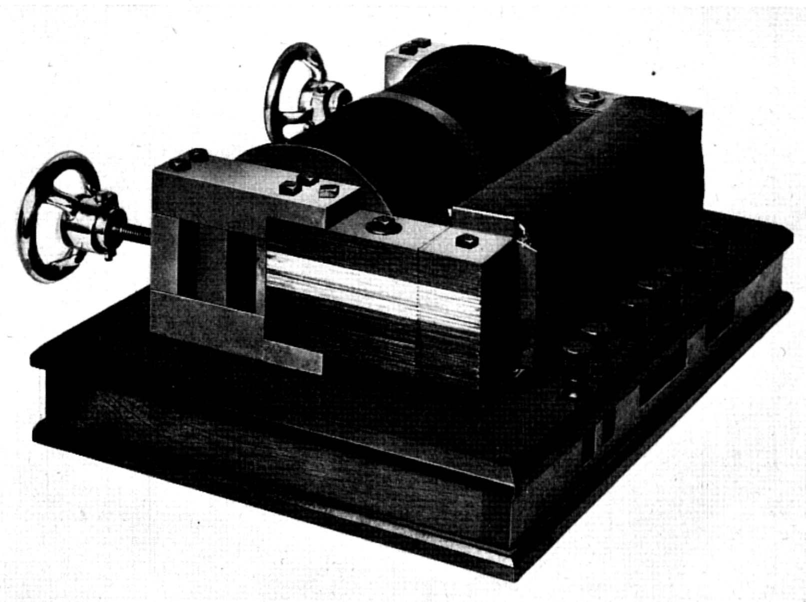

Fig. 2—The Babbitt permeameter. Used for making magnetic measurements on any kind of magnet steel.

Experimental Procedure. The test bars were hardened in the manner appropriate to each type of steel, except that some bars were purposely quenched from a high temperature in order to secure low values of B_r. After hardening, the demagnetization curve for each bar was determined by means of permeameter measurements, using the Babbitt[7] permeameter and a Grassot fluxmeter with lamp and scale. The control circuit of this permeameter is the conventional circuit for a ring test, as shown in Fig. 3. The magnetic data are all comparable since accurate measurements may be made with this permeameter on all of the above mentioned types of magnet steel.

Following the permeameter measurements, each bar was magnetized as a straight bar magnet in an air core solenoid. Field strengths of over 1,000 oersteds were applied to each bar. Upon removal from the solenoid the flux density at the middle of the bar was measured by means of a search coil and the Grassot fluxmeter with lamp and scale. It is the flux density measured in this way that is referred to in this discussion as the remanence of a magnet, or B_{rem}.

After the measurements at 12-in. lengths were completed, each bar was cut in succession to 10, 8, 6, 4, and 2-in. lengths by removing 1 inch of material from each end except that in the case of some of the bars of larger cross-section the lengths of 10, 6, and 2 inches were omitted.

Remanence measurements were made at each of the above lengths.

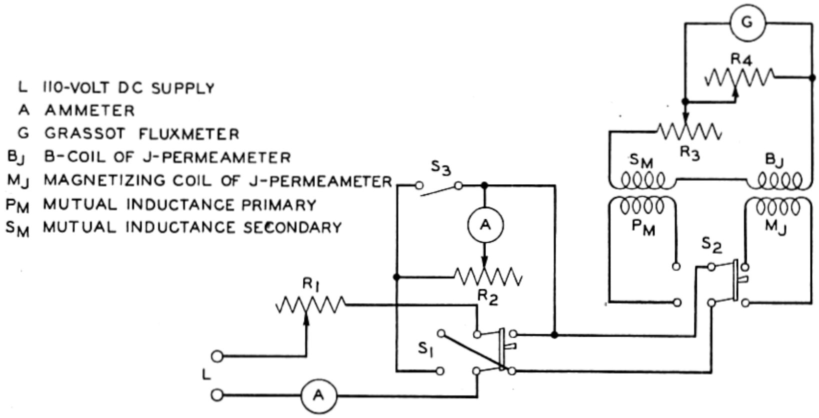

Fig. 3—Control circuit of the Babbitt permeameter. Note that the connections are those of a simple ring test.

The bars were remagnetized for each new measurement of B_{rem}.

The foregoing experiments made available for analysis a wide range of values of the associated variables B_{rem}, B_r, H_c, length L, and cross-sectional area A.

Analysis of Data. The values of remanence *versus* length for a number of the straight bar magnets are plotted in Fig. 4 to show the range of values existing in different bar magnets of the same length, when fully magnetized. The differences in B_{rem} for a particular length of bar are due to the differences in B_r, H_c, and cross-section of the various samples.

The chief characteristic that the curves of Fig. 4 have in common is a resemblance in shape to a normal induction curve and an asymptotic approach to limiting values of B_{rem} as the lengths increase.

In Fig. 5 is shown the result of plotting B_{rem}/B_r *vs.* $(L/D)\sqrt{H_c}(T/T_0)^2$ for the same set of bars. This combination of variables was arrived at purely by a cut and try method, but as shown, a surprisingly good correlation is obtained.

It should be mentioned in connection with Fig. 4 that it was found that the data relating to some of the bars did not fit the curve satisfactorily if B_{rem}/B_r was plotted against $L\sqrt{H_c}/D$, but in each case

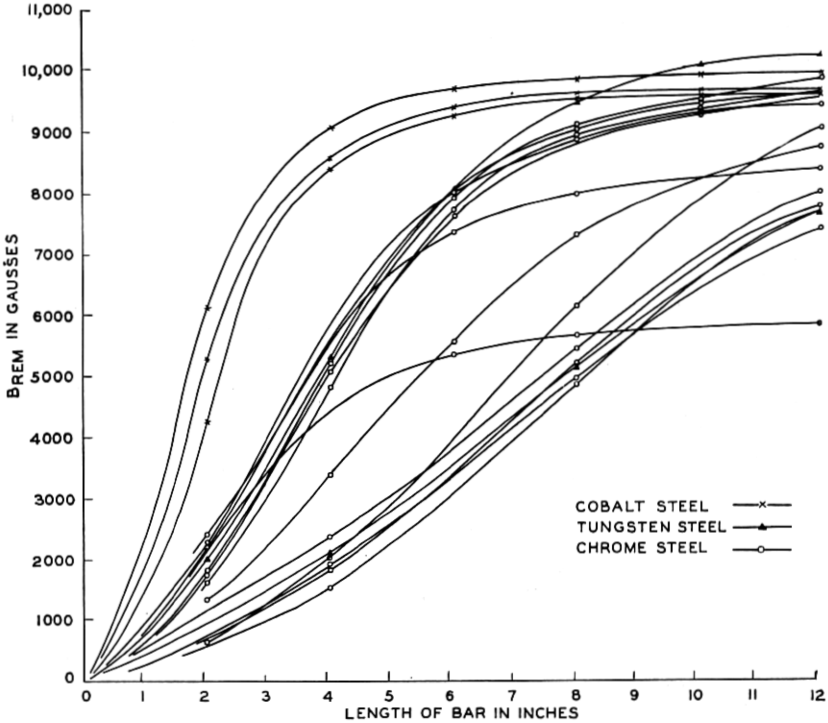

Fig. 4—Curves of remanence *vs.* length for straight bar magnets of different cross-sections and materials.

where this was so the bar had been quenched from above or below the optimum hardening temperature. It was found, however, that the data from these bars could be correlated with those from the properly hardened bars if B_{rem}/B_r were plotted against $(L/D)\sqrt{H_c}(T/T_0)^2$ in which T is the actual, and T_0 is the optimum hardening temperature on the absolute scale. The ratio T/T_0 is, of course, equal to unity for bars properly hardened. The definition of the optimum hardening temperature will be given later.

After this empirical correlation was obtained, it was suggested that B_{rem}/B_r *vs.* $L\sqrt{H_c}/D\sqrt{B_r}$ would be preferable as a choice of variables, from theoretical considerations based upon the assumption of uniform magnetization in the magnets. Although it can be shown that this assumption is not fulfilled in the actual case of straight bar magnets, this method of plotting gives as good a correlation as is shown in Fig. 5. Such a result might be expected, since the values of B_r commonly

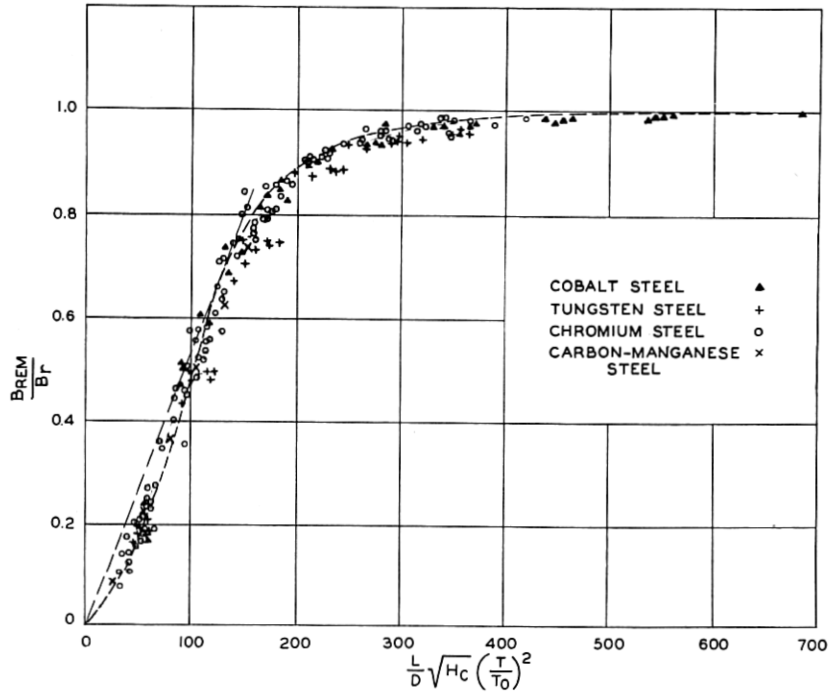

Fig. 5—The same data as in Fig. 4. Showing the correlation between B_{rem}/B_r and $(L/D)\sqrt{H_c}$ for bars of any kind of material, magnetic properties, length or cross-section.

encountered range from 9,000 gauss to 11,000 gauss, thus differing from 10,000 gauss by 10 per cent at most. The square root of B_r, therefore, differs from 100 by not more than 5 per cent in most cases, whence it follows that dividing $L\sqrt{H_c}/D$ by $\sqrt{B_r}$ shifts the relative position of the abscissas of the points in Fig. 5 by not more than 5 per cent in most cases. Fig. 6 illustrates the results obtained by the second method of plotting.

Dimensional considerations favor the use of the quantity $L\sqrt{H_c}/$

$D\sqrt{B_r}$, however, since this quantity, like B_{rem}/B_r, is dimensionless, whereas $L\sqrt{H_c}/D$ is not. For this reason it is felt that the method of plotting shown in Fig. 6 is to be preferred to that of Fig. 5 although the correlation obtained in Fig. 5 appears to be as good as that shown in Fig. 6. Further, the necessity for injecting the variable of temperature into the picture is done away with.

Inasmuch as the data from a large number of bars of widely different compositions, magnetic properties, dimensions and heat treatment are

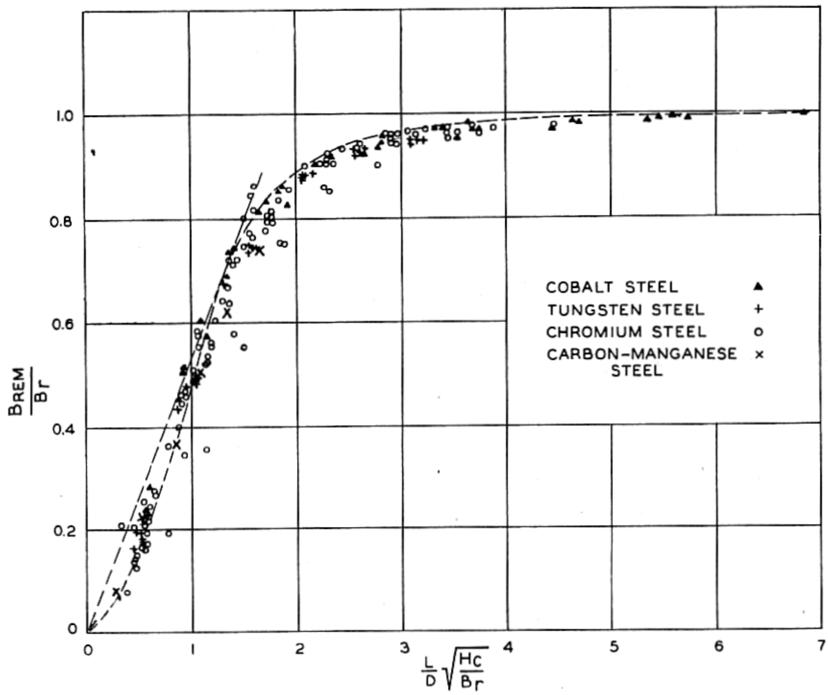

Fig 6.—The data of Fig. 4 with B_{rem}/B_r plotted against $L\sqrt{H_c}/D\sqrt{B_r}$.

all found to cluster quite closely along a single curve, it is felt that it is legitimate to use the curve as a basis for some generalizations, even though the curve was arrived at largely by empirical processes.

The type of curve that is obtained when B_{rem}/B_r is plotted against $L\sqrt{H_c}/D\sqrt{B_r}$ indicates that other factors being constant, the remanence of a magnet is roughly proportional to $\sqrt{B_r}$ and, for large values of $L\sqrt{H_c}/D\sqrt{B_r}$, in other words, for large values of dimension ratio or coercive force, the remanence is practically equal to B_r. This fact is of interest from a design standpoint.

It was to be expected, of course, that the values of B_{rem} would approach B_r as a limit for high values of length of bar, since the end effect diminishes as the length increases and the condition of an infinitely long bar or a closed ring is simulated. The functional relationship which the quantity $L\sqrt{H_c}/D\sqrt{B_r}$ bears to the ration B_{rem}/B_r is not known. A fairly good fit of the observed data is given by the expressions:

$$r = \frac{q^{1.5}}{2.15} \qquad q = O \text{ to } q = 1.25 \text{ and}$$

$$r = e^{-\frac{0.8}{q^{2.75}}} \qquad q = 1.25 \text{ to } q = \infty$$

in which $r = B_{rem}/B_r$ and $q = L\sqrt{H_c}/D\sqrt{B_r}$, but aside from the direct proportionality of B_{rem} and B_r, the equations appear to have little meaning. However, they do indicate that for a given dimension ratio, there is a practical upper limit to H_c, beyond which very large increases in H_c are necessary to produce small increases in the ratio of B_{rem} to B_r. Considerable interest attaches to the fact that the dimension ratio L/D and $\sqrt{H_c}/\sqrt{B_r}$ are of equal weight in affecting the remanence of a magnet. It is also worth noting that the value of B_{rem} is independent of the contour of the cross-sectional area. It is possible that this would not hold for dimension ratios less than one, but it does appear to hold for dimension ratios of practical importance. As shown in Fig. 5, a line drawn through the origin and tangent to the dotted curve of that figure has its point of tangency at values of B_{rem}/B_r and $L\sqrt{H_c}/D\sqrt{B_r}$ of approximately 0.65 and 1.25 respectively. It will be shown later that this is the point of maximum efficiency, *i.e.*, the point at which are obtained the highest values of B_{rem} or external magnetic energy per unit volume of steel.

It follows, if this is true, that magnets should be designed so that $L\sqrt{H_c}/D\sqrt{B_r} = 1.25$, and with this as a basis, the nomogram of Fig. 7 was laid out. The use of this chart to design a magnet for maximum efficiency, is illustrated by the dotted lines of the figure. In the case shown a total flux in the magnet of 3,000 maxwells is desired, and a steel with a B_r of 10,000 gauss and a coercive force of 54 oersteds is assumed. The correct values of A and L are found to be 0.461 cm.^{2} and 13 cm., respectively, and the dimension ratio is 17.

It should be borne in mind throughout this discussion that Figs. 4, 5, 6 and 7 apply directly only to straight bar magnets. Formed magnets with short air-gaps with or without pole pieces, will have higher remanence values than are shown by the curve of Fig. 6. This

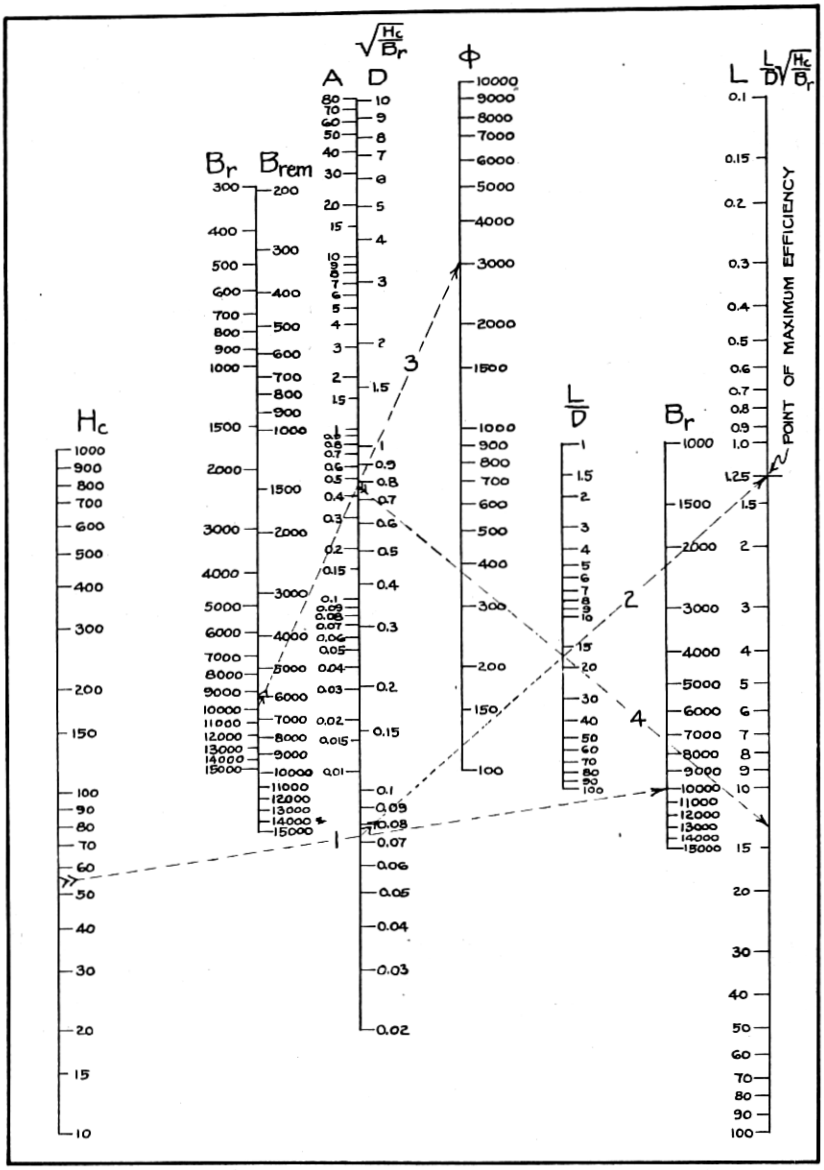

Fig. 7—Nomogram chart for permanent magnet design for maximum efficiency. Based on the curves of Figs. 6 and 10.

is because, compared with straight magnets, their effective lengths are greater than their actual lengths. The difference between effective and actual lengths is negligible, however, except for magnets whose poles are brought quite close together, either actually or by means of pole pieces of soft iron. Watt-hour meter damping magnets are good examples of magnets whose poles are brought thus closely together without the aid of pole pieces.

On the other hand, the general conclusions drawn from the curve of Fig. 6 are applicable to magnets of any shape if L is assumed to stand for the effective length of the magnet. Fig. 7 also is perfectly general on the same basis. By effective length is meant the length of a straight bar magnet of equal cross-section and magnetic properties, having the same remanance as the given magnet.

Correlation Between Straight and Bent Magnets

Subsequent to the establishment of the relationships shown in Fig. 6, a limited amount of data were secured which have a bearing upon the effect on B_{rem} of departure from the form of a straight bar magnet.

To get these data, one ¼ by ¼ in. rod each of cobalt steel and tungsten steel were formed into rings of 2-in. inside diameter, with the ends touching. Companion test bars were cut adjacent to each end of the rods used in forming the rings. All three pieces of each kind of steel received as nearly identical heat treatments as possible, even to heating and cooling the straight test bars at the time of forming the rings. After hardening, demagnetization curves were obtained on each of the two test bars of tungsten steel and of cobalt steel. Then data for B_{rem} *vs.* length curves were obtained from both the test bars and rings by the process of cutting off the ends and remeasuring B_{rem} already described.

The demagnetization curves and the B_{rem} *vs.* length curves, for the two straight cobalt steel bars are practically identical. Inasmuch as these bars were cut adjacent to the ends of the bar from which the ring was formed, and heat treated with it, it is safe to assume that the material in all three samples is alike magnetically, and that the demagnetization curve for the ring is the same as for the straight bars. The same conclusions hold for the tungsten steel, as shown in Fig. 9.

It follows then that it is valid to compare the B_{rem} *vs.* length curves for the straight bars with those for the rings, and conclude that all differences in the curves are due to differences in shape. It is interesting to note that the ring for cobalt steel yields the same curve for B_{rem} *vs.* length as the straight bar, but that, in the case of the tungsten steel, the proximity of the ends of the ring has an appreciable

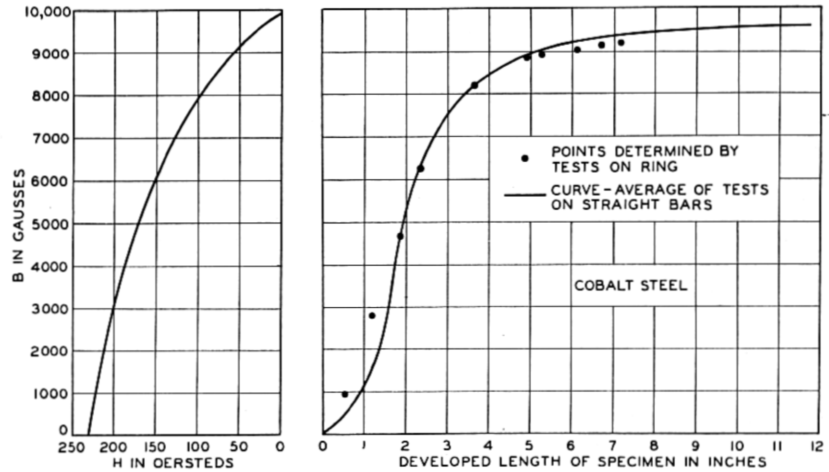

Fig. 8—Comparison of B_{rem} *vs.* length curves for straight bar and 2-in. I.D. ring of ¼-in. x ¼-in. cobalt steel in which B_r = 9,980 gauss and H_c = 230 oersteds.

effect on the value of B_{rem}, causing the curve for B_{rem} *vs.* length to turn upwards for those lengths corresponding to short air gaps.

Additional data along these lines would be very useful, and should include measurements on magnets with pole structures having well defined air gaps.

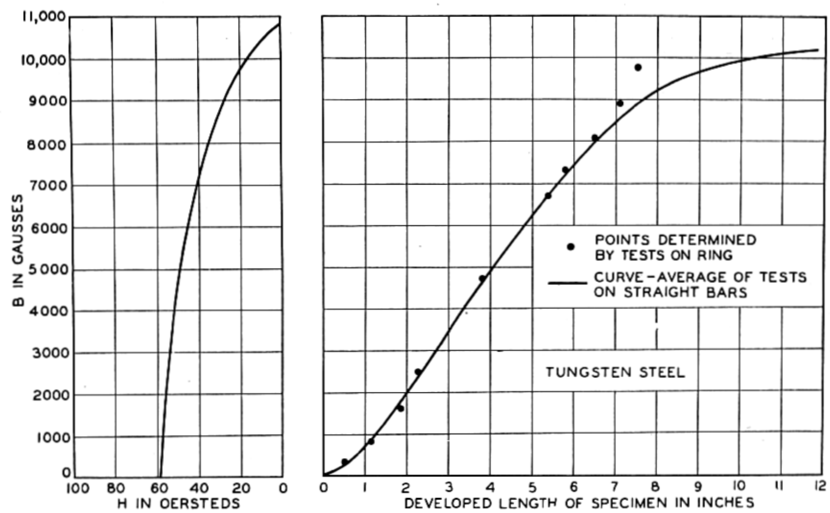

Fig. 9—Comparison of B_{rem} *vs.* length curves for straight bar and 2-in. I.D. ring of ¼-in. x ¼-in. tungsten steel in which B_r = 10,830 gauss and H_c = 59 oersteds.

Criteria of Magnetic Quality of Permanent Magnet Steels

A number of quantities determinable by magnetic measurement have been proposed at various times by different investigators for use as criteria of magnetic quality of magnet steels. Among these quantities may be listed $(BH)_{max}$[(1)], the maximum product of the coordinates of the demagnetization curve; and the quantities B_rH_c[(2)], B_r/B_{max}, B_rH_c/B_{max} and B_r/H_c[(3)]. High values of all of the proposed criteria except the last were considered desirable.

The form B_r/H_c has decided disadvantages, since a low value of the criterion is obtained when B_r is small as well as when H_c is large, and low values of B_r are distinctly undesirable in view of the fact that B_{rem} is directly proportional to B_r. The ratio B_r/B_{max} was found to be practically constant for some types of steel, regardless of the value of H_c, and thus was of no value as a criterion. The quantity $B_r\ H_c/B_{max}$ would accordingly amount to a constant times H_c and would give no actual weight to the value of B_r. These three criteria would therefore fail either to indicate a choice between different types of magnet steel or to aid in the selection of an optimum heat treatment for a given kind of steel.

This leaves the quantities $(BH)_{max}$ and $B_r\ H_c$ as the only ones not obviously faulty. The first was proposed by S. Evershed in a noteworthy paper in the *Journal* of the Institution of Electrical Engineers for September 1920, in which he showed that the quantity $(BH)_{max}$ is a measure of the maximum amount of external magnetic energy which can be supported per unit volume of a given magnet steel, and that this in turn defines the term "magnetic quality" as applied to magnet steel.

Evershed's derivation of his criterion of magnetic quality, given in the article noted above, is complete and convincing, but it was thought desirable to obtain experimental verification. The data from the bars of Fig. 4 were used to this end. In Fig. 10 are shown a number of curves each corresponding to a different type of magnet steel. The demagnetization curves for the bars are shown plotted in the usual way, and to the right of these are plotted the curves of (BH) *vs.* B. To the right of these are plotted further the experimentally determined curves of B_{rem} *vs.* length for the bars in question.

Fig. 10 shows that if tangents to the curves of B_{rem} *vs.* length are drawn through the origin the point of tangency in each case indicates the value of B_{rem} and length at which the ratio of B_{rem} per unit length or volume of steel is a maximum, and it is clearly demonstrated that in each case these values of B_{rem} coincide closely with the values of B for which the product (BH) is a maximum. The writer feels that this constitutes a sufficiently convincing verification of the correctness of Evershed's criterion.

The chief objection to the use of Evershed's criterion is that a number of points on the demagnetization curve for a piece of magnet steel must be known before the numerical value of the product $(BH)_{max}$ can be determined. The quantity $B_r H_c$ is more easily obtained than

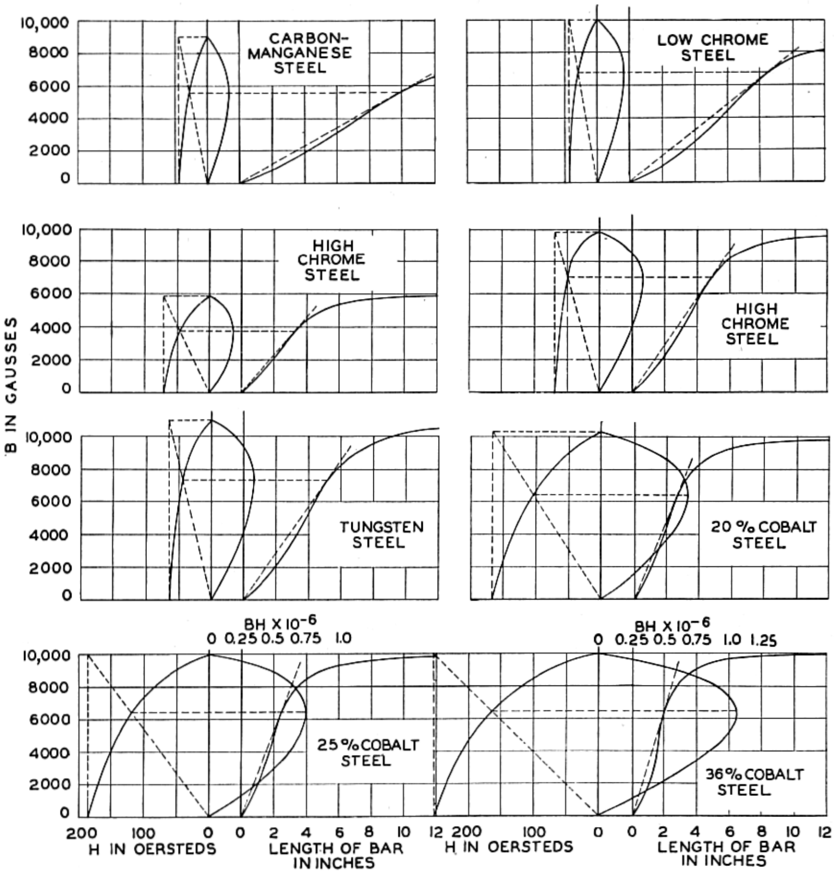

Fig. 10—Curves showing experimental verification of validity of Evershed's criterion for magnetic quality of permanent magnet steels.

$(BH)_{max}$ but its validity as a criterion cannot be established from theoretical considerations. Accordingly, the values of $(BH)_{max}$ and $B_r H_c$ were determined by actual measurement for a large number of bars of different kinds of magnet steel. In Fig. 11 are shown the values of $B_r H_c$ plotted against the corresponding values of $(BH)_{max}$.

It is felt that these points lie closely enough along the straight line in that figure to warrant the use of $B_r\ H_c$ as a criterion of magnetic quality of magnet steel for all practical purposes. In cases where very accurate comparisons are required, it may be necessary to go to the greater trouble of determining $(BH)_{max}$.

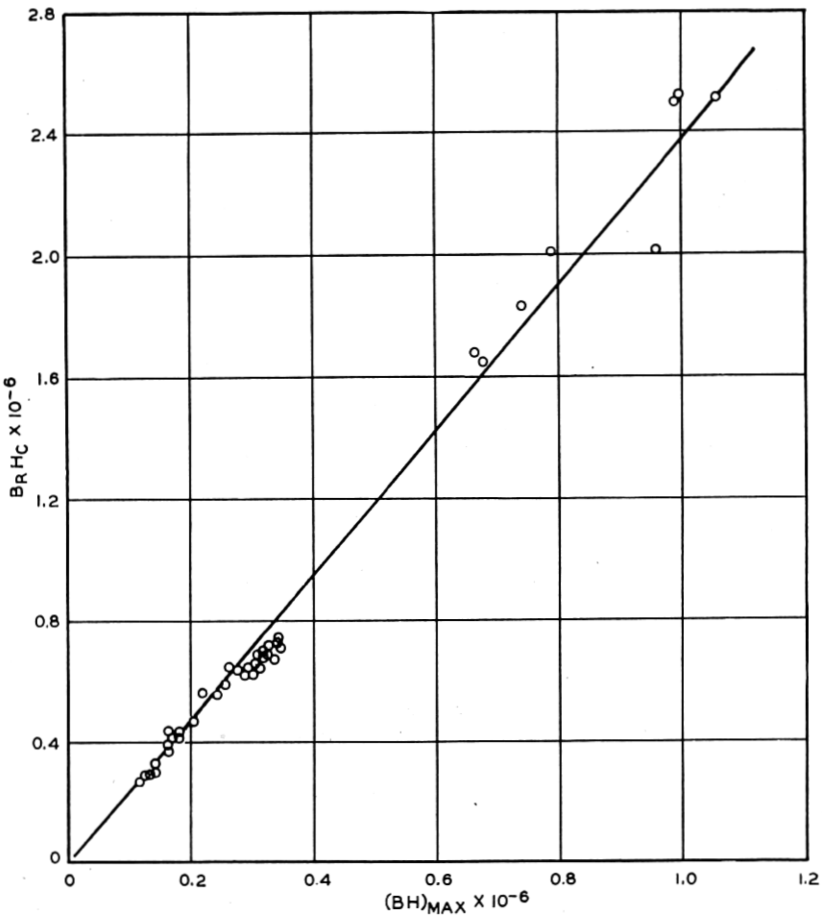

Fig. 11—Relation between the quantity $B_r \times H_c$ and Evershed's criterion, $(BH)_{max}$.

Referring to a previous mention of optimum hardening temperatures, it will be understandable now to state that the optimum hardening temperature for a given type of steel is that temperature at which a maximum value of $(BH)_{max}$ is obtained, or practically speaking, a maximum value of $B_r\ H_c$.

Fit of Demagnetization Curve by Hyperbola

It will be noted that the points on the demagnetization curves for which the product (BH) is a maximum are given quite accurately in each case by the intersection of the demagnetization curve with a line through the origin having the slope B_r/H_c. That this should be so follows from the fact that any demagnetization curve for magnet steel may be closely approximated by a rectangular hyperbola whose equation is $B = a - k/(H + b)$ in which a, b, and k are parameters of each particular curve. It is a mathematical property of the foregoing rectangular hyperbola that the coordinates whose product is a maximum are located by a line through the origin having a slope equal to the ratio of the intercepts of the hyperbola. This property of the

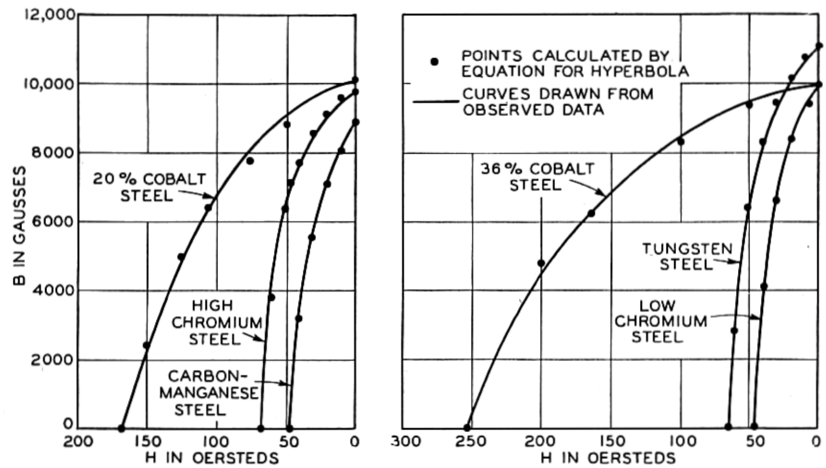

Fig. 12—Curves showing approximation of actual demagnetization curves by points on the general hyperbola, $B = a - k/(H + b)$.

hyperbola is a property of the demagnetization curves for magnet steel in so far as these curves can be closely fitted by appropriate hyperbolas. That the fit is quite good in all cases is shown by Fig. 12 in which hyperbolas calculated for each demagnetization curve are shown by dots and the observed points by a full line. The graphical method of Fig. 10 for determining the point on each demagnetization curve the product of whose coordinates is a maximum is more accurate than the method of plotting the curve of (BH) *vs.* B, because the latter curve is usually quite flat-topped and its maximum is hard to locate exactly. The fact that the curve is flat-topped explains why many methods of magnet design give good results. It is because about equal efficiencies are obtained in any case over a fairly wide range of values of B_{rem}.

A further application of the idea that the demagnetization curve can be represented by a hyperbola is made by Mr. E. A. Watson.[4] Watson applies a graphical construction for a hyperbola to the demagnetization curve and extends the construction into the region of positive values of H, thus providing a means of extrapolating the hysteresis loop to the saturation value of B. It has been the writer's experience that Watson's graphical construction cannot be legitimately extended into the first quadrant because the curve thus predetermined almost always lies considerably under the true curve, and yields too low a value for the saturation flux density. Fig. 13 illustrates the case in point.

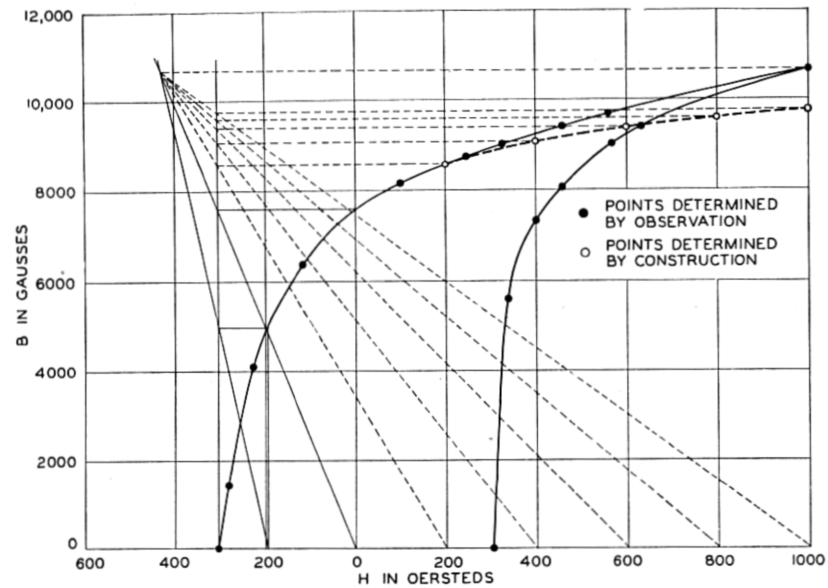

Fig. 13—Comparison of graphical construction of the descending portion of the hysteresis loop with actual measured data. Note the lack of agreement for positive values of H.

If, instead of potting B *vs.* H for the points of the demagnetization curve, we plot B/B_r *vs.* H/H_c, it is found that all test specimens yield points fitting fairly well the single curve represented by the equation $y = A(x + 1)/(x + A)$, in which $y = B/B_r$, $x = H/H_c$ and $A = \sqrt{2}$ approximately. It is possible to derive from this equation the relationship that $(BH)_{max} = 0.423\ B_r\ H_c$ which checks very well with the line drawn in Fig. 11. It can also be seen from the equation $y = A(x + 1)/(x + A)$ that the representation of the demagnetization curve by a hyperbola cannot be extrapolated to give the correct value for B_s since for values of H approaching infinity, the equation yields the result that $y = B_s/B_r = A = \sqrt{2}$ whence $B_s = \sqrt{2}\, B_r$, while values of B in excess of $\sqrt{2}\, B_r$ are commonly encountered at ordinary values of H.

Characteristic Curves for Magnet Steel

In Fig. 14 are shown demagnetization curves and (BH) curves for the various types of magnet steel with which the writer has had experience. It should be remembered that while these curves are more

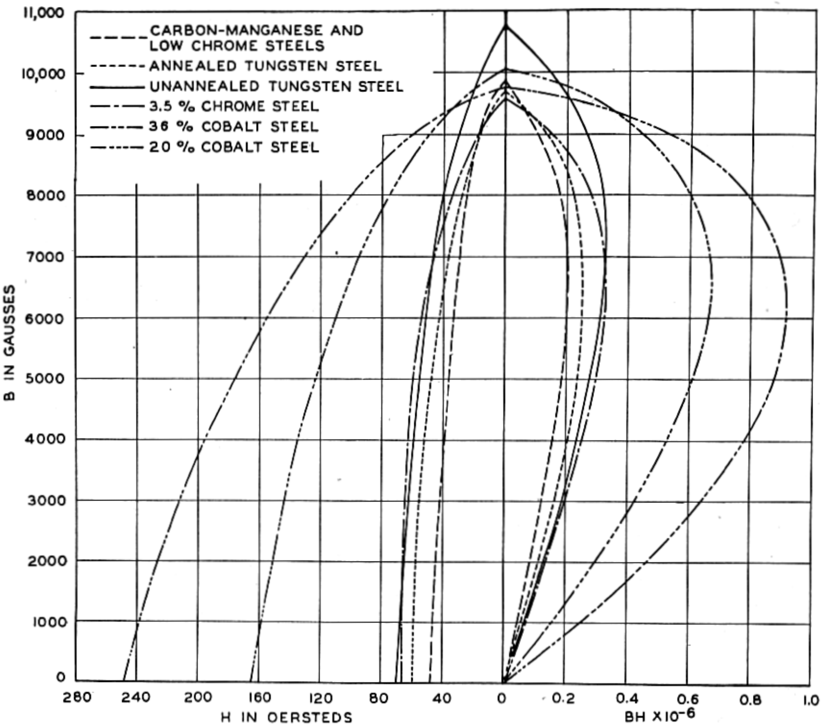

Fig. 14—Typical demagnetization curves and curves of BH *vs.* B for various kinds of magnet steel.

or less typical of the steels in question, there is no one curve which can be given as the curve for the material. There will be as many curves as there are samples tested, because magnetic characteristics of a piece of magnet steel, or any other magnetic material depend upon the whole previous thermal history of the sample from the time of melting to the time of testing, the amount of hot and cold working the sample has received, and the conditions of stress in the material at the time of testing. The relation of all of these variables to the remanence of a permanent magnet, fortunately, is focussed, as it were, in the values of B_r and H_c for the material, and the relation of these variables to B_{rem} is as shown by Fig. 6.

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