A General Theory of Electric Wave Filters *

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THE growth of the field of electric wave filters since their original discovery by Dr. G. A. Campbell shows the filtering action by no means inheres in any particular physical configuration. Filters have been built, for example, as recurrent or non-recurrent ladder structures, as lattices, as bridged-T's, and in a variety of combinations of transformers with ordinary elements. No general theory uniting all these configurations has, however, been developed. Each structure has been treated by methods which are primarily adapted to that configuration alone. In consequence, such questions as the relations between filters of different types and the possibilities of securing improved characteristics by going to a still wider variety of configurations have remained unsettled.

The present paper is an attempt to develop a general filter theory independent of any particular structure, by means of which these questions can be answered. For the sake of a rigorous discussion, the term "filter" has been used to signify a four-terminal network of ordinary lumped elements which, when terminated in its image impedances, transmits freely one continuous band of real frequencies and attenuates all other real frequencies. Since in actual operation the distinction between free transmission and attenuation is always more or less obscured by terminal effects and parasitic dissipation, this definition is necessarily somewhat arbitrary. It agrees, however, with common usage except in its rejection of multiple band-pass filters, which are rarely used in practice.

It follows from the definition that a "filter" can include only reactive elements. Otherwise, however, the structure considered may be an arbitrary four-terminal network and may include transformers as well as ordinary inductances and capacitances. The analysis is based upon a combination of the ordinary image parameter method of analyzing networks and the normal coordinate method familiar in the dynamics of vibrating systems. It is found that the conditions for filtering action can be expressed by means of relations between the set of normal coordinates of the network when it is short circuited at both

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ends and the sets obtained by open circuiting one or both ends. The same normal coordinate solutions also furnish convenient parameters in terms of which general expressions for the image parameters of the structure can be built up. For a band-pass filter, for example, the typical result is

$$\text{Tanh } \theta = k_1 \frac{\sqrt{a_{c_1}} a_{n+1} \cdots a_{p-1} \sqrt{a_{c_2}}}{a_n \cdots a_p},$$

$$Z_I = ik_2 f \frac{a_2 \cdots a_{m-1} \sqrt{a_{c1}}}{a_1 \cdots a_m} \frac{\sqrt{a_{c_2}} a_{q+1} \cdots a_r}{a_q \cdots a_{r-1}},$$

where a_j symbolizes a frequency factor of the form $1 - \frac{f^2}{f_j^2}$ and

$$0 \leq f_1 \leq f_2 \leq \cdots \leq f_m \leq f_{c_1} \leq f_n \leq f_{n+1} \leq \cdots \leq f_h \leq f_{c_2} \leq f_q \cdots \leq f_r \leqq \infty.$$

The formulæ are almost exactly similar to those familiar in the theory of the lattice, except that the quantities $f_1 \cdots f_n$ which are now natural frequencies of the network as determined under the previously described conditions have a different significance. As in the lattice, however, they fall into three groups: $f_1 \cdots f_m$ and $f_q \cdots f_r$, which affect Z_I only; $f_n \cdots f_p$, which affect θ only; and the cut-offs, f_{c_1} and f_{c_2}, which enter into both expressions. The formulæ can be extended to low-pass, high-pass and all-pass structures by allowing the cut-offs to assume the limiting values zero and infinity respectively.

Certain further restrictions upon the image impedance and transfer constant of physically realizable filters may be obtained from the consideration of another system of parameters, $r_1 \cdots r_n$, defined as the roots of the equation tanh $\theta = 1$. They are usually of either single or double multiplicity. The importance of the roots depends upon the fact that in combination with the cut-offs they are sufficient to determine θ at all frequencies. The restrictions to which they lead may be divided into two sets. The first affects the transfer constant alone and is expressed in terms of limitations on the allowable positions and multiplicities of the roots. The second is concerned with the restrictions which must be placed upon the relation between the transfer constant and the two image impedances. It may be expressed by the statement that when the transfer constant and one image impedance have been chosen as functions of frequency, the second image impedance is determined as a function of frequency to within a constant multiplier. The differences between the two image impedances depend only upon the roots of single multiplicity, so that if only double roots are involved the structure is necessarily symmetrical.

The second half of the paper is devoted to an interpretation of known filter theory in terms of these results and to an attempt to extend this theory until it affords a definite technique for the construction of any filter which the preceding analysis has shown to be physically admissible. The method followed depends upon the fact that when a number of filter structures with matched image impedances are connected in tandem, the roots, $r_1 \cdots r_n$, of the resulting filter will be the aggregate of the roots of all the individual units of which it is composed. This allows us to represent the general filter as a composite structure in which each constituent represents one or at most a few of the total number of roots. The resulting networks are very similar to the familiar Zobel type composite filter, especially when it is noticed that the various required roots can be obtained from simple prototype structures by transformations analogous to the m-derivation, and that the preceding classification into roots of single and double multiplicity corresponds in the composite filter to a classification of the constituent structures into half and full-sections.

In spite of these relations, the usual composite filter theory must be extended in several ways if the general filter is to be adequately represented. The first extension is demanded by the fact that in the general filter we must be able to assign one image impedance characteristic of each of the constituent sections in any form compatible with the preceding general equation. The required image impedances are not obtainable from ordinary ladder structures. When the constituent involved is a double root, or full-section, structure however, the required impedance can readily be realized by resorting to the lattice form. With half-section structures the procedure is more complicated. It is necessary to make use of a combination of Dr. Zobel's multiple m-derivation and a new transformation, described as an h-derivation, which alters the impedances of ladder type half-sections without affecting their transfer constants.

Similar extensions are also needed to provide the requisite variety of transfer constants. Roots falling within certain ranges can be provided by ordinary ladder structures, m-derived, in the usual way, with a real value of m less than one. In order to complete the list, however, it is also necessary to consider single sections derived with real values of m greater than one, which may be realized as lattices or as ladder structures with mutual inductance, and pairs of sections derived with conjugate complex values of m. By including all of these types of sections we can construct physical networks giving any filter characteristics falling within the general limitations discovered in the first part of the paper.

Aside from its purely theoretical interest, the analysis leads to two results of immediate practical value. The first is the introduction of new characteristics by the complex m and h-derived sections. The h-derived structures can be dismissed briefly. They are chiefly of interest for their impedance characteristics, which resemble those found in symmetrical lattices. They allow us, however, to extend these impedances to unsymmetrical structures and they also allow us to extend considerably the range of impedances, even of symmetrical structures, which can be realized in the ladder form. The complex m structures, on the other hand, are chiefly of interest for their novel phase and attenuation characteristics. The novel phase characteristics are particularly important since the elementary constituents of the linear phase shift filters described in the accompanying paper usually turn out to be complex m sections.

The second general result is an increase in our knowledge of the relations existing between filters of different physical configurations. This last point is particularly important because it allows us to convert filters from one type of configuration to an equivalent type which may be better suited for purposes of practical construction. For example, it allows us to convert the general lattice filter, which is very convenient for theoretical purposes but is very difficult to build practically, into a composite of simple lattices and ordinary ladder sections.