Further Extensions of the Theory of Multi-Electrode Vacuum Tube Circuits

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The response of circuits containing vacuum tubes with any number of electrodes due to impressed electromotive forces, and under such circumstances that the time of transit of the electrons is negligible, is discussed when arbitrary feedback is present between the circuits connected to the electrodes, each of which may carry conductive current. The use of the theory is illustrated by obtaining first and second order effects in typical three-electrode tube circuits.

In a previous paper in the *Bell System Technical Journal*, October, 1934, the treatment was restricted to three-electrode tube circuits in which it was assumed that the amplification factor of the tube was constant and that no conductive grid current was present. In the present paper these restrictions are removed.

Introduction

THE response in multi-electrode vacuum tube circuits due to impressed electromotive forces has been the subject of several papers. For the three-electrode vacuum tube circuit J. R. Carson [1] has used a method of successive approximations, assuming constant amplification factor and no conductive grid current. E. Peterson and H. P. Evans [2] removed the restriction on the amplification factor but maintained the assumption regarding the grid current, while F. B. Llewellyn [3] considered the general case with both plate and grid currents. Finally, J. G. Brainerd [4] has treated the general case of the four-electrode tube circuit. The theories given by these authors did not take into account any feedback between the circuits of the electrodes except in the first approximation.

In a previous paper [5] the theory given by Carson was extended to include the effects of feedback between plate and grid circuits not only in the first but also in the second and higher approximations. The aim of the present paper is to extend similarly the other theoretical work mentioned above[2, 3, 4] to circuits containing tubes with three, four, or any number of electrodes.

Theory of Three-Electrode Tube Circuits

We shall consider the three-electrode tube circuit shown in Fig. 1 where Z_1, Z_2, and Z_3 are impedances which may include inter-electrode

[1] J. R. Carson: *I. R. E. Proc.*, April, 1919, p. 187.
[2] E. Peterson and H. P. Evans: *B. S. T. J.*, July, 1927, p. 442.
[3] F. B. Llewellyn: *B. S. T. J.*, July, 1926, p. 433.
[4] J. G. Brainerd: *I. R. E. Proc.*, June, 1929, p. 1006.
[5] S. A. Levin and Liss C. Peterson: *B. S. T. J.*, October, 1934, p. 523.

admittances. The impressed variable electromotive forces are ϵ_p and ϵ_g in series with the impedances Z_p and Z_g, respectively. We will designate by E_p and I_p the total plate voltage and current, respectively, while the corresponding quantities for the grid are E_g and I_g. In the

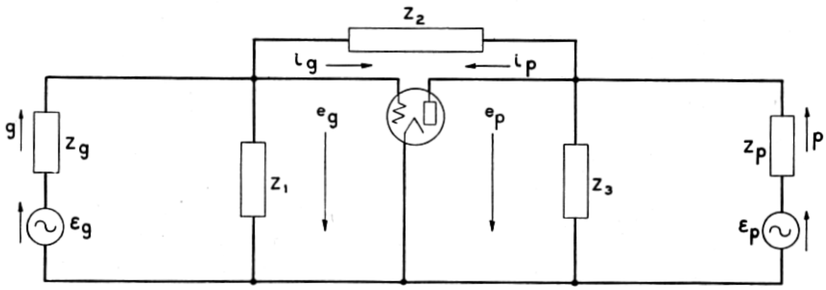


Fig. 1—Three-electrode vacuum tube and circuit.

absence of the variable electromotive forces the d-c. values of these voltages and currents are E_{p0}, I_{p0}, E_{g0}, and I_{g0}, respectively, while the increments due to the impressed forces are e_p, i_p, e_g, and i_g, respectively. Similarly, g and p denote the incremental voltages across Z_g and Z_p. All quantities referring to currents and voltages are instantaneous values.

We will now assume that I_p and I_g are functions of E_p and E_g, and that we can derive from these functions the expansions

$$i_p = \sum b_{mn} e_p{}^m e_g{}^n, \qquad i_g = \sum \beta_{mn} e_p{}^m e_g{}^n, \tag{1}$$

where

$$b_{mn} = \frac{1}{m!n!} \frac{\partial^{(m+n)} I_p}{\partial E_p{}^m \partial E_g{}^n}, \qquad \beta_{mn} = \frac{1}{m!n!} \frac{\partial^{(m+n)} I_g}{\partial E_p{}^m \partial E_g{}^n}, \tag{2}$$

evaluated at the operating point (E_{p0}, E_{g0}).

The important tube parameters are by definition

$$\left.\begin{aligned} \frac{1}{r_p} &= \frac{\partial I_p}{\partial E_p}, \quad \mu_p = \frac{\dfrac{\partial I_p}{\partial E_g}}{\dfrac{\partial I_p}{\partial E_p}} = -\left(\frac{dE_p}{dE_g}\right)_{I_p=\text{const.}} \quad S_{pg} = \frac{\partial I_p}{\partial E_g} = \frac{\mu_p}{r_p} \\ \frac{1}{r_g} &= \frac{\partial I_g}{\partial E_g}, \quad \mu_g = \frac{\dfrac{\partial I_g}{\partial E_p}}{\dfrac{\partial I_g}{\partial E_g}} = -\left(\frac{dE_g}{dE_p}\right)_{I_g=\text{const.}} \quad S_{gp} = \frac{\partial I_g}{\partial E_p} = \frac{\mu_g}{r_g} \end{aligned}\right\}, \tag{3}$$

where r denotes electrode resistances, μ the mu-factors, and S the transconductances.

It follows readily from (2) and (3) that

$$
\left.
\begin{aligned}
b_{10} &= \frac{1}{r_p} & \beta_{10} &= \frac{\mu_g}{r_g} \\
b_{01} &= \frac{\mu_p}{r_p} & \beta_{01} &= \frac{1}{r_g} \\
b_{20} &= -\frac{1}{2r_p^2}\frac{\partial r_p}{\partial E_p} = P_2 & \beta_{02} &= -\frac{1}{2r_g^2}\frac{\partial r_g}{\partial E_g} = T_2 \\
b_{11} &= \frac{1}{r_p}\frac{\partial \mu_p}{\partial E_p} + 2\mu_p P_2 & \beta_{11} &= \frac{1}{r_g}\frac{\partial \mu_g}{\partial E_g} + 2\mu_g T_2 \\
b_{02} &= \frac{1}{2r_p}\frac{\partial \mu_p}{\partial E_g} + \frac{\mu_p}{2r_p}\frac{\partial \mu_p}{\partial E_p} + \mu_p^2 P_2 & \beta_{20} &= \frac{1}{2r_g}\frac{\partial \mu_g}{\partial E_p} + \frac{\mu_g}{2r_g}\frac{\partial \mu_g}{\partial E_g} + \mu_g^2 T_2
\end{aligned}
\right\}, \tag{4}
$$

where P_2 and T_2 are new notations for b_{20} and β_{02}, respectively. Similar expressions may be derived for the coefficients b_{30}, β_{30}, etc.

If we now apply the circuital laws to the network external to the tube, we get a number of equations, two of which are

$$\epsilon_g = g + e_g, \qquad \epsilon_p = p + e_p. \tag{5}$$

To obtain a solution of (1) and (5) we utilize a method of successive approximations. Let

$$i_p = \sum i_{pk},\ i_g = \sum i_{gk},\ e_p = \sum e_{pk},\ e_g = \sum e_{gk},\ g = \sum g_k,\ p = \sum p_k, \tag{6}$$

where the summations extend from $k = 1$ to $k = \infty$. Let us further define the terms in the series (6) by the following equations:

$$
\left.
\begin{aligned}
r_p i_{p1} - e_{p1} &= \mu_p e_{g1} \\
r_g i_{g1} - e_{g1} &= \mu_g e_{p1} \\
\epsilon_g = g_1 + e_{g1}, \qquad \epsilon_p &= p_1 + e_{p1}
\end{aligned}
\right\}, \tag{7}
$$

$$
\left.
\begin{aligned}
r_p i_{p2} - e_{p2} &= \mu_p e_{g2} + r_p(b_{20}e_{p1}^2 + b_{11}e_{p1}e_{g1} + b_{02}e_{g1}^2) \\
r_g i_{g2} - e_{g2} &= \mu_g e_{p2} + r_g(\beta_{20}e_{p1}^2 + \beta_{11}e_{p1}e_{g1} + \beta_{02}e_{g1}^2) \\
0 = g_2 + e_{g2}, \qquad 0 &= p_2 + e_{p2}
\end{aligned}
\right\}, \tag{8}
$$

$$
\left.
\begin{aligned}
r_p i_{p3} - e_{p3} = \mu_p e_{g3} + r_p[&2b_{20}e_{p1}e_{p2} + b_{11}(e_{p1}e_{g2} + e_{p2}e_{g1}) + 2b_{02}e_{g1}e_{g2} \\
&+ b_{30}e_{p1}^3 + b_{21}e_{p1}^2e_{g1} + b_{12}e_{p1}e_{g1}^2 + b_{03}e_{g1}^3] \\
r_g i_{g3} - e_{g3} = \mu_g e_{p3} + r_g[&2\beta_{20}e_{p1}e_{p2} + \beta_{11}(e_{p1}e_{g2} + e_{p2}e_{g1}) + 2\beta_{02}e_{g1}e_{g2} \\
&+ \beta_{30}e_{p1}^3 + \beta_{21}e_{p1}^2e_{g1} + \beta_{12}e_{p1}e_{g1}^2 + \beta_{03}e_{g1}^3] \\
0 = g_3 + e_{g3}, \qquad 0 = p_3 + e_{p3}&
\end{aligned}
\right\} \tag{9}
$$

and so forth for subsequent terms.[5]

The physical interpretation of equations (7) to (9) is readily obtained. It follows from (7) that the equivalent circuit of Fig. 1 for first order quantities is given by Fig. 2. The equivalent circuit of

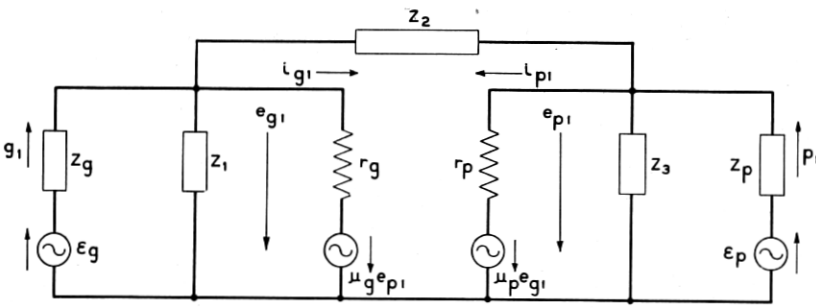


Fig. 2—Equivalent circuit—first-order effects.

Fig. 1 for second and third order effects are those shown in Fig. 3, and Fig. 4, respectively.

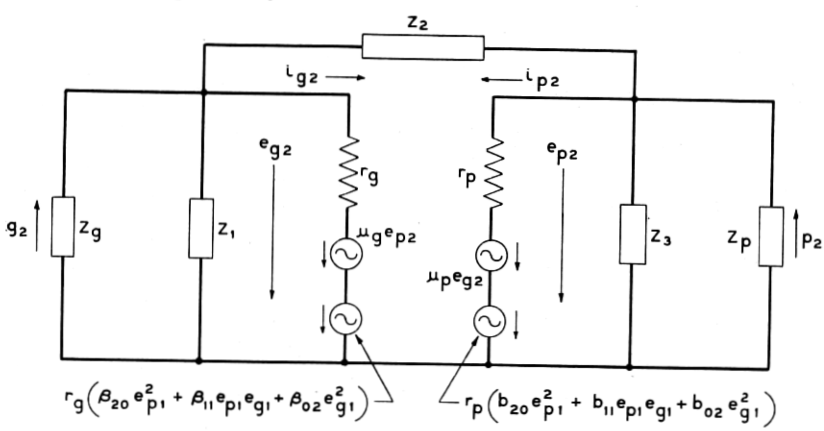


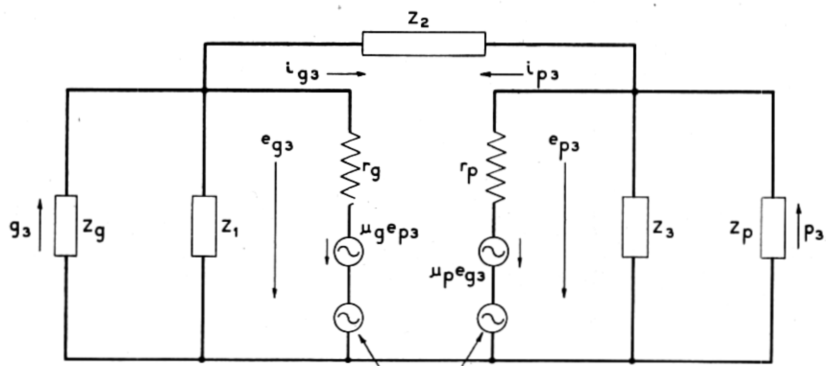
Fig. 3—Equivalent circuit—second-order effects.

It follows from (4) and (8) that

$$
\left\{
\begin{aligned}
& r_p(b_{20}e_{p1}^2 + b_{11}e_{p1}e_{g1} + b_{02}e_{g1}^2) \\
& \qquad = r_pP_2(e_{p1} + \mu_p e_{g1})^2 + \frac{1}{2}\left(\frac{\partial \mu_p}{\partial E_g} + \mu_p \frac{\partial \mu_p}{\partial E_p}\right) e_{g1}^2 + \frac{\partial \mu_p}{\partial E_p} e_{p1}e_{g1}, \\
& r_g(\beta_{20}e_{p1}^2 + \beta_{11}e_{p1}e_{g1} + \beta_{02}e_{g1}^2) \\
& \qquad = r_gT_2(e_{g1} + \mu_g e_{p1})^2 + \frac{1}{2}\left(\frac{\partial \mu_g}{\partial E_p} + \mu_g \frac{\partial \mu_g}{\partial E_g}\right) e_{p1}^2 + \frac{\partial \mu_g}{\partial E_g} e_{p1}e_{g1}.
\end{aligned}
\right. \tag{10}
$$

The corresponding terms in (9) can be expressed similarly.

Equations (7), (8) and (9) contain the general theory of the three-electrode vacuum tube circuit. In the special case when conductive grid current is absent it is only necessary to omit the second equation



$$r_g\left[2\beta_{20}e_{p1}e_{p2}+\beta_{11}(e_{p1}e_{g2}+e_{p2}e_{g1})+2\beta_{02}e_{g1}e_{g2}+\beta_{30}e_{p1}^3+\beta_{21}e_{p1}^2e_{g1}+\beta_{12}e_{p1}e_{g1}^2+\beta_{03}e_{g1}^3\right]$$

$$r_p\left[2b_{20}e_{p1}e_{p2}+b_{11}(e_{p1}e_{g2}+e_{p2}e_{g1})+2b_{02}e_{g1}e_{g2}+b_{30}e_{p1}^3+b_{21}e_{p1}^2e_{g1}+b_{12}e_{p1}e_{g1}^2+b_{03}e_{g1}^3\right]$$

Fig. 4—Equivalent circuit—third-order effects.

in each of the equations (7) to (10), inclusive, and to omit in each of the Figs. 2, 3, and 4, the branch containing r_g. If it is assumed that not only conductive grid current is absent, but also that μ_p is constant, the second plate e.m.f. in (8) reduces to $r_pP_2(e_{p1}+\mu_pe_{g1})^2$ as is seen from (10), and (8) thus becomes identical with the corresponding equation already obtained previously.[5] A similar reduction and correspondence occurs for the second plate e.m.f. in (9), as well as in subsequent equations.

Application to Steady State Solutions

In this section the use of the theory is illustrated by obtaining first and second-order effects assuming the circuit configuration to be that shown in Fig. 1. To avoid unnecessary complications the discussion is limited to steady-state solutions, and it is also assumed that no plate e.m.f. is impressed. We shall first obtain the solutions in the general case and then indicate how these are simplified in such special cases which have been treated by some previous investigators.[2, 3, 5]

General Case

Let the impressed grid e.m.f. be

$$\epsilon_g = \sum k_h \cos(\omega_h t + \kappa_h) = R\sum k_h e^{i(\omega_h t+\kappa_h)}, \qquad i = \sqrt{-1} \quad (11)$$

where the summation extends from $h = 1$ to $h = n$, and the letter R before an expression means its real part. Referring to Fig. 2 it may be shown that

$$\left.\begin{aligned} e_{g1} &= R\sum \frac{\alpha_1(\omega_h)}{Z(\omega_h)} k_h e^{i(\omega_h t+\kappa_h)} \\ e_{p1} &= R\sum -\frac{\alpha_2(\omega_h)}{Z(\omega_h)} k_h e^{i(\omega_h t+\kappa_h)} \\ (e_{p1} + \mu_p e_{g1}) &= R\sum r_p \frac{\alpha_3(\omega_h)}{Z(\omega_h)} k_h e^{i(\omega_h t+\kappa_h)} \\ (e_{g1} + \mu_g e_{p1}) &= R\sum \frac{\alpha_4(\omega_h)}{Z(\omega_h)} k_h e^{i(\omega_h t+\kappa_h)} \end{aligned}\right\}, \quad (12)$$

where

$$Z(\omega) = \frac{1}{(Z_g' + Z_2)r_g Z_2}\{[Z_2(r_g + Z_g') + r_g Z_g'] \times [Z_2(r_p + Z_p') + r_p Z_p'] - Z_g' Z_p'(\mu_g Z_2 - r_g)(\mu_p Z_2 - r_p)\}, \quad (13)$$

$$\alpha_1(\omega) = \frac{Z_1[Z_2(r_p + Z_p') + r_p Z_p']}{(Z_1 + Z_g)(Z_2 + Z_g')}, \quad (14)$$

$$\alpha_2(\omega) = \frac{Z_1 Z_p'(\mu_p Z_2 - r_p)}{(Z_1 + Z_g)(Z_2 + Z_g')}, \quad (15)$$

$$\alpha_3(\omega) = \frac{Z_1(\mu_p Z_2 + \mu_p Z_p' + Z_p')}{(Z_1 + Z_g)(Z_2 + Z_g')}, \quad (16)$$

$$\alpha_4(\omega) = \frac{Z_1 Z_2 r_p + Z_1 Z_2 Z_p'(1 - \mu_g \mu_p) + Z_1 r_p Z_p'(1 + \mu_g)}{(Z_1 + Z_g)(Z_2 + Z_g')}, \quad (17)$$

$$Z_p'(\omega) = \frac{Z_3 Z_p}{Z_3 + Z_p}, \qquad Z_g'(\omega) = \frac{Z_1 Z_g}{Z_1 + Z_g}. \quad (18)$$

The right-hand expressions in (13)–(18) are to be evaluated at ω. If we write

$$\frac{\alpha_k(\omega)}{Z(\omega)} = \left|\frac{\alpha_k(\omega)}{Z(\omega)}\right| e^{-i\varphi_k(\omega)}, \qquad k = 1, 2, 3, 4, \quad (19)$$

the equations (12) can be written

$$\left.\begin{aligned} e_{g1} &= \sum \left| \frac{\alpha_1(\omega_h)}{Z(\omega_h)} \right| k_h \cos [\omega_h t + \kappa_h - \varphi_1(\omega_h)] \\ e_{p1} &= -\sum \left| \frac{\alpha_2(\omega_h)}{Z(\omega_h)} \right| k_h \cos [\omega_h t + \kappa_h - \varphi_2(\omega_h)] \\ e_{p1} + \mu_p e_{g1} &= \sum r_p \left| \frac{\alpha_3(\omega_h)}{Z(\omega_h)} \right| k_h \cos [\omega_h t + \kappa_h - \varphi_3(\omega_h)] \\ e_{g1} + \mu_g e_{p1} &= \sum \left| \frac{\alpha_4(\omega_h)}{Z(\omega_h)} \right| k_h \cos [\omega_h t + \kappa_h - \varphi_4(\omega_h)] \end{aligned}\right\}. \tag{20}$$

This concludes our consideration of effects of the first order and we now turn to those of the second order. For this purpose we substitute the values given by (20) in the right-hand side of the expressions (10) for the grid and plate e.m.f.'s, and we then obtain two expressions, each of which is equal to a sum of sinusoidal terms. If we limit our attention to the terms of frequency $(\omega_1 - \omega_2)$, it is readily shown that the grid e.m.f. of this frequency is equal to the real part of

$$\begin{aligned} &\left[r_g T_2 \frac{\alpha_4(\omega_1)}{Z(\omega_1)} \left(\overline{\frac{\alpha_4(\omega_2)}{Z(\omega_2)}} \right) + \frac{1}{2} \left(\frac{\partial \mu_g}{\partial E_p} + \mu_g \frac{\partial \mu_g}{\partial E_g} \right) \frac{\alpha_2(\omega_1)}{Z(\omega_1)} \left(\overline{\frac{\alpha_2(\omega_2)}{Z(\omega_2)}} \right) \right. \\ &\left. - \frac{1}{2} \frac{\partial \mu_g}{\partial E_g} \left\{ \frac{\alpha_1(\omega_1)}{Z(\omega_1)} \left(\overline{\frac{\alpha_2(\omega_2)}{Z(\omega_2)}} \right) + \left(\overline{\frac{\alpha_1(\omega_2)}{Z(\omega_2)}} \right) \frac{\alpha_2(\omega_1)}{Z(\omega_1)} \right\} \right] k_1 k_2 e^{i((\omega_1 - \omega_2)t + \kappa_1 - \kappa_2)} \end{aligned} \tag{21}$$

and the plate e.m.f. is equal to the real part of

$$\begin{aligned} &\left[r_p{}^3 P_2 \frac{\alpha_3(\omega_1)}{Z(\omega_1)} \left(\overline{\frac{\alpha_3(\omega_2)}{Z(\omega_2)}} \right) + \frac{1}{2} \left(\frac{\partial \mu_p}{\partial E_g} + \mu_p \frac{\partial \mu_p}{\partial E_p} \right) \frac{\alpha_1(\omega_1)}{Z(\omega_1)} \left(\overline{\frac{\alpha_1(\omega_2)}{Z(\omega_2)}} \right) \right. \\ &\left. - \frac{1}{2} \frac{\partial \mu_p}{\partial E_p} \left\{ \frac{\alpha_1(\omega_1)}{Z(\omega_1)} \left(\overline{\frac{\alpha_2(\omega_2)}{Z(\omega_2)}} \right) + \left(\overline{\frac{\alpha_1(\omega_2)}{Z(\omega_2)}} \right) \frac{\alpha_2(\omega_1)}{Z(\omega_1)} \right\} \right] k_1 k_2 e^{i((\omega_1 - \omega_2)t + \kappa_1 - \kappa_2)}, \end{aligned} \tag{22}$$

where a bar over a quotient indicates its conjugate complex.

It follows from Fig. 3 by straightforward calculations that the currents i_{p2} and i_{g2} produced by the e.m.f.'s (21) and (22), are

$$\left.\begin{aligned} i_{p2}(\omega_1 - \omega_2) &= R \left[-\frac{[\epsilon_g]}{Z_b(\omega_1 - \omega_2)} + \frac{[\epsilon_p]}{Z_d(\omega_1 - \omega_2)} \right] \\ i_{g2}(\omega_1 - \omega_2) &= R \left[\frac{[\epsilon_g]}{Z_a(\omega_1 - \omega_2)} - \frac{[\epsilon_p]}{Z_c(\omega_1 - \omega_2)} \right] \end{aligned}\right\}, \tag{23}$$

where $[\epsilon_g]$ and $[\epsilon_p]$ are abbreviations for the complex quantities (21) and (22), respectively, and

$$\left.\begin{aligned} Z_a(\omega) &= Z(\omega)\frac{r_g(Z_g' + Z_2)}{Z_2(r_p + Z_p') + r_pZ_p'} = |Z_a(\omega)|e^{i\psi_a(\omega)} \\ Z_b(\omega) &= Z(\omega)\frac{r_g(Z_g' + Z_2)}{Z_g'(\mu_pZ_2 - r_p)} = |Z_b(\omega)|e^{i\psi_b(\omega)} \\ Z_c(\omega) &= Z(\omega)\frac{r_g(Z_2 + Z_g')}{Z_p'(\mu_gZ_2 - r_g)} = |Z_c(\omega)|e^{i\psi_c(\omega)} \\ Z_d(\omega) &= Z(\omega)\frac{r_g(Z_2 + Z_g')}{Z_2(r_g + Z_g') + r_gZ_g'} = |Z_d(\omega)|e^{i\psi_d(\omega)} \end{aligned}\right\}. \tag{24}$$

In (24) the introduction of the angles ψ is convenient when it is desired to evaluate the real parts of the expressions (23).

The expressions (21) to (24) can be used to obtain any second-order current of frequency $(\omega_a - \omega_b)$ by replacing ω_1 with ω_a and ω_2 with ω_b. The remaining second-order currents are found by a process similar to that above. For instance, $i_{p2}(2\omega_1)$ and $i_{g2}(2\omega_1)$ are given by the right-hand expressions in (23) provided the e.m.f.'s $[\epsilon]$ are those of frequency $(2\omega_1)$ and the impedances Z are evaluated at $(2\omega_1)$. In passing it may be remarked that equations similar to those in (23) and (24) also occur when third and higher-order effects are calculated.

Special Cases

If the impedances Z_1, Z_2, and Z_3 are infinite the case treated above reduces to that considered by Llewellyn,[3] and after proper simplifications the previous equations give results identical with those obtained by him. For instance, if we take the limiting values of e_{p1} in (12) as Z_1, Z_2, and Z_3 tend to infinity, and if we then divide the quantity inside the summation sign by $-Z_p(\omega_h)$, we get an expression for i_{p1} which may be shown to be identical with equation (33) in Llewellyn's paper, except for differences in notations. Similarly, the plate current $i_{p2}(\omega_1 - \omega_2)$ in (23) reduces to a value which may be shown to be equal to the sum of his equations (35) and (36), evaluated for this type of second-order current.

Another special case is that when the impedances Z_1, Z_2, and Z_3 are all finite but conductive grid current is absent. We then have μ_g equal to zero, and R_g equal to infinity, and the previous general equations are simplified correspondingly.

We arrive at the case treated by Peterson-Evans [2] by maintaining the assumption of no conductive grid current but by assuming Z_1, Z_2, and Z_3 to be infinite. For instance, if then i_{p1} and $i_{p2}(\omega_1 - \omega_2)$ are evaluated on this basis for a plate impedance Z_p equal to a pure resistance at all frequencies, it can be shown that the currents so obtained are identical with the corresponding currents given by equations (4) and (6) in the paper referred to.

Finally, if we assume finite values for Z_1, Z_2, and Z_3, no conductive grid current, and constant μ_p, we have the case treated in the previous paper.[5]

Theory of Four-Electrode Tube Circuits

Circuits with tubes having more than three electrodes can be treated by a process similar to that adopted above, as will be made clear by outlining the theory for the four-electrode tube circuit.

The circuit to be considered is shown in Fig. 5 where Z_1 to Z_6 are

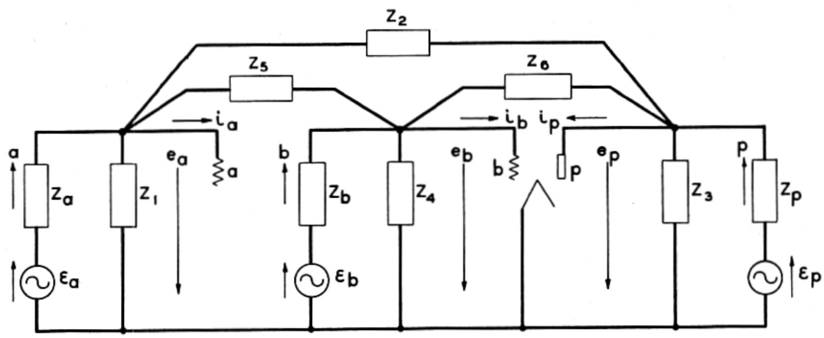


Fig. 5—Four-electrode vacuum tube and circuit.

impedances which may include inter-electrode admittances. On the electrodes denoted by a, b, and p are impressed the variable electromotive forces ϵ_a, ϵ_b, and ϵ_p in series with the impedances Z_a, Z_b, and Z_p, respectively. The significance of the quantities E_p, E_{p0}, e_p, I_p, I_{p0}, i_p and the corresponding quantities with indices a and b, is obvious from the preceding discussion of the three-electrode tube circuit. Let a, b, and p be the incremental voltages across the impedances Z_a, Z_b, and Z_p. As before, instantaneous values are implied.

For the currents we get the expansions

$$\left.\begin{aligned} i_p &= P_1e_a + P_2e_b + P_3e_p + P_4e_a^2 + P_5e_b^2 + P_6e_p^2 \\ &\qquad + P_7e_ae_b + P_8e_ae_p + P_9e_be_p + \cdots \\ i_a &= A_1e_a + A_2e_b + A_3e_p + A_4e_a^2 + A_5e_b^2 + A_6e_p^2 \\ &\qquad + A_7e_ae_b + A_8e_ae_p + A_9e_be_p + \cdots \\ i_b &= B_1e_a + B_2e_b + B_3e_p + B_4e_a^2 + B_5e_b^2 + B_6e_p^2 \\ &\qquad + B_7e_ae_b + B_8e_ae_p + B_9e_be_p + \cdots \end{aligned}\right\}, \qquad (25)$$

where

$$\left.\begin{aligned} &P_1 = \frac{\partial I_p}{\partial E_a},\ P_2 = \frac{\partial I_p}{\partial E_b},\ P_3 = \frac{\partial I_p}{\partial E_p},\ P_4 = \frac{1}{2}\frac{\partial^2 I_p}{\partial E_a{}^2},\ P_5 = \frac{1}{2}\frac{\partial^2 I_p}{\partial E_b{}^2} \\ &P_6 = \frac{1}{2}\frac{\partial^2 I_p}{\partial E_p{}^2},\ P_7 = \frac{\partial^2 I_p}{\partial E_a \partial E_b},\ P_8 = \frac{\partial^2 I_p}{\partial E_a \partial E_p},\ P_9 = \frac{\partial^2 I_p}{\partial E_b \partial E_p},\ \cdots \end{aligned}\right\} \tag{26}$$

and similar expressions hold for the A- and B-values.

The electrode resistances are by definition

$$\frac{1}{r_p} = \frac{\partial I_p}{\partial E_p}, \qquad \frac{1}{r_a} = \frac{\partial I_a}{\partial E_a}, \qquad \frac{1}{r_b} = \frac{\partial I_b}{\partial E_b}. \tag{27}$$

The mu-factors are

$$\left.\begin{aligned} \mu_{pa} &= \frac{\dfrac{\partial I_p}{\partial E_a}}{\dfrac{\partial I_p}{\partial E_p}} = -\left(\frac{dE_p}{dE_a}\right)_{E_b,\ I_p=\text{const.}} \\ \mu_{ap} &= \frac{\dfrac{\partial I_a}{\partial E_p}}{\dfrac{\partial I_a}{\partial E_a}} = -\left(\frac{dE_a}{dE_p}\right)_{E_b,\ I_a=\text{const.}} \\ \mu_{bp} &= \frac{\dfrac{\partial I_b}{\partial E_p}}{\dfrac{\partial I_b}{\partial E_b}} = -\left(\frac{dE_b}{dE_p}\right)_{E_a,\ I_b=\text{const.}} \\ \mu_{pb} &= \frac{\dfrac{\partial I_p}{\partial E_b}}{\dfrac{\partial I_p}{\partial E_p}} = -\left(\frac{dE_p}{dE_b}\right)_{E_a,\ I_p=\text{const.}} \\ \mu_{ab} &= \frac{\dfrac{\partial I_a}{\partial E_b}}{\dfrac{\partial I_a}{\partial E_a}} = -\left(\frac{dE_a}{dE_b}\right)_{E_p,\ I_a=\text{const.}} \\ \mu_{ba} &= \frac{\dfrac{\partial I_b}{\partial E_a}}{\dfrac{\partial I_b}{\partial E_b}} = -\left(\frac{dE_b}{dE_a}\right)_{E_p,\ I_b=\text{const.}} \end{aligned}\right\} \tag{28}$$

and the transconductances

$$\left.\begin{aligned} S_{pa} &= \frac{\partial I_p}{\partial E_a} = \frac{\mu_{pa}}{r_p} & S_{pb} &= \frac{\partial I_p}{\partial E_b} = \frac{\mu_{pb}}{r_p} \\ S_{ap} &= \frac{\partial I_a}{\partial E_p} = \frac{\mu_{ap}}{r_a} & S_{ab} &= \frac{\partial I_a}{\partial E_b} = \frac{\mu_{ab}}{r_a} \\ S_{bp} &= \frac{\partial I_b}{\partial E_p} = \frac{\mu_{bp}}{r_b} & S_{ba} &= \frac{\partial I_b}{\partial E_a} = \frac{\mu_{ba}}{r_b} \end{aligned}\right\}. \tag{29}$$

It can now be shown that

$$\left.\begin{aligned} P_1 &= \frac{\mu_{pa}}{r_p}, \qquad P_2 = \frac{\mu_{pb}}{r_p}, \qquad P_3 = \frac{1}{r_p} \\ P_6 &= -\frac{1}{2r_p^2}\frac{\partial r_p}{\partial E_p},\ P_8 = \frac{1}{r_p}\frac{\partial \mu_{pa}}{\partial E_p} + 2\mu_{pa}P_6,\ P_9 = \frac{1}{r_p}\frac{\partial \mu_{pb}}{\partial E_p} + 2\mu_{pb}P_6 \\ P_4 &= \frac{1}{2r_p}\frac{\partial \mu_{pa}}{\partial E_a} + \frac{\mu_{pa}}{2r_p}\frac{\partial \mu_{pa}}{\partial E_p} + \mu_{pa}^2P_6 \\ P_5 &= \frac{1}{2r_p}\frac{\partial \mu_{pb}}{\partial E_b} + \frac{\mu_{pb}}{2r_p}\frac{\partial \mu_{pb}}{\partial E_p} + \mu_{pb}^2P_6 \\ P_7 &= \frac{1}{r_p}\frac{\partial \mu_{pb}}{\partial E_a} + \frac{\mu_{pb}}{r_p}\frac{\partial \mu_{pa}}{\partial E_p} + 2\mu_{pb}\mu_{pa}P_6 \\ &= \frac{1}{r_p}\frac{\partial \mu_{pa}}{\partial E_b} + \frac{\mu_{pa}}{r_p}\frac{\partial \mu_{pb}}{\partial E_p} + 2\mu_{pa}\mu_{pb}P_6 \end{aligned}\right\}, \tag{30}$$

$$\left.\begin{aligned} A_1 &= \frac{1}{r_a}, \qquad A_2 = \frac{\mu_{ab}}{r_a}, \qquad A_3 = \frac{\mu_{ap}}{r_a} \\ A_4 &= -\frac{1}{2r_a^2}\frac{\partial r_a}{\partial E_a},\ A_7 = \frac{1}{r_a}\frac{\partial \mu_{ab}}{\partial E_a} + 2\mu_{ab}A_4,\ A_8 = \frac{1}{r_a}\frac{\partial \mu_{ap}}{\partial E_a} + 2\mu_{ap}A_4 \\ A_5 &= \frac{1}{2r_a}\frac{\partial \mu_{ab}}{\partial E_b} + \frac{\mu_{ab}}{2r_a}\frac{\partial \mu_{ab}}{\partial E_a} + \mu_{ab}^2A_4 \\ A_6 &= \frac{1}{2r_a}\frac{\partial \mu_{ap}}{\partial E_p} + \frac{\mu_{ap}}{2r_a}\frac{\partial \mu_{ap}}{\partial E_a} + \mu_{ap}^2A_4 \\ A_9 &= \frac{1}{r_a}\frac{\partial \mu_{ap}}{\partial E_b} + \frac{\mu_{ap}}{r_a}\frac{\partial \mu_{ab}}{\partial E_a} + 2\mu_{ap}\mu_{ab}A_4 \\ &= \frac{1}{r_a}\frac{\partial \mu_{ab}}{\partial E_p} + \frac{\mu_{ab}}{r_a}\frac{\partial \mu_{ap}}{\partial E_a} + 2\mu_{ab}\mu_{ap}A_4 \end{aligned}\right\}, \tag{31}$$

$$\left.\begin{aligned}
B_1 &= \frac{\mu_{ba}}{r_b}, \qquad B_2 = \frac{1}{r_b}, \qquad B_3 = \frac{\mu_{bp}}{r_b} \\
B_5 &= -\frac{1}{2r_b^2}\frac{\partial r_b}{\partial E_b},\ B_7 = \frac{1}{r_b}\frac{\partial \mu_{ba}}{\partial E_b} + 2\mu_{ba}B_5,\ B_9 = \frac{1}{r_b}\frac{\partial \mu_{bp}}{\partial E_b} + 2\mu_{bp}B_5 \\
B_4 &= \frac{1}{2r_b}\frac{\partial \mu_{ba}}{\partial E_a} + \frac{\mu_{ba}}{2r_b}\frac{\partial \mu_{ba}}{\partial E_b} + \mu_{ba}^2 B_5 \\
B_6 &= \frac{1}{2r_b}\frac{\partial \mu_{bp}}{\partial E_p} + \frac{\mu_{bp}}{2r_b}\frac{\partial \mu_{bp}}{\partial E_b} + \mu_{bp}^2 B_5 \\
B_8 &= \frac{1}{r_b}\frac{\partial \mu_{bp}}{\partial E_a} + \frac{\mu_{bp}}{r_b}\frac{\partial \mu_{ba}}{\partial E_b} + 2\mu_{bp}\mu_{ba}B_5 \\
&= \frac{1}{r_b}\frac{\partial \mu_{ba}}{\partial E_p} + \frac{\mu_{ba}}{r_b}\frac{\partial \mu_{bp}}{\partial E_p} + 2\mu_{ba}\mu_{bp}B_5
\end{aligned}\right\}. \tag{32}$$

The circuital laws applied to the external network furnish a number of equations, three of which are

$$\epsilon_a = a + e_a, \qquad \epsilon_b = b + e_b, \qquad \epsilon_p = p + e_p. \tag{33}$$

Let now

$$\left.\begin{aligned}
i_p &= \sum i_{pk}, & i_a &= \sum i_{ak}, & i_b &= \sum i_{bk} \\
e_p &= \sum e_{pk}, & e_a &= \sum e_{ak}, & e_b &= \sum e_{bk} \\
p &= \sum p_k, & a &= \sum a_k, & b &= \sum b_k
\end{aligned}\right\}. \tag{34}$$

We then obtain the equations

$$\left.\begin{aligned}
r_p i_{p1} - e_{p1} &= \mu_{pa}e_{a1} + \mu_{pb}e_{b1} \\
r_a i_{a1} - e_{a1} &= \mu_{ab}e_{b1} + \mu_{ap}e_{p1} \\
r_b i_{b1} - e_{b1} &= \mu_{ba}e_{a1} + \mu_{bp}e_{p1} \\
\epsilon_a = a_1 + e_{a1}, \qquad & \epsilon_b = b_1 + e_{b1}, \qquad \epsilon_p = p_1 + e_{p1}
\end{aligned}\right\}, \tag{35}$$

which show that the equivalent circuit for first order effects is that given in Fig. 6.

We get further for second-order quantities

$$\left.\begin{aligned}
r_p i_{p2} - e_{p2} &= \mu_{pa}e_{a2} + \mu_{pb}e_{b2} + r_p L \\
r_a i_{a2} - e_{a2} &= \mu_{ab}e_{b2} + \mu_{ap}e_{p2} + r_a M \\
r_b i_{b2} - e_{b2} &= \mu_{ba}e_{a2} + \mu_{bp}e_{p2} + r_b N \\
0 = a_2 + e_{a2}, \qquad & 0 = b_2 + e_{b2}, \qquad 0 = p_2 + e_{p2}
\end{aligned}\right\}, \tag{36}$$

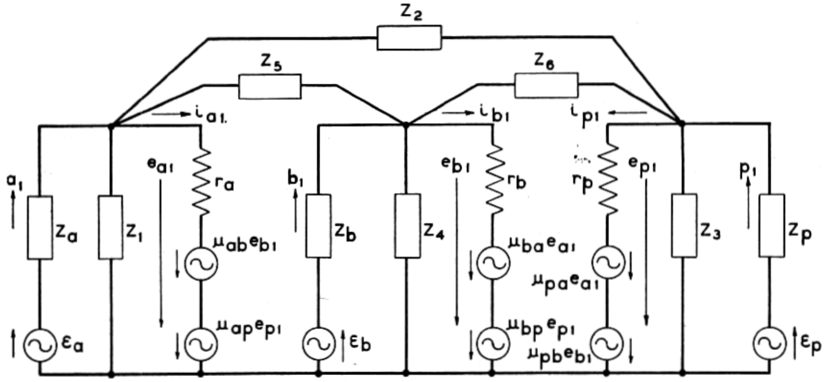


Fig. 6—Equivalent circuit—first-order effects.

where

$$\left.\begin{aligned} r_pL &= (P_4e_{a1}^2+P_5e_{b1}^2+P_6e_{p1}^2+P_7e_{a1}e_{b1}+P_8e_{a1}e_{p1}+P_9e_{b1}e_{p1})r_p \\ r_aM &= (A_4e_{a1}^2+A_5e_{b1}^2+A_6e_{p1}^2+A_7e_{a1}e_{b1}+A_8e_{a1}e_{p1}+A_9e_{b1}e_{p1})r_a \\ r_bN &= (B_4e_{a1}+B_5e_{b1}+B_6e_{p1}+B_7e_{a1}e_{b1}+B_8e_{a1}e_{p1}+B_9e_{b1}e_{p1})r_b \end{aligned}\right\}, \quad (37)$$

which in view of (30), (31) and (32) may be written

$$\left.\begin{aligned} r_pL = r_pP_6(\mu_{pa}e_{a1} + \mu_{pb}e_{b1} + e_{p1})^2 + \frac{1}{2}\left(\frac{\partial\mu_{pa}}{\partial E_a} + \mu_{pa}\frac{\partial\mu_{pa}}{\partial E_p}\right)e_{a1}^2 \\ + \frac{1}{2}\left(\frac{\partial\mu_{pb}}{\partial E_b} + \mu_{pb}\frac{\partial\mu_{pb}}{\partial E_p}\right)e_{b1}^2 + \left(\frac{\partial\mu_{pb}}{\partial E_a} + \mu_{pb}\frac{\partial\mu_{pa}}{\partial E_p}\right)e_{a1}e_{b1} \\ + \frac{\partial\mu_{pa}}{\partial E_p}e_{a1}e_{p1} + \frac{\partial\mu_{pb}}{\partial E_p}e_{b1}e_{p1} \end{aligned}\right\}, \quad (38)$$

$$\left.\begin{aligned} r_aM = r_aA_4(e_{a1} + \mu_{ab}e_{b1} + \mu_{ap}e_{p1})^2 + \frac{1}{2}\left(\frac{\partial\mu_{ab}}{\partial E_b} + \mu_{ab}\frac{\partial\mu_{ab}}{\partial E_a}\right)e_{b1}^2 \\ + \frac{1}{2}\left(\frac{\partial\mu_{ap}}{\partial E_p} + \mu_{ap}\frac{\partial\mu_{ap}}{\partial E_a}\right)e_{p1}^2 + \frac{\partial\mu_{ab}}{\partial E_a}e_{a1}e_{b1} \\ + \frac{\partial\mu_{ap}}{\partial E_a}e_{a1}e_{p1} + \left(\frac{\partial\mu_{ap}}{\partial E_b} + \mu_{ap}\frac{\partial\mu_{ab}}{\partial E_a}\right)e_{b1}e_{p1} \end{aligned}\right\}, \quad (39)$$

$$\left.\begin{aligned} r_bN = r_bB_5(\mu_{ba}e_{a1} + e_{b1} + \mu_{bp}e_{p1})^2 + \frac{1}{2}\left(\frac{\partial\mu_{ba}}{\partial E_a} + \mu_{ba}\frac{\partial\mu_{ba}}{\partial E_b}\right)e_{a1}^2 \\ + \frac{1}{2}\left(\frac{\partial\mu_{bp}}{\partial E_p} + \mu_{bp}\frac{\partial\mu_{bp}}{\partial E_b}\right)e_{p1}^2 + \frac{\partial\mu_{ba}}{\partial E_b}e_{a1}e_{b1} \\ + \left(\frac{\partial\mu_{bp}}{\partial E_a} + \mu_{bp}\frac{\partial\mu_{ba}}{\partial E_b}\right)e_{a1}e_{p1} + \frac{\partial\mu_{bp}}{\partial E_b}e_{b1}e_{p1} \end{aligned}\right\}. \quad (40)$$

From (36) it now follows that the equivalent circuit for second-order effects for the four-electrode tube circuit is as indicated in Fig. 7.

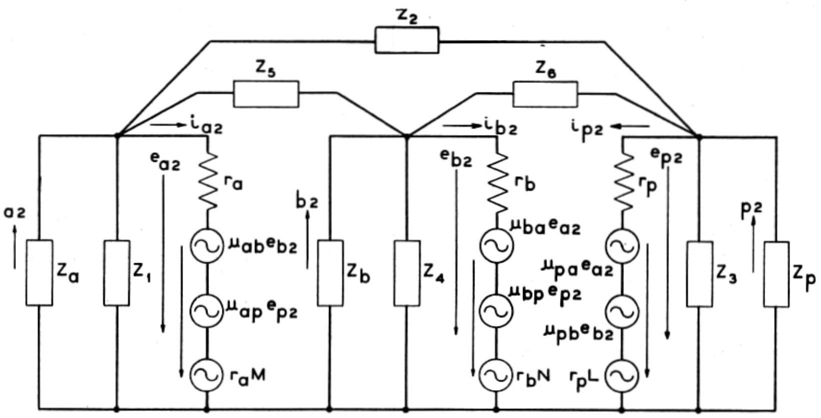


Fig. 7—Equivalent circuit—second-order effects.