# Constant Resistance Networks with Applications to Filter Groups

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The problem investigated is the determination of two finite networks such that, when connected in parallel, they will have a constant resistance at all frequencies. The admittance of any network may be written as the ratio of two polynomials in frequency. A network to be one of a constant resistance pair must have certain restrictions imposed on its admittance. In case the two networks are both filters of negligible dissipation, the expression for the input conductance of each may be written from a knowledge of the required loss characteristic.

The poles of the expression for the conductance are then found. They will be identical for the two networks. The networks are then built up by synthesis from those poles of the conductance which have negative real parts, these corresponding to real network elements.

The methods which have been developed for this last process are described in detail.

ONE of the most useful principles available to the network design engineer is that of constant resistance networks. The use of these networks is widespread in the telephone system for purposes of loss equalization and distortion correction, where they have the advantage of providing a means for altering the transmission properties of a circuit without affecting its impedance.[1] The three usual types of constant resistance networks are shown in Fig. 1, where, in all cases, $Z_1Z_2 = R^2$, a relationship which is always possible to fulfill if

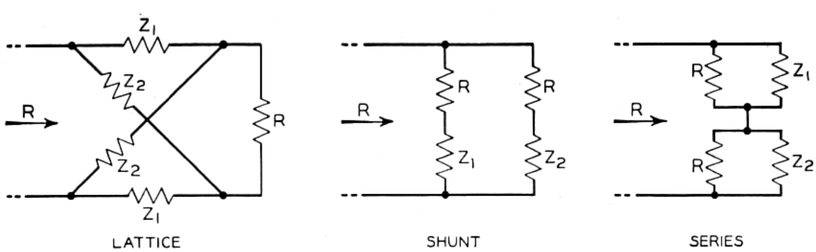


Fig. 1—The three fundamental forms of constant resistance networks.

[1] "Distortion Correction in Electrical Circuits with Constant Resistance Recurrent Networks," Otto J. Zobel, *Bell Sys. Tech. Jour.*, July 1928.

$Z_1$ and $Z_2$ are built up of resistive and reactive elements in the usual way.

The lattice type will not be considered here. The first step in extending the other two is shown in Fig. 2, where the networks shown

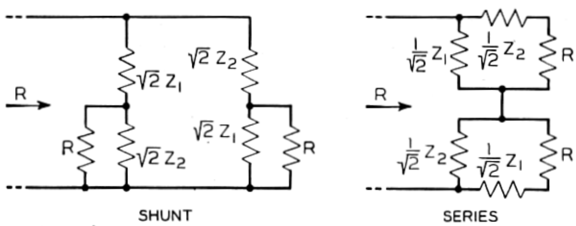


Fig. 2—The first step in extending the fundamental forms of the constant resistance networks.

have a constant resistance if $Z_1Z_2 = R^2$. The networks have now taken on the form of two half-section filters in parallel or series, provided that $Z_1$ and $Z_2$ are purely reactive. This suggests the possibility of an extension to more complicated configurations having the general properties of wave filters with constant resistance. Since the shunt and series types are analytically the same, only the former will be considered in detail.

Use will be made of the following theorem:

*Any finite network of linear elements having a constant conductance at all frequencies, and no purely reactive shunt across its terminals, has zero susceptance.*

The admittance may be written [2]

$$Y(\lambda) = \frac{A_0 + A_1\lambda + \cdots + A_m\lambda^m}{B_0 + B_1\lambda + \cdots + B_n\lambda^n},$$

where $\lambda = i(\omega/\omega_0)$ and $m$ is equal to or one greater or one less than $n$. $\omega_0$ is a constant which fixes the frequency scale. If the real part of $Y$ is to be a constant other than zero, $A_0$ cannot be zero and $m$ must be equal to or greater than $n$. If there is no purely reactive shunt across the terminals, $B_0$ cannot be zero and $m$ cannot be greater than $n$. The expression for the admittance may then be written

$$Y(\lambda) = G\frac{1 + A_1\lambda + \cdots + A_n\lambda^n}{1 + B_1\lambda + \cdots + B_n\lambda^n}.$$

[2] See "Synthesis of a Finite Two Terminal Network Whose Driving Point Impedance is a Prescribed Function of Frequency," Otto Brune, *M. I. T. Journal of Mathematics and Physics*, vol. 10, 1931.

By elementary methods it may be shown that if the real part of this expression is constant for all real frequencies then $A_1 = B_1$, $\cdots A_n = B_n$, and the imaginary part is zero. All other possibilities involve special relations between the $B$'s, which correspond to a $Y(\lambda)$ with poles on the imaginary axis. This has been excluded by the condition of no purely reactive shunt across the terminals. The study of networks of constant admittance may then be restricted to the study of the conditions for constant conductance.

We will consider, then, the problem of designing two passive networks of linear elements such that, when connected in parallel, they will have constant conductance. The value of the constant conductance may be taken as unity without loss of generality.

The conductance of a finite network may be written as a ratio of two polynomials in frequency. Its value must always be positive for real frequencies, and for the case under consideration it may never exceed unity, since otherwise the conductance of the second network to make up the constant resistance pair would be required to be negative. The expression for the conductance of the first network may be written in the form

$$G_1 = \frac{1}{1 + F(\lambda)}, \tag{1}$$

where $\lambda$ may be $i(\omega/\omega_0)$ as above, or it may be taken as any imaginary function of frequency which may be realized by the impedance of a reactive network, and $F(\lambda)$ is the ratio of two polynomials in even powers of $\lambda$. By subtracting $G_1$ from unity the required expression for $G_2$ may be obtained:

$$G_2 = \frac{1}{1 + \dfrac{1}{F(\lambda)}}. \tag{2}$$

An investigation of general networks of an arbitrary number of resistance and reactance elements fulfilling the relations (1) and (2) would take the present investigation too far from its main objective. If the networks are to have the general properties of wave filters with a minimum of loss in a band, they may be restricted to reactive networks having a single resistance. Furthermore, both resistances may be taken as unity, for, in cases where this is not necessary, a transformation to some other value may be made after the design is completed on the unit resistance basis. We assume, too, that when $\lambda = 0$, $F(\lambda) = 0$ and $G_1 = 1$, $G_2 = 0$. This implies the proper choice of the expression for $\lambda$.

With a voltage $E_0$ applied to the common terminals the power absorbed by the first network is $E_0^2G_1$ and by the second is $E_0^2G_2$. Since in both cases the power delivered to the network must be absorbed in the single resistance, the two insertion losses are given by

$$e^{-2\alpha_1} = G_1 = \frac{1}{1 + F(\lambda)}, \tag{3}$$

$$e^{-2\alpha_2} = G_2 = \frac{1}{1 + \dfrac{1}{F(\lambda)}}. \tag{4}$$

Since $F(\lambda)$ must be an even function of $\lambda$, the poles of (3) may be written $\pm c_m \pm id_n$ and (3) is

$$e^{-2\alpha_1} = -D^2 \frac{1}{\lambda + c_0} \frac{1}{\lambda - c_0} \prod_{1}^{m=(n-1)/2} \frac{1}{\lambda + c_m \pm id_m} \frac{1}{\lambda - c_m \pm id_m} \tag{5}$$

if the degree of $F(\lambda)$ is $2n$. If $n$ is even, the terms $\lambda + c_0$ and $\lambda - c_0$ are omitted and the product taken from $m = 1$ to $m = n/2$. The quantity $D^2$ is the denominator of $F(\lambda)$, a polynomial in $\lambda^2$.

Let $\beta_1$ be the phase angle between $E_0$ and the voltage $E_1$ across the resistance in the first network. The left side of equation (3) may then be factored in the form $e^{-2\alpha_1} = e^{-(\alpha_1+i\beta_1)}e^{-(\alpha_1-i\beta_1)}$. Similarly half of the factors on the right belong with $e^{-(\alpha_1+i\beta_1)}$ and half with $e^{-(\alpha_1-i\beta_1)}$. Now the terms with poles having a negative real part[3] must belong with $e^{-(\alpha_1+i\beta_1)}$ so that:

$$\begin{aligned} e^{-(\alpha_1+i\beta_1)} &= D\frac{1}{\lambda + c_0} \Pi \frac{1}{\lambda + c_m \pm id_m} \\ &= D\frac{1}{\lambda + c_0} \Pi \frac{1}{c_m^2 + d_m^2 + \lambda^2 + 2c_m\lambda}. \end{aligned} \tag{6}$$

Since $\lambda$ is an imaginary function of frequency, say $\lambda = ix$, and $D$ is real if $\lambda$ or $x$ is real, the phase $\beta_1$ is given by

$$\beta_1 = \tan^{-1}\frac{x}{c_0} + \sum_{m=1}^{m=(n-1)/2} \tan^{-1}\frac{2c_m x}{c_m^2 + d_m^2 - x^2}. \tag{7}$$

If $n$ is even the expression is

$$\beta_1 = \sum_{m=1}^{m=n/2} \tan^{-1}\frac{2c_m x}{c_m^2 + d_m^2 - x^2}. \tag{7a}$$

[3] These being the factors that correspond to physically realizable network elements, they belong with the physically realizable factor of the exponent.

The network can be designed from equation (6) or by making use of both (3) and (7). Both methods will be illustrated in two types of networks giving filter characteristics.

## Filters with Characteristics Similar to the "Constant K" Type of Filter

As the simplest form of $F(\lambda)$ take $F(\lambda) = [(\lambda)/(i)]^{2n}$. The poles of (3) are then simply the $2n$ roots of $(-1)^{n-1}$, which may be written $\pm \cos(m\pi/n) \pm i \sin(m\pi/n)$ if $n$ is odd, and $\pm \cos[(2m-1)\pi/2n] \pm i \sin[(2m-1)\pi/2n]$ when $n$ is even. In the first case $m$ varies between zero and $(n-1)/2$ and in the second case between unity and $n/2$. For the case of $n$ being odd, equation (6) may then be written

$$e^{-(\alpha_1+i\beta_1)} = \frac{1}{1+\lambda} \prod_{m=1}^{m=(n-1)/2} \frac{1}{1+\lambda^2+2\cos\frac{m\pi}{n}\lambda} \tag{8}$$

where the polynomial $D$ is unity in this case. The last equation expanded is in the form

$$e^{\alpha_1+i\beta_1} = 1 + A_1\lambda + A_2\lambda^2 + \cdots + A_n\lambda^n, \tag{9}$$

which is the form for the ratio $E_0/E_1$ for the network shown in Fig. 3. By writing out the ratio $E_0/E_1$ for this network and comparing terms with equation (8) expanded in the form of (9) the values of the $a$'s may be found to be [4]

$$\begin{aligned} a_1 &= \sin\frac{\pi}{2n}, \\ a_2 &= \frac{\sin\frac{3\pi}{2n}\sin\frac{\pi}{2n}}{a_1\cos^2\frac{\pi}{2n}}, \\ &\cdots\cdots\cdots\cdots \\ a_m &= \frac{\sin\frac{2m-1}{2n}\pi\,\sin\frac{2m-3}{2n}\pi}{a_{m-1}\cos^2\frac{m-1}{2n}\pi}, \\ a_n &= n\sin\frac{\pi}{2n}. \end{aligned} \tag{10}$$

[4] By the evaluation of the finite sums and products of the trigonometric terms. No short method has been found for obtaining the results.

The second network, which when connected in parallel with Fig. 3 will give a constant resistance, is obtained from the first by replacing $\lambda$ by $1/\lambda$. It is shown in Fig. 4.

These structures have been designed on the basis of $\lambda$ being a pure imaginary. Note, however, that the two structures will have a constant resistance provided that $\lambda$ is any function realizable by a combination of resistances and reactances. Equations (8) and (9) will still hold but (3) and (7) will no longer be true. Note, too, that for the simplest case of $n = 1$ the structures reduce to the usual form for constant resistance networks as shown in Fig. 1.

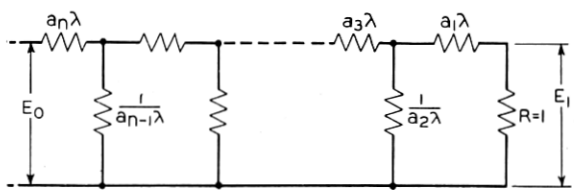


Fig. 3

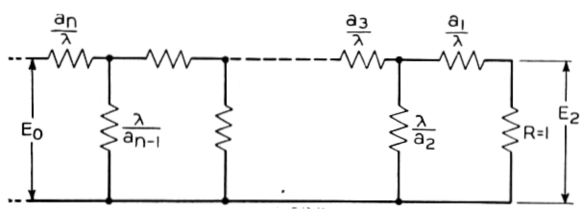


Fig. 4

Figs. 3–4—A pair of constant resistance networks of "constant K" configuration.

If $\lambda$ is taken of the form $i(f/f_0)$, the structure of Fig. 3 will be made up of series coils and shunt condensers in the form of a low-pass filter. The structure of Fig. 4 will be of the form of a high-pass filter with series condensers and shunt inductances. The loss of the first network is

$$e^{2\alpha_1} = 1 + \left(\frac{f}{f_0}\right)^{2n}$$

and of the second

$$e^{2\alpha_2} = 1 + \left(\frac{f_0}{f}\right)^{2n}.$$

With $f < f_0$ the loss of the first network will be small and the loss of the second network large. With $f > f_0$ the reverse is true. At $f = f_0$ each of the networks takes half of the available power, illustrating a necessary property of constant resistance networks of this type, of a three db loss at the cross-over frequency.

If $\lambda$ is taken of the form $i\dfrac{\left(\dfrac{f}{f_m}-\dfrac{f_m}{f}\right)}{\left(\dfrac{f_2}{f_m}-\dfrac{f_m}{f_2}\right)}$ the networks become band-pass and band-elimination filters, respectively. By taking other functions for $\lambda$ multiple band structures may be designed, subject always to the limitation that the combined bands of both filters must extend over the whole frequency range, with a three db loss at each cross-over point.

The evaluation of the elements is easily done from equations (10). The impedance denoted by $a_1\lambda$, for example, in the low-pass filter would have the value $i(f/f_0)\sin(\pi/2n)$, which is an inductance of a value $(1/2\pi f_0)[\sin(\pi/2n)]$. For a terminating resistance different from unity the value of the first inductance is $L_1 = (R_0/2\pi f_0)[\sin(\pi/2n)]$ or in general any inductance is $L_m = a_m L_0$ where $L_0 = R_0/2\pi f_0$. Similarly, any capacitance is $C_m = a_m C_0$ where $C_0 = 1/2\pi f_0 R_0$. The corresponding formulas for the second network are $C_m = C_0/a_m$ and $L_m = L_0/a_m$. The same formulas hold for $n$ even; in that case the networks of Fig. 3 and Fig. 4 would terminate on the right in a shunt arm with impedances of $1/a_1\lambda$ and $\lambda/a_1$, respectively. This is illustrated by Fig. 2 for $n = 2$.

## Filters with Characteristics Similar to Those of the "M-derived" Type

The networks shown in Fig. 3 and Fig. 4 have the same configuration and similar characteristics to constant $K$ filters. They are subject to the same objection of a relatively slow rate of cut-off and an excessive loss at frequencies remote from the cut-off. A type of characteristic similar to that obtained with $M$-derived filters, with points of infinite loss at finite frequencies, is necessary for an economical design in the majority of cases.

The loss characteristic of the network is of course fixed by the function $F(\lambda)$, a ratio of two polynomials in $\lambda$. It may be written

$$F(\lambda) = A_0\lambda^2\frac{1 + A_1\lambda^2 + \cdots + A_n\lambda^{2n-2}}{1 + B_1\lambda^2 + \cdots + B_n\lambda^{2n-2}}.$$

Now the first filter will have infinite loss points when the denominator is zero, and the second filter when the numerator is zero. If these peaks are to occur at real frequencies, $F(\lambda)$ must have poles at $\lambda^2 = -1/S_m^2$ and zeros at $\lambda^2 = -P_m^2$. Moreover, since $1/[1 + F(\lambda)]$ and $1/[1 + (1/F(\lambda)]$ must always be positive for real frequencies, the expression for $F(\lambda)$ when all its zeros and poles occur at real frequencies must be a perfect square. It may then be written

$$F(\lambda) = A_0\lambda^2 \frac{(P_1^2 + \lambda^2)^2 \cdots (P_{n-1}^2 + \lambda^2)^2}{(1 + S_1^2\lambda^2)^2 \cdots (1 + S^2{}_{(n-1)/2}\lambda^2)^2}.$$

In order to get an idea of the significance of the expression, let $\lambda = i(f/f_0)$ and restrict the $P$'s and the $S$'s to values less than unity. The first network will then have zero loss points at $f = 0$ and $f = P_m f_c$ and infinite loss points at $f = f_0/S_m$ and $f = \infty$. The second network will have infinite loss points when the loss of the first is zero, and zero loss points when the loss of the first is infinite. The first network is therefore a low-pass filter and the second a high-pass filter.

The following work is considerably simplified if $S_m = P_m$. This implies that the characteristic of the second filter is the same function of $1/\lambda$ that the first is of $\lambda$. If the cross-over point is fixed at $\lambda^2 = -1$, the value of $A_0$ is $-1$ and in order to write equation (6) or (7), it is necessary to find those zeros with a negative real part of

$$1 - \lambda^2 \frac{(P_1^2 + \lambda^2)^2 \cdots (P^2{}_{(n-1)/2} + \lambda^2)^2}{(1 + P_1^2\lambda^2)^2 \cdots (1 + P^2{}_{(n-1)/2}\lambda^2)^2}$$
$$= \left[1 + \lambda \frac{(P_1^2 + \lambda^2) \cdots}{(1 + P_1^2\lambda^2) \cdots}\right]\left[1 - \lambda \frac{(P_1^2 + \lambda^2) \cdots}{(1 + P_1^2\lambda^2) \cdots}\right].$$

Now since the zeros of the second factor on the right are the negatives of the zeros of the first factor, it will be sufficient to find all of the zeros of the first factor and reverse the signs when necessary to secure negative real parts. Consider, then, the equation

$$1 + \lambda \frac{(P_1^2 + \lambda^2) \cdots (P^2{}_{(n-1)/2} + \lambda^2)}{(1 + P_1^2\lambda^2) \cdots (1 + P^2{}_{(n-1)/2}\lambda^2)} = 0.$$

One root is $\lambda = -1$. It may be shown further that the magnitude of all of the roots is unity. Writing $\lambda = \rho e^{i\theta}$ as a root, the magnitude of the typical product term $(P^2 + \lambda^2)/(1 + P^2\lambda^2)$ may be written

$$\left|\frac{P^2 + \lambda^2}{1 + P^2\lambda^2}\right|^2 = 1 + \frac{\left(\frac{1}{P^2} - P^2\right)\left(\rho^2 - \frac{1}{\rho^2}\right)}{\left(P\rho + \frac{1}{P\rho}\right)^2 - 4\sin^2\theta}.$$

Now since the denominator of the expression on the right is always positive, and all of the $P$'s are less than unity, the magnitude of each of the product terms is greater than unity if $\rho$ is greater than unity and less than unity if $\rho$ is less than unity. Since, however, the magnitude of the complete product must be unity, the value of $\rho$ must be unity.

After dividing through by the factor $1 + \lambda$, the remaining function is a reciprocal equation in $\lambda$ and may be written as an equation in $p = \lambda + (1/\lambda)$. Since the magnitudes of the roots in $\lambda$ are all unity, the roots in $p$ must all be real and be in the region $-2, +2$.

The degree of the polynomial in $p$ is $(n-1)/2$. It may be shown further that if $(n-1)/2$ is even there are an equal number of positive and negative real roots, if the degree is odd there is one more positive than negative root.

The equations in $p$ for various values of $(n-1)/2$ are

$$\begin{aligned} \frac{n-1}{2} &= 1, \quad p - (1 - \Sigma_1) = 0 \\ &= 2, \quad p^2 - (1 - \Sigma_2)p - (1 - \Sigma_1 + \Sigma_2) = 0 \\ &= 3, \quad p^3 - (1 - \Sigma_3)p^2 - (2 - \Sigma_1 + \Sigma_3)p \\ &\qquad + (1 - \Sigma_1 + \Sigma_2 - \Sigma_3) = 0 \\ &= 4, \quad p^4 - (1 - \Sigma_4)p^3 - (3 - \Sigma_1 + \Sigma_4)p^2 \\ &\qquad + (2 - \Sigma_1 + \Sigma_3 - 2\Sigma_4)p \\ &\qquad + (1 - \Sigma_1 + \Sigma_2 - \Sigma_3 + \Sigma_4) = 0, \end{aligned}$$

where the $\Sigma$'s are the symmetric functions of the $P^2$'s, that is,

$$\begin{aligned} \Sigma_1 &= P_1^2 + P_2^2 + \cdots P^2{}_{(n-1)/2}, \\ \Sigma_2 &= P_1^2 P_2^2 + \cdots + P^2{}_{(n-3)/2} P^2{}_{(n-1)/2}. \end{aligned}$$

The equations in $p$ may also be written in trigonometric form as follows:

$$\begin{aligned} \frac{n-1}{2} &= 1, \quad \cos\frac{3}{2}\theta + \Sigma_1 \cos\frac{\theta}{2} = 0 \\ &= 2, \quad \cos\frac{5}{2}\theta + \Sigma_2 \cos\frac{3}{2}\theta + \Sigma_1 \cos\frac{\theta}{2} = 0 \\ &= 3, \quad \cos\frac{7}{2}\theta + \Sigma_3 \cos\frac{5}{2}\theta + \Sigma_1 \cos\frac{3}{2}\theta + \Sigma_2 \cos\frac{\theta}{2} = 0 \\ &= 4, \quad \cos\frac{9}{2}\theta + \Sigma_4 \cos\frac{7}{2}\theta + \Sigma_1 \cos\frac{5}{2}\theta + \Sigma_3 \cos\frac{3}{2}\theta \\ &\qquad + \Sigma_2 \cos\frac{\theta}{2} = 0. \end{aligned}$$

These equations include the root at $\lambda = -1$ corresponding to $\theta = \pi$. Excluding this they will each have $(n-1)/2$ roots between $\theta = 0$ and $\theta = \pi$. The roots in $p$ will then be given by $p = 2\cos\theta$.

Equation (7) becomes

$$\beta_1 = \tan^{-1} x + \sum_{1}^{(n-1)/2} \tan^{-1} \frac{p_m x}{1 - x^2}, \tag{11}$$

where the quantities $p_m$ are the roots of the above equations, without regard to sign.

We require also the value of $d\beta_1/dx$, which may be written

$$\frac{d\beta_1}{dx} = \frac{1}{1+x^2}\left[1 + \sum_{1}^{(n-1)/2} \frac{p_m}{1 - \frac{(4 - p_m^2)x^2}{(1+x^2)^2}}\right]. \tag{12}$$

A possible configuration for the first network is shown in Fig. 5 and for the second in Fig. 6.

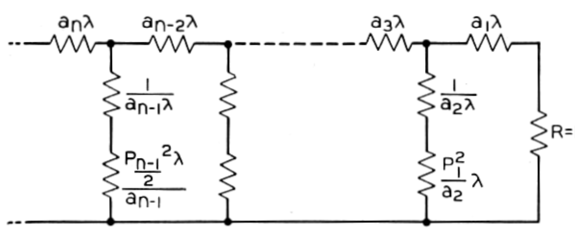


Fig. 5

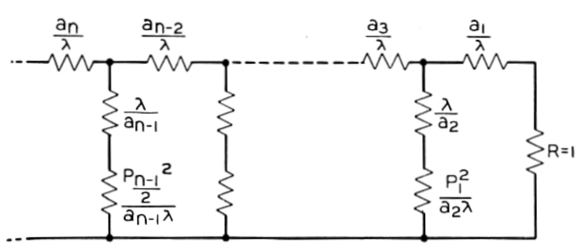


Fig. 6

Figs. 5–6—A pair of constant resistance networks of the "M-derived" configuration.

To find the elements it would be possible to expand the voltage ratio and solve for the $a$'s as was done in the constant $K$ illustration. Another method would be to find the input admittance of the network from the known input conductance, and find the $a$'s from this expression. A simpler method, however, takes advantage of the fact that each structure is a purely reactive network with the exception of the terminating resistance and finds the network elements in terms of the short circuit reactance as measured from the resistance end of the network.

Use may be made of the following theorem:

*With any four-terminal reactive network the reactance measured at terminals 3–4 with terminals 1–2 short-circuited is equal to the tangent of the phase shift between a voltage $E_0$ applied to terminals 1–2 and the resultant voltage $E_1$ across a unit resistance connected to terminals 3–4.*

The open-circuit voltage across 3–4 due to $E_0$ would be $\pm kE_0$, where $k$ is a real quantity, if the network contains only reactances. By Thévenin's Theorem, then,

$$E_1 = \frac{\pm kE_0}{1 + iX},$$

where $X$ is the reactance of the network from terminals 3–4. If $\beta$ is the phase shift between $E_0$ and $E_1$, $X = \tan\beta$.

Since this phase shift is given by (11) the short-circuit reactance is known. At a value of $\lambda = i(1/P_1)$ or $x = 1/P_1$, the impedance of the first shunt arm from the right of Fig. 5 is zero, so that the reactance of the filter is simply the reactance of the arm $a_1\lambda$, which gives the value of $a_1$ directly as

$$a_1 = P_1(\tan\beta_1)_1,$$

where $(\tan\beta_1)_1$ denotes the value of $\tan\beta_1$ when $x = 1/P_1$. The reactance of the network after subtracting $a_1x$ is $\tan\beta_1 - P_1(\tan\beta_1)_1x$. At values of $x$ very close to $1/P_1$ this is the reactance of the first shunt arm, or

$$a_2 = \frac{\dfrac{d}{dx}\left(P_1^2x - \dfrac{1}{x}\right)}{\dfrac{d}{dx}(\tan\beta_1 - a_1x)},$$

where, after differentiation, $x = 1/P_1$. Carrying through the differentiation,

$$a_2 = \frac{2P_1^2}{(1 + \tan^2\beta_1)_1\left(\dfrac{d\beta_1}{dx}\right)_1 - a_1}.$$

Similar formulas may be found for the rest of the elements. If $X_m$ denotes the reactance starting with the series arm $a_m\lambda$ or with the shunt arm $(P_{m/2}^2\lambda^2 + 1)/a_m\lambda$, then for $m$ odd, that is, for a series element,

$$a_m = P_{(m+1)/2}X_m \qquad \left(x = \frac{1}{P_{(m+1)/2}}\right)$$

and for a shunt arm, $m$ even,

$$\frac{1}{a_m} = \frac{1}{2P_{m/2}^2}\frac{dX_m}{dx} \qquad \left(x = \frac{1}{P_{m/2}}\right).$$

When $m = n$, or for the last series arm, a special relation is necessary, readily obtained by the limiting value of reactance as $x$ approaches zero. This gives

$$(a_1 + a_3 + \cdots + a_n)x = (1 + \Sigma p_m)x$$

or

$$a_n = 1 + \Sigma p_m - (a_1 + a_3 + \cdots + a_{n-2}).$$

To use these relations it is necessary to know the expression for $X_m$, the reactance to the left from the successive points in the network. To determine this in terms of the elements already known use may be made of the following theorem:

*If the impedance looking to the left into a network is $Z$, the impedance to the left from $A$, any point within the network is the negative of the impedance to the right from $A$ when the network is terminated on the right by an impedance $-Z$.*

For example, referring to Fig. 5, to determine $a_3$ it is necessary to know the reactance to the left starting with $a_3x$. By the theorem this is

$$X_3 = \frac{1}{\dfrac{a_2x}{1 - P_1^2x^2} - \dfrac{1}{a_1x - \tan\beta_1}}$$

and when $x = 1/P_2$ we have for $a_3$

$$\frac{1}{a_3} = \frac{a_2}{P_2^2 - P_1^2} - \frac{1}{a_1 - P_2(\tan\beta_1)_2}.$$

The impedance at that end of the filter terminated by the resistance is of interest. Its value of course depends upon the terminating impedance at the junction of the two filters, but assuming that this impedance and the separate terminating resistances are all $R_0$, the impedance from the load of the first filter is $R_0 \tanh(\alpha_2 + i\beta_2)$ if terminated in a series arm and $R_0 \coth(\alpha_2 + i\beta_2)$ if terminated in a shunt arm. Note that the impedance of the first filter depends upon the transfer constant of the second. The impedance from the load of the second filter depends in the same way upon the transfer constant of the first. The proof of these relations is based upon both networks being purely reactive.

## Applications

The use of the constant resistance pairs of filters is indicated wherever the impedance at the junction of two filters is of major importance. Another application which is of some importance is that of separating the energy in a band of frequencies into two or more channels, delivering all of the energy into one or the other of the loads.

The method may be extended to more than two networks in parallel or series to give a constant resistance. For example, the combination

$$G_1 = \frac{1}{[1 + F_1(\lambda)]\left[1 + \dfrac{1}{F_2(\lambda)}\right]},$$

$$G_2 = \frac{1}{\left[1 + \dfrac{1}{F_1(\lambda)}\right]\left[1 + \dfrac{1}{F_2(\lambda)}\right]},$$

$$G_3 = \frac{1}{1 + F_2(\lambda)},$$

will give a constant resistance for the three networks. Designs have been carried through on this basis where the networks are low-pass, high-pass and band-pass, respectively. This is one method of avoiding the limit of three db in the loss of the low-pass and the high-pass filters at their cross-over point, since in this case the band-pass filter will take up the power. A second method is to use a pair of low and high-pass filters, each terminated in another pair with different cross-over points. This method requires the use of both a low-pass and a high-pass filter as power absorbing networks but they would be simple structures and together would require no more elements than the single band-pass filter in a three-filter combination.

The two methods are illustrated in Fig. 7 and Fig. 8, respectively. The structures for the second type are given by Fig. 9. Note that the filters designated L.P. II, L.P. III, H.P. II and H.P. III have one series arm missing and are apparently terminated at a shunt point at the load end of the filter. This is a consequence of selecting the two $P$'s in such a way that the coefficient $a_1$ becomes zero, a matter of no particular difficulty in the case of a two-section filter.

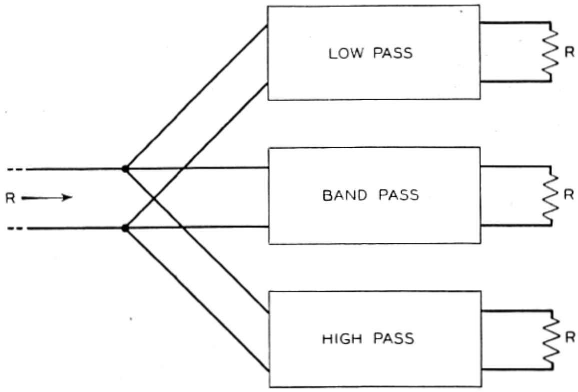


Fig. 7—A three-filter constant resistance combination.

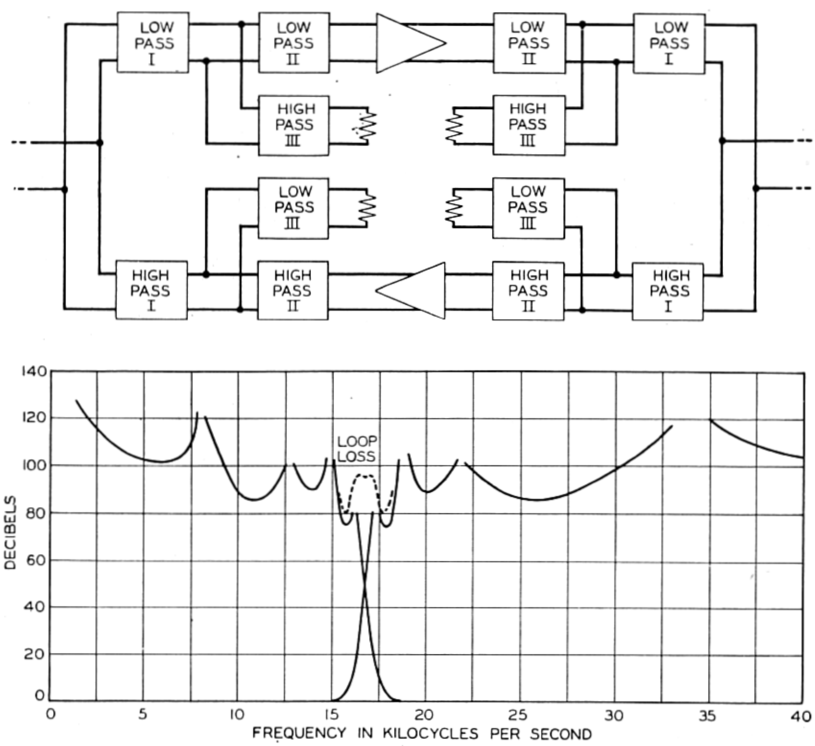


Fig. 8—Constant resistance networks used as directional filters.

It will be found that a filter of several sections of the type described in this paper will have somewhat less loss in the attenuated band than the usual type of design. On the other hand the loss in the band will, in general, be less unless additional elements are used in the standard type of filter to reduce reflection losses. A design for a pair of constant

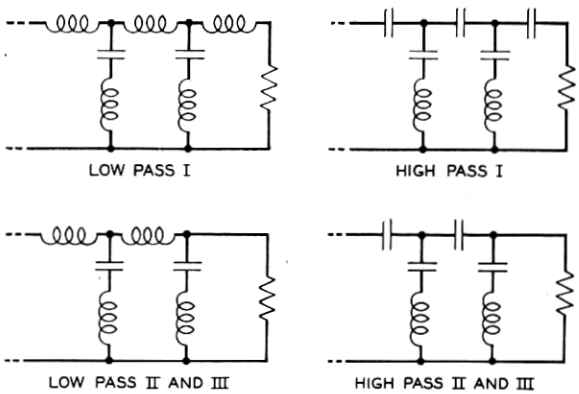


Fig. 9—The configuration of the filters of Fig. 8.

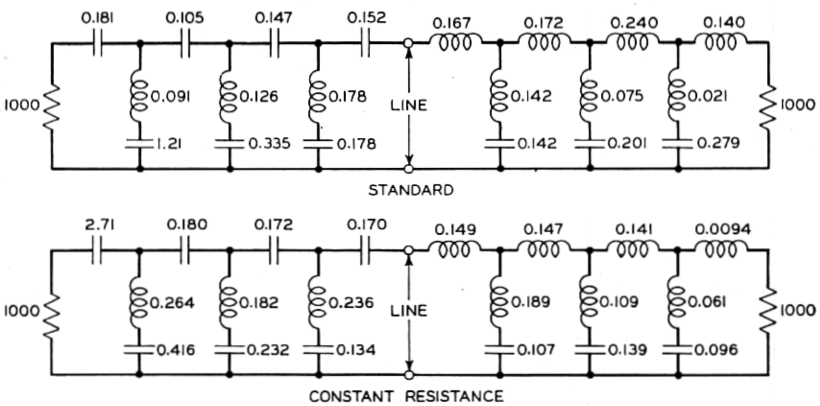


Fig. 10—Comparison of the elements of typical standard and constant resistance filters.

resistance filters having a cross-over frequency of 1000 cycles is compared with a design for a pair of standard filters in Fig. 10. No additional elements have been added to the standard type to improve the impedance.[5] The loss characteristics for the two low-pass filters

[5] "Impedance Correction of Wave Filters," E. B. Payne, and "A Method of Impedance Correction," H. W. Bode, *Bell Sys. Tech. Jour.*, October 1930.
"Extensions to the Theory and Design of Electric Wave Filters," Otto J. Zobel, *Bell Sys. Tech. Jour.*, April 1931.

are compared in Fig. 11. Note that in this case the constant resistance filters have only about sixty per cent of the loss of the standard

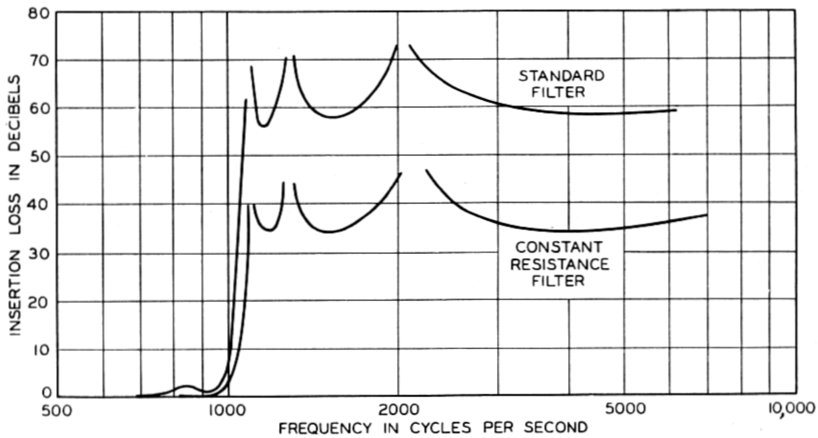


Fig. 11—Loss characteristics obtained by the filters of Fig. 10.

filters. The difference would not be as great for filters of less sharp discrimination.