Response of a Linear Rectifier to Signal and Noise*

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WHEN the input to a rectifier contains both signal and noise components, the resultant output is a complicated non-linear function of signal and noise. Given the spectra of the signal and noise input waves, the law of rectification, and the transmission characteristics of the input and output circuits of the rectifier, it should, in general, be possible to describe the spectrum of the resultant output wave. Before discussing the solution of the general problem, we shall derive some results of a simpler nature, which do not require a consideration of the distribution of the signal and noise energies as functions of frequency.

I. Direct-Current Component of Output

A quantity of considerable importance is the average value of the output amplitude. This is the quantity which would be read by a direct-current meter. Calculation of the average or d-c response can be performed in terms of the distribution of instantaneous output amplitudes in time. The distribution of output amplitude can be computed from the distribution of instantaneous input amplitudes and the law of rectification.

As an example, we shall compute the average current obtained from a linear rectifier when the input to the rectifier consists of a sinusoidal signal with random noise superposed upon it. The probability density function of the signal voltage is first determined, and the result given in (3). The corresponding probability density for the voltage of the noise is well known and is given in (4). The distribution of occurrence of the resultant instantaneous amplitudes of the combined noise and signal voltages is then computed by the rules of mathematical probability, and the result is shown in (7). The assumption that the rectifier is linear then leads directly to an integral which yields the average current obtained from the rectifier.

Let the signal voltage, E_s, be given by

$$E_s = P_o \cos \omega t. \tag{1}$$

The possible angular values of ωt are uniformly distributed throughout the range 0 to 2π. The range E_s to $E_s + dE_s$ corresponds to the range of values of ωt comprised in the interval.

$$\text{arc cos} \frac{E_s}{P_o} < \omega t < \text{arc cos} \frac{E_s + dE_s}{P_o} \tag{2}$$

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The angular width of this interval is $(P_o^2 - E_s^2)^{-1/2} dE_s$. There are two such intervals in the range $0 < \omega t < 2\pi$. Values of E_s outside the range $-P_o$ to P_o do not exist. Hence, the probability that the signal voltage lies in the interval dE_s at any particular E_s is given by

$$\Phi_s(E_s)dE_s = \begin{Bmatrix} 0, \; |E_s| > P_o \\ 2(P_o^2 - E_s^2)^{-1/2} \, dE_s/2\pi, \; |E_s| < P_o \end{Bmatrix} dE_s \tag{3}$$

Random noise as discussed in this section may be characterized by the fact that the instantaneous amplitudes are normally distributed in time; that is, if $\Phi_n(z)\,dz$ is the probability that the noise amplitude lies in the amplitude interval of width dz at z,

$$\Phi_n(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2\sigma^2} \tag{4}$$

where σ is the root mean square noise amplitude. The mean noise power dissipated in unit resistance is given by $W_n = \sigma^2$. The corresponding mean signal power is given by $W_s = P_o^2/2$. Let $\Phi_r(z)$ represent the probability density function of the instantaneous sum of the signal and noise amplitudes. Then

$$\Phi_r(z)dz = dz \int_{-\infty}^{\infty} \Phi_s(\lambda)\, \Phi_n(z - \lambda)d\lambda \tag{5}$$

or

$$\Phi_r(z) = \frac{1}{\pi\sigma\sqrt{2\pi}} \int_{-P_o}^{P_o} \frac{e^{-(z-\lambda)^2/2\sigma^2}\, d\lambda}{\sqrt{P_o^2 - \lambda^2}} \tag{6}$$

By the substitution $\lambda = P_o \cos\theta$, we may convert the integral to the form

$$\Phi_r(z) = \frac{1}{\pi\sigma\sqrt{2\pi}} \int_o^{\pi} e^{-(z-P_o\cos\theta)^2/2\sigma^2}\, d\theta \tag{7}$$

Suppose we insert a half-wave linear rectifier in series with the source of signal and noise, so that the current I is given in terms of the resultant instantaneous voltage E by

$$I = \begin{cases} 0, & E < 0 \\ \alpha E, & E > 0 \end{cases} \tag{8}$$

Then the average value of current flowing in the circuit is

$$\begin{aligned} \bar{I} &= \alpha \int_0^{\infty} z\Phi_r(z)\, dz \\ &= \frac{\alpha}{\pi\sigma\sqrt{2\pi}} \int_0^{\infty} z\, dz \int_o^{\pi} e^{-(z-P_o\cos\theta)^2/2\sigma^2}\, d\theta \end{aligned} \tag{9}$$

The value of this integral is shown in Appendix I to be

$$\bar{I} = \alpha \sqrt{\frac{W_n}{2\pi}}\, e^{-W_s/2W_n} \left\{ I_o(W_s/2W_n) + \frac{W_s}{W_n}\left[I_o\left(\frac{W_s}{2W_n}\right) + I_1\left(\frac{W_s}{2W_n}\right)\right]\right\} \tag{10}$$

This form is particularly convenient for calculation since Watson's Theory of Bessel Functions, Table II, gives $e^{-z}I_o(z)$ and $e^{-z}I_1(z)$ directly.

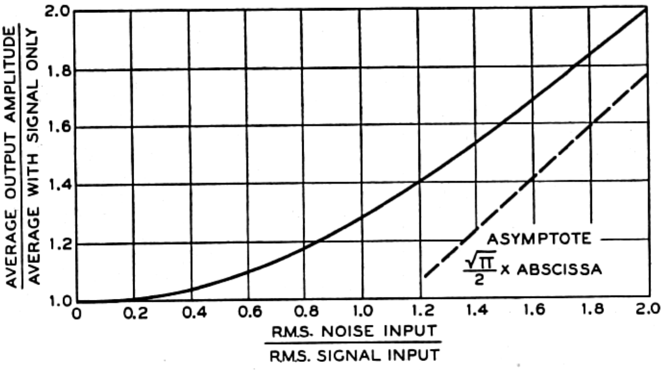

Fig. 1—Variation of direct-current component in response of linear rectifier with ratio of noise input to signal input.

Limiting forms of this equation may be expressed in terms of series in powers of W_s/W_n when the signal power is small compared with the noise power and in powers of W_n/W_s when the noise power is small compared with the signal power. The ascending series for small signal is:

$$\bar{I} = \alpha \sqrt{\frac{W_n}{2\pi}}\left[1 + \frac{1}{2(1!)^2}\frac{W_s}{W_n} + \frac{1(-1)}{2^2(2!)^2}\frac{(W_s)^2}{(W_n)} + \frac{1(-1)(-3)}{2^3(3!)^2}\frac{(W_s)^3}{(W_n)} + \cdots\right] = \alpha\sqrt{\frac{W_n}{2\pi}}\, {}_1F_1\left(\frac{-1}{2};1;\frac{-W_s}{W_n}\right) \tag{11}$$

The asymptotic series, which is available for computation when the signal is large, is

$$\bar{I} \sim \frac{\alpha\sqrt{2W_s}}{\pi}\left[1 + \frac{(-1)^2 W_n}{1!\,4W_s} + \frac{(-1)^2\cdot 1^2}{2!}\frac{(W_n)}{(4W_s)}^2 + \frac{(-1)^2\cdot 1^2\cdot 3^2}{3!}\frac{(W_n)}{(4W_s)}^3 + \frac{(-1)^2\cdot 1^2\cdot 3^2\cdot 5^2}{4!}\frac{(W_n)}{(4W_s)}^4 + \cdots\right] \tag{12}$$

Curves of $\bar{I}$ have been plotted in three ways. Fig. 1 shows the ratio of $\bar{I}$ to $\bar{I}_{so} = \alpha P_o/\pi$, the average current in the absence of noise, as a function of ratio of rms noise input to rms signal input. Figure 2 shows the ratio of $\bar{I}$ to $\bar{I}_{no} = \alpha\sigma/\sqrt{2\pi}$, the average current in the absence of signal, as a function of ratio of rms signal input to rms noise input. Figure 3 shows

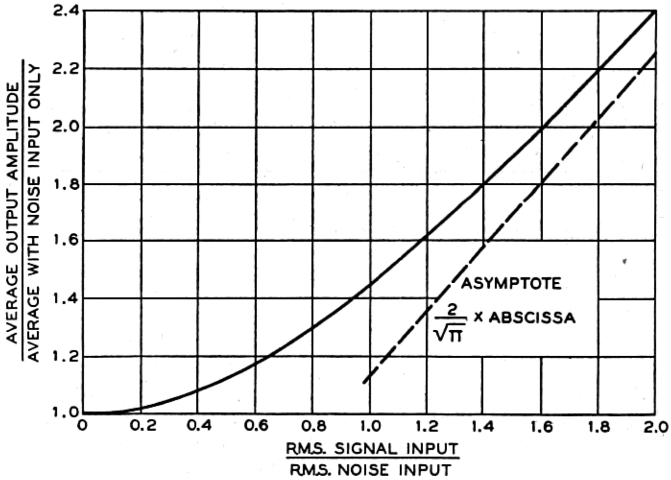

Fig. 2—Variation of direct-current component in response of linear rectifier with ratio of signal input to noise input.

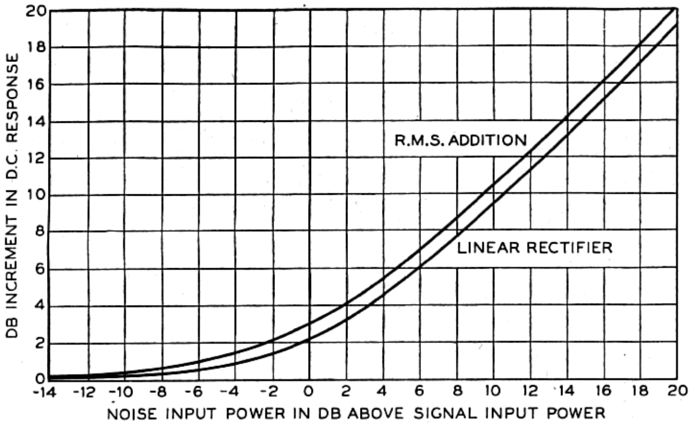

Fig. 3—Variation of direct-current component expressed in decibels, showing comparison between linear rectification and power addition of signal and noise.

the increment in d-c power output in decibels as varying amounts of noise expressed in decibels relative to the signal are added. The corresponding result for power addition is given for comparison.

II. Spectrum of Output

A much more powerful method of attack on this problem is obtained by the use of multiple Fourier series. In this section we shall use Fourier analysis to obtain not only the direct-current output of the rectifier, but also the spectral distribution of the sinusoidal components in the output of the rectifier. We represent the input spectrum by

$$E = P_0 \cos p_0 t + \sum_{n=1}^{N} P_n \cos p_n t \tag{13}$$

This representation is more general than that given by (4) in that a frequency spectrum as well as an amplitude distribution is defined; it may be shown that the probability density for the sum of N sinusoidal waves with incommensurable frequencies approaches (4) when N is large. The first term represents the sinusoidal signal; the mean power which would be dissipated by this signal in unit resistance is

$$W_s = P_0^2/2. \tag{14}$$

The noise is represented by a large number N of sinusoidal components with incommensurable frequencies (or commensurable frequencies with random phase angles) distributed along the frequency range f_1 to f_2 in such a way that the mean noise power in band width Δf is:

$$w(f)\Delta f = \frac{1}{2} \sum_{n=\nu(f-f_1)}^{\nu(f+\Delta f-f_1)} P_n^2 = \nu \Delta f \, P^2(f)/2 \tag{15}$$

Here ν is the number of components per unit band width and $P(f)$ represents the amplitude of a component in the neighborhood of frequency f. Note also that the mean total noise input power, W_n, is given by

$$W_n = \int_0^\infty w(f)\, df = \frac{\nu}{2} \int_0^\infty P^2(f)\, df \tag{16}$$

The linear rectifier is specified by the current-voltage relationship (8), which is equivalent to

$$I = -\frac{\alpha}{2\pi} \int_C e^{iEz} \frac{dz}{z^2} \tag{17}$$

where C is an infinite contour going from $-\infty$ to $+\infty$ with an indentation below the pole at the origin. We may expand I in the multiple Fourier series[1]

[1] Bennett and Rice, "Note on Methods of Computing Modulation Products," *Phil. Mag.*, Sept. 1934. The present application represents an extension to N variables of the theory there given for two.

$$I = \sum_{m_0=0}^{\infty} \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} a_{m_0 m_1 \cdots m_N} \cos m_0 x_0 \cos m_1 x_1 \cdots \cos m_N x_N \tag{18}$$

where

$$x_k = p_k t, \qquad k = 0, 1, 2, \cdots N \tag{19}$$

$$a_{m_0 m_1 \cdots m_N} = \frac{\epsilon_{m_0} \epsilon_{m_1} \cdots \epsilon_{m_N}}{\pi^{N+1}} \int_0^\pi dx_0 \int_0^\pi dx_1 \cdots \int_0^\pi I \cos m_0 x_0 \cos m_1 x_1 \cdots \cos m_N x_N \, dx_N \tag{20}$$

$$\epsilon_j = \begin{Bmatrix} 2, & j \neq 0 \\ 1, & j = 0 \end{Bmatrix} \tag{21}$$

The response of the rectifier is thus seen to consist of all orders of modulation products of signal and noise. In a typical case of interest the band of input frequencies is relatively narrow and centered about a high frequency while the output band includes only low frequencies. In such a case the important components in the output are the beats between signal and noise components and between noise components. The *d-c* component is present in the output only if the pass band of the system actually includes zero frequency; we have already computed its value in Section I, but we will derive it again by the method used here as a check.

The amplitude of the d-c component is in fact:

$$a_{00\cdots 0} = -\frac{\alpha}{2\pi} \int_C \frac{J_0(P_0 z) \prod_{n=1}^{N} J_0(P_k z)}{z^2} \, dz, \tag{22}$$

on substitution of the expression for E in the integral representation of I, substituting the result in (20) and interchanging the order of integration. When N is large, P_n is small, hence the principal contribution to the integral occurs near small values of z, where $J_o(P_n z)$ is nearly equal to unity, since the product of a large number of factors, all less than unity, will be small indeed unless each factor is only slightly less than unity. We therefore replace $J_o(P_n z)$ by a function which coincides with it near $z = 0$ and goes rapidly to zero as we depart from this region. Such an approximation (Laplace's process[2]) is

$$J_0(P_n z) \doteq e^{-P_n^2 z^2/4} \tag{23}$$

[2] Watson, "Theory of Bessel Functions," p. 421.

which is correct for the first two terms in the Taylor series expansion near $z = 0$. Therefore, when P_n approaches zero as N approaches infinity,

$$a_{00\cdots 0} = \bar{I} = -\frac{\alpha}{2\pi}\int_C J_0(P_0 z)e^{-\sum_{n=1}^{N} P_n^2 z^2/4}\frac{dz}{z^2}$$

$$= -\frac{\alpha}{2\pi}\int_C J_0(P_0 z)e^{-W_n z^2/2}\frac{dz}{z^2} \tag{24}$$

The contour integral cannot be replaced by a real integral directly because the integrand goes to infinity at the origin. However, since

$$\frac{J_0(u)}{u^2} = -\frac{J_1(u)}{u} - \frac{d}{du}\frac{J_0(u)}{u} \tag{25}$$

$$\frac{J_0(Pz)}{z^2} = -\frac{J_1(Pz)}{P^2z^2} - \frac{d}{d(Pz)}\frac{J_0(Pz)}{Pz} = -\frac{J_1(Pz)}{P^2z^2} - \frac{1}{P^2}\frac{d}{dz}\frac{J_0(Pz)}{z} \tag{26}$$

we can substitute (26) in the integral and perform an integration by parts to give the result.

$$\bar{I} = \frac{\alpha}{\pi}\int_0^\infty e^{-W_n z^2/2}\left[\frac{P_0 J_1(P_0 z)}{z} + W_n J_0(P_0 z)\right]dz$$

$$= \alpha\sqrt{\frac{W_n}{2\pi}}\left[{}_1F_1\left(\frac{1}{2};1;-\frac{W_s}{W_n}\right) + \frac{W_s}{W_n}\,{}_1F_1\left(\frac{1}{2};2;-\frac{W_s}{W_n}\right)\right] \tag{27}$$

by Hankel's formula.[3] But it may be shown that (see Appendix II)

$${}_1F_1\left(\tfrac{1}{2};1;-u\right) = e^{-u/2}I_0\left(\frac{u}{2}\right) \tag{28}$$

$${}_1F_1\left(\tfrac{1}{2};2;-u\right) = e^{-u/2}\left[I_0(u/2) - I_1(u/2)\right] \tag{29}$$

Hence,

$$\bar{I} = \alpha\sqrt{\frac{W_n}{2\pi}}\,e^{-W_s/2W_n}\left\{I_0(W_s/2W_n) + \frac{W_s}{W_n}\left[I_0(W_s/2W_n) + I_1(W_s/2W_n)\right]\right\} \tag{30}$$

which is identical with the result of Section I, noting that $\sigma = \sqrt{W_n}$. We point out that a resistance-capacity coupled amplifier will not pass this component since there is no transmission at zero frequency.

[3] Watson, "Theory of Bessel Functions," p. 393. As pointed out by Watson, in a footnote, the difficulty with singularities at the origin could be avoided by expressing Hankel's formula in terms of a contour integral instead of an ordinary integral along the real axis. This procedure would lead directly to the hypergeometric function given in (11).

The amplitude of the typical difference product between the signal and the rth noise component is

$$
\begin{aligned}
A_{sn} &= \tfrac{1}{2} a_{100\cdots010\cdots0} \\
&= \frac{\alpha}{\pi} \int dz \, \frac{J_1(P_0 z) J_0(P_1 z) J_0(P_2 z) \cdots J_1(P_n z) \cdots J_0(P_N z)}{z^2}
\end{aligned} \tag{31}
$$

Using the same process as before, we replace $J_1(P_n z)$ by

$$
J_1(P_n z) \doteq \frac{P_n z}{2} e^{-P_n^2 z^2/8} \tag{32}
$$

and obtain in the limit as N becomes indefinitely large

$$
\begin{aligned}
A_{sn} &= \frac{\alpha P_n}{\pi} \int_0^\infty \frac{J_1(P_0 z)}{z} e^{-W_n z^2/2} \, dz \\
&= \frac{\alpha P_n}{2} \sqrt{\frac{W_s}{\pi W_n}} \, {}_1F_1\left(\frac{1}{2}; 2; -\frac{W_s}{W_n}\right) \\
&= \frac{\alpha P_n}{2} \sqrt{\frac{W_s}{\pi W_n}} \, e^{-W_s/2W_n} \left[I_0\left(\frac{W_s}{2W_n}\right) + I_1\left(\frac{W_s}{2W_n}\right) \right]
\end{aligned} \tag{33}
$$

Relations between the ${}_1F_1$ function and Bessel functions are discussed in Appendix II.

The shape of the spectrum of the beats between P_0 and the noise input evidently consists of the superposition of the noise spectra above and below p_o, so that if we write $w_{sn}(f)\,\Delta f$ for the mean energy from this source in that part of the filter output lying in the band of width Δf at f,

$$
w_{sn}(f)\Delta f = \frac{\nu \Delta f}{2} [(A_{sn}^+)^2 + (A_{sn}^-)^2] \tag{34}
$$

$$
A_{sn}^+ = [A_{sn}]_{p_n = p_0 + 2\pi f} \tag{35}
$$

$$
A_{sn}^- = [A_{sn}]_{p_n = p_0 - 2\pi f} \tag{36}
$$

$$
P_n = \sqrt{\frac{2w(f_n)}{\nu}} \tag{37}
$$

$$
\begin{aligned}
w_{sn}(f) = \frac{\alpha^2 W_s}{4\pi W_n} e^{-W_s/W_n} \left[I_0\left(\frac{W_s}{2W_n}\right) + I_1\left(\frac{W_s}{2W_n}\right) \right]^2 \\
\times [w(f_0 + f) + w(f_0 - f)]
\end{aligned} \tag{38}
$$

The total noise from this source in the output of a particular filter of transfer admittance $Y(f)$ is obtained by integrating $w_{sn}(f)Y(f)df$ throughout the band of the filter. In the particular case in which the original band of noise is symmetrical about f_o and occupies the range $f_o - f_a$ to $f_0 + f_a$ and an ideal low pass filter cutting off at $f = f_a$ is used in the rectifier output, the total noise output from beats between signal and noise is

$$W_{sn} = 2\int_0^{f_a} w_{sn}(f)\,df = \frac{\alpha^2 w_s}{4\pi}\,e^{-W_s/W_n}[I_0(W_s/2W_n) + I_1(W_s/2W_n)]^2 \quad (39)$$

Next we shall calculate the spectrum of the energy resulting from beats between individual noise components. We write

$$\begin{aligned}
A_{nn} &= \tfrac{1}{2}\,a_{00\cdots010\cdots010\cdots0}\\
&= \frac{\alpha}{\pi}\int_C dz\,\frac{J_0(P_0 z)J_0(P_1 z)\cdots J_1(P_r z)\cdots J_1(P_s z)\cdots J_0(P_N z)}{z^2}\\
&= \frac{\alpha P_r P_s}{2\pi}\int_0^\infty J_0(P_0 z)e^{-W_n z^2/2}\,dz\\
&= \frac{\alpha P_r P_s}{2\sqrt{2\pi W_n}}\,{}_1F_1\left\{\frac{1}{2};\,1;\,-\frac{W_s}{W_n}\right\}\\
&= \frac{\alpha P_r P_s}{2\sqrt{2\pi W_n}}\,e^{-W_s/2W_n}\,I_0(W_s/2W_n)
\end{aligned} \quad (40)$$

To find the resulting spectrum $w_{nn}(f)df$ produced at f by the resultant of all such components, we note that we may sum over all components by beating each component of the primary band with the frequency f above it and adding the resultant power values. The result is

$$w_{nn}(f) = \frac{\alpha^2}{4\pi W_n}\,e^{-W_s/W_n}\,I_0^2(W_s/2W_n)\int_0^\infty w(\lambda)w(\lambda + f)\,d\lambda \quad (41)$$

In the particular case of a flat band of energy extending from f_1 to f_2,

$$\int_0^\infty w(\lambda)w(\lambda + f)\,d\lambda = \int_{f_1}^{f_2 - f}\frac{W_n^2}{(f_2 - f_1)^2}\,d\lambda = \frac{f_2 - f_1 - f}{(f_2 - f_1)^2}\,W_n^2, \qquad 0 < f < f_2 - f_1 \quad (42)$$

$$w_{nn}(f) = \frac{\alpha^2(f_2 - f_1 - f)W_n}{4\pi\,(f_2 - f_1)^2}\,e^{-W_s/W_n}\,I_0^2(W_s/2W_n), \qquad 0 < f < f_2 - f_1 \quad (43)$$

The total mean power of this type lying in the band 0 to f_b is

$$W_{nn}(f_b) = \int_0^{f_b} w_{nn}(f)\,df = \frac{\alpha^2 W_n(f_2 - f_1 - f_b/2)f_b}{4\pi(f_2 - f_1)^2}\,e^{-W_s/W_n}\,I_0^2(W_s/2W_n) \quad (44)$$

provided $f_b < f_2 - f_1$. The spectrum is confined to the region $0 < f < f_2 - f_1$. If f_b is equal to $f_2 - f_1$ so that the output filter passes all the noise of this type, we have

$$W_{nn}(f_2 - f_1) = W_{nn} = \frac{\alpha^2 W_n}{8\pi} e^{-W_s/W_n} I_0^2(W_s/2W_n) \tag{45}$$

This result seems to hold approximately for a considerable range of input spectra. For example, if we assume that the original noise is shaped like an error function about f_o, i.e.,

$$w_n(f) = W_n \sqrt{a/\pi}\, e^{-a(f-f_0)^2} \tag{46}$$

with f taken from $-\infty$ to $+\infty$ with small error for large f_o,

$$\int_{-\infty}^{\infty} w(\lambda)w(\lambda + f)\, d\lambda = W_n^2 \sqrt{a/2\pi}\, e^{-af^2/2} \tag{47}$$

$$\int_0^{\infty} df \int_{-\infty}^{\infty} w(\lambda)w(\lambda + f)\, d\lambda = W_n^2/2 \tag{48}$$

which is in agreement with (45).

The output of a half-wave linear rectifier contains fundamental components and all even order modulation products. In general, the amplitudes of the higher order products are small compared with the lower order. In a particular problem some consideration of where the principal products fall in the frequency band is required. The products just considered give a fair approximation for the problem of detection of a radio frequency band of signal and noise followed by audio amplification. Contain other products should also be added to obtain higher accuracy. We have calculated the products of order zero and two; the next ones of importance are the fourth order, since the third order products vanish in a perfectly linear rectifier. The fourth order products in this case which fall in the audio band are of frequency $2p_o - p_r - p_s$, $p_o + p_q - p_r - p_s$, and $p_n + p_q - p_r - p_s$, where the subscripts n, q, r, s refer to the original noise component frequencies. The latter is, however, less important than the sixth order product $3p_o - p_q - p_r - p_s$, which involves only three noise components, Expressions for the contributions from these products are given in Appendix III.

Figure 4 shows computed curves for the noise produced in an audio band by the various components. Curve A is $W_{sn} + W_{nn}$ and includes what are usually regarded as the principal contributors, the difference frequencies between signal and noise, and between individual noise components. Curve B is obtained by adding to Curve A, the contribution from the fourth order products $2p_o - p_r - p_s$ and $p_o + p_q - p_r - p_s$ and the sixth order products $3p_o - p_q - p_q - p_s$. Thus all products which include three or less noise fundamental components are included. The curves are plotted in terms of fraction of noise power received compared to the limiting noise when the mean signal input power is made indefinitely large compared to the mean input noise power. Some experimental points given by Williams[4] are shown for comparison. Williams gives the intercept at zero signal power as 35%; the theoretical value deduced here is $\pi/8$ or 39.27%. It will be noted that the inclusion of the higher order products improves the agreement between experimental and theoretical curves, even though the value of the intercept is unaffected by them. It shold also be stressed that our analysis applies

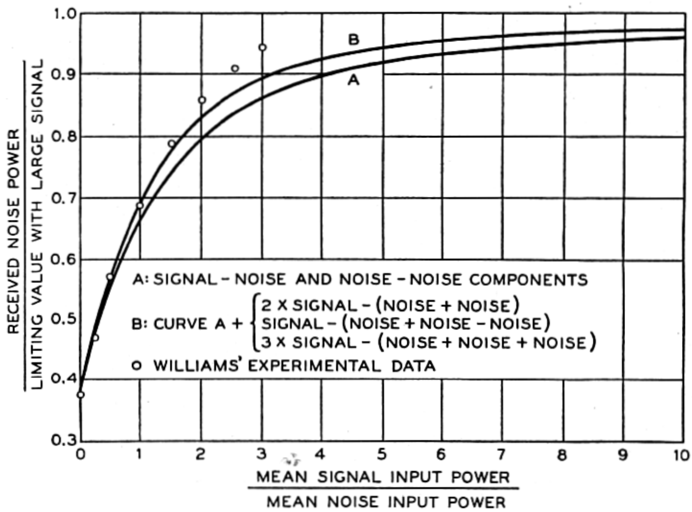

Fig. 4—Calculated noise power in audio band of output of linear rectifier when noise and signal are applied in a relatively narrow high-frequency band. The direct-current component is excluded.

strictly to purely resistive networks. The conventional radio detector circuit (which was used by Williams), in which a condenser is shunted across a resistance in series with a diode, departs from the conditions here assumed because of the reactive element, the condenser. The customary approximation made in treating this circuit is that the condenser has infinite impedance in the audio frequency range and zero impedance at the radio frequencies. This leads to a bias on the detector which depends on the signal. The methods given here may be applied, but the resulting formulas are much more difficult from the standpoint of numerical computation.

A recent paper by Ragazzini[5] gives an approximate solution based on

[4] F. C. Williams, "The Response of Rectifiers to Fluctuation Voltages," *Journal I. E. E.*, 1937, Vol. 80, pp. 218–226.

[5] John Ragazzini, "The Effect of Fluctuation Voltages on the Linear Detector," *Proc. I. R. E.*, June 1942, Vol. 30, p. 277–288.

expanding the envelope of the input wave by the binomial theorem and retaining only the first two terms. The validity depends on the noise amplitude being small compared with the sum of signal and noise, and hence the result should agree with our solution in the neighborhood of $W_n/W_s = 0$, which it does. When W_s/W_n is small, the error is appreciable. Ragazzini's result (Equation 15 of the paper) expressed in our notation is

$$W_{sn} + W_{nn} \doteq \frac{\alpha^2}{\pi^2} \frac{W_n(1 + \frac{1}{2} W_n/W_s)}{1 + W_n/W_s} \tag{49}$$

It will be seen by comparing the limiting values for $W_s/W_n = 0$ with that of $W_s/W_n = \infty$ from (49) that the intercept of the curve of Fig. 4 would be 50% instead of our value of 39.27%.

The results given in the present paper have been compiled from unpublished memoranda and notes by the author extending back as far as 1935. Discussions with colleagues have been of great aid, and in particular acknowledgment is made to Messrs. S. O. Rice and R. Clark Jones for many helpful suggestions.

APPENDIX I

EVALUATION OF INTEGRAL FOR I

Interchanging the order of integration in (9), we have

$$\bar{I} = \frac{\alpha}{\pi\sqrt{2\pi W_n}} \int_0^\pi d\theta \int_0^\infty e^{-(z - P_0 \cos\theta)^2/2W_n} z\, dz \tag{50}$$

By substituting $z = P_0 \cos\theta + u\sqrt{2W_n}$, we may evaluate the second integral in terms of the error function, obtaining

$$\bar{I} = \frac{\alpha}{\pi^{3/2}} \int_0^\pi d\theta \int_{-P\cos\theta/\sqrt{2W_n}}^\infty e^{-u^2}(u\sqrt{2W_n} + P_0 \cos\theta)\, du$$

$$= \frac{\alpha}{\pi} \frac{\sqrt{W_n}}{2\pi} \int_0^\pi e^{-P_0{}^2 \cos^2\theta/2W_n}\, d\theta$$

$$+ \frac{\alpha P_0}{2\pi} \int_0^\pi \operatorname{erf}(P_0 \cos\theta/\sqrt{2W_n}) \cos\theta\, d\theta$$

$$= \frac{\alpha}{\pi} \frac{\sqrt{W_n}}{2\pi} e^{-P_0^2/4W_n} \int_0^\pi e^{-\cos 2\theta/4W_n}\, d\theta$$

$$+ \frac{\alpha P_0}{2\pi} \int_0^\pi \frac{d}{d\theta}\left[\operatorname{erf}\left(\frac{P_0 \cos\theta}{\sqrt{2W_n}}\right) \sin\theta\right] d\theta - \frac{\alpha P_0}{2\pi} \int_0^\pi \sin\theta$$

$$\frac{d}{d\theta}\left(\operatorname{erf} \frac{P_0 \cos\theta}{\sqrt{2W_n}}\right) d\theta$$

$$= \frac{\alpha}{2\pi} \frac{\sqrt{W_n}}{2\pi} e^{-W_s/2W_n} \int_0^{2\pi} e^{-W_s \cos\Phi/2W_n} d\Phi$$

$$+ \frac{\alpha W_s}{2\pi\sqrt{2\pi W_n}} \int_0^{2\pi} e^{-W_s \cos\Phi/2W_n}(1 - \cos\Phi)\, d\Phi$$

$$= \alpha \sqrt{\frac{W_n}{2\pi}}\, e^{-W_s/2W_n} \Big(I_0(W_s/2W_n)$$

$$+ \frac{W_s}{W_n} [I_0(W_s/2W_n) + I_1(W_s/2W_n)]\Big) \tag{10}$$

In the above we have made use of the relations:

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-z^2}\, dz \tag{51}$$

$$\frac{d}{dz} \operatorname{erf} z = \frac{2}{\sqrt{\pi}} e^{-z^2} \tag{52}$$

$$\int_0^{2\pi} e^{-z\cos\Phi} \cos m\Phi\, d\Phi = (-)^m 2\pi I_m(z) \tag{53}$$

APPENDIX II

RELATIONS BETWEEN HYPERGEOMETRIC AND BESSEL FUNCTIONS

The modulation coefficients appearing in the linear rectification of noise are expressible in compact form in terms of the hypergeometric function:

$$\begin{aligned} {}_1F_1\,(a; c; -z) &= 1 - \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} - \cdots \\ &= \frac{\Gamma(c)}{\Gamma(a)} \sum_{m=0}^{\infty} \frac{\Gamma(a+m)}{\Gamma(c+m)m!} (-z)^m \end{aligned} \tag{54}$$

The ${}_1F_1$ function is a limiting case of the more familiar Gaussian hypergeometric function ${}_2F_1\,(a, b; c; z)$, viz.

$${}_1F_1(a; c, z) = \operatorname*{Limit}_{b=\infty} {}_2F_1(a, b; c; z/b) \tag{55}$$

In certain special cases this function may be expressed in terms of exponential and Bessel functions. For example, by a formula given by

Campbell and Foster, Fourier Integrals for Practical Application, Bell System Monograph B-584, p. 32 (also Watson, Theory of Bessel Functions, p. 191), we may show that

$$ {}_1F_1\left(\nu + \frac{1}{2}; 2\nu + 1; -z\right) = \frac{2^{2\nu}\Gamma(\nu + 1)e^{-z/2}}{(-z)^\nu} I_\nu(-z/2) \tag{56} $$

or setting $\nu = 0$

$$ {}_1F_1(\tfrac{1}{2}; 1; -z) = e^{-z/2} I_0 (z/2) \tag{57} $$

which is one of the functions appearing in our work.

We have also encountered the function ${}_1F_1$ $(1/2; 2; -z)$ which is not directly reducible by the above formula. The reduction may be effected in a number of ways. By making use of the relation obtained from (56) by setting $\nu = 1$,

$$ {}_1F_1(3/2; 3; -z) = \frac{4}{z} e^{-z/2} I_1(z/2) \tag{58} $$

and noting that

$$ \begin{aligned} &{}_1F_1(1/2; 2; -z) - {}_1F_1(1/2; 1; -z) \\ &= \frac{1}{\Gamma(1/2)} \sum_{m=0}^{\infty} \frac{\Gamma(m + 1/2)}{m!(m + 1)!} (-z)^m - \frac{1}{\Gamma(1/2)} \sum_{m=0}^{\infty} \frac{\Gamma(m + 1/2)}{(m!)^2} (-z)^m \\ &= \frac{-1}{\Gamma(1/2)} \sum_{m=0}^{\infty} \frac{\Gamma(m + 1/2)m}{m!(m + 1)!} (-z)^m \\ &= \frac{z}{\Gamma(1/2)} \sum_{m=0}^{\infty} \frac{\Gamma(m + 3/2)}{(m + 2)!m!} (-z)^m \\ &= \frac{z}{4} {}_1F_1(3/2; 3; -z), \end{aligned} \tag{59} $$

we find that[6]

$$ {}_1F_1 (1/2; 2; -z) = e^{-z/2} [I_0(z/2)+I_1(z/2)] \tag{60} $$

It may also be verified by integrating the series directly that

$$ \int_0^z {}_1F_1(1/2; 1; -z)\, dz = z\, {}_1F_1(1/2; 2; -z) \tag{61} $$

Combining this relation with (57) and (60) above, we deduce the indefinite integrals

[6] The relation (60) was brought to the attention of the author by Mr. R. M. Foster.

$$\left.\begin{aligned}
\int e^{x}I_0(x)\,dx &= xe^{x}[I_0(x) - I_1(x)] \\
\int e^{-x}I_0(x)\,dx &= xe^{-x}[I_0(x) + I_1(x)] \\
\int e^{x}I_1(x)\,dx &= e^{x}[(1 - x)I_0(x) + xI_1(x)] \\
\int e^{-x}I_1(x)\,dx &= e^{-x}[(1 + x)I_0(x) + xI_1(x)]
\end{aligned}\right\} \tag{62}$$

These integrals may be derived by differentiating the right hand members, and could, therefore, serve as a basis for an alternate derivation of (60).

In addition it was noted in Eq. (11) that the constant term in the modulation spectrum could be expressed in terms of ${}_1F_1\,(-1/2;\,1;\,-z)$; from the equations given, it follows that we must have the relation:

$${}_1F_1\,(-1/2; 1; -z) = e^{-z/2}\,[(1 + z)\,I_0(z/2) + z\,I_1(z/2)] \tag{63}$$

Another interesting set of formulas which can be obtained as a by-product from (62) by setting $x = iy$ is:

$$\left.\begin{aligned}
\int J_0(y)\cos y\,dy &= y[J_0(y)\cos y + J_1(y)\sin y] \\
\int J_0(y)\sin y\,dy &= y[J_0(y)\sin y - J_1(y)\cos y] \\
\int J_1(y)\cos y\,dy &= yJ_1(y)\cos y - J_0(y)(y\sin y - \cos y) \\
\int J_1(y)\sin y\,dy &= yJ_1(y)\sin y + J_0(y)(y\cos y - \sin y)
\end{aligned}\right\} \tag{64}$$

The hypergeometric notation is particularly convenient in determining series expansions for the coefficients to be used for calculation when the variable z is either very small or very large. For small values of z, the form (54) suffices; for large values of z, we may use the general asymptotic expansion formula[7] for the real part of z positive:

$$\begin{aligned}
{}_1F_1(a;\,c;\,-z) &= \frac{\Gamma(c)}{\Gamma(c-a)z^{a}}\;{}_2F_0(a,\,1+a-c;\,1/z) \\
&= \frac{\Gamma(c-a)z^{a}}{\Gamma(c)}\Bigg[1 + \frac{a(1+a-c)}{1!z} \\
&\qquad + \frac{a(a+1)(1+a-c)(2+a-c)}{2!z^{2}} + \cdots\Bigg]
\end{aligned} \tag{65}$$

Copson, "Functions of a Complex Variable," pp. 264–5.

The series expansions required here could also be obtained from the appropriate series for Bessel functions. It will be noted, however, that the typical modulation coefficient can be expressed in terms of either a single ${}_1F_1$ function or several Bessel functions, so that manipulations must be performed on the series for the latter to give the final result. The Bessel functions on the other hand are more convenient for numerical computations because of the excellent tables available.

Reduction formulas for certain other hypergeometric functions are needed in evaluating the higher order products. They are:

$$ {}_1F_1(3/2; 1; -z) = e^{-z/2}[(1 - z)I_0(z/2) + I_1(z/2)] \tag{66} $$

$$ {}_1F_1(3/2; 2; -z) = e^{-z/2}[I_0(z/2) - I_1(z/2)] \tag{67} $$

$$ {}_1F_1(5/2; 4; -z) = \frac{4}{z} e^{-z/2}\left[\left(\frac{4}{z} + 1\right)I_1(z/2) - I_0(z/2)\right] \tag{68} $$

Derivation of these is facilitated by the use of the easily demonstrated relations:

$$ {}_1F_1(a; 1; -z) = \frac{d}{dz}[z\,{}_1F_1(a; 2; -z)] \tag{69} $$

$$ 2z\,{}_1F_1(a; 2; -z) = \frac{d}{dz}[z^2\,{}_1F_1(a; 3; -z)] \tag{70} $$

$$ {}_1F_1(3/2; 3; -z) - {}_1F_1(3/2; 2; -z) = \frac{z}{4}\,{}_1F_1(5/2; 4; -z) \tag{71} $$

APPENDIX III

HIGHER ORDER PRODUCTS

The methods described in Section II may be applied to calculate the general expression for the general modulation coefficient. The result is for the amplitude of the term $\cos mp_0t \cos p_{n_1}t \cos p_{n_2}t \cdots \cos p_{n_M}t$:

$$ a_{mM} = \frac{(-)^{\frac{m+M}{2}+1} P_{n_1}P_{n_2}\cdots P_{n_M}}{\pi(W_n/2)^{(M-1)/2}m!}\Gamma\left(\frac{m+M-1}{2}\right)\frac{(W_s)^{m/2}}{(W_n)} \times {}_1F_1\left(\frac{m+M-1}{2}; m+1; \frac{-W_s}{W_n}\right) \tag{72} $$

The coefficient of the term $\cos(mp_0 \pm p_{n_1} \pm p_{n_2} \pm \ldots p_{n_M})\,t$ is a_{mM} divided by $2^{M-1}\,\epsilon_m$. The number of terms of a particular type falling in a particular frequency interval can be calculated by a method previously described by the author.[8] Under the assumed conditions that the original noise spectrum is either flat throughout a limited range, or falls off like an error function, and that the audio amplifier passes all the difference components in question, we find the following results:

$$2p_0 - p_r - p_s :$$

$$W_{2s,nn} = \frac{\alpha^2 W_n}{8\pi} e^{-W_s/W_n} I_1^2(W_s/2W_n) \tag{73}$$

$$p_0 + p_q - p_r - p_s :$$

$$W_{sn,nn} = \frac{\alpha^2 W_s}{32\pi} e^{-W_s/W_n} [I_0(W_s/2W_n) - I_1(W_s/2W_n)]^2 \tag{74}$$

$$3p_0 - p_q - p_r - p_s :$$

$$W_{3s,nnn} = \frac{\alpha^2 W_s}{32\pi} e^{-W_s/W_n} [(1 + 4W_n/W_s)I_1(W_s/2W_n) - I_0(W_s/2W_n)]^2 \tag{75}$$

This includes all beats containing not more than three noise fundamentals. The reductions of hypergeometric functions to exponential and Bessel functions given in Appendix II have been used in deriving the above results.

[8] Bennett, "Cross-Modulation in Multichannel Amplifiers," *Bell Sys. Tech. Jour.*, Oct. 1940, Vol. XIX, pp. 587–610.