The Biased Ideal Rectifier

By W. R. BENNETT

Methods of solution and specific results are given for the spectrum of the response of devices which have sharply defined transitions between conducting and non-conducting regions in their characteristics. The input wave consists of one or more sinusoidal components and the operating point is adjusted by bias, which may either be independently applied or produced by the rectified output itself.

Introduction

THE concept of an ideal rectifier gives a useful approximation for the analysis of many kinds of communication circuits. An ideal rectifier conducts in only one direction, and by use of a suitable bias may have the critical value of input separating non-conduction from conduction shifted to any arbitrary value, as illustrated in Fig. 1. A curve similar to Fig. 1 might represent for example the current versus voltage relation of a biased diode. By superposing appropriate rectifying and linear characteristics with different conducting directions and values of bias, we may approximate the characteristic of an ideal limiter, Fig. 2, which gives constant response when the input voltage falls outside a given range. Such a curve might approximate the relationship between flux and magnetizing force in certain ferromagnetic materials, or the output current versus signal voltage in a negative-feedback amplifier. The abrupt transitions from non-conducting to conducting regions shown are not realizable in physical circuits, but the actual characteristics obtained in many devices are much sharper than can be represented adequately by a small number of terms in a power series or in fact by any very simple analytic function expressible in a reasonably small number of terms valid for both the non-conducting and conducting regions.

In the typical communication problem the input is a signal which may be expressed in terms of one or more sinusoidal components. The output of the rectifier consists of modified segments of the original resultant of the individual components separated by regions in which the wave is zero or constant. We are not so much interested in the actual wave form of these chopped-up portions, which would be very easy to compute, as in the frequency spectrum. The reason for this is that the rectifier or limiter is usually followed by a frequency-selective circuit, which delivers a smoothly varying function of time. Knowing the spectrum of the chopped input to the selective network and the steady-state response as a function of

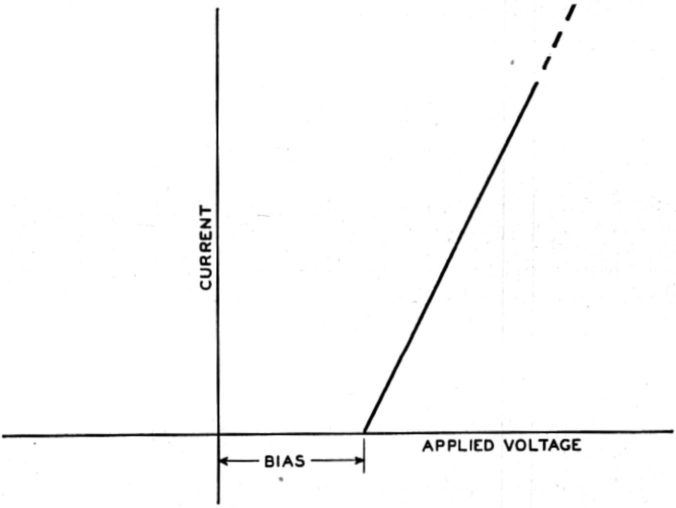

Fig. 1.—Ideal biased linear rectifier characteristic.

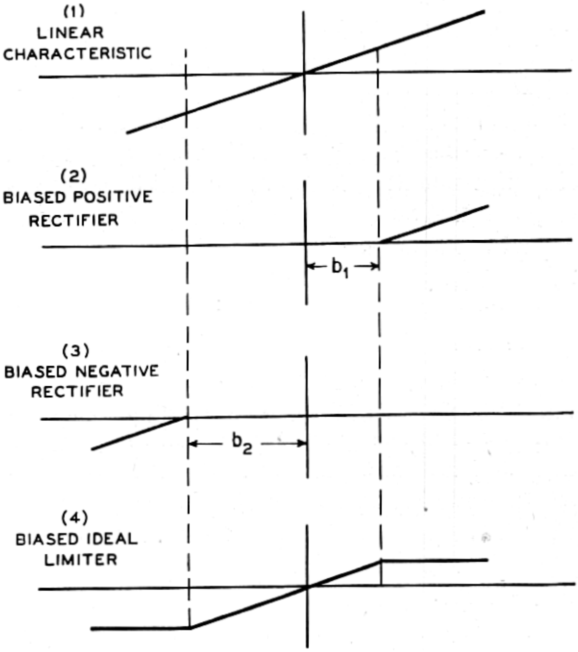

Fig. 2.—Synthesis of limiter characteristic.

frequency of the network, we can calculate the output wave, which is the one having most practical importance. The frequency selectivity may in many cases be an inherent part of the rectifying or limiting action so that discrete separation of the non-linear and linear features may not actually be possible, but even then independent treatment of the two processes often yields valuable information.

The formulation of the analytical problem is very simple. The standard theory of Fourier series may be used to obtain expressions for the amplitudes of the harmonics in the rectifier output in the case of a single applied frequency, or for the amplitudes of combination tones in the output when two or more frequencies are applied. These expressions are definite integrals involving nothing more complicated than trigonometric functions and the functions defining the conducting law of the rectifier. If we were content to make calculations from these integrals directly by numerical or mechanical methods, the complete solutions could readily be written down for a variety of cases covering most communication needs, and straightforward though often laborious computations could then be based on these to accumulate eventually a sufficient volume of data to make further calculations unnecessary.

Such a procedure however falls short of being satisfactory to those who would like to know more about the functions defined by these integrals without making extensive numerical calculations. A question of considerable interest is that of determining under what conditions the integrals may be evaluated in terms of tabulated functions or in terms of any other functions about which something is already known. Information of this sort would at least save numerical computing and could be a valuable aid in studying the more general aspects of the communication system of which the rectifier may be only one part. It is the purpose of this paper to present some of these relationships that have been worked out over a considerable period of time. These results have been found useful in a variety of problems, such as distortion and cross-modulation in overloaded amplifiers, the performance of modulators and detectors, and effects of saturation in magnetic materials. It is hoped that their publication will not only make them available to more people, but also stimulate further investigations of the functions encountered in biased rectifier problems.

The general forms of the integrals defining the amplitudes of harmonics and side frequencies when one or two frequencies are applied to a biased rectifier are written down in Section I. These results are based on the standard theory of Fourier series in one or more variables. Some general relationships between positive and negative bias, and between limiters and biased rectifiers are also set down for further reference. Some discussion is given of the modifications necessary when reactive elements are used in the circuit.

Section II summarizes specific results on the single-frequency biased rectifier case. The general expression for the amplitude of the typical harmonic is evaluated in terms of a hypergeometric function for the power law case with arbitrary exponent.

Section III takes up the evaluation of the two-frequency modulation products. It is found that the integer-power-law case can be expressed in finite form in terms of complete elliptic integrals of the first, second, and third kind for almost all products. Of these the first two are available in tables, directly, and the third can be expressed in terms of incomplete integrals of the first and second kinds, of which tables also exist. No direct tabulation of the complete elliptic integrals of the third kind encountered here is known to the author. They are of the hyperbolic type in contrast to the circular ones more usual in dynamical problems. Imaginary values of the angle β would be required in the recently published table by Heuman[1].

A few of the product amplitudes depend on an integral which has not been reduced to elliptic form, and which is a transcendental function of two variables about which little is known. Graphs calculated by numerical integration are included.

The expressions in terms of elliptic integrals, while finite for any product, show a rather disturbing complexity when compared with the original integrals from which they are derived. It appears that elliptic functions are not the most natural ones in which the solution to our problem can be expressed. If we did not have the elliptic tables available, we would prefer to define new functions from our integrals directly, and the study of such functions might be an interesting and fruitful mathematical exercise.

Solutions for more than two frequencies are theoretically possible by the same methods, although an increase of complexity occurs as the first few components are added. When the number of components becomes very large, however, limiting conditions may be evaluated which reduce the problem to a manageable simplicity again. The case of an infinite number of components uniformly spaced along an appropriate frequency range has been used successfully as a representation of a noise wave, and the detected output from signal and noise inputs thus evaluated[2]. The noise problem will not be treated in the present paper.

I. The General Problem

Let the biased rectifier characteristic, Fig. 1, be expressed by

$$I = \begin{pmatrix} 0, & E < b \\ f(E - b), & b < E \end{pmatrix} \tag{1.1}$$

[1] Carl Heuman, Tables of Complete Elliptic Integrals, *Jour. Math. and Physics*, Vol. XX, No. 2, pp. 127–206, April, 1941.

[2] W. R. Bennett, Response of a Linear Rectifier to Signal and Noise, *Jour. Acous. Soc. Amer.*, Vol. 15, pp. 164–172, Jan. 1944.

Then if a single frequency wave defined by

$$E = P \cos pt, \quad - P < b < P, \tag{1.2}$$

is applied as input, the output contains only the tips of the wave, as shown in Fig. 3. It is convenient to place the restrictions on P and b given in Eq. (1.2). The sign of P is taken as positive since a change of phase may be introduced merely by shifting the origin of time and is of trivial interest. If the bias b were less than $-P$, the complete wave would fall in the conducting region and there would be no rectification. If b were greater than

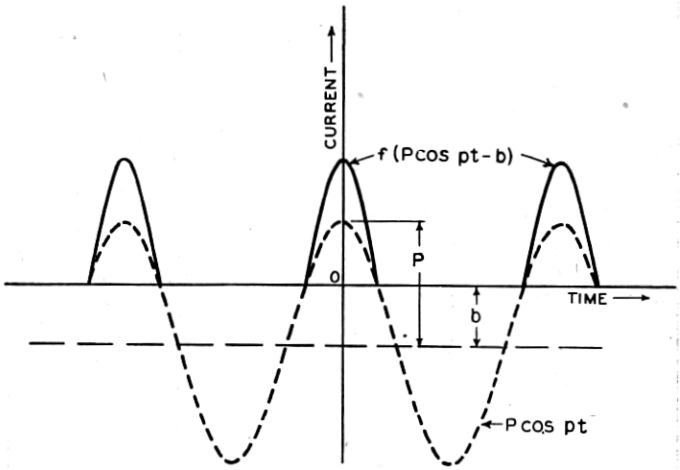

Fig. 3.—Response of biased rectifier to single-frequency wave.

P, the output would be completely suppressed. Applying the theory of Fourier series to (1.1) and (1.2), we have the results

$$I = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\, pt \tag{1.3}$$

$$a_n = \frac{2}{\pi} \int_0^{\text{arc cos } b/P} f(P \cos x - b) \cos nx \, dx \tag{1.4}$$

When two frequencies are applied, the output may be represented by a double Fourier series. The typical coefficient may be found by the method explained in an earlier paper by the author[3]. The problem is to obtain the double Fourier series expansion in x and y of the function $g(x,y)$ defined by:

$$g(x, y) = \begin{pmatrix} 0, & P \cos x + Q \cos y < b \\ f(P \cos x + Q \cos y - b), & b < P \cos x + Q \cos y \end{pmatrix} \tag{1.5}$$

[3] W. R. Bennett, New Results in the Calculation of Modulation Products, *B.S.T.J.*, Vol. XII, pp. 228–243, April, 1933.

We substitute the special values $x = pt, y = qt$ after obtaining the expansion. Let

$$k_1 = Q/P, \; k_0 = -b/P \tag{1.6}$$

The mcst general conditions of interest are comprised in the ranges:

$$0 < k_1 < 1, \; -2 < k_0 < 2 \tag{1.7}$$

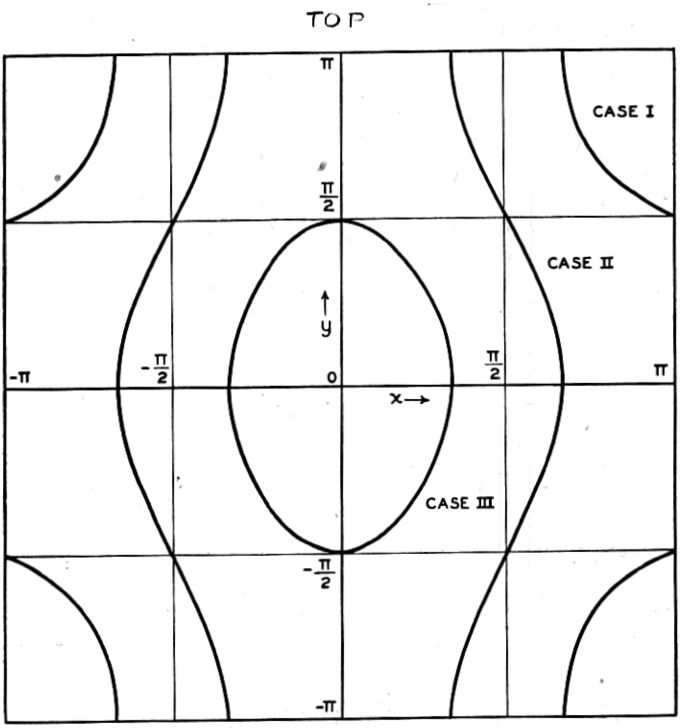

Fig. 4.—Regions in xy-plane bounded by $k_0 + \cos x + k_1 \cos y = 0$.

The regions in the xy-plane in which $g(x,y)$ does not vanish are bounded by the various branches of the curve:

$$k_0 + \cos x + k_1 \cos y = 0 \tag{1.8}$$

We need to consider only one period rectangle bounded by $x = \pm\pi, y = \pm\pi$, since the function repeats itself at intervals of 2π in both x and y. The shape of the curve (1.8) within this rectangle may have three forms, which are depicted in Fig. 4. In Case I, $k_0 + k_1 > k$, $k_0 - k_1 < 1$, the curve divides into four branches which are open at both ends of the x- and y-axes. In Case (2), $k_0 + k_1 < 1$, $k_0 - k_1 > -1$, the curve has two branches open at the ends of the y-axis. In Case (3), $-1 < k_0 + k_1 < 1$, $k_0 - k_1 < -1$, a single closed curve is obtained. The limits of integration must be chosen to fit the proper case. The Fourier series expansion of $g(x,y)$ may be written:

$$g(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos mx \cos ny \tag{1.9}$$

where a_{mn} is found from integrals of the form:

$$A = \frac{\epsilon_m \epsilon_n}{\pi^2} \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} f(P \cos x + Q \cos y - b) \cos mx \cos ny \, dx \tag{1.10}$$

Here, as usual, ϵ_m is Neumann's discontinuous factor equal to two when m is not zero and unity when m is zero. The values of the limits for the different cases are:

Case I, $a_{mn} = A_1 + A_2$

$$A_1 = A \text{ with limits} \begin{pmatrix} (x_1 = 0, & x_2 = \text{arc cos } (-k_0 - k_1 \cos y) \\ y_1 = \text{arc cos } \dfrac{1 - k_0}{k_1}, & y_2 = \pi \end{pmatrix} \tag{1.11}$$

$$A_2 = A \text{ with limits} \begin{pmatrix} x_1 = 0, & x_2 = \pi \\ y_1 = 0, & y_2 = \text{arc cos } \dfrac{1 - k_0}{k_1} \end{pmatrix} \tag{1.12}$$

Case II, $a_{mn} = A$

$$\text{Limits} \begin{pmatrix} x_1 = 0, & x_2 = \text{arc cos } (-k_0 - k_1 \cos y \\ y_1 = 0, & y_2 = \pi \end{pmatrix} \tag{1.13}$$

Case III, $a_{mn} = A$

$$\text{Limits} \begin{pmatrix} (x_1 = 0, & x_2 = \text{arc cos } (-k_0 - k_1 \cos y) \\ (y_1 = 0, & y_2 = \text{arc cos } \left(- \dfrac{1 + k_0}{k_1} \right) \end{pmatrix} \tag{1.14}$$

For a considerable variety of rectifier functions f, the inner integration may be performed at once leaving the final calculation in terms of a single definite integral.

A somewhat different point of view is furnished by evaluating the integral (1.4) for the biased single-frequency harmonic amplitude, and then replacing the bias by a constant plus a sine wave having the second frequency. When each harmonic of the first frequency is in turn expanded in a Fourier series in the second frequency, the two-frequency modulation coefficients are obtained. Some early calculations carried out graphically in this way are the source of the curves plotted in Figs. 18 to 21 inclusive, for which I am indebted to Dr. E. Peterson.

If reactive elements are used in the rectifier circuit, the voltage across the rectifying element may depart from the input wave shape applied to the complete network. The solution then loses its explicit nature since the rectifier current is expressed in terms of input voltage components which in turn depend on voltage drops produced in the remainder of the network by the rectifier currents. Practical solutions can be worked out when relatively few components are important.

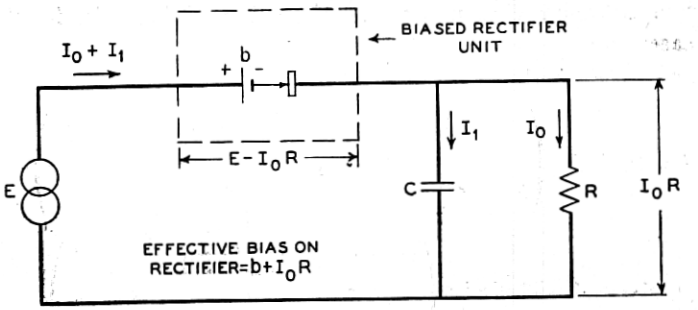

Fig. 5.— Biased rectifier in series with RC network.

As an example consider the familiar case of a parallel combination of resistance R and capacitance C in series with the biased rectifier, Fig. 5. If C has negligible impedance at all frequencies of importance in the rectifier circuit except zero, we may assume that the voltage across R is constant and equal to I_0R, where I_0 is the d-c. component of the rectifier current. The voltage across the rectifier unit is then $E - I_0R$. The effect is a change in the value of bias from b to $b + I_0R$. If the d-c component in the output is calculated for bias $b + I_0R$, we obtain the value of I_0 in terms of $b + I_0R$, an implicit equation defining I_0. If this equation can be solved for I_0, the bias $b + I_0R$ can then be determined and the remaining modulation products calculated.

A more important case is that of the so-called envelope detector, in which the impedance of the condenser is very small at all frequencies contained in the input signal, but is very large at frequencies comparable with the band width of the spectrum of the input signal. These are the usual conditions prevailing in the detection of audio or video signals from modulated r-f or i-f waves. The solution depends on writing the input signal in the form of a slowly varying positive valued envelope function multiplying a rapidly oscillating cosine function. That is, if the input signal can be represented as

$$E = A\,(t) \cos \phi\,(t), \tag{1.15}$$

where $A\,(t)$ is never negative and has a spectrum confined to the frequency range in which $2\pi fC$ is negligibly small compared with $1/R$, while $\cos \phi(t)$ has a spectrum confined to the frequency range in which $1/R$ is negligibly small compared with $2\pi fC$, we divide the components in the detector output into two groups, viz.:

1. A low-frequency group I_{lf} containing all the frequencies comparable with those in the spectrum of $A\,(t)$. The components of this group flow through R.

2. A high-frequency group I_{hf} containing all the frequencies comparable to and greater than those in the spectrum of $\cos \phi\,(t)$. The components of this group flow through C and produce no voltage across R.

The instantaneous voltage drop across R is therefore equal to $I_{lf}R$, and hence the bias on the rectifier is $b + I_{lf}R$. If A and ϕ were constants, we could make use of (1.3) and (1.4) to write:

$$I_{lf} + I_{hf} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\,\theta \tag{1.16}$$

$$a_n = \frac{2}{\pi} \int_0^{\arccos\,[(b+I_{lf}R)/A]} f(A \cos x - b - I_{lf}\,R) \cos nx\; dx \tag{1.17}$$

If A and ϕ are variable, the equation still holds provided $I_{lf}R < A$ at all times. Assuming the latter to be true (keeping in mind the necessity of checking the assumption when I_{lf} is found), we note that terms of the form $a_n \cos n\,\theta$ consist of high frequencies modulated by low frequencies and hence the main portion of their spectra must be in the high-frequency range. Hence we must have as a good approximation when the envelope frequencies are well separated from the intermediate frequencies,

$$I_{lf} = \frac{a_0}{2} = \frac{1}{\pi} \int_0^{\arccos\,[(b+I_{lf}R)/A]} f(A \cos x - b - I_{lf}R)dx \tag{1.18}$$

This equation defines I_{lf} as a function of A, and if it is found that the condition $b + I_{lf}R < A$ is satisfied by the resulting value of I_{lf}, the problem is solved. If the condition is not satisfied, a more complicated situation exists requiring separate consideration of the regions in which $b + I_{lf}R < A$ and $b + I_{lf}R > A$.

To be specific, consider the case of a linear rectifier with forward conductance $\alpha = 1/R$, and write $V = I_{lf}R$. Then

$$\frac{\pi R_0}{R} V = \sqrt{A - (b + V)^2} - (b + V) \arccos \frac{b + V}{A} \tag{1.19}$$

When $b = 0$ (the case of no added bias), this equation may be satisfied by setting

$$V = cA, 0 \leq c \leq 1, \tag{1.20}$$

which leads to

$$\frac{\pi R_0}{R} = \sqrt{\frac{1}{c^2} - 1} - \text{arc cos } c, \tag{1.21}$$

defining c as a function of R_0/R. The value of c approaches unity when the ratio of rectifier resistance to load resistance approaches zero and falls off to zero as R_0/R becomes large. The curve may be found plotted elsewhere[4]. This result justifies the designation of this circuit as an envelope detector since with the proper choice of circuit parameters the output voltage is proportional to the envelope of the input signal.

The equations have been given here in terms of the actual voltage applied to the circuit. The results may also be used when the signal generator contains an internal impedance. For example, a nonreactive source independent of frequency may be combined with the rectifying element to give a new resultant characteristic. If the source impedance is a constant pure resistance r_0 throughout the frequency range of the signal input but is negligibly small at the frequencies of other components of appreciable size flowing in the detector, we assume the voltage drop in r_0 is $r_0 a_1 \cos \phi\ (t)$. We then set $n = 1$ in (1.17) and replace a_1 by $(A_0 - A)/r_0$, where A_0 is the voltage of the source. The value of I_{lf} in terms of A from (1.18) is then substituted, giving an implicit relation between A and A_0.

A further noteworthy fact that may be deduced is the relationship between the envelope and the linearly rectified output. By straightforward Fourier series expansion, the positive lobes of the wave (1.15), may be written as:

$$E_r = \begin{pmatrix} E, & E > 0 \\ 0, & E < 0 \end{pmatrix} = A(t) \left[\frac{1}{\pi} + \frac{1}{2} \cos \phi\ (t) - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-)^m \cos 2m\ \phi(t)}{4m^2 - 1} \right] \tag{1.22}$$

Hence if we represent the low-frequency components of E_r by E_{lf}, we have:

$$E_{lf} = \frac{A(t)}{\pi} \tag{1.23}$$

or

$$A\ (t) = \pi E_{lf} \tag{1.24}$$

[4] See, for example, the top curve of Fig. 9–25, p. 311, H. J. Reich, Theory and Applications of Electron Tubes, McGraw-Hill, 1944.

Equation (1.23) expresses the fact that we may calculate the signal component in the output of a half-wave linear rectifier by taking $1/\pi$ times the envelope. Equation (1.24) shows that we may calculate the response of an envelope detector by taking π times the low-frequency part of the Fourier series expansion of the linearly rectified input. Thus two procedures are in general available for either the envelope detector or linear rectifier solution, and in specific cases a saving of labor is possible by a proper choice between the two methods. The final result is of course the same, although there may be some difficulty in recognizing the equivalence. For example, the solution for linear rectification of a two-frequency wave $P \cos pt + Q \cos qt$ was given by the author in 1933[3], while the solution for the envelope was given by Butterworth in 1929[5]. Comparing the two expressions for the direct-current component, we have:

$\overline{E}_{lf} = \frac{2P}{\pi^2}[2E - (1 - k^2)\,K]$, where K and E are complete elliptic integrals of the first and second kinds with modulus $k = Q/P$

$\overline{A\ (t)} = \frac{2P}{\pi}\,(1 + k)\,E_1$, where E_1 is a complete elliptic intregal of the second kind with modulus $k_1 = 2\sqrt{k}/(1 + k)$. Equation (1.24) implies the existence of the identity

$$(1 + k)\,E_1 = 2E - (1 - k^2)\,K \tag{1.25}$$

The identity can be demonstrated by making use of Landen's transformation in the theory of elliptic integrals.

2. Single-Frequency Signal

The expression for the harmonic amplitudes in the output of the rectifier can be expressed in a particularly compact form when the conducting part of the characteristic can be described by a power law with arbitrary exponent. Thus in (1.4) if $f(z) = \alpha z^\nu$, we set $\lambda = b/P$ and get

$$a_n = \frac{2\alpha P^\nu}{\pi}\int_0^{\operatorname{arc\,cos}\lambda} (\cos x - \lambda)^\nu \cos nx\, dx$$

$$= \frac{2^{\frac{1}{2}}\,\Gamma(\nu + 1)\alpha P^\nu(1 - \lambda)^{\nu+\frac{1}{2}}}{\pi^{\frac{1}{2}}\,\Gamma\left(\nu + \frac{3}{2}\right)}\; F\left(\frac{1}{2} + n, \frac{1}{2} - n; \nu + \frac{3}{2}; \frac{1 - \lambda}{2}\right) \tag{2.1}$$

[5] S. Butterworth, Apparent Demodulation of a Weak Station by a Stronger One *Experimental Wireless*, Vol. 6, pp. 619–621, Nov. 1929.

The equation holds for all real values of ν greater than -1. The symbol F represents the Gaussian hypergeometric function[6]:

$$F(a, b; c; z) = 1 + \frac{a\,b}{c\,1!}\,z + \frac{a(a+1)\,b\,(b+1)}{c(c+1)\,2!}\,z^2 + \cdots \tag{2.2}$$

The derivation of (2.1) requires a rather long succession of substitutions, expansions, and rearrangements, which will be omitted here.

When ν is an integer, the hypergeometric function may be expressed in finite algebraic form, either by performing the integration directly, or by making use of the formulas:

$$\left.\begin{aligned} F(\mu/2, -\mu/2; 1/2; z) &= \cos(\mu \arcsin z), \\ F\left(\frac{1+\mu}{2}, \frac{1-\mu}{2}; \frac{3}{2}; z^2\right) &= \frac{\sin(\mu \arcsin z)}{\mu z}, \end{aligned}\right. \tag{2.3}$$

together with recurrence formulas for the F-function. When ν is an odd multiple of one half, the F-function may be expressed in terms of complete elliptic integrals of the first and second kind with modulus $[(1-\lambda)/2]^{1/2}$ by means of the relations,

$$\left.\begin{aligned} F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) &= \frac{2}{\pi}\,K, \\ F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) &= \frac{2}{\pi}\,E, \end{aligned}\right\} \tag{2.4}$$

and the recurrence formulas for the F-function. For the case of zero bias, we set $\lambda = 0$, and apply the formula

$$F(a, 1-a; c; 1/2) = \frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{c+1}{2}\right)}{\Gamma\left(\frac{c+a}{2}\right)\Gamma\left(\frac{1+c-a}{2}\right)} \tag{2.5}$$

obtaining the result:

$$a_n = \frac{2^{\frac{1}{2}}\,\Gamma(\nu+1)\Gamma\left(\frac{2\nu+3}{4}\right)\Gamma\left(\frac{2\nu+5}{4}\right)\alpha P^\nu}{\pi^{\frac{1}{2}}\,\Gamma(\nu+3/2)\Gamma\left(\frac{2+\nu+n}{2}\right)\Gamma\left(\frac{2+\nu-n}{2}\right)} \tag{2.6}$$

We point out that the above results may be applied not only when the applied signal is of the form $P \cos pt$ with P and p constants, but to signals

[6] For an account of the properties of the hypergeometric function, see Ch. XIV of Whittaker and Watson, Modern Analysis, Cambridge, 1940. A discussion of elliptic integrals is given in Ch. XXII of the same book.

in which P and p are variable, provided that P is always positive. We thus can apply the results to detection of an ordinary amplitude-modulated wave or to the detection of a frequency-modulated wave after it has passed through a slope circuit.

A case of considerable practical interest is that of an amplitude-modulated wave detected by a diode in series with a parallel combination of resistance R and capacitance C. The value of C is assumed to be sufficiently large so that the voltage across R is equal to the $a_0/2$ component of the current through the diode multiplied by the resistance. This is the condition for envelope detection mentioned in Part 1. The diode is assumed to follow Child's law, which gives $\nu = 3/2$. We write

$$I_0 = \frac{V}{R} = a_0/2 = \frac{\Gamma(5/2)(1-\lambda)^2 \alpha P^{3/2}}{(2\pi)^{\frac{1}{2}}\Gamma(3)} F\left(\tfrac{1}{2}, \tfrac{1}{2}; 3; \frac{1-\lambda}{2}\right) \tag{2.7}$$

where $\lambda = V/P$. Note that V is a constant equal to the direct-voltage output if P is constant. If P varies slowly with time compared with the high-frequency term $\cos pt$, V represents the slowly varying component of the output and hence is the recovered signal.
But

$$F(\tfrac{1}{2}, \tfrac{1}{2}; 3; k^2) = \frac{16}{9\pi k^4}\left[2(2k^2 - 1)E + (2 - 3k^2)(1 - k^2)K\right] \tag{2.8}$$

where K and E are complete elliptic integrals of the first and second kind with modulus k. Hence

$$\frac{3\pi}{R_1\sqrt{2P}} = \rho = \frac{(1 + 3\lambda)(1 + \lambda)}{\lambda} K - 8E \tag{2.9}$$

where the modulus of K and E is $\sqrt{(1 - \lambda)/2}$. This equation defines ρ as a function of λ, and hence by inversion gives λ as a function of ρ. The resulting curve of λ vs. ρ is plotted in Fig. 6 and may be designated as the function $\lambda = g(\rho)$. If we substitute $\lambda = V/P$ we then have

$$V = P\, g\, (3\pi/R\alpha\sqrt{2P}) \tag{2.10}$$

This enables us to plot V as a function of P, for various values of $R\alpha$, Fig. 7. Since P may represent the envelope of an amplitude-modulated (or differentiated FM) wave, and V the corresponding recovered signal output voltage, the curves of Fig. 7 give the complete performance of the circuit as an envelope detector. In general the envelope would be of form $P = P_0[1 + c\, s(t)]$, where $s(t)$ is the signal. We may substitute this value of P directly in (2.10) provided the absolute value of $c\, s(t)$ never exceeds unity.

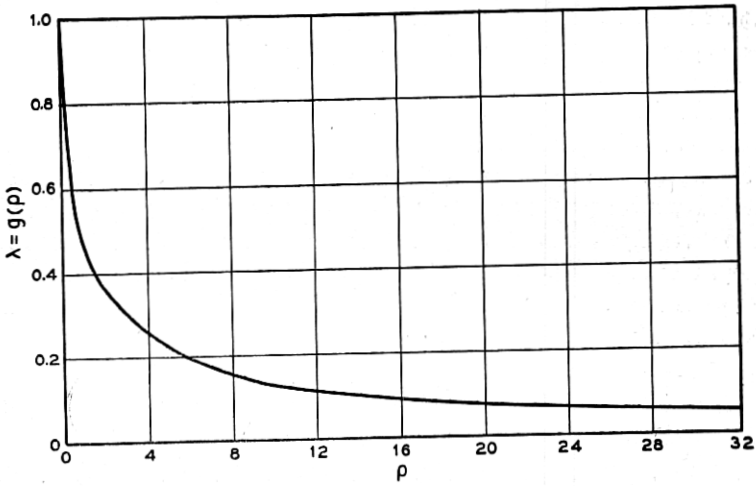

Fig. 6.—The Function $\lambda = g(\rho)$ defined by Eq. (2.9).

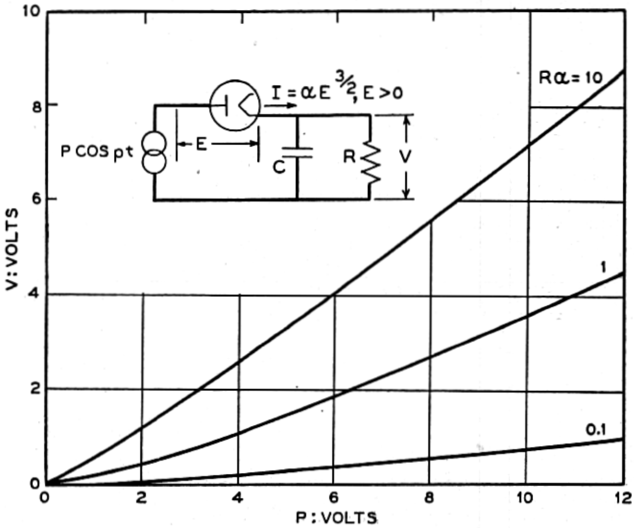

Fig. 7.—Performance of 3/2—power-law rectifier as an envelope detector with low-impedance signal generator.

To express the output in terms of a source voltage P_0 in series with an impedance equal to the real constant value r_0 at the signal frequency and zero at all other frequencies, we write

$$\frac{P_0 - P}{r_0} = a_1 = \frac{3\alpha P^{3/2}(1 - \lambda)^2}{4\sqrt{2}} F\left(\tfrac{3}{2}, -\tfrac{1}{2}; 3; \frac{1 - \lambda}{2}\right) \qquad (2.11)$$

or

$$P_0 = \left(1 + \frac{r_0}{R} H\right) P, \tag{2.12}$$

where

$$\begin{aligned} H &= \frac{3R\alpha(1-\lambda)^2 p^{\frac{1}{2}}}{4\sqrt{2}} F\left(\tfrac{3}{2}, -\tfrac{1}{2}; 3; \frac{1-\lambda}{2}\right) \\ &= \frac{8R\alpha}{5\pi}\sqrt{2P}[2(1-k^2+k^4)E - (2-k^2)(1-k^2)K]. \end{aligned} \tag{2.13}$$

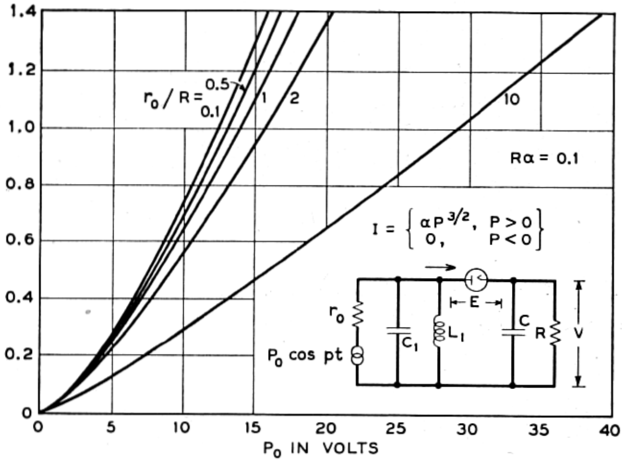

Fig. 8.—Performance of 3/2—power-law rectifier as an envelope detector with impedance of signal generator low except in signal band.

By combining the curves of Fig. 7 giving V in terms of P with the above equations giving the relation between P and P_0, we obtain the curves of Figs. 8, 9, 10, giving V as a function of P_0. The curves approach linearity as $R\alpha$ is made large. On the assumption that the curves are actually linear, we define the conversion loss D of the detector in *db* in terms of the ratio of maximum power available from the source to the power delivered to the load:

$$D = 10 \log_{10} \frac{P_0^2/8r_0}{V^2/R} = 10 \log_{10} \left(\frac{P_0}{V}\right)^2 \frac{R}{8r_0} \tag{2.14}$$

Curves of D vs r_0/R are given in Figs. 11 and 12. The optimum relation between r_0 and R when the forward resistance of the rectifier vanishes has long been known to be $r_0/R = .5$. The curves show a minimum in this region when $R\alpha$ is large. In the limit as $R\alpha$ approaches infinity, we may show that the relation between P_0 and V approaches:

$$P_0 = V\left(1 + \frac{2r_0}{R}\right) \tag{2.15}$$

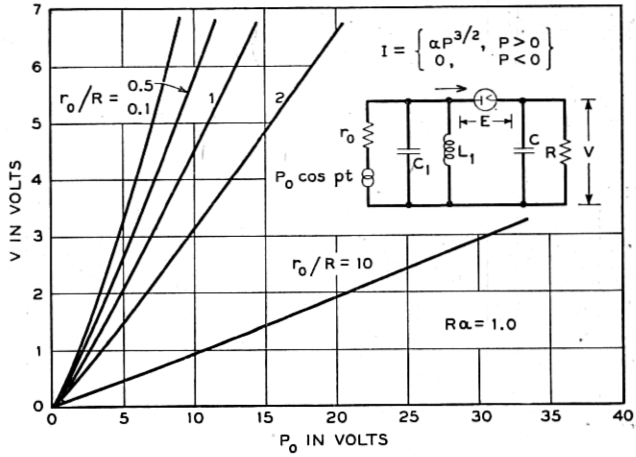

Fig. 9.—Performance of 3/2—power-law rectifier as an envelope detector with impedance of signal generator low except in signal band.

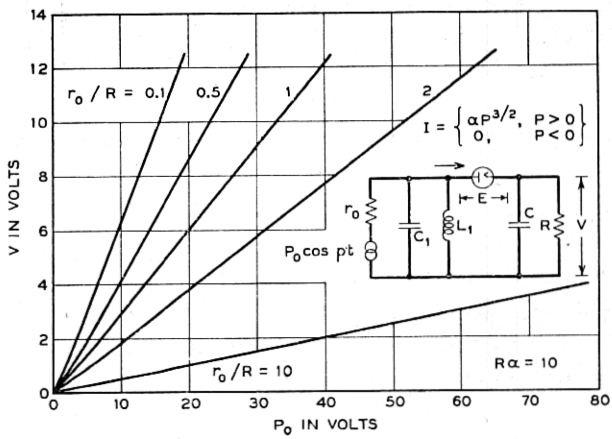

Fig. 10.—Performance of 3/2—power-law rectifier as an envelope detector with impedance of signal generator low except in signal band.

The corresponding limiting formula for D is

$$D = 10 \log_{10} \frac{R}{8r_0}\left(1 + \frac{2r_0}{R}\right)^2 \tag{2.16}$$

The minimum value of D is then found to occur at $r_0 = R/2$ and is zero *db*. We note from the curves that the minimum loss is 1.2 *db* when $R\alpha =$ 10 and 0.4 when $R\alpha = 100$.

This example is intended mainly as illustrative rather than as a complete tabulation of possible detector solutions. The methods employed are sufficiently general to solve a wide variety of problems, and the specific evaluation process included should be sufficiently indicative of the procedures required. Cases in which various other selective networks are associated with the detector have been treated by Wheeler[7].

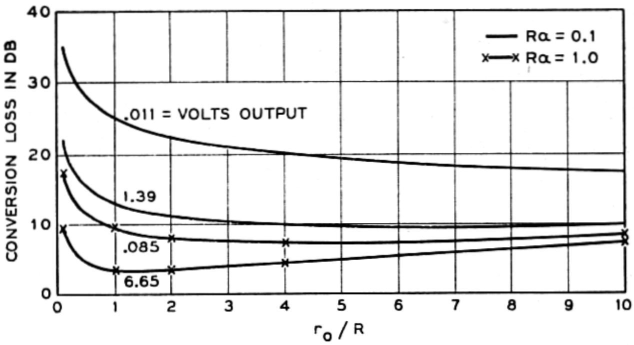

Fig. 11.—Conversion loss of 3/2—power-law rectifier as envelope detector with impedance of signal generator low except in signal band.

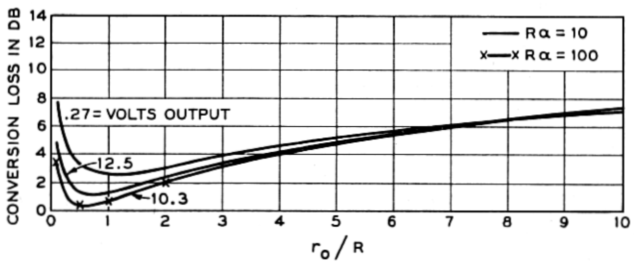

Fig. 12.—Conversion loss of 3/2—power-law rectifier as envelope detector with impedance of signal generator low except in signal band.

3. Two-Frequency Inputs

The general formula for the coefficients in the two-frequency case depends on a double integral as indicated by (1.10). In many cases one integration may be performed immediately, thereby reducing the problem to a single definite integral which may readily be evaluated by numerical or mechanical

[7] H. A. Wheeler, Design Formulas for Diode Detectors, *Proc. I. R. E.*, Vol. 26, pp. 745–780, June 1938.

means. It appears likely in most cases that the expression of these results in terms of a single integral is the most advantageous form for practical purposes, since the integrands are relatively simple, while evaluations in terms of tabulated functions, where possible, often lead to complicated terms. Numerical evaluation of the double integral is also a possible method in cases where neither integration can be performed in terms of functions suitable for calculation.

One integration can always be accomplished for the integer power-law case, since the function $f(P \cos x + Q \cos y - b)$ in (1.12) then becomes a polynomial in $\cos x$ and $\cos y$. Cases of most practical interest are the zero-power, linear, and square-law detectors, in which $f(z)$ is proportional to z^0, z^1, and z^2 respectively. The zero-power-law rectifier is also called a total limiter, since it limits on infinitesimally small amplitudes. We shall tabulate here the definite integrals for a few of the more important low-order

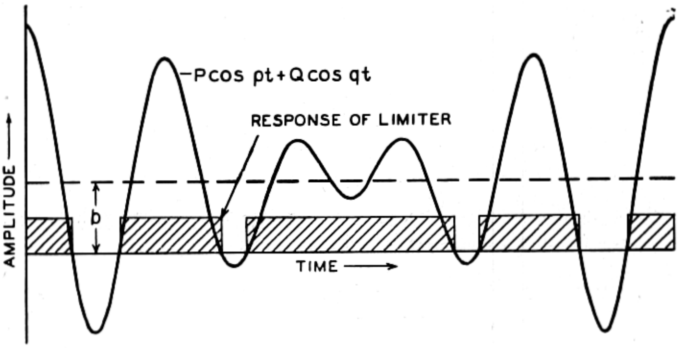

Fig. 13.—Response of biased total limiter to two-frequency wave.

coefficients. To make the listing uniform with that of our earlier work, we express results in terms of the coefficient A_{mn}, which is the amplitude of the component of frequency $mp \pm nq$. The coefficient A_{mn} is half of a_{mn} when neither m nor n is zero. When m or n is zero, we take $A_{mn} = a_{mn}$ and drop the component with the lower value of the $\pm$ sign. When both m and n are zero, we use the designation $A_{00}/2$ for a_{00}, the d-c term. In the tabulations which follow we have set $f(z) = \alpha z^\nu$ with ν taking the values of zero and unity.

We first consider the biased zero-power-law rectifier or biased total limiter. This is the case in which the current switches from zero to a constant value under control of two frequencies and a bias as illustrated by Fig. 13. The results are applicable to saturating devices when the driving forces swing through a large range compared with the width of the linear region. It is also to be noted that the response of a zero-power-law rectifier may be regarded as the Fourier series expansion of the conductance of a linear rectifier under control of two carrier frequencies and a bias. The results may therefore be applied to general modulator problems based on the method described by Peterson and Hussey[8]. We may also combine the Fourier series with proper multiplying functions to analyze switching between any arbitrary forms of characteristics. We give the results for positive values of k_0. The corresponding coefficients for $-k_0$ can be obtained from the relations:

$$\left.\begin{aligned} \frac{A_{00}^-}{2} &= \alpha - \frac{A_{00}^+}{2} \\ A_{mn}^- &= (-)^{m+n+1} A_{mn}^+, \qquad m + n > 0 \end{aligned}\right\} \tag{3.1}$$

Here we have used plus and minus signs as superscripts to designate coefficients with bias $+k_0$ and $-k_0$ respectively. We thus obtain a reduction in the number of different cases to consider, since Case III consists of negative bias values only, and these can now be expressed in terms of positive bias values falling in Cases I and II. It is convenient to define an angle θ by the relations:

$$\theta = \begin{pmatrix} \text{arc cos}\, \dfrac{1 - k_0}{k_1}, & k_0 + k_1 > 1,\ k_0 - k_1 < 1 \quad \text{(Case I)} \\ 0 & ; k_0 + k_1 < 1,\ k_0 - k_1 > -1 \quad \text{(Case II)} \end{pmatrix} \tag{3.2}$$

ZERO-POWER RECTIFIER OR TOTAL-LIMITER COEFFICIENTS

Setting $f(z) = \alpha$ in (1.10),

$$\left.\begin{aligned} \frac{A_{00}}{2\alpha} &= 1 - \frac{1}{\pi^2}\int_\theta^\pi \text{arc cos}\,(k_0 + k_1 \cos y)\, dy \\ \frac{A_{10}}{\alpha} &= \frac{2}{\pi^2}\int_\theta^\pi \sqrt{1 - (k_0 + k_1 \cos y)^2}\, dy \\ \frac{A_{01}}{\alpha} &= \frac{2k_1}{\pi^2}\int_\theta^\pi \frac{\sin^2 y\, dy}{\sqrt{1 - (k_0 + k_1 \cos y)^2}} \\ \frac{A_{11}}{\alpha} &= \frac{2}{\pi^2}\int_\theta^\pi \cos \sqrt{1 - (k_0 + k_1 \cos y)^2}\, dy \\ \frac{A_{20}}{\alpha} &= -\frac{2}{\pi^2}\int_\theta^\pi (k_0 + k_1 \cos y)\sqrt{1 - (k_0 + k_1 \cos y)^2}\, dy \\ \frac{A_{02}}{\alpha} &= \frac{2k_1}{\pi^2}\int_\theta^\pi \frac{\sin^2 y \cos y\, dy}{\sqrt{1 - (k_0 + k_1 \cos y)^2}} \\ \frac{A_{21}}{\alpha} &= -\frac{4}{\pi^2}\int_\theta^\pi (k_0 + k_1 \cos y) \cos y \sqrt{1 - (k_0 + k_1 \cos y)^2}\, dy \end{aligned}\right\} \tag{3.3}$$

[8] E. Peterson and L. W. Hussey, Equivalent Modulator Circuits, *B. S. T. J.*, Vol. 18, pp. 32–48, Jan. 1939.

Similarly for a linear rectifier:

$$\left.\begin{aligned} \frac{A_{00}^-}{2} &= \frac{A_{00}^+}{2} + \alpha b \\ A_{10}^- &= \alpha P - A_{10}^+ \\ A_{01}^- &= \alpha Q - A_{01}^+ \\ A_{mn}^- &= (-)^{m+n} A_{mn}^+, \qquad m + n > 1 \end{aligned}\right\} \tag{3.4}$$

We have shown in Fig. 2 how an ideal limiting characteristic, which transmits linearly between the upper and lower limits, may be synthesized from two biased linear rectification characteristics. Equation (3.4) shows how to calculate the corresponding modulation coefficients, when the coefficients for bias of one sign are known. The limiter characteristic is equal to $\alpha z - f_1(z) - f_2(z)$, where

$$f_1(z) = \alpha \begin{pmatrix} z - b_1, & z > b_1 \\ 0, & z < b_1 \end{pmatrix}, \quad f_2(z) = \alpha \begin{pmatrix} 0, & z > -b_2 \\ z + b_2, & z < -b_2 \end{pmatrix} \tag{3.5}$$

The expression for $f_2(z)$ may also be written:

$$f_2(z) = \alpha(z + b_2) - \alpha \begin{pmatrix} z - (-b_2), & z > -b_2 \\ 0, & z < -b_2 \end{pmatrix} \tag{3.6}$$

Hence the modulation coefficient A_{mn} for the limiter may be expressed in terms of $A_{mn}(b_1)$ and $A_{mn}(-b_2)$ as follows:

$$A_{mn} = -A_{mn}(b_1) + (-)^{m+n} A_{mn}(b_2), \; m + n \neq 1 \tag{3.7}$$

If the limiter is symmetrical ($b_1 = b_2$), the even-order products vanish and the odd orders are doubled. The terms αP, αQ are to be added to the dexter of (3.7) for A_{10}, A_{01} respectively. The odd linear-rectifier coefficients, when multiplied by two, thus give the modulation products in the output of a symmetrical limiter with maximum amplitude k_0, as may be seen by substituting $b_1 = b_2 = -k_0$ in (3.7). For the fundamental components αP and αQ respectively must be subtracted from twice the A_{10} and A_{01} coefficients for k_0.

Linear Rectifier Coefficients

D.C.

$$\frac{A_{00}}{2}/\alpha P = k_0 + \frac{1}{\pi^2}\int_\theta^\pi [\sqrt{1 - (k_0 + k_1 \cos y)^2} - (k_0 + k_1 \cos y) \operatorname{arc}(\cos k_0 + k_1 \cos y)]\, dy \tag{3.8}$$

FUNDAMENTALS

$$A_{10}/\alpha P = 1 + \frac{1}{\pi^2}\int_\theta^\pi \left[(k_0 + k_1 \cos y)\sqrt{1 - (k_0 + k_1 \cos y)^2} - \operatorname{arc\,cos}(k_0 + k_1 \cos y)\right] dy \tag{3.9}$$

$$A_{01}/\alpha P = k_1 + \frac{2}{\pi^2}\int_\theta^\pi \left[\sqrt{1 - (k_0 + k_1 \cos y)^2} - (k_0 + k_1 \cos y)\operatorname{arc\,cos}(k_0 + k_1 \cos y)\right] \cos y\, dy \tag{3.10}$$

SUM AND DIFFERENCE PRODUCTS—Second Order

$$A_{11} = \frac{\alpha P}{\pi^2}\int_\theta^\pi \left[(k_0 + k_1 \cos y)\sqrt{1 - (k_0 + k_1 \cos y)^2} - \operatorname{arc\,cos}(k_0 + k_1 \cos y)\right] \cos y\, dy \tag{3.11}$$

SUM AND DIFFERENCE PRODUCTS—Third Order

$$A_{21} = \frac{2\alpha P}{3\pi^2}\int_\theta^\pi [1 - (k_0 + k_1 \cos y)^2]^{3/2} \cos y\, dy \tag{3.12}$$

The above products are the ones usually of most interest. Others can readily be obtained either by direct integration or by use of recurrence formulas. The following set of recurrence formulas were originally derived by Mr. S. O. Rice for the biased linear rectifier:

$$\left.\begin{aligned}
&2n\, A_{mn} + k_1 (n - m - 3)\, A_{m+1,n-1} \\
&\qquad + k_1 (m + n + 3) A_{m+1,n-1} + 2k_0 n\, A_{m+1,n} = 0 \\
&2n\, A_{mn} + k_1 (n + m - 3)\, A_{m-1,n+1} \\
&\qquad + k_1 (n - m + 3) A_{m-1,n+1} + 2k_0 n\, A_{m-1,n} = 0 \\
&2m\, k_1\, A_{mn} + (m - n - 3)\, A_{m-1,n+1} \\
&\qquad + (m + n + 3) A_{m+1,n+1} + 2k_0 m\, A_{m,n+1} = 0 \\
&2\, m\, k_1\, A_{mn} + (m + n - 3)\, A_{m-1,n-1} \\
&\qquad + (m - n + 3) A_{m+1,n-1} + 2k_0 m\, A_{m,n-1} = 0
\end{aligned}\right\} \tag{3.13}$$

By means of these relations, all products can be expressed in terms of A_{00}, A_{10}, A_{01}, and A_{11}. The following specific results are tabulated:

$$\left.\begin{aligned}
A_{20} &= \tfrac{1}{3}(A_{00} - 2k_1 A_{11} - 2k_0 A_{10}) \\
A_{02} &= \frac{1}{3k_1}(k_1 A_{00} - 2A_{11} - 2k_0 A_{01})
\end{aligned}\right\} \tag{3.14}$$

$$\left.\begin{aligned} A_{21} &= \tfrac{1}{2}(A_{01} - k_1 A_{10} - k_0 A_{11}) \\ A_{12} &= \frac{1}{2k_1}(k_1 A_{10} - A_{01} - k_0 A_{11}) \end{aligned}\right\} \tag{3.15}$$

$$\left.\begin{aligned} A_{30} &= -k_0 A_{20} - k_1 A_{21} \\ A_{03} &= -\frac{1}{k_1}(k_0 A_{02} + A_{12}) \end{aligned}\right\} \tag{3.16}$$

The third-order product A_{21} is of considerable importance in the design of carrier amplifiers and radio transmitters, since the $(2p - q)$-product is the cross-product of lowest order falling back in the fundamental band when overload occurs. Figure 14 shows curves of A_{21} calculated by Mr. J. O. Edson from Eq. (3.12) by mechanical integration.

We point out also that the linear-rectifier coefficients give the Fourier series expansion of the admittance of a biased square-law rectifier when two frequencies are applied.

We shall next discuss the problem of reduction of the integrals appearing above to a closed form in terms of tabulated elliptic integrals[9]. This can be done for all the coefficients above except the d-c for the zero-power law and for the d-c and two fundamentals for the linear rectifier. These contain the integral

$$\Xi(k_0, k_1) = \int_\theta^\pi \text{arc cos } (k_0 + k_1 \cos y)\, dy \tag{3.17}$$

which has been calculated separately and plotted in Fig. 22. When the arc cos term is accompanied by $\cos my$ as a multiplier with $m \neq 0$, an integration by parts is sufficient to reduce the integrand to a rational function of $\cos y$ and the radical $\sqrt{1 - (k_0 + k_1 \cos y)^2}$, which may be reduced at once to a recognizable elliptic integral by the substitution $z = \cos y$. It is found that all the integrals except that of (3.17) appearing in the results can be expressed as the sum of a finite number of integrals of the form:

$$Z_m = \int_{-1}^{\cos\theta} \frac{z^m\, dz}{\sqrt{(1 - z^2)[1 - (k_0 + k_1 z)^2]}}, \quad m = 0, 1, 2, \cdots \tag{3.18}$$

By differentiating the expression $z^{m-3}\sqrt{(1 - z)^2[1 - (k_0 + k_1 z^2]}$ with respect to z, we may derive the recurrence formula:

$$\begin{aligned} Z_m = -\frac{1}{(m - 1)k_1^2}[&(2m - 3)k_0 k_1 z_{m-1} \\ &+ (m - 2)(k_0^2 - k_1^2 - 1)Z_{m-2} \\ &- (2m - 5)k_0 k_1 Z_{m-3} + (m - 3)(1 - k_0^2)Z_{m-4}] \end{aligned} \tag{3.19}$$

[9] Power series expansions of coefficients such as treated here have been given by A. G. Tynan, Modulation Products in a Power Law Modulator, *Proc. I. R. E.*, Vol. 21, pp. 1203–1209, Aug. 1933.

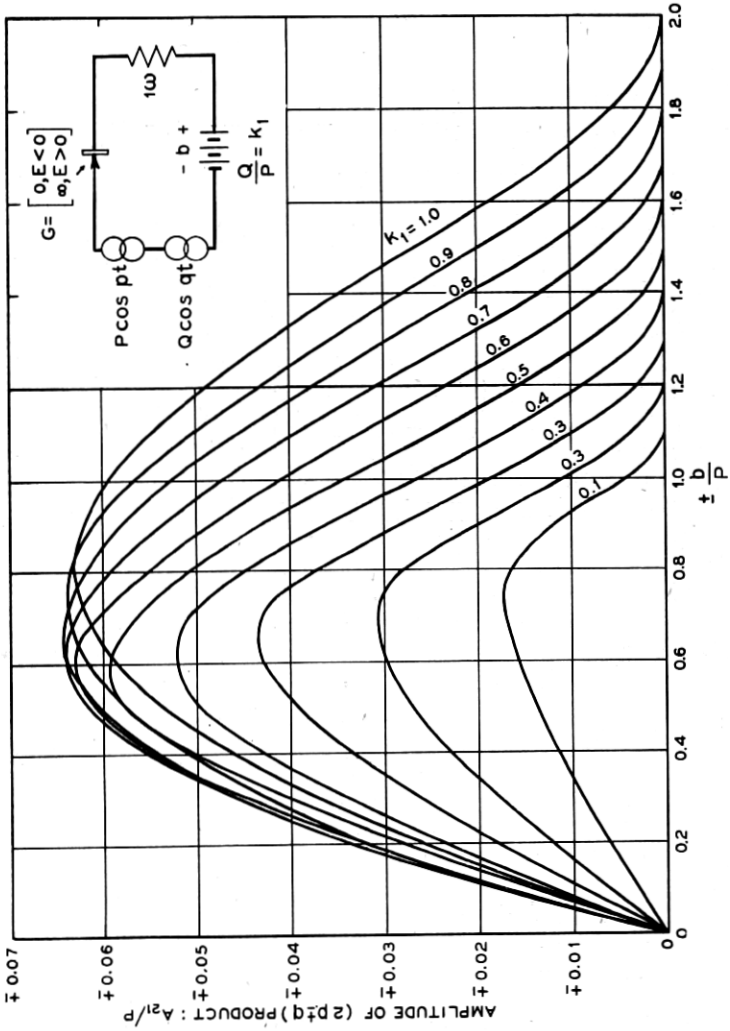

Fig. 14.—$(2p \pm q)$—Product from linear rectifier with bias.

It thus is found that the value of Z_m for all values of m greater than 2 can be expressed in terms of Z_0, Z_1, and Z_2.

Eq. (3.18) may be written in the form:

$$Z_m = \frac{1}{k_1} \int_{z_2}^{z_3} \frac{z^m \, dz}{\sqrt{(z - z_1)(z - z_2)(z_3 - z)(z_4 - z)}} \tag{3.20}$$

$$\left.\begin{aligned} z_1 &= -(1 + k_0)/k_1 , \; z_2 = -1 \\ z_3 &= \begin{pmatrix} (1 - k_0)/k_1 , \text{ Case I)} \\ 1, \text{ Case II} \end{pmatrix} \\ z_4 &= \begin{pmatrix} 1, \text{ Case I} \\ (1 - k_0)/k_1, \text{ Case II}, \end{pmatrix} \end{aligned}\right\} \tag{3.21}$$

The substitution

$$z = \frac{z_2(z_3 - z_1) - z_1(z_3 - z_2)u^2}{z_3 - z_1 - (z_3 - z_2)u^2} \tag{3.22}$$

reduces the integral to

$$Z_m = \frac{2}{k_1 \sqrt{(z_4 - z_2)(z_3 - z_1)}} \int_0^1 \frac{\left(z_1 + \dfrac{z_2 - z_1}{1 - \eta u^2}\right)^m du}{\sqrt{(1 - u^2)(1 - \kappa^2 u^2)}} \tag{3.23}$$

where:

$$\eta = \frac{z_3 - z_2}{z_3 - z_1} \tag{3.24}$$

$$\kappa^2 = \frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)} \tag{3.25}$$

Hence if K, E and Π represent respectively complete elliptic integrals of the first, second, and third kinds with modulus κ, and in the case of third kind with parameter $-\eta$, we have immediately:

$$Z_0 = \frac{2K}{k_1 \sqrt{(z_4 - z_2)(z_3 - z_1)}} \tag{3.26}$$

$$Z_1 = \frac{2[z_1 K + (z_2 - z_1)\,\Pi]}{k_1 \sqrt{(z_4 - z_2)(z_3 - z_1)}} \tag{3.27}$$

$$Z_2 = \frac{2}{k_1 \sqrt{(z_4 - z_2)(z_3 - z_1)}} \Bigg[z_1^2 K + 2z_1(z_2 - z_1)\Pi + (z_2 - z_1)^2 \int_0^1 \frac{du}{(1 - \eta u^2)^2 \sqrt{(1 - u^2)(1 - \kappa^2 u^2)}} \Bigg] \tag{3.28}$$

To complete the evaluation of Z_2, assume a relation of the following type with undetermined constants C_1, C_2, C_3, C_4:

$$\int_0^z \frac{du}{(1-\eta u^2)^2\sqrt{(1-u^2)(1-\kappa^2u^2)}} = C_1\int_0^z \frac{du}{\sqrt{(1-u^2)(1-\kappa^2u^2)}}$$

$$+ C_2\int_0^z \sqrt{\frac{1-\kappa^2u^2}{1-u^2}}\,du + C_3\int_0^z \frac{du}{(1-\eta u^2)\sqrt{(1-u^2)(1-\kappa^2u^2)}}$$

$$+ C_4\,\frac{z\sqrt{(1-z^2)(1-\kappa^2z^2)}}{1-\eta z^2} \qquad (3.29)$$

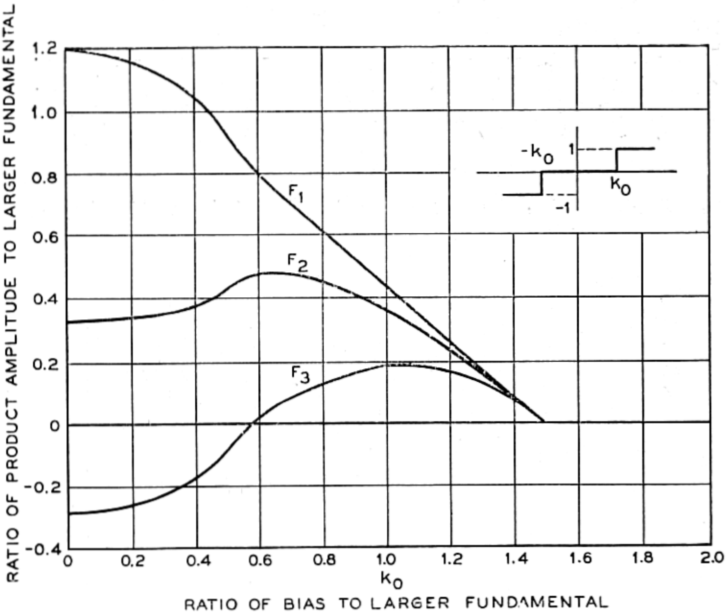

Fig. 15.—Fundamentals and $(2p \pm q)$—product from full-wave biased zero-power-law rectifier with ratio of applied fundamental amplitudes equal to 0.5. F_1 = larger fundamental, F_2 = smaller fundamental, $F_3 = (2p \pm q)$—product.

Differentiate both sides with respect to z, set $z = 1$, and clear fractions. Equating coefficients of like powers of z separately then gives four simultaneous equations in C_1, C_2, C_3, C_4. Solving for C_1, C_2, C_3 and setting $z = 1$ in (3.29) gives

$$\int_0^1 \frac{du}{(1-\eta u^2)^2\sqrt{(1-u^2)(1-\kappa^2u^2)}} = \frac{1}{2(\eta-1)}\left[K + \frac{\eta E}{\kappa^2-\eta}\right.$$

$$\left. + \frac{(2\eta-3)\,\kappa^2 - \eta(\eta-2)}{\kappa^2-\eta}\,\Pi\right] \qquad (3.30)$$

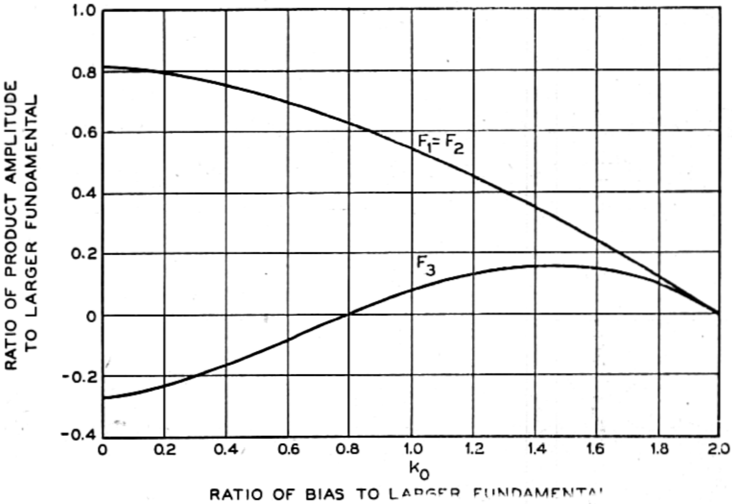

Fig. 16.—Fundamentals and $(2p \pm q)$—product from full-wave biased zero-power-law rectifier with equal applied fundamental amplitudes.

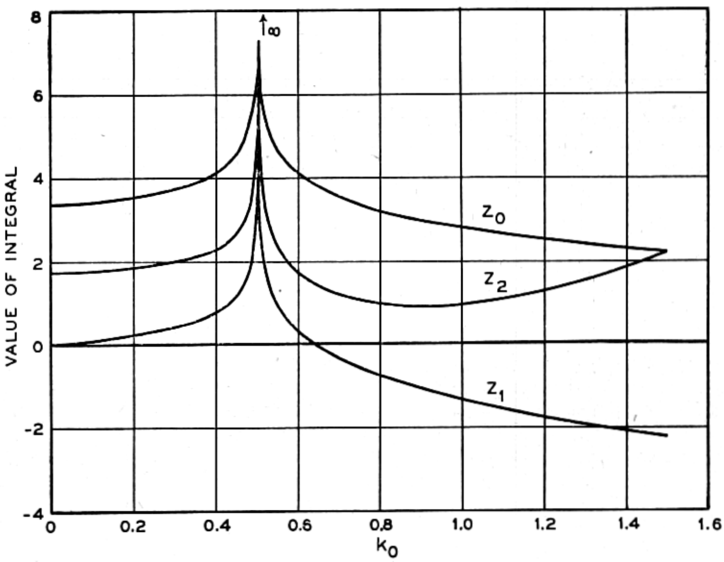

Fig. 17.—The integral Z_m with $k_1 = 0.5$.

Since the necessary tables of Π are not available, we make use of Legendre's Transformation,[10] which in this case gives:

[10] Legendre, Traités des Fonctions Elliptiques, Paris, 1825–28, Vol. I, Ch. XXIII.

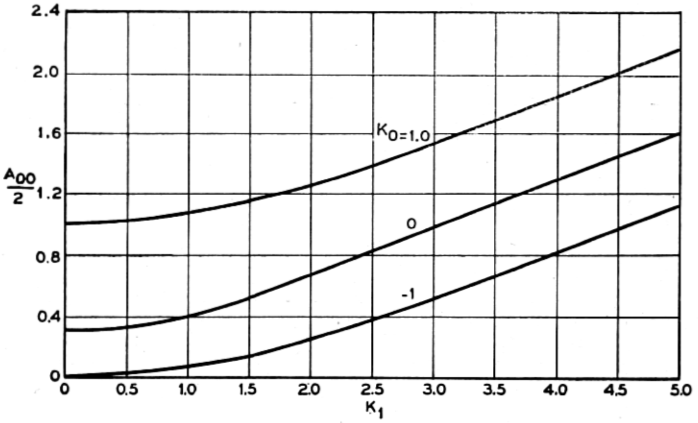

Fig. 18.—D-c. term in linear rectifier output with two applied frequencies.

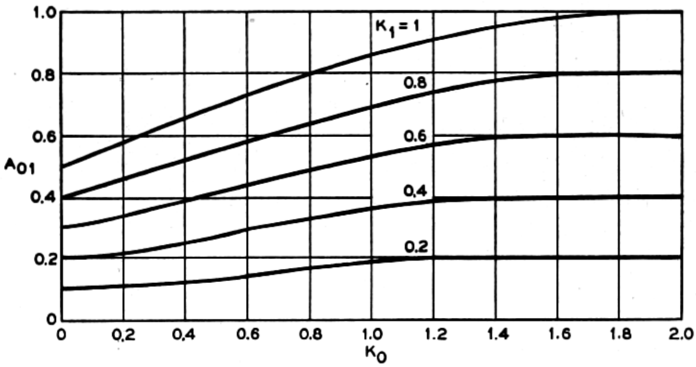

Fig. 19.—Smaller fundamental in biased linear rectifier output.

$$\Pi = K + \frac{\tan\phi}{\sqrt{1-\kappa^2\sin^2\phi}}\,[KE(\phi) - EF(\phi)] \tag{3.31}$$

$$\phi = \text{arc}\sin\frac{\eta^{1/2}}{\kappa} \tag{3.32}$$

$$F(\phi) = \int_0^\phi \frac{d\theta}{\sqrt{1-\kappa^2\sin^2\theta}} \tag{3.33}$$

$$E(\phi) = \int_0^\phi \sqrt{1-\kappa^2\sin^2\theta}\,d\theta \tag{3.34}$$

The functions $F(\phi)$ and $E(\phi)$ are incomplete elliptic integrals of the first and second kinds. They are tabulated in a number of places. Fairly good tables, e.g. the original ones of Legendre, are needed here since the difference between $KE(\phi)$ and $EF(\phi)$ is relatively small.

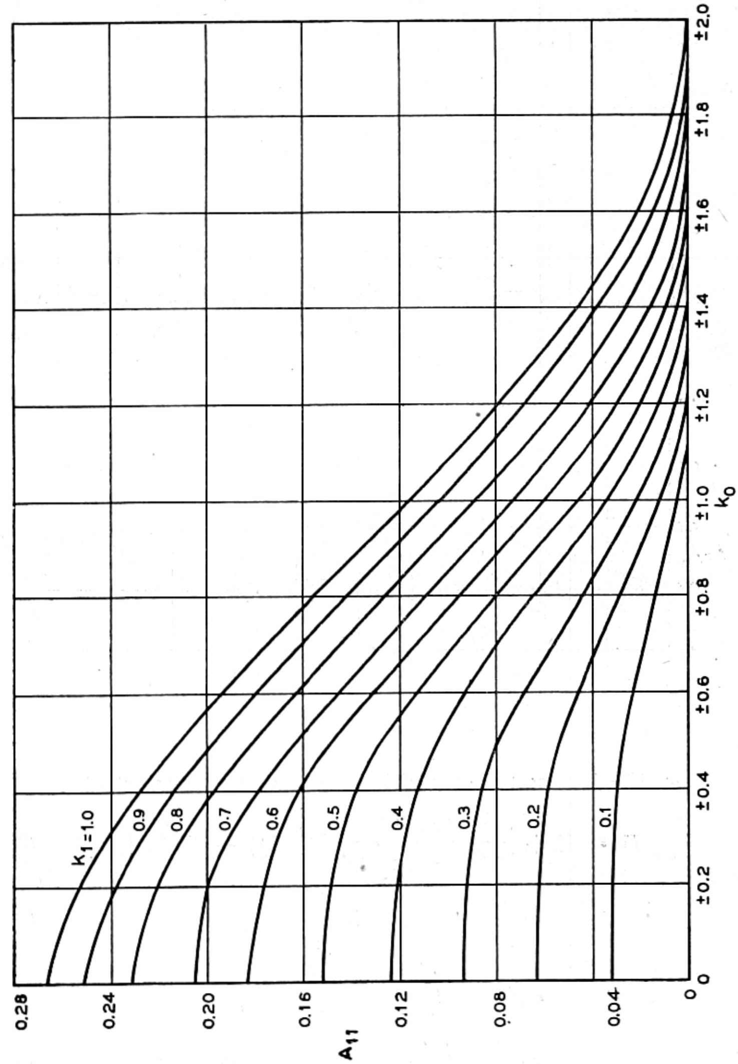

Fig. 20.—Second-order sideband amplitude in biased linear rectifier output.

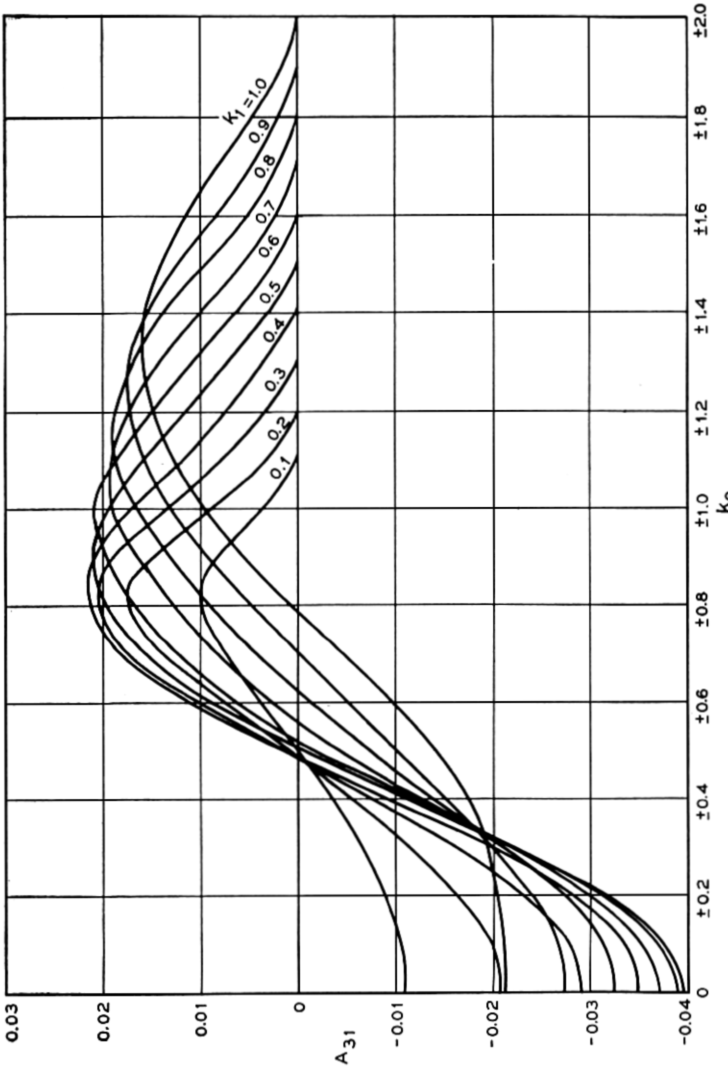

Fig. 21.—$(3p \pm q)$—product in biased linear rectifier output.

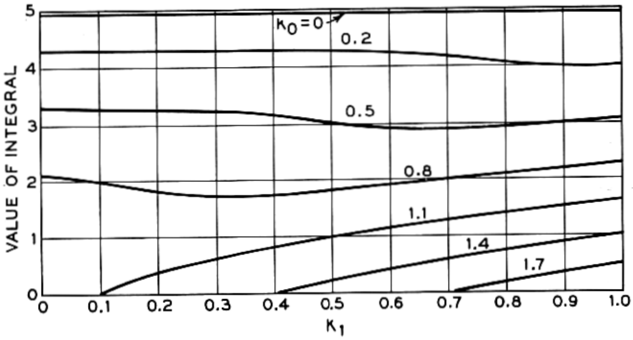

Fig. 22.—Graph of the integral $\Xi\ (k_0,\ k_1)$.

Summarizing:

$$\left.\begin{aligned} &\text{Case I, } k_0 + k_1 > 1, \qquad k_0 - k_1 < 1 \\ &Z_0 = \frac{K}{\sqrt{k_1}} \\ &Z_1 = \frac{2}{k_1}\,[KE(\phi) - EF(\phi)] - Z_0 \\ &Z_2 = \frac{1}{k_1}\left[Z_0 - k_0 Z_1 - \frac{2}{\sqrt{k_1}}\,E\right] \\ &\kappa = \sqrt{\frac{(k_1 + 1)^2 - k_0^2}{4k_1}} \\ &\phi = \text{arc sin}\sqrt{\frac{2k_1}{1 + k_0 + k_1}} \end{aligned}\right\} \qquad (3.35)$$

Case II, $k_0 + k_1 < 1$, $k_0 - k_1 > -1$

$$\left.\begin{aligned} &Z_0 = \frac{2K}{\sqrt{(1 + k_1)^2 - k_0^2}} \\ &Z_1 = \frac{2}{k_1}\,[KE(\phi) - EF(\phi)] - Z_0 \\ &Z_2 = \frac{1}{2k_1^2}\left[(1 + k_1^2 - k_0^2)Z_0 - 2k_0 k_1 Z_1 - 2E\sqrt{(1 + k_1)^2 - k_0^2}\right] \\ &\kappa = \sqrt{\frac{4k_1}{(1 + k_1)^2 - k_0^2}} \\ &\phi = \text{arc sin}\sqrt{\frac{1 - k_0 + k_1}{2}} \end{aligned}\right\} \qquad (3.36)$$

The values of the fundamentals and third-order sum and difference products for the biased zero-power-law rectifier have been calculated by the formulas above for the cases $k_1 = .5$ and $k_1 = 1$. The resulting curves are shown in Fig. (15) and (16). The values of the auxiliary integrals Z_0, Z_1, and Z_2 are shown for $k_1 = .5$ in Fig. (17). These integrals become infinite at $k_0 = 1 - k_1$ so that the formulas for the modulation coefficients become indeterminate at this point. The limiting values can be evaluated from the integrals (3.3), etc., directly in terms of elementary functions when the relation $k_0 = 1 - k_1$ is substituted, except for the Ξ-function.

Limiting forms of the coefficients when k_0 is small are of value in calculating the effect of a small signal superimposed on the two sinusoidal components in an unbiased rectifier. By straightforward power-series expansion in k_0, we find:

Zero-Power-Law Rectifier, k_0 Small:

$$\left.\begin{aligned}
A_{10} &= \frac{4}{\pi^2}E - \frac{2E}{\pi^2(1-k^2)}k_0^2 + \cdots \\
A_{01} &= \frac{4}{\pi^2 k_1}[E - (1-k_1^2)K] + \frac{2}{\pi^2 k_1}\left(\frac{E}{1-k_1^2} - K\right)k_0^2 + \cdots \\
A_{21} &= \frac{4}{3\pi^2 k_1}[(1-2k_1^2)E - (1-k_1^2)K] \\
&\quad + \frac{2}{\pi^2 k_1}\left[K - \frac{1-2k_1^2}{1-k_1^2}E\right]k_0^2 + \cdots
\end{aligned}\right\} \qquad (3.37)$$

In the above expressions, the modulus of K and E is k_1. When $k_0 = 0$, these coefficients reduce to half the values of the full-wave unbiased zero-power-law coefficients, which have been tabulated in a previous publication.[11]

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[11] R. M. Kalb and W. R. Bennett, Ferromagnetic Distortion of a Two-Frequency Wave, *B. S. T. J.*, Vol. XIV, April 1935, Eq. (21), p. 336.