Reflection from Corners in Rectangular Wave Guides—Conformal Transformation*

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A conformal transformation method is used to obtain approximate expressions for the reflection coefficients of sharp corners in rectangular wave guides. The transformation carries the bent guide over into a straight guide filled with a non-uniform medium. The reflection coefficient of the transformed system can be expressed in terms of the solution of an integral equation which may be solved approximately by successive substitutions. When the corner angle is small and the corner is not truncated the required integrations may be performed and an explicit expression obtained for the reflection coefficient. Although applied here only to corners, the method has an additional interest in that it is applicable to other types of irregularities in rectangular wave guides.

INTRODUCTION

THE propagation of electromagnetic waves around a rectangular corner has been studied in two recent papers, one by Poritsky and Blewett[1] and the other by Miles[2]. Poritsky and Blewett make use of Schwarz' "alternating procedure" in which a sequence of approximations is obtained by going back and forth between two overlapping regions. Miles derives an equivalent circuit by using solutions of the wave equation in rectangular coordinates. Several papers giving experimental results have been published. Of these, we mention one due to Elson[3] who gives values of reflection coefficients for various types of corners.

Here we shall deal with the more general type of corner shown in Fig. 1 by transforming, conformally, the bent guide (in which the propagation "constant" of the dielectric is constant) into a straight guide in which the propagation "constant" is a function of position—its greatest deviation from the original value being in the vicinity of points corresponding to the corner. This type of corner has been chosen for our example because it possesses a number of features common to problems which may be treated by the transformation method.

The essentials of the procedure used are due to Routh[4] who studied the vibration of a membrane of irregular shape by transforming it into a rectangle. After the transformation the density (analogous to the propagation constant in the guide) was no longer constant but this disadvantage was more than offset by the simplification in shape.

Until this paper was presented at the Symposium I was unaware of any

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[1] See list of references at end of paper.

other wave guide work based on conformal transformations (as described above) except that of Krasnooshkin[5]. At the meeting I learned that the transformation method had also been discovered (but not yet published) by Levine and by Piloty independently of each other. Levine has studied the same corner, see Fig. 1, as is done here. However, his method of approach is quite different in that he obtains expressions for the elements in the equivalent pi network representing the corner, whereas here the reflection coefficient is considered directly. This is discussed in more detail at the beginning of Section 6. Piloty's work is closely related to the material presented in a companion paper[6] and is discussed in its introduction.

In this paper the partial differential equation resulting from the transformation, together with the boundary conditions, is converted into a rather complicated integral equation. Numerical work indicates that satisfactory values of the reflection coefficient, in which we are primarily interested, may be obtained by solving this integral equation by the method of successive substitutions. However, the question of convergence is not investigated.

Although they are here applied only to corners, the equations of Sections 3, 4 and 5 are quite general. In order to test their generality they were used to check the expression[7] for the reflection coefficient of a gentle circular bend in a rectangular wave guide, E being in the plane of the bend. The work has been omitted because of its length. It was found that the essential parts of the transformation may be obtained by regarding the inner and outer walls of the guide system as the two plates of a condenser, solving the corresponding electrostatic problem (using series of the Fourier type), and utilizing the relation between two-dimensional potentials and the theory of conformal mapping.

When the angle of the corner is small we may obtain the series (7-5) and (7-11) for the reflection coefficients corresponding to simple (i.e. not truncated) E and H corners, respectively (a corner having the electric intensity E in the plane of the bend will be called an E corner or an electric corner. H corners are defined in a similar manner). When the angle of the general E corner shown in Fig. 1 is small we may use the series (7-18).

The series (7-5) and (7-11) giving the reflection from small angle corners are related to the series giving the reflection coefficients for gentle circular bends. In fact, if the radii of curvature of the latter be held constant while the angle of bend is made small, the series for the circular bends reduce to those for the corners.

As for the limitations of the method, note first that it can be used only for wave guide systems in which the dimension normal to the plane of transformation is constant throughout. Moreover, the integral equations of the present paper, except for the work of Appendix III, are derived on the assumption that the dimensions of the guide approach constant values at minus infinity and the same values at plus infinity. When this assumption is not met, a conformal transformation may still be used to carry the system into a straight guide. However, there appears to be some doubt as to the best way of dealing with the resulting partial differential equation. One method, discussed in the companion paper[6], leads to an infinite set of ordinary linear differential equations of the second order. Again, possibly the Green's functions appearing in Sections 3 and 5 may be replaced by suitable approximations.

1. Representation of Field for Corner or Bend in Rectangular Guide

Quite often waves in rectangular wave guides are classed as "transverse electric" or "transverse magnetic". However, for our purposes it is more convenient to class them as "electrically oriented" or "magnetically oriented" waves.[8,9] Thus, the electric and magnetic intensities are obtained by multiplying

$$\begin{aligned} E_x &= \frac{1}{i\omega\epsilon}\frac{\partial^2 A}{\partial x\partial\zeta} - \frac{\partial B}{\partial y} & H_x &= \frac{\partial A}{\partial y} + \frac{1}{i\omega\mu}\frac{\partial^2 B}{\partial x\partial\zeta} \\ E_y &= \frac{1}{i\omega\epsilon}\frac{\partial^2 A}{\partial y\partial\zeta} + \frac{\partial B}{\partial x} & H_y &= -\frac{\partial A}{\partial x} + \frac{1}{i\omega\mu}\frac{\partial^2 B}{\partial y\partial\zeta} \\ E_\zeta &= -i\omega\mu A + \frac{1}{i\omega\epsilon}\frac{\partial^2 A}{\partial\zeta^2} & H_\zeta &= -i\omega\epsilon B + \frac{1}{i\omega\mu}\frac{\partial^2 B}{\partial\zeta^2} \end{aligned} \tag{1-1}$$

by $e^{i\omega t}$ and taking the real part. Here ω, μ, and ϵ are the radian frequency, the permeability of the medium filling the guide ($\mu = 1.257 \times 10^{-6}$ henries per meter for air), and the dielectric constant of the same ($\epsilon = 8.854 \times 10^{-12}$ farads per meter for air), respectively. x, y, and ζ constitute a right-handed set of rectangular coordinates in which the ζ axis is normal to the plane of the bend. Equations (1-1) may be verified by substituting them in Maxwell's equations.

The potentials A and B satisfy the wave equation

$$\begin{gathered} \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial\zeta^2} = \sigma^2 A \\ \sigma = i\omega\sqrt{\mu\epsilon} = i2\pi/\lambda_0 \end{gathered} \tag{1-2}$$

where λ_0 is the wave length in free space corresponding to the radian frequency ω.

When the electric vector lies in the plane of the bend, as shown in Fig. 1, and the incident wave contains only the dominant mode we set

$$A = 0, \qquad B = Q\sin(\pi\zeta/a) \tag{1-3}$$

where a is the wide dimension of the rectangular cross-section, the guide walls normal to the ζ axis are at $\zeta = 0$ and $\zeta = a$, and Q is a function of x and y such that

$$\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} - \Gamma_{10}^2 Q = 0$$
$$\Gamma_{10} = i2\pi\lambda_0^{-1}(1 - \lambda_0^2 a^{-2}/4)^{1/2} \qquad (1\text{-}4)$$

The guide walls are assumed to be perfect conductors and hence the tangential component of E must vanish at the walls. This requires the normal derivative of Q to vanish at those walls which are perpendicular to the plane of the bend:

$$\frac{\partial Q}{\partial n} = 0. \qquad (1\text{-}5)$$

When the magnetic vector lies in the plane of the bend and the incident wave consists of the dominant mode, we set

$$A = P, \qquad B = 0 \qquad (1\text{-}6)$$

where P is a function of x and y such that

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} - \Gamma_{00}^2 P = 0, \qquad \Gamma_{00} = i2\pi/\lambda_0 \qquad (1\text{-}7)$$

and

$$P = 0 \qquad (1\text{-}8)$$

at the walls perpendicular to the plane of the bend. In this case the guide walls parallel to the plane of the bend are at $\zeta = 0$ and $\zeta = b$.

2. *Electric Vector in Plane of Bend*

Figure 1 shows a section of the bend taken parallel to the electric vector. b is the narrow dimension of the guide. Let the frequency and the wide dimension a of the guide (measured normal to the plane of Fig. 1) be such that only the dominant mode is freely propagated. The position of any point in this section is specified by the complex number $z = x + iy$ where the origin and the orientation of the axes have been chosen somewhat arbitrarily.

The constant k and related propagation constants which appear in the formulas dealing with Q and electric bends are given by

$$k = (2b/\lambda_0)\,[1 - (\lambda_0/2a)^2]^{1/2} = -i\Gamma_{10}b/\pi$$
$$\gamma_m^2 = m^2 - k^2; \qquad m = 0, 1, 2, \cdots; \qquad \gamma_0 = ik \qquad (2\text{-}1)$$
$$\lambda_0 = \text{free space wavelength}$$

Since, by assumption, only the dominant mode is freely propagated, k and γ_m for $m > 0$ are real and positive.

We imagine an incident wave of unit amplitude coming down from z_5 in the upper left portion of Fig. 1. What are the amplitudes of the reflected wave traveling back toward z_5 and the transmitted wave traveling outward to the right towards z_3? Our task is to find a $Q(x, y)$, satisfying the wave equation (1-4) and the boundary condition (1-5), which represents a disturbance of the assumed type.

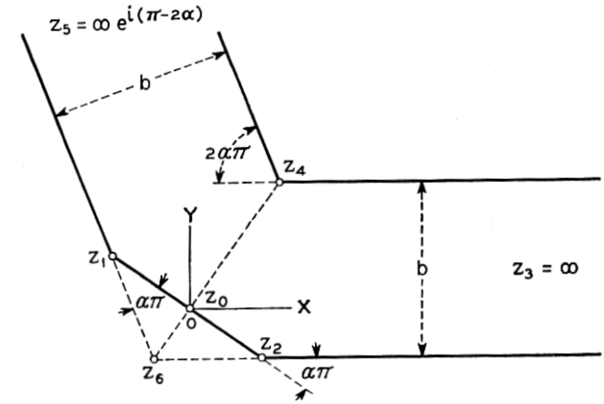

FIG. 1

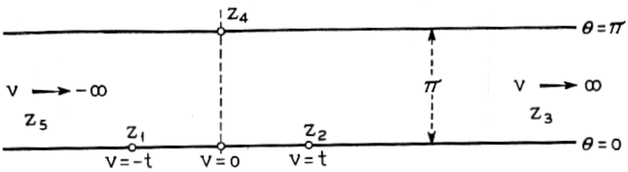

FIG. 2

The first step is to find the conformal transformation

$$z = x + iy = f(v + i\theta) = f(w) \tag{2-2}$$

which carries the bent guide (shown in Fig. 1) in the (x, y) plane over into the straight guide (shown in Fig. 2) in the (v, θ) plane. This may be done by the Schwarz-Christoffel method discussed in Appendix I. This transformation carries the wave equation (1-4) and the boundary condition (1-5) into

$$\frac{\partial^2 Q}{\partial v^2} + \frac{\partial^2 Q}{\partial \theta^2} + [1 + g(v, \theta)]k^2 Q = 0, \tag{2-3}$$

$$\frac{\partial Q}{\partial \theta} = 0 \text{ at } \theta = 0 \text{ and } \theta = \pi \tag{2-4}$$

where the upper and lower guide walls are carried into $\theta = 0$ and $\theta = \pi$, respectively, and $g(v, \theta)$ is given by

$$1 + g(v, \theta) = |f'(v + i\theta)|^2 \pi^2/b^2 \tag{2-5}$$

$$g(v, \theta) = \frac{[ch\, v + \cos\theta]^{2\alpha}}{[ch(v - t) - \cos\theta]^{\alpha}[ch(v + t) - \cos\theta]^{\alpha}} - 1 \tag{2-6}$$

Here ch denotes the hyperbolic cosine, $f'(v + i\theta)$ denotes the first derivative of $f(w)$, and from Appendix I, $2\pi\alpha$ is the total angle of the bend. t is a parameter which depends upon α and the ratio d/d_0 where $d = |z_4 - z_0|$ and $d_0 = |z_4 - z_6|$ in Fig. 1. A table giving values of t for a 90° bend ($\alpha = 1/4$) appears in Appendix I.

That the propagation constant is no longer uniform in the transformed guide shows up through the fact that the coefficient of k^2Q in (2-3) is now a function of the coordinates (v, θ). $g(v, \theta)$ measures the deviation of the propagation constant from its value at $v = -\infty$. For example, if we consider a wave front coming down from z_5 we expect it to get past z_4 before it reaches z_0. In Fig. 2 the same wave front is tilted forward corresponding to a high phase-velocity (or small propagation constant) at z_4 where $v = 0$ and $\theta = \pi$. This is in line with the fact that the coefficient of k^2Q in (2-3) vanishes at z_4 by virtue of (2-6). Similar considerations hold at z_1 and z_2.

What is our reflection problem in terms of the transformed guide? In addition to satisfying the two equations (2-3) and (2-4) Q must behave properly at infinity. For large negative values of v, Q must represent an incident wave plus a reflected wave. The incident wave is of unit amplitude and the reflected wave is of the, as yet, unknown value R_E. For large positive values of v Q must represent an outgoing wave. Thus Q must also satisfy the two equations

$$Q = e^{-ikv} + R_E e^{ikv}, \qquad v \to -\infty \tag{2-7}$$

$$Q = T_E e^{-ikv} \qquad , \qquad v \to \infty \tag{2-8}$$

where the subscript E appears on the "reflection coefficient" R_E and the "transmission coefficient" T_E to indicate that here we are dealing with an electric corner.

Our problem is now to take the four equations (2-3, 4, 7, 8) and somehow or other obtain the value of R_E. We are not so much interested in T_E because it does not have the practical importance of the reflection coefficient. There are at least two different ways we may proceed from here. One is to transform the differential equation plus the boundary conditions into an integral equation which may be solved approximately by iteration. Another way is to assume Q to be a Fourier cosine series in θ whose coefficients are functions of v. Substitution of the assumed series in the differential equation (2-3) gives rise to a set of ordinary differential equations having v as the independent variable and the coefficients as the dependent variables. The integral equation method is used in this paper. The second method is discussed in the companion paper.[6]

3. *Conversion of Differential Equation into an Integral Equation*

The differential equation (2-3) may be converted into an integral equation by using the appropriate Green's function in the conventional manner. The only modifications necessary are essentially those given by Poritsky and Blewett[1] in a similar procedure.

The conversion is based upon Green's theorem in the form

$$\int \left(Q \frac{\partial G}{\partial n} - G \frac{\partial Q}{\partial n} \right) ds = \iint (Q \nabla^2 G - G \nabla^2 Q) \, dv \, d\theta \qquad (3\text{-}1)$$

where the integration on the right extends over the rectangular region $v_1 < v < v_2$, $0 < \theta < \pi$ (inside the straight guide associated with (v, θ), i.e. the guide of Fig. 2) except for a very small circle surrounding the point (v_0, θ_0). $G \equiv G(v_0, \theta_0; v, \theta)$ is the Green's function corresponding to

$$\frac{\partial^2 V}{\partial v^2} + \frac{\partial^2 V}{\partial \theta^2} + k^2 V = 0 \qquad (3\text{-}2)$$

in the region $-\infty < v < \infty$, $0 < \theta < \pi$ subject to the boundary condition $\partial V/\partial n = 0$ on the walls ($\partial V/\partial \theta = 0$ at $\theta = 0$ and $\theta = \pi$). G becomes infinite as $-\log r$ when $r \to 0$, r being the distance between the variable point (v, θ) and the fixed point (v_0, θ_0). Poritsky and Blewett* have shown that, in the notation (2-1),

$$G = \sum_{m=0}^{\infty} \epsilon_m \gamma_m^{-1} \cos m\theta_0 \cos m\theta \, \epsilon^{-|v - v_0| \gamma_m} \qquad (3\text{-}3)$$

$$\epsilon_0 = 1, \; \epsilon_m = 2 \quad \text{for} \quad m = 1, 2, 3 \cdots$$

Equation (3-1) leads to

$$2\pi Q(v_0, \theta_0) + \int_0^{\pi} \left[-Q \frac{\partial G}{\partial v} + G \frac{\partial Q}{\partial v} \right]_{v_1} d\theta + \int_0^{\pi} \left[Q \frac{\partial G}{\partial v} - G \frac{\partial Q}{\partial v} \right]_{v_2} d\theta = k^2 \int_{v_1}^{v_2} dv \int_0^{\pi} d\theta \, g(v, \theta) QG \qquad (3\text{-}4)$$

from which the required integral equation for Q is found to be $Q(v_0, \theta_0) =$

$$e^{-ikv_0} + \frac{k^2}{2\pi} \int_{-\infty}^{\infty} dv \int_0^{\pi} d\theta \, g(v, \theta) Q(v, \theta) \sum_{m=0}^{\infty} \epsilon_m \gamma_m^{-1} \cos m\theta_0 \cos m\theta \, e^{-|v - v_0| \gamma_m} \qquad (3\text{-}5)$$

* We have replaced their i by $-i$ since here we assume the time to enter through the factor $e^{i\omega t}$ instead of $e^{-i\omega t}$.

where γ_m is given by (2-1). The term e^{-ikv_0} comes from the first integral on the left side of (3-4) as $v_1 \to -\infty$. Equation (3-5) is a general equation which may be applied to a number of wave guide problems by choosing a suitable function $g(v, \theta)$. For the corner of Fig. 1 $g(v, \theta)$ is given by (2-6).

If $g(v, \theta)$ approaches zero when $|v|$ becomes large, as it does for the corner, expressions for the reflection coefficient R_E and the amplitude T_E of the transmitted wave may be obtained by letting $v_0 \to \pm\infty$ in (3-5). For very large values of $|v_0|$ the contributions of all the terms in the summation except the first ($m = 0$) vanish. Comparison of the resulting expression for $Q(v_0, \theta_0)$ with the limiting forms (2-7) and (2-8) defining R_E and T_E gives

$$R_E = -\frac{ik}{2\pi}\int_{-\infty}^{\infty} dv \int_0^{\pi} d\theta\, g(v, \theta)\, Q(v, \theta) e^{-ikv} \tag{3-6}$$

$$T_E = 1 - \frac{ik}{2\pi}\int_{-\infty}^{\infty} dv \int_0^{\pi} d\theta\, g(v, \theta)\, Q(v, \theta) e^{ikv} \tag{3-7}$$

Since the integrands involve the as yet unknown $Q(v, \theta)$ these expressions are not immediately applicable. In fact, if we knew $Q(v, \theta)$ it would not be necessary to use these integrals for R_E and T_E—we could simply let $v \to \pm\infty$ and use (2-7) and (2-8). Nevertheless, (3-6) and (3-7) are useful in obtaining approximations to R_E and T_E when approximations to Q are known.

In Appendix IV it is shown that R_E is the stationary value, with respect to variations of the function Q, of an expression made up of integrals containing Q in their integrands. From the integral equation it follows that when $k \to 0$, i.e., when the frequency decreases toward the cut-off frequency of the dominant mode, Q becomes approximately $\exp(-ikv)$. Furthermore, R_E approaches zero. This is in contrast to the apparent behavior of R_H which, according to the discussion given in Section 5, may possibly approach -1 under the same circumstances. Thus reflections from the two types of corners, or more generally, irregularities in the E plane and in the H plane, appear to behave quite differently as the cut-off frequency is approached.

R_E and T_E are not independent. Since the energy in the incident wave is equal to the sum of the energies in the reflected and transmitted waves we expect

$$R_E R_E^* + T_E T_E^* = 1, \tag{3-8}$$

where the asterisk denotes the conjugate complex quantity. In addition, there is a relation between R_E and T_E which for a symmetrical irregularity, i.e. for $g(v, \theta)$ an even function of v, states that the phase of R_E differs from that T_E by $\pm\pi/2$. In this special case T_E is determined to within a plus or minus sign when R_E is given. These relations may be proved by substituting various solutions of equation (2-3) for Q and $\hat{Q}$ in the equation

$$\left[Q\frac{\partial \hat{Q}}{\partial v} - \hat{Q}\frac{\partial Q}{\partial v}\right]_{v_1} = \left[Q\frac{\partial \hat{Q}}{\partial v} - \hat{Q}\frac{\partial Q}{\partial v}\right]_{v_2} \tag{3-9}$$

where v_1 and v_2 are large enough (v_1 negative and v_2 positive) to ensure that Q and $\hat{Q}$ have reduced to exponential functions of v. Equation (3-9) follows from Green's theorem. When Q is taken to be the solution for which (2-7) and (2-8) holds and $\hat{Q}$ its conjugate complex Q^*, equation (3-8) is obtained. Keeping the same solution for Q but now letting $\hat{Q}$ denote the solution corresponding to an incident wave of unit amplitude coming in from the right:

$$Q_1 = e^{ikv} + R_1 e^{-ikv}, \qquad v \to \infty$$

$$Q_1 = T_1 e^{ikv} \qquad , \qquad v \to -\infty$$

gives $T = T_1$ where we have dropped the subscript E and have assumed that $g(v, \theta)$ may be unsymmetrical. Taking $\hat{Q}$ to be Q_1^* gives

$$RT_1^* + R_1^* T = 0$$

which is the relation sought. In the symmetrical case $R = R_1$, $R/T + R^*/T^*$ is zero and hence R/T is purely imaginary as was mentioned above. The same relations hold for R_H and T_H. These results are special cases of a more general result which states that the "scattering matrix" is symmetrical and unitary for a lossless junction.[10]

4. *Approximate Solution of Integral Equation*

A first approximation to the solution of the integral equation (3-5) is obtained when we assume that the non-uniformity of the propagation constant has no effect on Q. Thus we put

$$Q^{(1)}(v, \theta) = e^{-ikv} \tag{4-1}$$

in the integral on the right and obtain an expression for the second approximation $Q^{(2)}(v, \theta)$, and so on. Here we shall not go beyond $Q^{(2)}(v, \theta)$.

It is convenient to expand $g(v, \theta)$ in a Fourier cosine series

$$g(v, \theta) = \sum_{n=0}^{\infty} a_n(v) \cos n\theta$$

$$a_n(v) = \frac{\epsilon_n}{\pi}\int_0^{\pi} g(v, \theta) \cos n\theta \, d\theta, \qquad \epsilon_0 = 1; \epsilon_n = 2, n > 0. \tag{4-2}$$

The second approximation, obtained by substituting (4-1) in (3-5), may then be written as

$$Q^{(2)}(v_0, \theta_0) = e^{-ikv_0} + k^2 2^{-1} \sum_{m=0}^{\infty} \gamma_m^{-1} \cos m\theta_0 \cdot \int_{-\infty}^{\infty} a_m(v) e^{-ikv - |v - v_0|\gamma_m}\, dv. \tag{4-3}$$

The nth approximation $R_E^{(n)}$ to the reflection coefficient (when the electric vector lies in the plane of the bend) is defined in terms of $Q^{(n)}$ by

$$\operatorname*{Limit}_{v \to -\infty} Q^{(n)}(v, \theta) = e^{-ikv} + R_E^{(n)} e^{ikv} \tag{4-4}$$

$R_E^{(n)}$ is also equal to the integral obtained by replacing Q in (3-6) by $Q^{(n-1)}$. We have

$$R_E^{(1)} = 0, \qquad R_E^{(2)} = -ik2^{-1} \int_{-\infty}^{\infty} a_0(v) e^{-2ikv}\, dv,$$

$$R_E^{(3)} = R_E^{(2)} - ik^3 \sum_{m=0}^{\infty} (4\gamma_m \epsilon_m)^{-1} \cdot \int_{-\infty}^{\infty} dv_0\, a_m(v_0) \int_{-\infty}^{\infty} dv\, a_m(v) e^{-ik(v+v_0) - |v - v_0|\gamma_m} \tag{4-5}$$

where γ_m is given by (2-1).

The results of this section have the same generality as the integral equation (3-5) in that they are not restricted to corners.

5. *Truncated Corner—Magnetic Vector in Plane of Bend*

When the magnetic vector lies in the plane of the bend the reflection may be calculated by a similar procedure. The wide dimension a of the wave guide now replaces the narrow dimension b in Fig. 1. We shall call the result of making this change the "modified Fig. 1". We again assume the frequency to be such that only the dominant mode is propagated without attenuation. In place of equations (1–3, 4, 5) involving Q we have those of (1–6, 7, 8) involving P.

The conformal transformation which carries the modified Fig. 1 into Fig. 2 leads to

$$\frac{\partial^2 P}{\partial v^2} + \frac{\partial^2 P}{\partial \theta^2} + [1 + g(v, \theta)]\kappa^2 P = 0 \tag{5-1}$$

$$P = 0 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = \pi$$

where

$$\kappa = 2a/\lambda_0 = -i\Gamma_{00}a/\pi, \qquad c = (\kappa^2 - 1)^{1/2} = ak/b$$
$$\delta_m^2 = m^2 - 1 - c^2 = m^2 - \kappa^2, \qquad \delta_1 = ic \tag{5-2}$$
$$\lambda_0 = \text{free space wave length}, \qquad m = 1, 2, 3 \cdots$$

and

$$| f'_{\text{mod}}(v + i\theta) |^2 \pi^2/a^2 = 1 + g(v, \theta). \tag{5-3}$$

Here $f_{\text{mod}}(w)$ pertains to the modified Fig. 1. Since the expression for $f'(w)$ given in Appendix I is proportional to b and since the modified transformation contains a in place of b, it follows that $g(v, \theta)$ for the magnetic corner is exactly the same function, given by (2-6), as for the electric corner.

It is again assumed that the incident wave coming down from the left in the modified Fig. 1 is of unit amplitude and of the dominant mode. At large distances from the corner

$$\begin{aligned} P &= [e^{-icv} + R_H e^{icv}] \sin \theta, \qquad v \to -\infty \\ P &= T_H e^{-icv} \sin \theta, \qquad v \to +\infty \end{aligned} \tag{5-4}$$

which serve to define the coefficients of reflection and transmission. The subscript H on the reflection and transmission coefficients indicate that here we are dealing with a magnetic corner.

The conversion of the differential equation into the integral equation now employs the Green's function

$$G = 2 \sum_{m=1}^{\infty} \delta_m^{-1} \sin m\theta_0 \sin m\theta e^{-|v-v_0|\delta_m} \tag{5-5}$$

which corresponds to

$$\frac{\partial^2 V}{\partial v^2} + \frac{\partial^2 V}{\partial \theta^2} + \kappa^2 V = 0$$
$$V = 0 \quad \text{at} \quad \theta = 0 \quad \text{and } \theta = \pi$$

The integral equation for P is found to be

$$\begin{aligned} P(v_0, \theta_0) &= e^{-icv_0} \sin \theta_0 \\ &+ \frac{\kappa^2}{2\pi} \int_{-\infty}^{+\infty} dv \int_0^{\pi} d\theta\, g(v, \theta) P(v, \theta) \sum_{m=1}^{\infty} 2\delta_m^{-1} \sin m\theta_0 \sin m\theta e^{-|v-v_0|\delta_m} \end{aligned} \tag{5-6}$$

where the parameters are given by (5-2). This is a general equation. For the corner of the modified Fig. 1 $g(v, \theta)$ is given by (2-6).

By letting $v_0 \to -\infty$ we obtain the exact expression

$$R_H = -\frac{i\kappa^2}{\pi c}\int_{-\infty}^{\infty} dv \int_0^{\pi} d\theta\, g(v, \theta)P(v, \theta)e^{-icv}\sin\theta \tag{5-7}$$

When dealing with the electric corner we saw that $R_E \to 0$ as $k \to 0$. The presence of c in the denominator of (5-7) suggests the possibility that $R_H \to -1$ as $c \to 0$. For R_H must remain finite and this may perhaps come about through $P(v, \theta) \to 0$ in the region, say around $v = 0$, where $g(v, \theta)$ is appreciably different from zero. This and the fact that $P(v, \theta)$ must contain a unit incident wave suggest that for $v < 0$ the dominant portion of $P(v, \theta)$ is $2i \sin cv$ which gives $R_H = -1$. Incidentally, it is apparent that the approximations for $P(v, \theta)$ given below in (5-8) and (5-10) (and therefore also the approximations (5-11) for R_H) fail when c becomes small.

The first approximation to the solution of the integral equation (5-6) is

$$P^{(1)}(v, \theta) = e^{-icv}\sin\theta \tag{5-8}$$

When we introduce the coefficients

$$\begin{aligned} b_n(v) &= \frac{2}{\pi}\int_0^{\pi} g(v, \theta)\sin\theta\sin n\theta\, d\theta \\ \sin\theta g(v, \theta) &= \sum_{n=1}^{\infty} b_n(v)\sin n\theta \end{aligned} \tag{5-9}$$

$$b_1(v) = a_0(v) - a_2(v)/2, \qquad b_n(v) = [a_{n-1}(v) - a_{n+1}(v)]/2, \qquad n > 1$$

we find that the second approximation is

$$\begin{aligned} P^{(2)}(v_0, \theta_0) = e^{-icv_0}\sin\theta_0 + \kappa^2 2^{-1}\sum_{m=1}^{\infty}\delta_m^{-1}\sin m\theta_0 \\ \cdot\int_{-\infty}^{\infty} b_m(v)e^{-icv-|v-v_0|\delta_m}\, dv. \end{aligned} \tag{5-10}$$

The successive approximations to the reflection coefficient are

$$\begin{aligned} R_H^{(1)} &= 0, \qquad R_H^{(2)} = -\frac{i\kappa^2}{2c}\int_{-\infty}^{+\infty} dv\, b_1(v)e^{-2icv} \\ R_H^{(3)} &= R_H^{(2)} - i\kappa^4\sum_{m=1}^{\infty}(4c\delta_m)^{-1}\int_{-\infty}^{+\infty} dv_0\, b_m(v_0) \\ &\quad \cdot\int_{-\infty}^{+\infty} dv\, b_m(v_0)e^{-ic(v+v_0)-|v-v_0|\delta_m}. \end{aligned} \tag{5-11}$$

6. Series for $R^{(2)}$ When Corner Has No Truncation

The integrals which appear in the approximations for the reflection coefficients are difficult to evaluate in general. This section serves to put on record several expressions which have been obtained for $R^{(2)}$ when the corner is not truncated. Corresponding evaluations of $R^{(3)}$ would be welcome since the work of Section 7 for small angle corners indicates that $R^{(3)} - R^{(2)}$ is of the same order as $R^{(2)}$. However, I have been unable to go much beyond the results shown here.

As mentioned in the introduction, H. Levine has studied the effect of a corner in a wave guide by representing it as an equivalent pi network having an inductance for the series element and two equal condensers for the shunt elements. Early in 1947 he derived the following expressions (in our notation) for the elements corresponding to a simpe E corner:*

$$B_a/Y_0 = k\left[\Psi\left(\frac{\beta - 1}{2}\right) - \Psi(-\tfrac{1}{2})\right]$$

$$B_b/Y_0 = (k\pi)^{-1} \cot (\beta\pi/2)$$

where Y_0 is the characteristic admittance of the straight guide, iB_a the admittance of one of the two equal shunt condensers, $-iB_b$ the admittance of the series inductance, $\Psi(x)$ the logarithmic derivative of $\Gamma(x + 1)$, and $\beta\pi$ is the total angle of the simple corner (for no truncation we set $\beta = 2\alpha$).

When the reflection coefficient for the corner is computed from the equivalent network for the case $\beta \to 0$ it is found to lie between the approximate value $R_E^{(2)}$ given by (7-3) and the considerably more accurate value $R_E^{(3)}$ given by (7-5). All three approximations are of the form $A\beta^2 + O(\beta^3)$ where A differs from approximation to approximation but is independent of β, and $O(\beta^3)$ denotes correction terms of order β^3. Since $R_E^{(3)}$ gives the exact value of A, it may be regarded as the standard when the three approximations are compared. If this comparison be taken as a guide, it suggests that the rather cumbersome expressions (6-2) and (6-5) for $R_E^{(2)}$ given below are not as accurate as the simpler expressions resulting from Levine's work. Dr. Levine has also obtained corresponding results for the general E-corner of Fig. 1. It is hoped that his work will be published soon.

When the corner is not truncated it is convenient, as mentioned above, to replace 2α by β so that $\beta\pi$ is the total angle of the bend. For no truncation $t = 0$ and (2-6) becomes

$$g(v, \theta) = \left[\frac{chv + \cos\theta}{chv - \cos\theta}\right]^\beta - 1. \tag{6-1}$$

* I am indebted to Dr. Levine for communicating these expressions to me.

From (4-2) and (4-5), or from (3-6),

$$\begin{aligned} R_E^{(2)} &= -\frac{ik}{2\pi}\int_{-\infty}^{\infty} dv \int_0^{\pi} d\theta g(v, \theta) e^{-2ikv} \\ &= -ik \sum_{n=1}^{\infty} \frac{\Gamma(n - ik)\Gamma(n + ik)}{n!(n-1)!2} \sum_{m=0}^{n} \frac{(-2\beta)_{2m}(\beta)_{n-m}}{(2m)!(n-m)!} \end{aligned} \tag{6-2}$$

where we have expanded $g(v, \theta)$ as given by (6-1) in powers of $\cos \theta/ch\, v$ and integrated termwise. The notation is $(\alpha)_0 = 1$, $(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$.

For a right angle corner $\beta = 1/2$, and a more rapidly convergent series may be obtained by subtracting the sum of the series corresponding to $k = 0$, namely

$$\log 2 = \sum_{n=1}^{\infty} \frac{(1/2)_n}{n!2n} \tag{6-3}$$

Thus for $\beta = 1/2$

$$\begin{aligned} R_E^{(2)} &= -ik\left[\log_e 2 - \sum_{n=1}^{\infty} \frac{(1/2)_n}{n!2n}(1 - A_n)\right], \\ A_1 &= \pi k/\sinh \pi k, \qquad A_n = A_1 \prod_{m=1}^{n-1}(1 + k^2 m^{-2}), \qquad n > 1 \end{aligned} \tag{6-4}$$

The rate of convergence of the more general series (6-2) may be increased in a somewhat similar way. It is found that

$$\begin{aligned} R_E^{(2)} &= -\frac{ik}{2}\left[J - 2\beta^2(1 - A_1) - \frac{\beta^2}{3}(2 + \beta^2)(1 - A_2)\right. \\ &\qquad \left. - \frac{2\beta^2}{135}(23 + 20\beta^2 + 2\beta^4)(1 - A_3) - \cdots\right] \\ J &= K + L \\ K &= \sum_{n=1}^{\infty} \frac{(\beta)_n}{n!n} = \frac{1}{1-\beta} - \Psi(1 - \beta) - .5772 \\ L &= \sum_{m=1}^{\infty} \frac{(-2\beta)_{2m}}{(2m)!} \sum_{n=m}^{\infty} \frac{(\beta)_{n-m}}{(n-m)!n} = -\beta \sum_{m=1}^{\infty} \frac{(1/2 - \beta)_m}{(1/2)_m m(m - \beta)} \end{aligned} \tag{6-5}$$

where $.5772 \cdots$ is Euler's constant, $\Psi(x)$ is the logarithmic derivative of $\Pi(x) = \Gamma(x + 1)$, and A_n is given by (6-4).

The results corresponding to $R_H^{(2)}$ are quite similar. When the corner is not truncated

$$R_H^{(2)} = -i\kappa^2 \sum_{n=1}^{\infty} \frac{\Gamma(n - ic)\Gamma(n + ic)}{(n + 1)!(n - 1)!2c} \sum_{m=0}^{n} \frac{(-2\beta)_{2m}(\beta)_{n-m}}{(2m)!(n - m)!}$$

$$= -\frac{i\kappa^2}{2c}\left[\hat{J} - \beta^2(1 - \hat{A}_1) - \frac{\beta^2}{3}(2 + \beta^2)(1 - \hat{A}_2) \right. \qquad (6\text{-}6)$$

$$\left. - \frac{\beta^2}{270}(23 + 20\beta^2 + 2\beta^4)(1 - \hat{A}_3) - \cdots\right]$$

$$\hat{J} = 1 - \Psi(1 - \beta) - .5772$$

$$- \beta(1 - \beta) \sum_{m=1}^{\infty} \frac{(1/2 - \beta)_m}{(1/2)_m m(m - \beta)(m - \beta + 1)}$$

in which $\hat{A}_n$ is obtained by replacing c by k in the expression (6-4) for A_n.

The evaluation of the integrals for $R_E^{(2)}$ and $R_H^{(2)}$ for general values of t appears to be difficult although it is possible to obtain approximate expressions for the case when t is large.

7. *Reflection from Small Angle Corners*

The expressions for $R^{(2)}$ and $R^{(3)}$ may be evaluated approximately when the angle of the corner is small. It turns out that, for $t = 0$, they are of the same order of magnitude and both of them must be considered. Moreover $R^{(n)}$ for $n > 3$ differs from $R^{(3)}$ by terms of the same order as those neglected in our approximations so that there is no point in going to the higher values of n.

We first obtain the approximation for R_E for a corner with no truncation having the total angle $\pi\beta$. Since β is very small (6-1) may be written as

$$g(v, \theta) = \exp[\beta\varphi] - 1 = \beta\varphi + \beta^2\varphi^2/2! + 0(\beta^3)$$
$$\varphi = \log(chv + \cos\theta) - \log(chv - \cos\theta) \qquad (7\text{-}1)$$

where $0(\beta^3)$ denotes terms of order β^3. The expression φ becomes very large near the two points (0, 0) and (0, π) (the coordinates being (v, θ)). The following considerations indicate that this does not invalidate our procedure. The remainder, denoted by $0(\beta^3)$, in (7-1) is less than $|\beta\varphi|^3 \exp|\beta\varphi|$. Near (0, 0) φ is approximately equal to $2\log(2/r)$ where $r^2 = v^2 + \theta^2$. Consequently the remainder is less than $(2\beta \log 2/r)^3 (2/r)^{2\beta}$. When the expression (7-1) for $g(v, \theta)$ is set in the integral equation it is seen that all terms, and in particular the remainder term (by virtue of the inequality just stated), of the double integral converge at (0, 0). Hence the contribution of the remainder term is of order β^3 even in the worst case when the

Green's function is replaced by $-\log r$. A similar result holds for the other point in question, namely $(0, \pi)$.

Integrating (7-1) from $\theta = 0$ to $\theta = \pi$ and using equations (A2-1, 3) of Appendix II gives

$$\begin{aligned} a_0(v) &= \beta^2[I_1(v, v) - I_2(v, v)] + 0(\beta^3) \\ &= 4\beta^2 \sum_{n=1,3,5,\cdots}^{\infty} n^{-2} e^{-2nv} + 0(\beta^3) \end{aligned} \tag{7-2}$$

where $v > 0$. We consider only positive values of v since $g(v, \theta)$ and the $a_n(v)$'s are even functions of v. Thus (4-5) yields

$$R_E^{(2)} = -ik2\beta^2 \sum_{n=1,3,5,\cdots} n^{-1}(n^2 + k^2)^{-1} + 0(\beta^3) \tag{7-3}$$

This is an approximation to the exact value given by the double series in (6-2). Comparison of (6-5) and (7-3) when β and k approach zero gives, incidentally,

$$\sum_{m=1}^{\infty} m^{-2} \sum_{n=1}^{m} (n - 1/2)^{-1} = \frac{7}{2} \sum_{m=1}^{\infty} m^{-3}.$$

From (4-2), (7-1) and the expansions (A2-2) of $\log(chv \pm \cos\theta)$ it follows that

$$\begin{aligned} a_m(v) &= 4\beta m^{-1} e^{-m|v|} + 0(\beta^2), \qquad m = 1, 3, 5, \cdots \\ a_m(v) &= 0(\beta^2) \qquad\qquad\qquad\quad , \qquad m = 0, 2, 4, 6 \cdots \end{aligned} \tag{7-4}$$

Equations (4-5), (A2-4), the relation $\gamma_m^2 = m^2 - k^2$, and (A2-8) give us the answer we seek:

$$\begin{aligned} R_E^{(3)} &= R_E^{(2)} - ik^3 2\beta^2 \sum_{m=1,3,5,\cdots} \gamma_m^{-1} m^{-2} J(m, m, k, \gamma_m, 0, 0) + 0(\beta^3) \\ &= -ik2\beta^2 \sum_{m=1,3,5,\cdots} \gamma_m^{-1} m^{-2} + 0(\beta^3) \end{aligned} \tag{7-5}$$

It is not necessary to go to $R_E^{(4)}$ because it differs from $R_E^{(3)}$ by only $0(\beta^3)$.

When H lies in the plane of the bend the reflection from a small angle corner with no truncation may be obtained by much the same procedure. For brevity we shall not write down the order of magnitude of the remainder terms. From (5-9), (A2-1), and (A2-3)

$$\begin{aligned} b_1(v) &= a_0(v) - a_2(v)/2 \qquad\qquad (7\text{-}6) \\ &= \beta^2[I_1 - I_2 - (I_3 - I_4)/2] \\ &= \beta^2\Big[2e^{-2v} + 4 \sum_{n=3,5,\cdots} n^{-2} e^{-2nv} - 4 \sum_{n=2,4,\cdots} (n^2 - 1)^{-1} e^{-2nv}\Big] \end{aligned}$$

where we have written I_m for $I_m(v, v)$ and assumed $v > 0$. Then, using (5-11),

$$R_H^{(2)} = -i\kappa^2 c^{-1}\beta^2[\kappa^{-2} + 2 \sum_{n=3,5,\cdots} n^{-1}(n^2 + c^2)^{-1} - 2 \sum_{n=2,4,\cdots} n(n^2 - 1)^{-1}(n^2 + c^2)^{-1}] \tag{7-7}$$

When we put

$$b_n(v) = [a_{n-1}(v) - a_{n+1}(v)]/2, \qquad n > 1 \tag{7-8}$$

$$b_n(v) = 2\beta[(n - 1)^{-1}e^{-(n-1)|v|} - (n + 1)^{-1}e^{-(n+1)|v|}], \qquad n = 2, 4, 6, \cdots$$

$$b_n(v) = 0(\beta^2), \qquad n = 1, 3, 5, \cdots$$

in (5-11) and use the results of Appendix II we obtain

$$R_H^{(3)} = R_H^{(2)} - i\kappa^4 c^{-1}\beta^2 \sum_{n=2,4,6\cdots} \delta_n^{-1}[(n - 1)^{-2} J(n - 1, n - 1, \delta_n, 0, 0) + (n + 1)^{-2} J(n + 1, n + 1, c, \delta_n, 0, 0) - 2(n^2 - 1)^{-1} J(n - 1, n + 1, c, \delta_n, 0, 0)] \tag{7-9}$$

The values of the first two J's, obtained by setting $m = n \pm 1$ in (A2-7), may be simplified by using

$$c^2 + (n \pm 1 + \delta)^2 = 2(n \pm 1)(n + \delta)$$

where we have dropped the subcript n from δ_n. In order to eliminate δ from the denominator we multiply both numerator and denominator by $n - \delta$ and use

$$(n - \delta)\ (\delta + 2n \pm 2) = (n \pm 1)^2 + c^2 - \delta(n \pm 2)$$

$$n^2 - \delta^2 = 1 + c^2 = \kappa^2$$

Setting in the value, given by (A2-9), of the last J and separating the terms (into those which contain the first power of δ and those which do not) enable us to write the term within the square brackets in (7-9) as

$$\frac{4n^2}{\kappa^4(n^2 - 1)^2} - \frac{\delta_n}{\kappa^2}\left[\frac{(n - 1) - 1}{(n - 1)^2\{c^2 + (n - 1)^2\}} + \frac{(n + 1) + 1}{(n + 1)^2\{c^2 + (n + 1)^2\}} + \frac{2n\{2(n^2 + c^2) - \kappa^2(n^2 - 1)\}}{\kappa^2(n^2 - 1)^2(n^2 + c^2)}\right] \tag{7-10}$$

It is found that when (7-10) is put in (7-9), the contribution of the first two terms within the square bracket of (7-10) exactly cancels the summation which is taken over 3, 5, 7 ⋯ in the expression (7-7) for $R_H^{(2)}$. Moreover, if we make use of

$$\sum_{n=2,4,6,\cdots} 4n(n^2 - 1)^{-2} = 1$$

we see that the contribution of the last term within the square brackets of (7-10) cancels the remaining terms in $R_H^{(2)}$. Only the contribution of the first term in (7-10) remains and it gives

$$R_H^{(3)} = -i4\beta^2 c^{-1} \sum_{n=2,4,6,\cdots} n^2(n^2 - 1)^{-2}\delta_n^{-1} + 0(\beta^3) \tag{7-11}$$

The relative simplicity of this result indicates that there may be another method of derivation which avoids the lengthy algebra of our method.

Recently approximate expressions for the reflection coefficient of gentle circular bends have been published[7]. In our present notation these may be written as

$$R_E \approx -ib^2\rho_1^{-2}\left[\frac{\sin u}{24} - 4k \sum_{m=1,3,5,\cdots} \frac{\cos u - e^{-u\gamma_m/k}}{\pi^4 m^4 \gamma_m}\right]$$

$$R_H \approx -ia^2\rho_1^{-2}\left[\frac{\sin u}{8\pi^2 c^2} - 8 \sum_{n=2,4,6,\cdots} \frac{\cos u - e^{-u\delta_n/c}}{\pi^4 c\delta_n} \frac{n^2}{(n^2 - 1)^3}\right]$$

where $\beta\pi$ is the angle of the bend, ρ_1 is the radius of curvature of the center line of the guide and u is 2π times the length of the center line in the bend divided by the wavelength in the guide:

$$u = \beta\pi^2 k\rho_1/b = \beta\pi^2 c\rho_1/a$$

The first expression for u is to be used in R_E and the second in R_H. If we now let $\beta \to 0$, keeping ρ_1 fixed, then $u \to 0$. The trigonometric and exponential terms may be approximated by the first few terms in their power series expansions, and part of the series which make their appearance may be replaced by their sums given, for example, by equations (4.1-7) and(4.1-8) of reference[7]. After some cancellation, the above expression for R_E and R_H, which hold for gentle circular bends, reduce to (7-5) and (7-11), respectively, which hold for the sharp corners. In other words, the reflection coefficients for both the sharp and the circular bends approach zero as $\beta \to 0$, and furthermore their ratio approaches unity.

We shall merely outline the derivation of the approximation $R_E^{(3)}$ for a truncated corner. Instead of (7-1) we have from (2-6),

$$g(v, \theta) = \exp[\alpha\varphi] - 1 = \alpha\varphi + \alpha^2\varphi^2/2! + 0(\alpha^3), \tag{7-12}$$

$$\varphi = 2\log[chv + \cos\theta] - \log[ch(v - t) - \cos\theta] - \log[ch(v + t) - \cos\theta]$$

The Fourier coefficients of $g(v, \theta)$ may be obtained by using the results of Appendix II. Assuming $v > 0$, $m > 0$,

$$a_0(v) = 2\alpha(v - t)\psi(v) + 2^{-1}\alpha^2\{4I_1(v, v) + I_1(v - t, v - t)$$
$$+ I_1(v + t, v + t) - 4I_2(v - t, v) - 4I_2(v + t, v) + 2I_1(v - t, v + t)\},$$
$$a_m(v) = 2\alpha m^{-1}[-2(-)^m e^{-m|v|} + e^{-m|v-t|} + e^{-m|v+t|}] \tag{7-13}$$

where $\psi(v) = 1$ when $0 < v < t$ and $\psi(v) = 0$ when $v > t$. Substitution of the values (A2-3) for I_1 and I_2 gives

$$a_0(v) = [2\alpha(v - t) + 2\alpha^2(v - t)^2]\psi(v) \tag{7-14}$$
$$+ \alpha^2 \sum_{n=1}^{\infty} n^{-2}[4e^{-2nv} + e^{-2nv-2nt} - 4(-)^n e^{-2nv-nt}$$
$$+ e^{-2n|v-t|} + 2e^{-n|v-t|-n|v+t|} - 4(-)^n e^{-n|v-t|-nv}]$$

The second approximation to the reflection coefficient is

$$R_E^{(2)} = i\alpha k^{-1}\sin^2 kt - i\alpha^2 k^{-2}2^{-1}(2kt - \sin 2kt)$$
$$- ik\alpha^2 \sum_{n=1}^{\infty} n^{-1}(n^2 + k^2)^{-1}\{2-(-)^n 2e^{-nt} \tag{7-15}$$
$$+ [1 - 2(-)^n e^{-nt} + e^{-2nt}]\cos 2kt$$
$$+ nk^{-1}[e^{-2nt} - (-)^n 2e^{-nt}]\sin 2kt\}$$

The typical term in the summation (4-5) for $R_E^{(3)}$ is

$$-\frac{ik^3}{4\gamma_m\epsilon_m}\int_{-\infty}^{+\infty} dv_0\, a_m(v_0)\int_{-\infty}^{+\infty} dv\, a_m(v)e^{-ik(v+v_0)-|v-v_0|\gamma_m} \tag{7-16}$$

When $m = 0$, $\epsilon_0 = 1$, $\gamma_0 = ik$, and $a_0(v)$ is $2\alpha(v - t) + 0(\alpha^2)$ for $0 < v < t$ and is $0(\alpha^2)$ for $v > t$. The integral may then be approximated by replacing the upper limit ∞ in (A2-14) by t. The value of (7-16) for $m = 0$ is found to be, to within $0(\alpha^2)$,

$$2^{-1}\alpha^2 t^2(e^{-2ikt} - 1) - (3/4)i\alpha^2 k^{-2}(\sin 2kt - 2kt) \tag{7-17}$$

When $m > 0$, $\epsilon_m = 2$, $\gamma_m^2 = m^2 - k^2$, and the substitution of the value (7-13) for $a_m(v)$ enables us to express (7-16) as the sum of six J's where J is defined by (A2-4). The J's may be evaluated with the help of (A2-7) and (A2-8). Substitution of this value of (7-16) and the value (7-17) for $m = 0$, together with $R_E^{(2)}$ given by (7-15), in the expression (4-5) for $R_E^{(3)}$ gives our final result

$$R_E^{(3)} = i\alpha k^{-1}\sin^2 kt + \alpha^2 t^2 2^{-1}(e^{-2ikt} - 1) \tag{7-18}$$
$$+ i\alpha^2[4^{-1}k^{-2}(2kt - \sin 2kt) - B\sin 2kt + k\sum_{n=1}^{\infty} n^{-2}\gamma_n^{-1}A_n]$$

where

$$B = \sum_{n=1}^{\infty} n^{-2}[e^{-2nt} - 2(-)^n e^{-nt}]$$

$$A_n = \cos 2kt - [2\cos kt - (-)^n e^{-\gamma_n t}]^2$$

Equation (7-18) is an approximation, to within terms of order α^2, for the reflection coefficient of a truncated corner which turns through a small angle $2\pi\alpha$. The electric vector lies in the plane of the bend. When $t = 0$, (7-18) reduces to (7-5) by virtue of $2\alpha = \beta$.

APPENDIX I

Conformal Transformation of Truncated Corner

We shall use a Schwarz-Christoffel transformation* to carry the guide of Fig. 1 into the straight guide of Fig. 2. The first step is to transform the interior of Fig. 1 into the upper half of an auxiliary complex plane which we shall denote by ζ. Let the points z_1, z_2, z_3, z_4, z_5 in Fig. 1 correspond to the points $-h$, h, 1, ∞, -1 in the ζ plane. A suitable transformation is then

$$z = D + E\int_0^{\zeta} (\tau + h)^{-\alpha}(\tau - h)^{-\alpha}(\tau - 1)^{-1}(\tau + 1)^{-1}d\tau \qquad \text{(A1-1)}$$

where D, E and h are to be determined from the geometry of Fig. 1. Because of the symmetry of our transformation about the line joining z_0 and z_4 it follows that $z = z_0$ corresponds to $\zeta = 0$. Hence $D = z_0$. As ζ travels from $1 - \epsilon$ to $1 + \epsilon$, ϵ being very small and positive, along a semicircular indentation above $\zeta = 1$, z as given by (A1-1) increases by

$$E(1 - h^2)^{-\alpha} 2^{-1} \int_{1-\epsilon}^{1+\epsilon} (\tau - 1)^{-1}\, d\tau = \frac{-iE\pi}{2}(1 - h^2)^{-\alpha}$$

while, according to Fig. 1, it increases from $\infty + i0$ to $\infty + ib$. Hence we set the real part of E equal to $-2b\pi^{-1}(1 - h^2)^{\alpha}$. We have tacitly assumed the factors in (A1-1) to have their principal values at $\tau = 1 + \epsilon$ and also that $0 < h < 1$. As z goes from z_1 to z_2, ζ goes from $-h$ to $+h$. In this range $\arg(\tau + h) = 0$ and $\arg(\tau - h) = \pi$.
Consequently, if $| z_2 - z_1 | = \ell$, then

$$z_2 - z_1 = \ell e^{-i\alpha\pi} = -Ee^{-i\alpha\pi}\int_{-h}^{h} (h^2 - \tau^2)^{-\alpha}(1 - \tau^2)^{-1}d\tau$$

* See, for example, S. A. Schelkunoff, Electromagnetic Waves, New York (1943) pp. 184–187.

and we see that E is purely real. Hence

$$\ell = 2b\pi^{-1}(1 - h^2)^\alpha \int_{-h}^{h} (h^2 - \tau^2)^{-\alpha}(1 - \tau^2)^{-1}\, d\tau$$

is an equation from which h may be determined as a function of ℓ. Setting $\tau^2 = h^2 x$, expanding $(1 - h^2 x)^{-1}$ in powers of h^2 and integrating termwise leads to

$$\begin{aligned} \frac{\ell}{2b} &= \frac{\pi^{-\frac{1}{2}}\Gamma(1 - \alpha)}{\Gamma(\frac{3}{2} - \alpha)}(1 - h^2)^\alpha h^{1-2\alpha} F(1, \tfrac{1}{2}; \tfrac{3}{2} - \alpha; h^2) \\ &= \frac{\pi^{-\frac{1}{2}}\Gamma(1 - \alpha)}{\Gamma(\frac{3}{2} - \alpha)} h^{1-2\alpha} F(\tfrac{1}{2} - \alpha, 1 - \alpha; \tfrac{3}{2} - \alpha; h^2) \qquad \text{(A1-2)} \\ &= \frac{1}{\sin \pi\alpha} + \frac{\pi^{-\frac{1}{2}}\Gamma(-\alpha)}{\Gamma(\frac{1}{2} - \alpha)} h^{1-2\alpha}(1 - h^2)^\alpha F(1, \tfrac{1}{2}; 1 + \alpha; 1 - h^2) \end{aligned}$$

where we have used relations from the theory of hypergeometric functions. The term $1/\sin \pi\alpha$ is the reduced form of an original term containing a hypergeometric function which has been evaluated by the binomial theorem. The second and third expressions are suited to calculation when $h^2 < 1/2$ and $h^2 > 1/2$ respectively.

Now that the guide of Fig. 1 has been transformed into the upper half of the ζ plane, the next step is to transform this upper half into the straight guide of Fig. 2. We want $\zeta = -1$, i.e. z_5, to go into $v = -\infty$ and $\zeta = 1$, i.e. z_3, to go into $v = +\infty$. Again using the Schwarz-Christoffel formula with $w = v + i\theta$ (the exterior angles at $v = \pm\infty$ are equal to π)

$$w = D_1 + E_1 \int_0^\zeta (\tau + 1)^{-1}(\tau - 1)^{-1}\, d\tau \qquad \text{(A1-3)}$$

We take the point z_0 in Fig. 1 to correspond to $v = 0$, $\theta = 0$ in Fig. 2. Since this corresponds to $\zeta = 0$, D_1 must be zero. Also $dw/d\zeta$ is real because w traverses the walls of the guide of Fig. 2 as ζ moves along the real axis in the ζ plane. Hence E_1 is real. As ζ goes from $1 - \epsilon$ to $1 + \epsilon$ around a small circular indentation above $\zeta = 1$, w changes from ∞ to $\infty + i\pi$. Thus

$$i\pi = E_1 2^{-1}(-i\pi) \quad \text{or} \quad E_1 = -2 \qquad \text{(A1-4)}$$

When (A1-3) is integrated, (A1-4) inserted, and the result solved for ζ we obtain

$$\zeta = \tanh w/2 \qquad \text{(A1-5)}$$

The function we require is obtained by differentiating (A1-1) and (A1-3):

$$
\begin{aligned}
f'(v + i\theta) &= f'(w) = \frac{dz}{dw} = \frac{dz}{d\zeta} \Big/ \frac{dw}{d\zeta} \\
&= E(\zeta^2 - h^2)^{-\alpha}(\zeta^2 - 1)^{-1}E_1^{-1}(\zeta^2 - 1) \\
&= (1 - h^2)^{\alpha}(\zeta^2 - h^2)^{-\alpha}b/\pi \\
&= \frac{b}{\pi}\left[\frac{ch^2 w/2}{sh\tfrac{1}{2}(w - t)\ sh\tfrac{1}{2}(w + t)}\right]^{\alpha} \\
&= \frac{b}{\pi}\left[\frac{(e^w + 1)^2}{(e^{w-t} - 1)(e^{w+t} - 1)}\right]^{\alpha}
\end{aligned}
\qquad \text{(A1-6)}
$$

where

$$h = \tanh t/2 \qquad \text{(A1-7)}$$

For a 90 degree corner $\alpha = 1/4$ and

$$\frac{\ell}{2b} = 2^{1/2}(1 - d/d_0) \qquad \text{(A1-8)}$$

where, in Fig. 1, $d = |z_4 - z_0|$ and $d_0 = |z_4 - z_6|$. In order to obtain the relation between t, defined by (A1-7), and d/d_0 various values of h^2 were picked and the corresponding values of t and d/d_0 (using (A1-2) and (A1-8)) computed. Representative values are given in the following table.

d/d_0	t	d/d_0	t
1.000	0	.5796	1.2302
.9041	.0633	.5385	1.4910
.8565	.1417	.5000	1.7594
.8292	.2007	.4615	2.0634
.7745	.3500	.3727	2.8872
.7196	.5421	.2804	4.0096
.6919	.6549	.1708	5.987
.6273	.9624	.0959	8.294

APPENDIX II

Integrals Associated with Corners of Small Angle

The derivation of the integrals encountered in Sections 7 and 8 will be outlined here. The first ones are

$$I_1(u, v) = \frac{1}{\pi}\int_0^{\pi} \log(ch\ u - \cos\theta)\log(ch\ v - \cos\theta)\, d\theta$$

$$I_2(u, v) = \frac{1}{\pi} \int_0^\pi \log(ch\, u - \cos\theta) \log(ch\, v + \cos\theta)\, d\theta$$

$$I_3(u, v) = \frac{2}{\pi} \int_0^\pi \cos 2\theta \log(ch\, u - \cos\theta) \log(ch\, v - \cos\theta)\, d\theta \tag{A2-1}$$

$$I_4(u, v) = \frac{2}{\pi} \int_0^\pi \cos 2\theta \log(ch\, u - \cos\theta) \log(ch\, v + \cos\theta)\, d\theta$$

Assuming u and v to be positive and using the expansions

$$\log(ch\, u - \cos\theta) = \log(e^u/2) - 2\sum_{n=1}^{\infty} n^{-1}e^{-nu}\cos n\theta$$

$$\log(ch\, u + \cos\theta) = \log(e^u/2) - 2\sum_{n=1}^{\infty} (-)^n n^{-1}e^{-nu}\cos n\theta \tag{A2-2}$$

leads to

$$I_1(u, v) = \log(e^u/2)\log(e^v/2) + 2\sum_{n=1}^{\infty} n^{-2}e^{-nu-nv}$$

$$I_2(u, v) = \log(e^u/2)\log(e^v/2) + 2\sum_{n=1}^{\infty} (-)^n n^{-2}e^{-nu-nv}$$

$$I_3(u, v) = -e^{-2u}\log(e^v/2) - e^{-2v}\log(e^u/2) + 2e^{-u-v} + 2\sum_{n=1}^{\infty} n^{-1}(n+2)^{-1}e^{-nu-nv}(e^{-2u} + e^{-2v}) \tag{A2-3}$$

$$I_4(u, v) = -e^{-2u}\log(e^v/2) - e^{-2v}\log(e^u/2) - 2e^{-v-u} + 2\sum_{n=1}^{\infty} (-)^n n^{-1}(n+2)^{-1}e^{-nu-nv}(e^{-2u} + e^{-2v})$$

When u or v are negative they are to be replaced by their absolute values in the expressions (A2-2, 3).

Now we consider the double integral

$$J(\mu, m, c, \delta; r, s) = \int_{-\infty}^{+\infty} dv_0 \int_{-\infty}^{+\infty} dv \cdot \exp\left[-\mu\,|v_0 - r| - m\,|v - s| - ic(v + v_0) - \delta\,|v - v_0|\right] \tag{A2-4}$$

in which μ, m, c, δ are real and positive and r and s are real. The double integral may be reduced to a single integral by substituting

$$e^{-\delta|v-v_0|} = \frac{\delta}{\pi} \int_{-\infty}^{+\infty} (\delta^2 + x^2)^{-1} e^{ix(v-v_0)}\, dx, \tag{A2-5}$$

interchanging the order of integration, and integrating with respect to v and v_0. Assuming $s - r \geq 0$, the integral is then evaluated by closing the path of integration by an infinite semicircle in the upper half plane and calculating the residues of the integrand at the poles $i\delta$, $c + im$, $-c + i\mu$:

$$\begin{aligned} J(\mu, m, c, \delta; r, s) &= \int_{-\infty}^{\infty} \frac{4\delta\mu m e^{ix(s-r)-ic(r+s)}\,dx}{\pi(\delta^2 + x^2)[\mu^2 + (x+c)^2][m^2 + (x-c)^2]} \\ &= 4\delta\mu m \left[\frac{e^{-(\delta+ic)s+(\delta-ic)r}}{\delta[\mu^2 + (c+i\delta)^2][m^2 + (c-i\delta)^2]} \right. \\ &\quad + \frac{e^{-ms+(m-2ic)r}}{m[\delta^2 + (c+im)^2][\mu^2 + (2c+im)^2]} \\ &\quad \left. + \frac{e^{\mu r-(\mu+2ic)s}}{\mu[\delta^2 + (c-i\mu)^2][m^2 + (2c-i\mu)^2]} \right] \end{aligned} \tag{A2-6}$$

Substituting special values for the parameters gives the results required in the text. Thus,

$$\begin{aligned} J(m, m, k, \gamma; t, t) &= e^{-2ikt} J(m, m, k, \gamma; 0, 0) \\ J(m, m, k, \gamma; -t, t) &= e^{2ikt}\, J(m, m, k, \gamma; 0, 2t) \\ J(m, m, k, \gamma; -t, 0) &= e^{2ikt}\, J(m, m, k, \gamma; 0, t) \\ J(m, m, c, \delta; 0, 0) &= \frac{2m(\delta + 2m)}{(c^2 + m^2)[c^2 + (m+\delta)^2]} \end{aligned} \tag{A2-7}$$

which hold irrespective of any relations between the parameters. The derivation of the last result is simplified by setting $\alpha = c + im$, $\bar{\alpha} = c - im$ and factoring the denominators in (A2-6) so as to obtain terms of the form $\bar{\alpha} \pm i\delta$, $\alpha \pm i\delta$.

When $\gamma^2 = m^2 - k^2$ considerable simplification is possible and we obtain

$$\begin{aligned} J(m, m, k, \gamma; 0, 0) &= \frac{\gamma}{k^2}\left[\frac{1}{\gamma} - \frac{m}{m^2 + k^2}\right] \\ J(m, m, k; \gamma, 0, t) &= \frac{\gamma e^{-ikt}}{k^2}\left[\frac{e^{-\gamma t}}{\gamma} - \frac{e^{-mt}(m\cos kt - k\sin kt)}{m^2 + k^2}\right] \end{aligned} \tag{A2-8}$$

If we put $\mu = n - 1$, $m = n + 1$, and set $\delta^2 = n^2 - 1 - c^2 = n^2 - \kappa^2$ where $\kappa^2 = 1 + c^2$, (A2-6) yields, after some reduction,

$$\begin{aligned} J(n-1, n+1, c, \delta; 0, 0) &= \frac{n^2 - 1}{(cn + i\delta)^2} + \frac{(n-1)\delta}{2(n+1)(1-ic)^2(n-ic)} \\ &\quad + \frac{(n+1)\delta}{2(n-1)(1-ic)^2(n+ic)} \\ &= \frac{c^2n^2 - \delta^2}{\kappa^4(n^2-1)} + \frac{n\delta[2(n^2 + c^2) - \kappa^2(n^2-1)]}{\kappa^4(n^2-1)(n^2+c^2)} \end{aligned} \tag{A2-9}$$

The form of the final expression has been chosen so as to be suited to the use we shall make of it.

Another double integral which appears in our work is

$$I(k, \gamma) = \int_{-\infty}^{+\infty} dv_0 \, a(v_0) \cdot \int_{-\infty}^{\infty} dv \, a(v) \exp[-ik(v + v_0) - \gamma \mid v - v_0 \mid] \tag{A2-10}$$

where $a(v)$ is an even function of v and is such that all of the integrals encountered converge. We begin our transformation by dividing the interval of integration $(-\infty, \infty)$ for v_0 into $(-\infty, 0)$ and $(0, \infty)$. Making the change of variable $v_0 = -v_0'$, $v = -v'$ in the first interval, dropping the primes and using $a(-v) = a(v)$ leads to

$$I(k, \gamma) = 2 \int_0^\infty dv_0 \, a(v_0) \int_{-\infty}^{\infty} dv \, a(v) e^{-\gamma |v - v_0|} \cos k(v + v_0) \tag{A2-11}$$

We now split the interval of integration of v in (A2-11) into the intervals $(-\infty, 0)$, $(0, v_0)$, (v_0, ∞). In $(-\infty, 0)$ we change the variable from v to $-v'$, drop the prime, and use $a(-v) = a(v)$. By paying attention to the sign of $v - v_0$ we may remove the absolute value sign. By changing the order of integration in the double integral arising from the third interval (in which $0 \leq v_0 \leq \infty$, $v_0 \leq v \leq \infty$) we may show that it is equal to the double integral arising from the second interval. Thus

$$\begin{aligned} I(k, \gamma) &= 2 \int_0^\infty dv_0 \, a(v_0) \int_0^\infty dv \, a(v) e^{-\gamma v - \gamma v_0} \cos k(v_0 - v) \\ &+ 4 \int_0^\infty dv_0 \, a(v_0) \int_0^{v_0} dv \, a(v) e^{-\gamma v_0 + \gamma v} \cos k(v_0 + v) \end{aligned} \tag{A2-12}$$

When $a(v)$, γ and k are real we may write (A2-12) as

$$\begin{aligned} I(k, \gamma) &= 2 \left| \int_0^\infty dv \, a(v) e^{-\gamma v - ikv} \right|^2 \\ &+ 4 \text{ Real} \int_0^\infty dv_0 \, a(v_0) e^{-\gamma v_0 + ikv_0} \int_0^{v_0} dv \, a(v) \, e^{\gamma v + ikv} \end{aligned} \tag{A2-13}$$

and when $\gamma = ik$ we have

$$\begin{aligned} I(k, ik) &= 2 \int_0^\infty dv_0 \, a(v_0) \int_0^\infty dv \, a(v) e^{-2ikv} \\ &+ 2 \int_0^\infty dv_0 \, a(v_0) \int_0^{v_0} dv \, a(v) [e^{2ikv} + e^{-2ikv_0}]. \end{aligned} \tag{A2-14}$$

APPENDIX III

Integral Equation When Guides Entering and Leaving Irregularity Are of Different Sizes

Here we shall indicate how the integral equation method may be extended to cover the case mentioned in the above title. It is supposed that only the dominant mode is propagated freely in both guides.

E in Plane of Irregularity

Let the notation for the guide carrying the incident wave be the same as for the E-corner. b denotes the narrow dimension of the guide and the quantities k and γ_m are given by (2-1). Both guides have the same wide

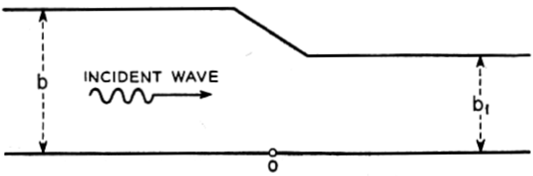

Fig. 3

dimension a. The narrow dimension of the guide shown on the right of Fig. 3 is b_1. We introduce the new quantity

$$k_1 = [(2b_1/\lambda_0)^2 - (b_1/a)^2]^{1/2} \tag{A3-1}$$

to correspond to k. Since, by assumption, only the dominant mode is freely propagated in both guides both k and k_1 are real positive quantities less than unity.

Let $z = f(w)$ carry the system of Fig. 3 into a straight guide of width π in the $w = v + i\theta$ plane (see Fig. 2), and let $g(v, \theta)$ be defined by

$$1 + g(v, \theta) = |f'(w)|^2.$$

The behavior of $g(v, \theta)$ at infinity is shown by the table

v	dz/dw	$g(v, \theta)$
$-\infty$	b/π	0
$+\infty$	b_1/π	$k_1^2k^{-2} - 1$

where $b_1/b = k_1/k$ has been used. It is convenient to introduce the approximation $\hat{g}(v)$ to $g(v, \theta)$. $\hat{g}(v)$ may be chosen at our convenience subject only to the conditions that it be differentiable, $\hat{g}(-\infty) = 0$, and $\hat{g}(\infty) = k_1^2k^{-2} - 1$.

When we define G by equation (3-3) so that, as before, it is the Green's function corresponding to a guide of width b, we may use equation (3-4) to derive the new integral equation

$$Q(v_0, \theta_0) = e^{-ikv_0} + \frac{k^2}{2\pi} \int_{-\infty}^{\infty} dv \int_0^{\pi} d\theta \cdot \{g(v, \theta)Q(v, \theta) - \hat{g}(v)T_E e^{-ik_1 v}\}G(v_0, \theta_0; v, \theta) + T_E F(v_0) \quad \text{(A3-2)}$$

in which

$$\begin{aligned} F(v_0) &= e^{-ik_1 v_0}\hat{g}(v_0)/\hat{g}(\infty) - e^{-ikv_0}N^-(v_0) - e^{ikv_0}N^+(v_0) \\ N^-(v_0) &= 2^{-1}k(k_1 - k)^{-1} \int_{-\infty}^{v_0} \hat{g}'(v)e^{-i(k_1-k)v}\, dv \\ N^+(v_0) &= 2^{-1}k(k_1 + k)^{-1} \int_{v_0}^{\infty} \hat{g}'(v)e^{-i(k_1+k)v}\, dv \end{aligned} \quad \text{(A3-3)}$$

Here $\hat{g}'(v)$ denotes $d\hat{g}(v)/dv$. Equation (A3-2) and

$$\operatorname*{Limit}_{v\to\infty} Q(v, \theta) = T_E e^{-ik_1 v} \quad \text{(A3-4)}$$

are to be solved for the unknown function $Q(v, \theta)$ and the unknown quantity T_E. The method of successive approximations may be used in somewhat the same fashion as in the simpler case but we shall not give a general discussion.

The first approximations are found to be

$$T_E^{(1)} = 1/N^-(\infty), \qquad R_E^{(1)} = -N^+(-\infty)/N^-(\infty) \quad \text{(A3-5)}$$

where the N's may be obtained by setting $v_0 = \pm\infty$ in equations (A3-3).

One of the simplest choices for $\hat{g}(v)$ is to let it be zero for negative values of v and to have the value $\hat{g}(\infty) = k_1^2k^{-2} - 1$ for positive values of v. Then

$$T_E^{(1)} = 2k(k_1 + k)^{-1}, \qquad R_E^{(1)} = (k - k_1)(k + k_1)^{-1} \quad \text{(A3-6)}$$

These are quite similar to the corresponding expressions for a transmission line which have been used extensively in wave guide work.

In working with these formulas, when k is small, it is sometimes convenient to use the result

$$\int_{v_1}^{v_2} dv \int_0^{\pi} d\theta\, g(v, \theta) = \pi^2 b^{-2} \int_{v_1}^{v_2} dv \int_0^{\pi} d\theta\, |f'(w)|^2 - (v_2 - v_1)\pi \quad \text{(A3-7)}$$

where the evaluation of the double integral on the right is made easier by the fact that it represents the area in the original guide (in the (x, y) plane) enclosed by the lines corresponding to $v = v_1$ and $v = v_2$. v_2 and v_1 are chosen to be moderately large positive and negative numbers, respectively. It turns out that, when k_1 and k are very small, this is related to the "excess capacity" localized at the irregularity whose effect must be added to that of the mismatch, indicated by (A3-6).

When the entering and leaving guides are of the same size it is still possible to use the formulas of this appendix. $N^-(v_0)$ may be replaced by an expression which now has for its limiting value

$$N^-(\infty) = 1 + i(k/2) \int_{-\infty}^{\infty} \hat{g}(v)\, dv \tag{A3-8}$$

H in Plane of Irregularity

Let the figure corresponding to the irregularity be Fig. 3 with b and b_1 replaced by a and a_1, respectively. In addition to the quantities c and κ defined by equations (5-2) we define

$$\kappa_1 = 2a_1/\lambda_0\,, \qquad c_1 = (\kappa_1^2 - 1)^{1/2} \tag{A3-9}$$

where we assume κ and κ_1 to lie between 1 and 2. At $v = -\infty$ $P(v, \theta)$ still consists of the unit incident wave plus the reflected wave given by the first of equations (5-4) and $g(v, \theta)$ is still zero. However, now, at $v = \infty$,

$$\begin{aligned} P(v, \theta) &= T_H e^{-ic_1 v} \sin\theta \\ \hat{g}(\infty) &= \kappa_1^2\kappa^{-2} - 1 = \kappa^{-2}(c_1^2 - c^2) \end{aligned} \tag{A3-10}$$

The integral equation for $P(v, \theta)$ and T_H is

$$\begin{aligned} P(v_0, \theta_0) = e^{-icv_0} \sin\theta_0 + \frac{\kappa^2}{2\pi} \int_{-\infty}^{\infty} dv \int_0^{\pi} \\ \cdot d\theta\{g(v, \theta)P(v, \theta) - \hat{g}(v)T_H e^{-ic_1 v} \sin\theta\}G(v_0, \theta_0; v, \theta) \\ + T_H \sin\theta_0 F_H(v_0) \end{aligned} \tag{A3-11}$$

in which

$$\begin{aligned} F_H(v_0) &= e^{-ic_1 v_0} \hat{g}(v_0)/\hat{g}(\infty) - e^{-icv_0} M^-(v_0) - e^{icv_0} M^+(v_0) \\ M^-(v_0) &= \kappa^2(2c)^{-1}(c_1 - c)^{-1} \int_{-\infty}^{v_0} \hat{g}'(v)e^{-i(c_1-c)v}\, dv \\ M^+(v_0) &= \kappa^2(2c)^{-1}(c + c_1)^{-1} \int_{v_0}^{\infty} \hat{g}'(v)e^{-i(c_1+c)v}\, dv \end{aligned} \tag{A3-12}$$

First approximations are

$$T_H^{(1)} = 1/M^-(\infty), \qquad R_H^{(1)} = -M^+(-\infty)/M^-(\infty) \tag{A3-13}$$

which, when we choose $\mathfrak{g}(v)$ to be zero for $v < 0$ and $\kappa_1^2\kappa^{-2} - 1$ for $v > 0$, become

$$T_H^{(1)} = 2c(c_1 + c)^{-1}, \qquad R_H^{(1)} = (c - c_1)(c_1 + c)^{-1} \tag{A3-14}$$

which again agrees with results obtained from transmission line considerations. When the entering and leaving guides are the same size we may use

$$M^-(\infty) = 1 + i\kappa^2(2c)^{-1} \int_{-\infty}^{\infty} \mathfrak{g}(v)\, dv \tag{A3-15}$$

It seems difficult to give any general rules for the choice of $\mathfrak{g}(v)$. Since for R_H and T_H, the factor $\sin\theta$ reduces the effect of the singularities on the walls of the transformed guide, the choice $\mathfrak{g}(v) = g(v, \pi/2)$ suggests itself. The factor $\sin\theta$ is not present in the formulas for R_E and T_E and regions near the walls are more important. In this case the selection

$$\mathfrak{g}(v) = \pi^{-1} \int_0^{\pi} g(v, \theta)\, d\theta$$

may be useful, especially since it allows us to use the result (A3-7) when k and k_1 become small.

APPENDIX IV

Variational Expressions for Reflection Coefficients

The reflection coefficients are proportional to the stationary values of certain forms associated with the integral equations. In order to obtain these forms we proceed as follows. It is readily seen that the values of x_1 and x_2 which satisfy the symmetrical set of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{12}x_1 + a_{22}x_2 &= b_2 \end{aligned} \tag{A4-1}$$

are the ones which make

$$J = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 - 2b_1x_1 - 2b_2x_2 \tag{A4-2}$$

stationary when x_1 and x_2 are given small arbitrary increments. This stationary value of J is

$$J_s = -b_1x_1 - b_2x_2$$

If we take the integral equation to be the analogue of the set of linear equations, the reflection coefficient turns out to be proportional to J_s. In order to set down the actual expressions it is convenient to write r for

(v, θ) and dS for the element of area $dvd\theta$ so that the integral equation (3-5) for $Q(v, \theta)$ may be written as

$$Q(r_0) = e^{-ikv_0} + k^2(2\pi)^{-1} \int g(r)Q(r)G(r_0, r)\, dS \qquad \text{(A4-3)}$$

where the integration extends over the interior of the guide and $G(r_0, r)$ denotes the Green's function (3-3).

If the number of equations in the set (A4-1) were increased from two to a large number N, the set of x's would correspond, say, to the values of $Q(r)$ or of $g(r)Q(r)$, and the b's would correspond to the values of $\exp(-ikv_0)$. In any event, we take the analogue of J to be

$$J_E = \int g(r)Q(r)[Q(r) - 2e^{-ikv}]\, dS - k^2(2\pi)^{-1} \int\int g(r)Q(r)g(r_0)Q(r_0)G(r_0, r)\, dS_0\, dS \qquad \text{(A4-4)}$$

where the subscript E indicates that we are dealing with an electric corner. It may be verified,* by giving $Q(r)$ a small variation $\delta Q(r)$, that the function $Q(r)$ which makes J_E stationary is the one which satisfies the integral equation (A4-3). Furthermore, when we assume $Q(r)$ to satisfy the integral equation, the expression for J_E reduces to an integral which is proportional to the integral (3-6) for the reflection coefficient R_E. More precisely, R_E is given by

$$R_E = \frac{ik}{2\pi} [\text{Stationary value of } J_E] \qquad \text{(A4-5)}$$

It follows that if, by some means, we have obtained a fairly good approximation to Q, we may obtain a better approximation to R_E by computing J_E and using the formula

$$R_E \approx ik(2\pi)^{-1}J_E$$

When we use the first approximation $\exp(-ikv)$ for Q to compute J_E it turns out that the above formula gives the third approximation, $R_E^{(3)}$, to the reflection coefficient.

The magnetic corner may be treated in much the same way. The integral equation (5-6) for $P(v, \theta)$ becomes, in the notation of this appendix,

$$P(r_0) = e^{-icv_0} \sin \theta_0 + \kappa^2(2\pi)^{-1} \int g(r)P(r)G(r_0, r)\, dS \qquad \text{(A4-6)}$$

in which the v in $dS = dvd\theta$ is integrated from $-\infty$ to $+\infty$ and θ from 0 to π,

* See Courant and Hilbert, Methoden der Mathematischen Physik, Julius Springer, Berlin (1931), page 176, where a similar problem is treated.

as before, and $G(r_0, r)$ now denotes the Green's function (5-5). We define J_H by

$$J_H = \int g(r)P(r)[P(r) - 2e^{-icv} \sin \theta]dS$$
$$- \kappa^2(2\pi)^{-1} \iint g(r)P(r)g(r_0)P(r_0)G(r_0, r)\, dS_0\, dS. \quad \text{(A4-7)}$$

J_H is stationary with respect to small variations in $P(r)$ when $P(r)$ satisfies the integral equation (A4-6). Furthermore, from the integral (5-7) for R_H,

$$R_H = i\kappa^2(\pi c)^{-1} [\text{Stationary value of } J_H] \quad \text{(A4-8)}$$

which may be used in the same way as equation (A4-5) for R_E.

J. Schwinger has used variational methods with considerable success to deal with obstacles in wave guides.* However, his variational equations differ somewhat from those given here. Some light on the relation between Schwinger's equations and the present one may be obtained by returning to the simple algebraic equations (A4-1) and (A4-2). A rough analogue of the expression required to be stationary in Schwinger's theory is

$$(a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2)/(b_1x_1 + b_2x_2)^2 \quad \text{(A4-9)}$$

The essential point here is that the stationary value of the expression corresponding to (A4-9) gives the value of an impedance or combination of impedances appearing in some equivalent circuit. Expression (A4-9) may be obtained by expressing J, defined by (A4-2), as a function of x_1 and $y = x_2/x_1$. J is still to be made stationary but now it is a function of x_1 and y. Solving $\partial J/\partial x_1 = 0$ for x_1 and setting this value of x_1 in J gives the following function of y

$$-(b_1 + b_2y)^2 (a_{11} + 2a_{12}y + a_{22}y^2)^{-1},$$

which is the stationary value of J with respect to variations in x_1 when y is held constant. This function is still required to be stationary with respect to y. The same is true of its reciprocal which becomes (A4-9) when both numerator and denominator are multiplied by x_1^2 and the definition of y used. When (A4-1) is replaced by a larger number of equations similar considerations lead to a generalized form of (A4-9). The expression required to be stationary by Schwinger is obtained when the sums in the generalized form are replaced by integrals.

* An account of the method together with applications is given in "Notes on Lectures by Julian Schwinger: Discontinuities in Waveguides" by David S. Saxon. An account is also given by John W. Miles.[11]

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