A Set of Second-Order Differential Equations Associated with Reflections in Rectangular Wave Guides—Application to Guide Connected to Horn*

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In dealing with corners and similar irregularities in rectangular wave guides it is sometimes helpful to transform the system, conformally, into a straight guide. Propagation in the straight guide may then be studied by an integral equation method, as is done in a companion paper, or by a more general method based upon a certain set of ordinary differential equations. Here the second method is developed and applied to determine the reflection produced at the junction of a straight guide and a sectoral horn—a problem the first method is unable to handle. The *WKB* approximation for a single second-order differential equation is extended to a set of equations and approximate expressions for the reflection coefficient are derived.

IN A companion paper[1] the disturbance produced by a corner in a rectangular wave guide is examined by transforming the system, conformally, into a straight guide. Although the medium in the straight guide is no longer uniform, an integral equation may be set up and approximate solutions obtained.

In that paper the wave guide is assumed to have the same cross-section at $+\infty$ as at $-\infty$. When this is not so, a conformal transformation may still be used to transform the system into a straight guide provided one dimension of the original cross-section is constant. However, now some advantage appears to be gained by replacing the integral equation by a set of differential equations. Since two cases appear, corresponding to E and H corners, there are two sets of equations to be considered.

These two sets of equations are studied in the present paper. After their derivation in Sections 1 and 2 several remarks are made in Section 3 concerning their solution, special emphasis being laid on the problem of determining the reflection coefficient. In the remainder of the paper the general theory is applied to a system formed by joining a rectangular wave guide to a horn (with plane sides) flared in one direction. The reflection coefficients for sectoral horns flared in the planes of the electric and magnetic intensity, respectively, are given approximately by equations (6-1) and (7-1). These approximations assume the angle of flare to be small so that, as it turns out, only the first equations of the respective sets need be considered.

As was mentioned in the companion paper, Robert Piloty has recently made use of conformal transformations in wave guide problems. In his

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[1]See list of references at end of paper.

method the propagation function $g(v, \theta)$ is derived graphically from the geometry of the wave guide irregularities and the result used in one or the other of two sets of differential equations which are equivalent to those derived below. Piloty's work is scheduled to appear soon in the *Zeitschrift für angewandte Physik* under the title "Ausbreitung el.-magn. Wellen in inhomogenen Rechteckrohren."

1. *Differential Equations when Electric Vector is in (x, y) Plane*

The partial differential equation to be solved is, from equation (2-3) of the companion paper[1],

$$\frac{\partial^2 Q}{\partial v^2} + \frac{\partial^2 Q}{\partial \theta^2} + [1 + g(v, \theta)]k^2 Q = 0 \tag{1-1}$$

where

$$\frac{\partial Q}{\partial \theta} = 0 \text{ at } \theta = 0 \text{ and } \theta = \pi$$

$$1 + g(v, \theta) = 1 + \sum_{n=0}^{\infty} a_n \cos n\theta = | f'(v + i\theta) |^2 \pi^2/b^2 \tag{1-2}$$

$$k = [(2b/\lambda_0)^2 - (b/a)^2]^{\frac{1}{2}}, \qquad \lambda_0 = \text{free space wavelength}$$

In (1-2), $z = x + iy = f(v + i\theta)$ is the transformation which carries the wave guide system in the (x, y) plane into the straight guide of width $\theta = \pi$ in the (v, θ) plane. For the sake of simplicity we shall always assume that far to the left the system becomes a straight wave guide of dimensions a, b ($b < a$) such that only the dominant mode is propagated without attenuation. This insures that the a_n's (which are functions of v) will approach zero as $v \to -\infty$. The dimension (of our system) normal to the (x, y) plane is a throughout.

Since the normal derivative of Q vanishes on the walls at $\theta = 0$ and $\theta = \pi$ we assume

$$Q = F_0 + F_1 \cos \theta + F_2 \cos 2\theta + \cdots, \tag{1-3}$$

where $F_1, F_2, \cdots$ are functions of v, and substitute it together with the Fourier series (1-2) for $1 + g(v, \theta)$ in (1-1).

The equations obtained by setting the coefficients of the resulting cosine series to zero are

$$F_0'' + (1 + a_0)k^2 F_0 + \frac{k^2}{2} \sum_{n=1}^{\infty} a_n F_n = 0 \tag{1-4}$$

$$F_m'' + [(1 + a_0 + a_{2m}/2)k^2 - m^2]F_m + a_m k^2 F_0 + \frac{k^2}{2} \sum_{n=1}^{\infty}{}' (a_{|n-m|} + a_{n+m})F_n = 0 \tag{1-5}$$

where $m = 1, 2, 3, \cdots$, $F_m'' = d^2F_m/dv^2$, and the prime on Σ indicates that the term $n = m$ is to be omitted. In grouping the terms we have assumed that F_0 is the major part of Q.

The principal problem is to solve equations (1-4) and (1-5) when the fundamental mode F_0 is of the form

$$\begin{aligned} F_0 &= e^{-ikv} + R_E e^{ikv}, & v \to -\infty \\ F_0 &= T_E(v), & v \to +\infty \end{aligned} \tag{1-6}$$

in which R_E is a constant and $T_E(v)$ represents a wave traveling towards $v = \infty$. At $v = \pm\infty$ $F_1, F_2, \cdots$ have the form of waves traveling (or being attenuated) away from the region around $v = 0$. As before, we shall be mainly interested in determining the reflection coefficient R.

It is assumed that only the dominant mode is propagated without attenuation in the straight wave guide far to the left and hence $F_1, F_2, \cdots$ all become zero as $v \to -\infty$.

2. *Differential Equations when Magnetic Vector is in (x, y) Plane*

The partial differential equation is now given by equation (5-1) of the companion paper[1]

$$\frac{\partial^2 P}{\partial v^2} + \frac{\partial^2 P}{\partial \theta^2} + [1 + g(v, \theta)]\kappa^2 P = 0 \tag{2-1}$$

where the dimension of the system normal to the (x, y) plane is now b, a is the dimension (in the (x, y) plane) of the straight guide at the far left and

$$P = 0 \text{ at } \theta = 0 \text{ and } \theta = \pi$$

$$1 + g(v, \theta) = 1 + \sum_{n=1}^{\infty} a_n \cos n\theta \tag{2-2}$$

$$\kappa = 2a/\lambda_0, \ \lambda_0 = \text{free space wavelength}$$

$$c = (\kappa^2 - 1)^{1/2}$$

Since $P = 0$ at $\theta = 0$ and $\theta = \pi$ we assume

$$P = \sum_{n=1}^{\infty} F_n \sin n\theta \tag{2-3}$$

where the F's are functions of v to be determined by the equations

$$F_1'' + [\kappa^2(1 + a_0 - a_2/2) - 1]F_1 + \frac{\kappa^2}{2} \sum_{n=2}^{\infty} (a_{n-1} - a_{n+1})F_n = 0 \tag{2-4}$$

$$\begin{aligned} F_m'' + [\kappa^2(1 + a_0 - a_{2m}/2) - m^2]F_m + \frac{\kappa^2}{2}(a_{m-1} - a_{m+1})F_1 \\ + \frac{\kappa^2}{2} \sum_{n=2}^{\infty}{}' (a_{|m-n|} - a_{m+n})F_n = 0 \end{aligned} \tag{2-5}$$

in which $m = 2, 3, 4, \cdots$ and the primes on F_m and $\sum$ have the same significance as in (1-4) and (1-5).

The principal problem here is to solve equations (2-4) and (2-5) simultaneously subject to

$$\begin{aligned} F_1 &= e^{-icv} + R_H e^{icv}, \qquad v \to -\infty \\ F_1 &= T_H(v), \qquad v \to +\infty \end{aligned} \tag{2-6}$$

which again corresponds to a unit wave in the dominant mode incident from the left. $T_H(v)$ and the remaining F's correspond to outward traveling waves as before. $F_2, F_3, \cdots$ all approach zero as $v \to -\infty$.

3. *Remarks on Solving the Equations of Sections 1 and 2 for the Reflection Coefficient*

Suppose that we have a system in which the wave propagation is governed by the single differential equation

$$\frac{d^2 y}{dv^2} - h^2 y = 0 \tag{3-1}$$

where $h \equiv h(v)$ is a positive imaginary function of v, twice differentiable and such that $h \to ic$, c being a constant; as $v \to -\infty$. We desire the solution of (3-1) which, together with its first derivative, is continuous everywhere and at $\pm\infty$ satisfies the conditions

$$y = e^{-icv} + Re^{icv}, \qquad v \to -\infty \tag{3-2}$$

$$y' + (h + h'/(2h))y \to 0, \; v \to \infty \tag{3-3}$$

The constant R (the reflection coefficient) is to be determined. Condition (3-3), in which the primes denote differentiation with respect to v, is suggested by the fact that we want y to represent a wave traveling in the positive v direction (the factor exp $(i\omega t)$ is suppressed). In writing (3-3) we have assumed that h is such that for large values of v the two solutions of (3-1) are asymptotically proportional to*

$$y = h^{-\frac{1}{2}} e^{\pm\xi}, \tag{3-4}$$

$$\xi \equiv \xi(v) = icv + \int_{-\infty}^{v} (h - ic)\, dv. \tag{3-5}$$

Physical considerations suggest that solutions satisfying (3-2) and (3-3) exist in most cases of practical importance. However, if the function h is picked arbitrarily the corresponding solutions may be incapable of satisfying

* S. A. Schelkunoff[2] mentions that this approximation, sometimes designated by "WKB", goes back to Liouville. The ideas we shall use are quite similar to those in Schelkunoff's paper.

the conditions. For example, if $h = ic/(1 + \exp v)$ then $h \rightarrow ic \exp(-v)$ as $v \rightarrow \infty$, and the solutions of (3-1) behave like Bessel functions of order zero and argument $c \exp(-v)$. It may be verified that these solutions do not satisfy (3-3). Again, condition (3-3) may be satisfied without y having much resemblance to an outgoing wave at $v = \infty$. Thus if $h \rightarrow ia/v$ as $v \rightarrow \infty$, y increases like v^n where $n^2 - n - a^2 = 0$. When $0 < a < 1/2$ both values of n lie between 0 and 1, and both solutions satisfy (3-3). Despite these shortcomings it still seems best to retain (3-3) to specify the behavior of y at $v = \infty$.

It should be mentioned that P. S. Epstein[3] has obtained the reflected wave by transforming the hypergeometric differential equation into the form (3-1). This method has been extended by K. Rawer[4] who gives a number of references in which the approximation (3-4) is used to study propagation in a medium having a variable dielectric "constant". An interesting paper on the general subject of reflection in non-uniform transmission lines has been written by L. R. Walker and N. Wax[5].

1. When most of the reflection occurs in a short interval, say near $v = 0$' R may be obtained by numerical integration of (3-1). One method is to start at $v = 0$ with the initial conditions $y = 1$, $y' = 0$ and work outwards in both directions. Let $Y_a(v)$ denote this solution and $Y_b(v)$ the solution obtained by starting with $y = 0$, $y' = 1$. The general solution is

$$y = C_1 Y_a(v) + C_2 Y_b(v). \tag{3-6}$$

C_1 and C_2 are to be determined by the conditions

$$y = (\text{constant})\, h^{-1/2} e^{-\xi} \quad , \quad v > v_2 \tag{3-7}$$

$$y = (ic/h)^{1/2}[e^{-\xi} + Re^{\xi}] \quad , \quad v < v_1 \tag{3-8}$$

where v_1 and v_2 are large negative and positive values, respectively, of v. These conditions lead to equations for C_1, C_2, R:

$$\begin{aligned} &[y' + \theta^{+} y]_{v=v_2} = 0 \\ &[y' - \theta^{-} y + 2(ich)^{1/2}\, e^{-\xi}]_{v=v_1} = 0 \\ &[y' + \theta^{+} y - 2(ich)^{1/2} Re^{\xi}]_{v=v_1} = 0 \end{aligned} \tag{3-9}$$

in which ξ is given by (3-5) and

$$\theta^{\pm} = h \pm h'/(2h). \tag{3-10}$$

The required value of R is obtained by letting $v_1 \rightarrow -\infty$, $v_2 \rightarrow \infty$ in the expressions, which follow from (3-9),

$$\gamma = C_2/C_1 = -[(Y_a' + \theta^+ Y_a)/(Y_b' + \theta^+ Y_b)]_{v=v_2}$$

$$\Gamma = [y'/y]_{v=v_1} = [(Y_a' + \gamma Y_b')/(Y_a + \gamma Y_b)]_{v=v_1} \tag{3-11}$$

$$R = [(\theta^+ + \Gamma)/(\theta^- - \Gamma)]_{v=v_1} \exp\left[-2icv_1 - 2\int_{-\infty}^{v_1} (h - ic)\, dv\right]$$

where the arguments of $Y_a(v)$ and $Y_b(v)$ have been omitted for brevity.

If h should change from a positive imaginary quantity to a positive real quantity in (v_1, v_2) and remain greater than some fixed positive number for $v > v_2$ it may be shown that $|R| = 1$ (γ and Γ are real and $\text{Im}\,\theta^+ = \text{Im}\,\theta^-$, $\text{Real}\,\theta^+ = -\text{Real}\,\theta^-$ at $v = v_1$). This complete reflection is to be expected from physical consideration.

2. An exact expression for the reflection coefficient which holds when h satisfies the conditions following (3-1) (in particular it must not pass through zero anywhere in $-\infty < v < \infty$) is

$$R = \tfrac{1}{2}(ic)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\xi} y(v) \frac{d^2}{dv^2} h^{-\frac{1}{2}}\, dv \tag{3-12}$$

where ξ is given by (3-5). Before this integral for R may be evaluated $y(v)$, and hence R itself, must be known. Nevertheless, when R is small a useful approximation may be obtained by using the *WKB* approximation

$$y(v) \approx (ic/h)^{1/2} e^{-\xi} \tag{3-13}$$

Thus

$$\begin{aligned} R &\approx \frac{1}{2}\int_{-\infty}^{\infty} e^{-2\xi} h^{-\frac{1}{2}} \frac{d^2}{dv^2} h^{-\frac{1}{2}}\, dv \\ &= \frac{1}{2i}\int_{-\infty}^{\infty} e^{-2\xi}\left[\frac{5}{16}K^{-5/2}\left(\frac{dK}{dv}\right)^2 - \frac{1}{4}K^{-3/2}\frac{d^2K}{dv^2}\right] dv \end{aligned} \tag{3-14}$$

in which $K = -h^2$.

The expression (3-12) for R is obtained by letting $v_0 \to -\infty$ in the integral equation

$$\begin{aligned} y(v_0) &= (ic/h_0)^{\frac{1}{2}} e^{-\xi_0} - \int_{-\infty}^{+\infty} G_a(v_0, v) y(v) h^{\frac{1}{2}} \frac{d^2}{dv^2} h^{-\frac{1}{2}}\, dv, \\ G_a(v_0, v) &= -\tfrac{1}{2} h_0^{-\frac{1}{2}} h^{-\frac{1}{2}} \begin{cases} e^{\xi - \xi_0}, & v < v_0 \\ e^{\xi_0 - \xi}, & v > v_0 \end{cases} \\ \xi_0 - \xi &= \int_v^{v_0} h\, dv, \qquad h_0 \equiv h(v_0), \qquad \xi_0 \equiv \xi(v_0). \end{aligned} \tag{3-15}$$

$G_a(v_0, v)$ is the approximate Green's function suggested by (3-13). The integral equation may be obtained from the differential equation (3-1) and the boundary conditions (3-2) and (3-3) by the one-dimensional analogue of the method used in Section 3 of the companion paper[1]. If we multiply both sides of

$$\frac{d^2 y}{dv^2} - h^2 y = s(v) \tag{3-16}$$

(where $s(v)$ has been added for generality) by $G_a(v_0, v)$, integrate twice by parts over the intervals $(v_1, v_0 - \epsilon)$, $(v_0 + \epsilon, v_2)$ with $\epsilon > 0$ and $v_1 < v_0 < v_2$, and finally let $\epsilon \to 0$ we obtain

$$\begin{aligned} y(v_0) = \int_{v_1}^{v_2} G_a(v_0, v) \left[s(v) - y(v) h^{\frac{1}{2}} \frac{d^2}{dv^2} h^{-\frac{1}{2}} \right] dv \\ + G_a(v_0, v_1)[y' - \theta^- y]_{v=v_1} - G_a(v_0, v_2)[y' + \theta^+ y]_{v=v_2} . \end{aligned} \tag{3-17}$$

Equation (3-15) follows when we put $s(v) = 0$ and let $v_1 \to -\infty$, $v_2 \to \infty$. It will be recognized that (3-17) and (3-15) are closely related to integral equations occurring in the work of R. E. Langer[6] and E. C. Titchmarsh[9].

When h has, for example, one or more simple zeros in $-\infty < v < \infty$ the integral in (3-15) contains a factor which becomes infinite and the integral equation fails. However, we shall not concern ourselves with this case beyond remarking that it involves results obtained by H. Jeffreys[10], Langer[7], Furry[11] and others.

3. So far we have been considering the solution of only one equation whereas we really require the solution of a set of equations. If it is apparent that most of the disturbance is given by the first equation of the set it may be possible to proceed by successive approximations, each of the remaining equations being of the form (3-16) with $s(v)$ determined by the solution of the first equation.

Another method of dealing with a system of N equations is that of numerical integration. As a contribution towards obtaining the boundary conditions at large positive and negative values of v we shall state a generalized form of the *WKB* solution. Although this solution is related to the general results obtained by Birkhoff[12], Langer[8], and Newell[13] concerning the asymptotic forms assumed by the solutions of a system of ordinary linear differential equations of the first order, it is worth mentioning explicitly.

Let the mth equation of the set be

$$\ddot{y}_m = \sum_{n=1}^{N} A_{mn} y_n, \qquad m = 1, 2, \cdots, N \tag{3-18}$$

where the A_{mn}'s are relatively slowly varying functions of v (see equations

(3-22) for a more precise statement of the assumptions) and the dots denote differentiation with respect to v. We shall reserve primes to denote transposition of matrices. It is supposed that $A_{mn} = A_{nm}$ (equations(2-4) plus (2-5) satisfy this condition and (1-4) plus (1-5) may be made to do so by setting $\tilde{F}_0 = 2^{1/2}F_0$).

The solution of (3-18) is approximately

$$y_m \approx \sum_{\ell=1}^{N} S_{m\ell}[e^{\xi_\ell} d_\ell^- + e^{-\xi_\ell} d_\ell^+] \tag{3-19}$$

where the $d_\ell^\pm$ are the $2N$ constants of integration and

$$\begin{aligned} \varphi_\ell^2 S_{m\ell} &= \sum_{n=1}^{N} A_{mn} S_{n\ell} \\ \varphi_\ell \sum_{n=1}^{N} S_{n\ell}^2 &= 1 \\ \xi_\ell &= \int_{v_{3\ell}}^{v} \varphi_\ell \, dv \end{aligned} \tag{3-20}$$

serve to determine φ_ℓ, ξ_ℓ, and $S_{m\ell}$ (the last to within a plus or minus sign). We assume the N roots $\varphi_1^2, \varphi_2^2, \cdots \varphi_N^2$ of the determinantal equation arising from the first of equations (3-20) to be unequal, and denote by φ_ℓ that square root of φ_ℓ^2 which has a positive real part or, if the real part be zero, which has a positive imaginary part. $v_{3\ell}$ is any convenient constant.

The approximation (3-19) may be obtained by setting the assumed form

$$y_m = g_m e^{\pm\xi}, \qquad \xi = \int_{v_3}^{v} \varphi \, dv$$

in (3-18). The result is a set of N equations of which the mth is

$$\ddot{g}_m \pm 2\dot{g}_m \varphi \pm g_m \dot{\varphi} + g_m \varphi^2 = \sum_n A_{mn} g_n . \tag{3-21}$$

We also assume

$$\begin{aligned} |\dot{\varphi}| \ll |\varphi^2|, \ |\ddot{g}_m| &\ll |\dot{g}_m \varphi| \ll |g_m \varphi^2| \\ g_m &= g_{m0} + g_{m1} + g_{m2} + \cdots \end{aligned} \tag{3-22}$$

where g_{mr} and its first two derivatives satisfy inequalities of the type

$$|g_{m0}| \gg |g_{m1}| \gg |g_{m2}| \cdots$$

The first and second order terms in (3-21) give, respectively,

$$g_{m0} \varphi^2 - \sum_n A_{mn} g_{n0} = 0 \tag{3-23}$$

Although the *WKB* approximation has the same form as (3-30) in the region where v is finite, we regard (3-30) and (3-31) as being the exact limiting forms of y. Hence, g^+ may differ from f^+.

Letting $v_0 \to -\infty$ in (3-29) and comparing the result with (3-30) gives the exact result

$$f^- = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-\Xi} [\ddot{S}' - 2\Phi \dot{S}' - \dot{\Phi} S'] y(v) \, dv \tag{3-32}$$

which leads to an approximation for the reflected wave when $y(v)$ is known approximately.

The integral equation (3-29) may be obtained by premultiplying both sides of $\ddot{y} = Ay$ by the transpose of the approximate Green's matrix

$$G_a(v_0, v) = \begin{cases} -\tfrac{1}{2} S e^{\Xi - \Xi_0} S_0', & v < v_0 \\ -\tfrac{1}{2} S e^{\Xi_0 - \Xi} S_0', & v > v_0 . \end{cases}$$

and integrating by parts twice. It is seen that each column of $G_a(v_0, v)$ is an approximate solution of $\ddot{y} = Ay$, in which the columnar constants of integration are the columns of S_0', and represents a wave traveling away from v_0 in both directions. $G_a(v_0, v)$ is continuous at $v = v_0$ and

$$\left[\frac{\partial}{\partial v} G_a(v_0, v)\right]_{v=v_0+0} - \left[\frac{\partial}{\partial v} G_a(v_0, v)\right]_{v=v_0-0} = S_0 \Phi_0 S_0' = I$$

Thus the nth column of $G_a(v_0, v)$ gives the approximate values of $y_1(v)$, $y_2(v)$, $\cdots$, $y_n(v)$, subject to the conditions that all these and all of their first derivatives are continuous at $v = v_0$ except $\dot{y}_n(v)$ which has the jump $\dot{y}_n(v_0 + 0) - \dot{y}_n(v_0 - 0) = 1$.

The presence of

$$\begin{aligned} 2\Phi \dot{S}' + \dot{\Phi} S' &= \Phi \dot{S}' - \Phi S' \dot{S} S^{-1} \\ &= \Phi(\dot{S}' S - S' \dot{S}) S^{-1} \end{aligned}$$

in (3-29) and (3-32) makes the N variable case somewhat different from the case $N = 1$.

5. When Z_{mn} and Y_{mn} are slowly varying functions of v the approximate solution of the transmission line equations

$$\begin{aligned} \frac{dV_m}{dv} &= -\sum_{n=1}^{N} Z_{mn} J_n \\ \frac{dJ_m}{dv} &= -\sum_{n=1}^{N} Y_{mn} V_n \end{aligned} \tag{3-32}$$

where $Z_{mn} = Z_{nm}$ and $Y_{mn} = Y_{nm}$ is, as in (3-19),

$$V_m \approx \sum_{\ell=1}^{N} S_{m\ell} \, [e^{\xi_\ell} d_\ell^- + e^{-\xi_\ell} d_\ell^+]$$

$$J_m \approx \sum_{\ell=1}^{N} T_{m\ell} \, [e^{\xi_\ell} d_\ell^- - e^{-\xi_\ell} d_\ell^+]. \tag{3-33}$$

Here ξ_ℓ is the integral of φ_ℓ as given by (3-20), and φ_ℓ is determined by setting the determinant of the matrix $\varphi^2 I - ZY$ to zero. When φ_ℓ is known, $S_{m\ell}$ and $T_{m\ell}$ are determined (to within a plus or minus sign which may be absorbed by the constants $d_\ell^\pm$ of integration) by the relations

$$\varphi_\ell S_{m\ell} = -\sum_{n=1}^{N} Z_{mn} T_{n\ell}$$

$$\varphi_\ell T_{m\ell} = -\sum_{n=1}^{N} Y_{mn} S_{m\ell} \tag{3-34}$$

$$\sum_{n=1}^{N} S_{m\ell} T_{m\ell} = 1$$

The last condition, which arises from the condition that the equations for the second-order terms be consistent, may be regarded as a generalization of Slater's[15] result for the case $N = 1$.

4. *Transformation for Wave Guide Plus Horn*

The system to which we shall apply some of the preceding equations consists of a straight wave guide starting at $x = -\infty$ and running to $x = 0$ where it is connected to a sectoral horn. The horn is flared in the (x, y) plane only. The dimension of the system normal to the (x, y) plane is constant and equal to a or b according to whether the electric or magnetic vector is in the plane of the horn.

One might expect that the field in this system may also (in addition to our method) be determined by an alternating procedure of the type described by Poritsky and Blewett[16] using the equations obtained by Barrow and Chu[17] for transmission in the horn. However, we shall not investigate this possibility as we are primarily interested in using the system as an example to which we may apply the foregoing equations.

If the total angle of the horn is $2\alpha\pi$, and if the sides of the straight guide are at $y = 0$ and $y = b$, (assuming the electric vector to be in the plane of the horn), the equation of the lower side, i.e., the continuation of the side $y = 0$, of the horn is $y = -x \tan \alpha\pi$ and that of the upper side is $y = b + x \tan \alpha\pi$. If $z = x + iy$ and $w = v + i\theta$ then the Schwarz-Christoffel transformation

$z = f(w)$ which carries the guide plus horn in the z plane into the straight guide with walls at $\theta = 0$, $\theta = \pi$ in the w plane may be obtained from

$$\frac{dz}{dw} = (1 - e^{2w})^{\alpha} \, b/\pi \tag{4-1}$$

This gives, upon setting

$$\left| \frac{dz}{dw} \right|^2 = | f'(v + i\theta) |^2 = [1 - 2e^{2v} \cos 2\theta + e^{4v}]^{\alpha} \, b^2/\pi^2 ,$$

the relation (4-2)

$$1 + g(v, \theta) = [1 - 2e^{2v} \cos 2\theta + e^{4v}]^{\alpha}$$

from which the a_n's may be obtained in accordance with (1-2).

5. *Expressions for the a_n's for Horn*

The Fourier coefficients of $1 + g(v, \theta)$ appearing in (1-2) and (2-2) are the same. It may be shown from (4-2) that

$$1 + a_0 = \begin{cases} e^{4\alpha v} F(-\alpha, -\alpha; 1; e^{-4v}) & , \quad v > 0 \\ \Gamma(1 + 2\alpha)/\Gamma^2(1 + \alpha) & , \quad v = 0 \\ F(-\alpha, -\alpha; 1; e^{4v}) & , \quad v < 0 \end{cases} \tag{5-1}$$

and

$$a_{2r} = \begin{cases} 2e^{4\alpha v - 2rv}(-\alpha)_r F(-\alpha, r - \alpha; r + 1; e^{-4v})/r! \, , & v > 0 \\ 2(-\alpha)_r (1 + a_0)_{v=0}/(1 + \alpha)_r \quad , & v = 0 \\ 2e^{2rv}(-\alpha)_r F(-\alpha, r - \alpha; r + 1; e^{4v})/r! \, , & v < 0 \end{cases} \tag{5-2}$$

where the F's denote hypergeometric functions, $r = 1, 2, \cdots$ and we have used the notation

$$(\beta)_0 = 1, \; (\beta)_r = \beta(\beta + 1) \cdots (\beta + r - 1) \tag{5-3}$$

When n is odd, $a_n = 0$ because of symmetry about $\theta = \pi/2$. The expressions for $v > 0$ in (5-1) and (5-2) may be verified by expanding the two factors in

$$1 + g(v, \theta) = e^{4\alpha v}(1 - e^{2i\theta - 2v})^{\alpha}(1 - e^{-2i\theta - 2v})^{\alpha}$$

by the binominal theorem and picking out the terms containing $e^{2ri\theta}$. When $v < 0$ we use the relation $1 + g(v, \theta) = e^{4\alpha v}[1 + g(-v, \theta)]$, and when $v = 0$ we may sum the hypergeometric series.

Differentiation of (5-1) and (5-2) leads to

$$\frac{d}{dv}(1 + a_0) = \begin{cases} 4\alpha e^{4\alpha v} F(-\alpha, 1 - \alpha; 1; e^{-4v}) \, , & v > 0 \\ 2\alpha(1 + a_0)_{v=0} \, , & v = 0 \\ 4\alpha^2 e^{4v} F(1 - \alpha, 1 - \alpha; 2; e^{4v}), & v < 0 \end{cases} \tag{5-4}$$

$$\frac{d^2}{dv^2}(1 + a_0) = \begin{cases} 16\alpha^2 e^{4\alpha v}(1 - e^{-4v})^{2\alpha-1} F(\alpha, \alpha; 1; e^{-4v}), & v > 0 \\ 16\alpha^2 e^{4v}(1 - e^{4v})^{2\alpha-1} F(\alpha, \alpha; 1; e^{4v}), & v < 0 \end{cases} \tag{5-5}$$

where in obtaining (5-5) use was made of Euler's transformation

$$F(a, b; c; x) = (1 - x)^{c-a-b} F(c - a, c - b; c; x)$$

It is seen that $d(1 + a_0)/dv$ is continuous at $v = 0$ but the second derivative becomes infinite as $v^{2\alpha-1}$.

When $1 + a_0$ and a_2 are expressed as the customary integrals defining the Fourier coefficients it is seen that one of the coefficients occurring in equation (2-4) for F_1 is given by

$$\begin{aligned} 1 + a_0 - a_2/2 &= \frac{2}{\pi}\int_0^\pi (1 - 2e^{2v}\cos 2\theta + e^{4v})^\alpha \sin^2\theta \, d\theta \\ &= (e^{2v} + 1)^{2\alpha} F(-\alpha, \tfrac{1}{2}; 2; \operatorname{sech}^2 v) \end{aligned} \tag{5-6}$$

At $v = 0$, $1 + a_0 - a_2/2$ and its first and second derivatives are continuous, their values being

$$\frac{\Gamma(2 + 2\alpha)}{\Gamma(1 + \alpha)\Gamma(2 + \alpha)}, \quad \frac{2\alpha\Gamma(2 + 2\alpha)}{\Gamma(1 + \alpha)\Gamma(2 + \alpha)}, \quad \frac{4\alpha(2\alpha^2 + 2\alpha + 1)\Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)\Gamma(2 + \alpha)}, \tag{5-7}$$

respectively. These may be obtained by differentiating the integral in (5-6) and setting $v = 0$.

A second expression for $1 + a_0 - a_2/2$ follows from (5-1) and (5-2):

$$1 + a_0 - a_2/2 = \begin{cases} e^{4\alpha v}[F(-\alpha, -\alpha; 1; e^{-4v}) + \alpha e^{-2v} F(-\alpha, 1 - \alpha; 2; e^{-4v})], & v > 0 \\ F(-\alpha, -\alpha; 1; e^{4v}) + \alpha e^{2v} F(-\alpha, 1 - \alpha; 2; e^{4v}), & v < 0. \end{cases} \tag{5-8}$$

6. *Approximation to Reflection Coefficient of Horn, Electric Vector in (x, y) Plane*

When the flare angle $2\alpha\pi$ of the horn is very small the reflection coefficient may be shown to be

$$R_E = \frac{i\alpha}{2k} + 0(\alpha^2) \tag{6-1}$$

where $0(\alpha^2)$ denotes correction terms of the order α^2. This result is based upon the fact that when terms of order α^2 are neglected the set of differential equations (1-4) and (1-5) reduce to the single equation

$$F_0'' + (1 + a_0)k^2F_0 = 0 \tag{6-2}$$

where, from (5-1, 4, 5),

	$v > 0$	$v < 0$
$1 + a_0$	$e^{4\alpha v}$	1
$\frac{d}{dv}(1 + a_0)$	$4\alpha e^{4\alpha v}$	0
$\frac{d^2}{dv^2}(1 + a_0)$	$16\alpha^2 e^{4\alpha v}(1 - e^{-4v})^{2\alpha - 1}$	$16\alpha^2 e^{4v}(1 - e^{4v})^{2\alpha-1}$

The reflection coefficient (6-1) is the one corresponding to the differential equation (6-2) and may be computed by setting

$$(1 + a_0)k^2 = -h^2 = K \tag{6-3}$$

in the integrals (3-14).

The expression (6-1) for R_E may be obtained quickly (but the procedure is not trustworthy) by assuming that the principal contribution to the first integral in (3-14) comes from the region close to $v = 0$, say in $-\epsilon < v < \epsilon$, where the second derivative of $h^{-1/2}$ is infinite but integrable. When the integration is performed approximately by replacing the second derivative by the first, (3-14) gives

$$R_E \approx \frac{1}{2}\left[h^{-\frac{1}{2}} \frac{d}{dv} h^{-\frac{1}{2}}\right]_{-\epsilon}^{+\epsilon} \tag{6-4}$$

$$\approx \frac{1}{2ik}\left[\frac{d}{dv}(1 + a_0)^{-\frac{1}{4}}\right]_{-\epsilon}^{\epsilon} = \frac{i\alpha}{2k}$$

where ϵ is assumed to be so small that $1 + a_0$ is effectively unity and $d(1 + a_0)/dv$ changes from 0 at $-\epsilon$ to 4α at $+\epsilon$.

A more careful investigation based on the second integral in (3-14) also leads to the value (6-1) for R_E. It further suggests that possibly most of the correction term, denoted by $0(\alpha^2)$ in (6-1), is given by

$$\frac{\alpha^2}{2ik}\int_0^\infty e^{-2\xi - 2\alpha v}\, dv = \frac{1}{4ix} + \frac{e^{ix}}{4i}[Si(x) - \pi/2 + iCi(x)] \tag{6-5}$$

with $x = k/\alpha$ and $2\xi = ix[\exp(2\alpha v) - 1]$. $Si(x)$ and $Ci(x)$ denote the integral sine and cosine functions. Incidentally, the rather curious result

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{mn(m+n)} = 2\sum_{n=1}^{\infty} n^{-3}$$

turned up in the investigation of the orders of magnitude of the various terms.

7. *Approximation to Reflection Coefficient of Horn, Magnetic Vector in (x, y) Plane*

The work of this section is quite similar to that in Section 6 except that here we enter into more of the details. We shall show that when α is small the reflection coefficient appearing in equation (2-6) is

$$R_H = \frac{i\alpha}{2c^3} + 0(\alpha^2). \tag{7-1}$$

From (2-4) the analogue of the differential equation (3-1) is

$$F_1'' + [\kappa^2(1 + a_0 - a_2/2) - 1]F_1 = 0 \tag{7-2}$$

and the K appearing in the second of equations (3-14) is now

$$K = -h^2 = \kappa^2(1 + a_0 - a_2/2) - 1 \tag{7-3}$$

The largest terms in the expression (5-8) for $1 + a_0 - a_2/2$ yield, to within terms of $0(\alpha)$,

$$\begin{aligned} K &= \kappa^2(e^{4\alpha v} + \alpha e^{-2v}) - 1\,, \quad v > 0 \\ \dot{K} &= \kappa^2(4\alpha\, e^{4\alpha v} - 2\alpha e^{-2v}) \\ \ddot{K} &= \kappa^2(16\alpha^2 e^{4\alpha v} + 4\alpha e^{-2v}) \end{aligned} \tag{7-4}$$

$$\begin{aligned} K &= \kappa^2(1 + \alpha e^{2v}) - 1 = c^2 + \kappa^2\alpha e^{2v}\quad , \quad v < 0 \\ \dot{K} &= 2\alpha\kappa^2 e^{2v} \\ \ddot{K} &= 4\alpha\kappa^2 e^{2v} \end{aligned} \tag{7-5}$$

where the dots denote differentiation with respect to v and $c^2 = \kappa^2 - 1$. We have retained the α^2 in $\ddot{K}$ as given by (7-4) because at this stage we do not know whether it may be neglected or not.

When $v < 0$, the definition (3-5) of ξ and (7-5) yield

$$\begin{aligned} \xi &= icv + i\int_{-\infty}^{v} (K^{\frac{1}{2}} - c)\, dv \\ &= icv + ic\int_{-\infty}^{v} [(1 + \kappa^2 c^{-2}\alpha e^{2v})^{\frac{1}{2}} - 1]\, dv = icv + 0(\alpha) \end{aligned} \tag{7-6}$$

and we have

$$\int_{-\infty}^{0} e^{-2\xi} K^{-5/2} \dot{K}^2 \, dv = \int_{-\infty}^{0} e^{-2icv} c^{-5} 4\alpha^2 \kappa^4 e^{4v} \, dv$$
$$= 0(\alpha^2) \tag{7-7}$$

which may be neglected. The other integral suggested by (3-14) is

$$\int_{-\infty}^{0} e^{-2\xi} K^{-3/2} \ddot{K} \, dv = \int_{-\infty}^{0} e^{-2icv} c^{-3} 4\alpha\kappa^2 e^{2v} \, dv$$
$$= 2\alpha\kappa^2 c^{-3}/(1 - ic) \tag{7-8}$$

When $v > 0$,

$$\xi = \xi_{v=0} + i \int_0^v [\kappa^2(e^{4\alpha v} + \alpha e^{-2v}) - 1]^{\frac{1}{2}} \, dv$$
$$= i \int_0^v (\kappa^2 e^{4\alpha v} - 1)^{\frac{1}{2}} \, dv + 0(\alpha), \tag{7-9}$$
$$= \frac{i}{2\alpha} [x - \tan^{-1} x - c + \tan^{-1} c] + 0(\alpha),$$
$$x = (\kappa^2 e^{4\alpha v} - 1)^{\frac{1}{2}}, \qquad 2\alpha \, dv = x(1 + x^2)^{-1} \, dx$$

In the integrals containing exp $(-2v)$ as a factor, ξ may be taken to be icv since the integrand becomes negligibly small by the time icv differs significantly from (7-9). We have

$$\int_0^\infty e^{-2\xi} K^{-5/2} \dot{K}^2 \, dv = \int_0^\infty e^{-2\xi}(\kappa^2 e^{4\alpha v} - 1)^{-\frac{5}{2}} \kappa^4 \alpha^2 (4e^{4\alpha v} - 2e^{-2v})^2 \, dv$$
$$= \int_0^\infty e^{-2\xi} x^{-5} \kappa^4 \alpha^2 16 e^{8\alpha v} \, dv \tag{7-10}$$
$$= 8\alpha \int_c^\infty e^{-i[x - \tan^{-1} x - c + \tan^{-1} c]/\alpha} (x^{-4} + x^{-2}) \, dx$$

where the integrals containing e^{-2v} and e^{-4v} have been neglected since their contribution is $0(\alpha^2)$. When α becomes exceedingly small the exponential term oscillates rapidly and the last line of (7-10) is likewise $0(\alpha^2)$. This may be verified by integrating by parts, starting with

$$\exp Y \, dx = i\alpha \, x^{-2}(1 + x^2) d(\exp Y),$$
$$Y = -i(x - \tan^{-1} x)/\alpha$$

The last integral which must be considered is

$$\int_0^\infty e^{-2\xi} K^{-3/2} \ddot{K}\, dv = \int_0^\infty e^{-2\xi}(\kappa^2 e^{4\alpha v} - 1)^{-\frac{3}{2}} \kappa^2[16\alpha^2 e^{4\alpha v} + 4\alpha e^{-2v}]\, dv$$

$$= 16\kappa^2 \alpha^2 \int_0^\infty e^{-2\xi} x^{-3} e^{4\alpha v}\, dv$$

$$+ \int_0^\infty e^{-2icv-2v} c^{-3} \kappa^2 4\alpha\, dv \qquad (7\text{-}11)$$

$$= 8\alpha \int_c^\infty e^{-i[x-\tan^{-1}x-c+\tan^{-1}c]/\alpha} x^{-2}\, dx + 2\alpha\kappa^2 c^{-3}/(1 + ic)$$

$$= 0(\alpha^2) + 2\alpha\kappa^2 c^{-3}/(1 + ic).$$

That the integral having x as the variable of integration is $0(\alpha^2)$ may be shown as in (7-10).

When we combine our results in accordance with (3-14) we obtain

$$R_H = \frac{1}{2i} \int_{-\infty}^\infty e^{-2\xi}[\tfrac{5}{16}K^{-5/2} \dot{K}^2 - \tfrac{1}{4}K^{-3/2} \ddot{K}]\, dv$$

$$= -\frac{\alpha\kappa^2 c^{-3}}{4i}\left[\frac{1}{1 - ic} + \frac{1}{1 + ic}\right] + 0(\alpha^2) \qquad (7\text{-}12)$$

$$= i\alpha/(2c^3) + 0(\alpha^2)$$

which is (7-1).

If, instead of discarding (7-10) because it is $0(\alpha^2)$, we retain it and the corresponding integral in (7-11) (in the hope that they represent most of the difference between the approximate value (7-1) for R_H and the true value) we obtain the approximation

$$R_H = \frac{i\alpha}{2c^3} - \frac{i\alpha}{4} \int_c^\infty e^{-i(x-\tan^{-1}x-c+\tan^{-1}c)/\alpha}(5x^{-4} + x^{-2})\, dx \qquad (7\text{-}13)$$

in which the integral may be evaluated by numerical integration.

The approximations (6-1) and (7-1) for the reflection coefficients may also be obtained from an equation given by N. H. Frank.[18] However, care must be taken to suitably define the wave guide characteristic impedance which appears in his expression.

8. *Speculation on the Reflection Obtained from Horn Flared in Both Directions*

All the work from Section 4 onward applies only to a horn flared in one plane. Nevertheless, it is interesting to speculate on how close an estimate of the reflection from a three-dimensional horn may be obtained by superposing the two reflection coefficients (6-1) and (7-1). It must be kept in mind that the flare angles (the α's) may be different in the two directions, that k is given by (1-2) and c by (2-2), and finally the difference (not the sum) of R_E and R_H must be taken. In (6-1) R_E is the reflection coefficient of the component of the magnetic vector normal to the (x, y) plane (which is proportional to Q), while in (7-1) R_H is the reflection coefficient of the transverse electric vector (which is proportional to P) and there is a difference in sign just as in the case of voltage and current reflection coefficients. If $a > b$ and λ_0 is the wavelength in free space, the superposition gives the following expression for the reflection coefficient of the electric vector:

$$R = R_H - R_E$$
$$= \frac{i}{2} [(2a/\lambda_0)^2 - 1]^{-\frac{1}{2}} (\alpha_H / [(2a/\lambda_0)^2 - 1] - a\alpha_E/b) \tag{8-1}$$

where $2\pi\alpha_H$ and $2\pi\alpha_E$ are the total horn angles in the planes of H and E, respectively. Of course this approximation can be expected to hold only when α_H and α_E are small.

9. *Numerical Calculations—R_H for 60° Horn*

The value of R_H, the reflection coefficient when the magnetic vector lies in the plane of the flare, was computed on the assumption that only the dominant mode need be considered.* Thus, instead of the system of equations (2-4) and (2-5), only their simplified version, namely the single second order differential equation (7-2), was used. This equation may be written as

$$\frac{d^2 F_1}{dv^2} + KF_1 = 0 \tag{9-1}$$

where, according to (5-6),

$$K = -h^2 = \kappa^2(1 + a_0 - a_2/2) - 1 \tag{9-2}$$
$$1 + a_0 - a_2/2 = (e^{2v} + 1)^{2\alpha} F(-\alpha, 1/2; 2; \operatorname{sech}^2 v).$$

The problem was to obtain the R_H appearing in that solution F_1 of (9-1) which satisfies the boundary conditions (2-6).

No computations for R_E were made.

In the first method of calculation the integrals in the approximation (3-14), namely

$$R_H = \frac{1}{2i} \int_{-\infty}^{\infty} e^{-2\xi} K^{-1/4} \frac{d^2}{dv^2} K^{-1/4} \, dv \tag{9-3}$$
$$2\xi = 2icv + 2i \int_{-\infty}^{v} (K^{\frac{1}{2}} - c) \, dv,$$

* I am indebted to Miss M. Darville for carrying out the computations of this section.

were evaluated by Simpson's rule. The second derivative of $K^{-1/4}$ was computed from the even order central differences of $K^{-1/4}$.[19] For $\alpha = 1/6$, corresponding to an angle of $\pi/3$ between the two sides of the horn, calculations at two representative wave lengths led to the table

λ_0	c	$\kappa^2 = 1+c^2$	R_H	(9-4)
1.549a	.8173	1.6680	$-.0420 + i.0724$	
1.610a	.7376	1.5441	$-.0551 + i.0878$	

An idea of the variation of K may be obtained from its values at $-\infty$, $-.6$, 0, .6, 1.8, 3.6 which are approximately .67, .76, .98, 1.62, 4.56, 17.4, respectively. The range of integration was $-3 \leq v \leq 4.4$.

The second method of computation used the formulas (3-11) with F_1 playing the role of y. The differential equation (9-1) was integrated by the Kutte-Runge method, the interval between successive values of v being 0.2. For $c = .8173$ the values obtained were

v_1	v_2	γ	Γ	R_H	(9-5)
$-.6$	.6	$-.202 - i.981$	$-.142 - i.794$	$-.0167 + i.0658$	
-1.2	1.2	$-.218 - i1.004$	$-.049 - i.696$	$-.0525 + i.0754$	
-1.8	1.8	$-.225 - i.989$	$+.086 - i.716$	$-.0512 + i.0753$	
-2.4	2.4	$-.220 - i1.000$	$.136 - i.842$	$-.0424 + i.0722$	

In order to gain an idea of the meaning of these values of v it should be recalled that $w = v + i\theta$ and the walls of the guide are at $\theta = 0$ and $\theta = \pi$. An interval of length $\pi = 3.14 \cdots$ in the v direction therefore corresponds roughly to a distance equal to the width of the guide. The above table indicates that, loosely speaking, most of the reflection occurs close to the junction of the horn and wave guide.

The last value of R_H in (9-5) agrees quite well with the value $-.0420 + i.0724$ obtained from the approximate expression (9-3). It appears that the method leading to (9-5) is superior to the one based on (9-3) since, in theory, it may be made as accurate (insofar as the single equation (9-1) may replace the set of equations (2-4, 5)) as desired. Moreover, less actual work seems to be required.

The approximation (7-1) yields, for $c = .8173$,

$$R_H = \frac{i\alpha}{2c^3} = \frac{i(\frac{1}{6})}{2(.8173)^3} = i.153$$

which is considerably in error, as we might expect, since $\alpha = 1/6$ is not small. However, if we use the approximation (7-13) and evaluate the integral by Simpson's rule we obtain

$$R_H = i.153 - (.061 + i.077)$$
$$= -.061 + i.076$$

which is in better agreement with the earlier values of R_H.

No similar computations have been made to test the corresponding approximation for R_E obtained when the correction term (6-5) is added to the leading term in (6-1). However, it appears that for $\alpha = 1/6$ and the representative value $k = .38$, (6-5) is only about one sixth as large as $i\alpha/(2k)$ and hence is relatively unimportant.

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