A New Type of High-Frequency Amplifier

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This paper describes a new amplifier in which use is made of an electron flow consisting of two streams of electrons having different average velocities. When the currents or charge densities of the two streams are sufficient, the streams interact to give an increasing wave. Conditions for an increasing wave and the gain of the increasing wave are evaluated for a particular geometry of flow.

1. Introduction

IN CENTIMETER range amplifiers involving electromagnetic resonators or transmission circuits as, in klystrons and conventional traveling-wave tubes, it is desirable to have the electron flow very close to the metal circuit elements, where the radio-frequency field of the circuit is strong, in order to obtain satisfactory amplification. It is, however, difficult to confine the electron flow close to metal circuit elements without an interception of electrons, which entails both loss of efficiency and heating of the circuit elements. This latter may be extremely objectionable at very short wavelengths for which circuit elements are small and fragile.

In this paper the writers describe a new type of amplifier. In this amplifier the gain is not obtained through the interaction of electrons with the field of electromagnetic resonators, helices or other circuits. Instead, an electron flow consisting of two streams of electrons having different average velocities is used. When the currents or charge densities of the two streams are sufficient, the streams interact so as to give an increasing wave. Electromagnetic circuits may be used to impress a signal on the electron flow, or to produce an electromagnetic output by means of the amplified signal present in the electron flow. The amplification, however, takes place in the electron flow itself, and is the result of what may be termed an electromechanical interaction.[1,2]

While small magnetic fields are necessarily present because of the motions of the electrons, these do not play an important part in the amplification.

[1] Some electro-mechanical waves with a similar amplifying effect are described in "Possible Fluctuations in Electron Streams Due to Ions," J. R. Pierce, *Jour. App. Phys.*, Vol. 19, pp. 231–236, March 1948.

[2] While this paper was in preparation a classified report by Andrew V. Haeff entitled "The Electron Wave Tube—A Novel Method of Generation and Amplification of Microwave Energy" was received from the Naval Research Laboratory. Dr. Haeff's report (now declassified) contains a similar analysis of interaction of electron streams and in addition gives experimental data on the performance of amplifying tubes built in accordance with the new principle. We understand that similar work has been done at the RCA Laboratories.

The important factors in the interaction are the electric field, which stores energy and acts on the electrons, and the electrons themselves. The charge of the electrons produces the electric field; the mass of the electrons, and their kinetic energy, serve much as do inductance and stored magnetic energy in electromagnetic propagation.

By this sort of interaction, a traveling wave which increases as it travels, i.e., a traveling wave of negative attenuation, may be produced. To start such a wave, the electron flow may be made to pass through a resonator or a short length of helix excited by the input signal. Once initiated, the wave grows exponentially in amplitude until the electron flow is terminated or until non-linearities limit the amplitude. An amplified output can be obtained by allowing the electron flow to act on a resonator, helix or other output circuit at a point far enough removed from the input circuit to give the desired gain.

There are several advantages of such an amplifier. Because the electrons interact with one another, the electron flow need not pass extremely close to complicated circuit elements. This is particularly advantageous at very short wavelengths. Further, if we make the distance of electron flow between the input and output circuits long enough, amplification can be obtained even though the input and output circuits have very low impedance or poor coupling to the electron flow. Even though the region of amplification is long, there is no need to maintain a close synchronism between an electron velocity and a circuit wave velocity, as there is in the usual traveling-wave tube.

A companion paper by Dr. A. V. Hollenberg of these laboratories describes an experimental "double stream" amplifier tube consisting of two cathodes which produce concentric electron streams of somewhat different average velocity, and short helices serving as input and output circuits. No further physical description of double stream amplifiers will be given in this paper. Rather, a theoretical treatment of such devices will be presented.

2. Simple Theory

For simplicity we will assume that the flow consists of coincident streams of electrons of d-c. velocities u_1 and u_2 in the x direction. It will be assumed that there is no electron motion normal to the x direction. The treatment will be a small-signal or perturbation theory, in which products of a-c. quantities are neglected. M.K.S. units will be used. All quantities will be assumed to vary with time and distance $\exp j(\omega t - \beta x)$. The wavelength in the stream, λ_s, is then related to β by

$$\beta = 2\pi/\lambda_s \tag{1}$$

The following additional nomenclature will be used:

ϵ_0	dielectric constant of vacuum $\epsilon_0 = 8.85 \times 10^{-12}$ farad/meter
η	charge-to-mass ratio of the electron $\eta = 1.76 \times 10^{11}$ coulomb/kilogram
J_1, J_2	d-c. current densities
u_1, u_2	d-c. velocities
ρ_{01}, ρ_{02}	d-c. charge densities $\rho_{01} = -J_1/u_1,\ \rho_{02} = -J_2/u_2$
ρ_1, ρ_2	a-c. charge densities
v_1, v_2	a-c. velocities
V_1, V_2	d-c. voltages with respect to the cathode
V	a-c. potential

$\beta_1 = \omega/u_1,\ \beta_2 = \omega/u_2$

Although the small-signal equations relating charge density to voltage V have been derived many times, it seems well to present them for the sake of completeness. For one stream of electrons the first-order force equation is

$$\frac{dv_1}{dt} = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x} u_1 = \eta \frac{\partial V}{\partial x}$$

$$(\omega - \beta u_1) v_1 = -\eta \beta V$$

$$v_1 = \frac{-\eta \beta V}{u_1(\beta_1 - \beta)} \tag{2}$$

From the conservation of charge we obtain to the first order

$$\frac{\partial \rho_1}{\partial t} = -\frac{\partial}{\partial x}(\rho_{01} v_1 + \rho_1 u_1)$$

$$\omega \rho_1 = \rho_{01} \beta v_1 + u_1 \beta \rho_1$$

$$\rho_1 = \frac{\rho_{01} \beta v_1}{u_1(\beta_1 - \beta)}$$

$$\rho_1 = -\frac{J_1 \beta v_1}{u_1^2(\beta_1 - \beta)} \tag{3}$$

From (2) and (3) we obtain

$$\rho_1 = \frac{\eta J_1 \beta^2 V}{u_1^3(\beta_1 - \beta)^2} \tag{4}$$

We would find similarly

$$\rho_2 = \frac{\eta J_2 \beta^2 V}{u_2^3(\beta_2 - \beta)^2} \tag{5}$$

It will be convenient to call the fractional velocity separation b, so that

$$b = \frac{2(u_1 - u_2)}{u_1 + u_2} \tag{6}$$

It will also be convenient to define a sort of mean velocity u_0

$$u_0 = \frac{2u_1 u_2}{u_1 + u_2} \tag{7}$$

We may also let V_0 be the potential drop specifying a velocity u_0, so that

$$u_0 = \sqrt{2\eta V_0} \tag{8}$$

It is further convenient to define a phase constant based on u_0

$$\beta_0 = \frac{\omega}{u_0} \tag{9}$$

We see from (6), (7) and (9) that

$$\beta_1 = \beta_0(1 - b/2) \tag{10}$$

$$\beta_2 = \beta_0(1 + b/2) \tag{11}$$

We shall treat only a special case, that in which

$$\frac{J_1}{u_1^3} = \frac{J_2}{u_2^3} = \frac{J_0}{u_0^3} \tag{12}$$

Here J_0 is a sort of mean current which, together with u_0, specifies the ratios J_1/u_1^3 and J_2/u_2^3, which appear in (4) and (5).

In terms of these new quantities, the expression for the total a-c. charge density ρ is, from (4) and (5) and (8)

$$\rho = \rho_1 + \rho_2 = \frac{J_0\beta^2}{2u_0 V_0} \cdot \left[\frac{1}{\left[\beta_0\left(1 - \frac{b}{2}\right) - \beta\right]^2} + \frac{1}{\left[\beta_0\left(1 + \frac{b}{2}\right) - \beta\right]^2} \right] V \tag{13}$$

Equation (13) is a *ballistical* equation telling what charge density ρ is produced when the flow is bunched by a voltage V. To solve our problem, that is, to solve for the phase constant β, we must associate (13) with a *circuit* equation which tells us what voltage V the charge density produces. We assume that the electron flow takes place in a tube too narrow to propagate a wave of the frequency considered. Further, we assume that the wave velocity is much smaller than the velocity of light. Under these circumstances the circuit problem is essentially an electrostatic problem. The a-c. voltage will be of the same sign as, and in phase with, the a-c. charge density ρ. In other words, the "circuit effect" is purely capacitive.

Let us assume at first that the electron stream is very narrow compared with the tube through which it flows, so that V may be assumed to be constant over its cross section. We can easily obtain the relation between

V and ρ in two extreme cases. If the wavelength in the stream, λ_s, is very short (β large), so that transverse a-c. fields are negligible, then from Poisson's equation we have

$$\rho = -\epsilon_0 \frac{\partial^2 V}{\partial x^2}$$

$$\rho = \epsilon_0 \beta^2 V \tag{14}$$

If, on the other hand, the wavelength is long compared with the tube radius (β small) so that the fields are chiefly transverse, the lines of force running from the beam outward to the surrounding tube, we may write

$$\rho = CV \tag{15}$$

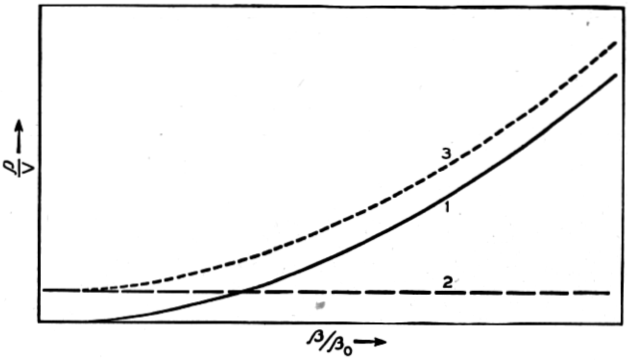

Fig. 1—A "circuit" curve for a narrow electron stream in a tube. The ratio of the a-c. charge density ρ to the a-c. voltage V produced by the charge density is plotted vs. a parameter β/β_0, which is inversely proportional to the wavelength λ_s in the flow. Curve 1 holds for very large values of β/β_0; curve 2 holds for very small values of β/β_0, and curve 3 over-all shows approximately how ρ/V varies for intermediate values of β/β_0.

Here C is a constant expressing the capacitance per unit length between the region occupied by the electron flow and the tube wall.

We see from (14) and (15) that if at some particular frequency we plot ρ/V vs. β/β_0 for real values of β, ρ/V will be constant for small values of β and will rise as β^2 for large values of β, approximately as shown in Fig. 1. For another frequency, β_0 would be different and, as ρ/V is a function of β, the horizontal scale of the curve would be different.

Now, we have assumed that the charge is produced by the action of the voltage, according to the ballistical equation (10). This relation is plotted in Fig. 2, for a relatively large value of J_0/u_0V_0 (curve 1) and for a smaller value of J_0/u_0V_0 (curve 2). There are poles at $\beta/\beta_0 = 1 \pm \frac{b}{2}$, and a minimum between the poles. The height of the minimum increases as J_0/u_0V_0 is increased.

A circuit curve similar to that of Fig. 1 is also plotted on Fig. 2. We see that for the small-current case (curve 2) there are four intersections, giving *four real* values of β and hence *four unattenuated* waves. However, for the larger current (curve 1) there are only two intersections and hence two unattenuated waves. The two additional values of β satisfying both the circuit equation and the ballistical equation are complex conjugates, and represent waves traveling at the same speed, but with equal positive and negative attenuations.

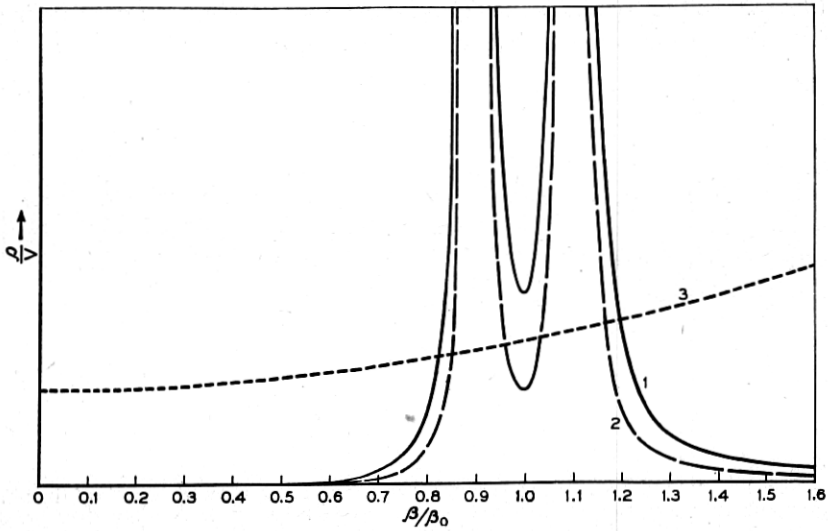

Fig. 2—Curve 3 is a circuit curve similar to that of Fig. 1. Curves 1 and 2 are based on a ballistical equation telling how much charge density ρ is produced when the voltage V acts to bunch a flow consisting of electrons of two velocities. The abscissa, β/β_0, is proportional to phase constant. Intersections of the circuit curve with a ballistical curve represent waves. Curve 2 is for a relatively small current. In this case intersections occur for four real values of β, so the four waves are unattenuated. For a larger current (curve 1) there are two intersections (two unattenuated waves). For the other two waves β is complex. There are an increasing and a decreasing wave.

Thus we deduce that, as the current densities in the electron streams are raised, a wave with negative attenuation appears for current densities above a certain critical value.

We can learn a little more about these waves by assuming an approximate expression for the circuit curve of Fig. 1. Let us merely assume that over the range of interest (near $\beta/\beta_0 = 1$) we can use

$$\rho = \alpha^2 \epsilon_0 \beta^2 V \tag{16}$$

Here α^2 is a factor greater than unity, which merely expresses the fact that the charge density corresponding to a given voltage is somewhat greater than if there were field in the x direction only and equation (11) were valid. Combining (16) with (13) we obtain

$$\frac{1}{\left(\beta_0\left(1-\frac{b}{2}\right)-\beta\right)^2}+\frac{1}{\left(\beta_0\left(1+\frac{b}{2}\right)-\beta\right)^2}=\frac{1}{\beta_0^2 U^2} \tag{17}$$

where

$$U = \frac{J_0}{2\alpha^2 \epsilon_0 \beta_0^2 u_0 V_0} \tag{18}$$

In solving (17) it is most convenient to represent β in terms of β_0 and a new variable δ

$$\beta = \beta_0(1+\delta) \tag{19}$$

Thus, (14) becomes

$$\frac{1}{\left(\delta-\frac{b}{2}\right)^2}+\frac{1}{\left(\delta+\frac{b}{2}\right)^2}=\frac{1}{U^2} \tag{20}$$

Solving for δ, we obtain

$$\delta = \pm\left(\frac{b}{2}\right)\left[\left(\frac{2U}{b}\right)^2 + 1 \pm \left(\frac{2U}{b}\right)\sqrt{\left(\frac{2U^2}{b}\right)+4}\right]^{1/2}. \tag{21}$$

The positive sign inside of the brackets always gives a real value of δ and hence unattenuated waves. The negative sign inside the brackets gives unattenuated waves for small values of U/b. However, when

$$\left(\frac{U}{b}\right)^2 > \frac{1}{8} \tag{22}$$

there are two waves with a phase constant β_0 and with equal and opposite attenuation constants.

Suppose we let U_M be the minimum value of U for which there is gain. From (22),

$$U_{M^2} = b^2/8 \tag{23}$$

From (21) we have for the increasing wave

$$\delta = j\frac{b}{2}\left[\frac{1}{2}\left(\frac{U}{U_M}\right)^2\left(\sqrt{1+8\left(\frac{U}{U_M}\right)^2}-1\right)-1\right]^{1/2} \tag{24}$$

The gain in db/wavelength is

$$\begin{aligned} \text{db/wavelength} &= 20(2\pi)\log_{10} e\,|\,\delta\,| \\ &= 54.6\,|\,\delta\,| \end{aligned} \tag{25}$$

We see that by means of (24) and (25) we can plot db/wavelength per unit b vs. $(U/U_M)^2$. This is plotted in Fig. 3. Because U^2 is proportional to current, the variable $(U/U_M)^2$ is the ratio of the actual current to the current which will just give an increasing wave. If we know this ratio, we can obtain the gain in db/wavelength by multiplying the corresponding ordinate from Fig. 3 by b.

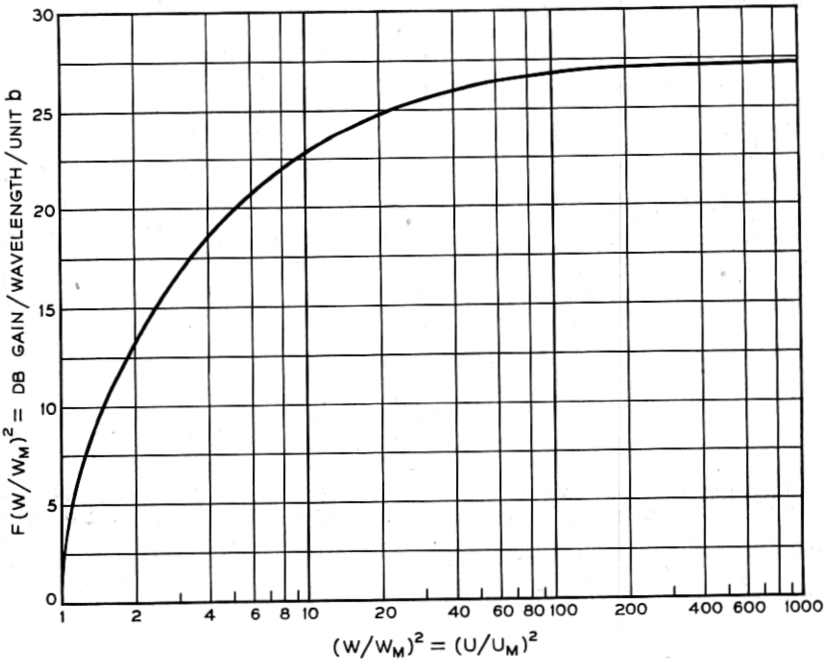

Fig. 3—The parameter $(W/W_M)^2 = ((U/U_M)^2$ is proportional to current. As the current is increased above a critical value for which $(W/W_M)^2 = 1$, there is an increasing wave of increasing gain. In this curve the gain per wavelength per unit b, called $F(W/W_M)^2$, is plotted vs. $(W/W_M)^2$. For large values of $(W/W_M)^2$, $F(W/W_M)^2$ approaches 27.3 and the gain per wavelength approaches 27.3 b.

We see that, as the current is increased, the gain per wavelength at first rises rapidly and then rises more slowly, approaching a value of $27.3b$ db/wavelength for very large values of $(U/U_M)^2$.

We now have some idea of the variation of gain per wavelength with velocity separation b and with current $(U/U_M)^2$. A more complete theory would require the evaluation of the lower limiting current for gain (or of U_M^2) in terms of physical dimensions and an investigation of the boundary conditions to show how strong an increasing wave is set up by a given input signal. The latter problem will not be considered in this paper; the former is dealt with in the third section and in the appendix.

3. Design Curves

It is proposed to present in this section material for actually evaluating the gain of the increasing wave for a particular geometry of electron flow. In this section there is some repetition from earlier sections, so that the material presented can be used without referring unduly to section 2. In order to avoid confusion, much of the mathematical work on which the section is based has been put in the appendix.

The flow considered is one in which electrons of two velocities, u_1 and u_2, corresponding to accelerating voltages V_1 and V_2, are intermingled, the corresponding current densities J_1 and J_2 being constant over the flow. The flow occupies a cylindrical space of radius a. It is assumed that the surrounding cylindrical conducting tube is so remote as to have negligible effect on the a-c. fields.

It will be assumed, according to (12), that the current densities and the voltages V_1 and V_2 are specified in terms of a "mean" current J_0 and a "mean" voltage V_0 corresponding to a velocity u_0, by

$$\frac{J_1}{V_1^{3/2}} = \frac{J_2}{V_2^{3/2}} = \frac{J_0}{V_0^{3/2}} \tag{12a}$$

The gain will depend on the beam radius, the free-space wavelength λ, and on J_0 and V_0, and on the fractional velocity separation

$$b = \frac{2(u_1 - u_2)}{u_1 + u_2} \tag{6}$$

The wavelength in the beam, λ_s, which is associated with the voltage V_0 is given by

$$\lambda_s = \lambda \frac{u_0}{c} = \lambda \frac{\sqrt{2\eta V_0}}{c}$$

$$\lambda_s = 1.98 \times 10^{-3} \lambda \sqrt{V_0} \tag{26}$$

Here c is the velocity of light.

A dimensionless parameter W is defined to be

$$W^2 = \frac{\omega_e^2}{\omega^2} = \frac{J_0}{\epsilon_0 u_0 \omega^2} \tag{27}$$

$$W^2 = 8.52 \times 10^6 \frac{J_0}{f\sqrt{V_0}} \tag{28}$$

Here ω_e is the electron plasma frequency associated with the average space charge density J_0/u_0, and ω is the radian frequency corresponding to the wavelength λ. In (28), the constant is adjusted so that J_0 is expressed in amperes per square centimeter rather than in amperes per square meter, while f is expressed in megacycles.

Below a minimum value of W, which will be called W_M, there is no gain. W_M is a function of the velocity separation b and of the ratio of the beam radius a to the beam wavelength, λ_s. A plot of $(W_M/b)^2$ as a function of (a/λ_s) is shown in Fig. 4.

The variation of gain in the interval, $W_M \lessgtr W < \infty$, is shown in Fig. 3 where "Decibels gain/wavelength/unit b" is plotted as a function of $(W/W_M)^2$. This is the same curve which was derived in section 2. The

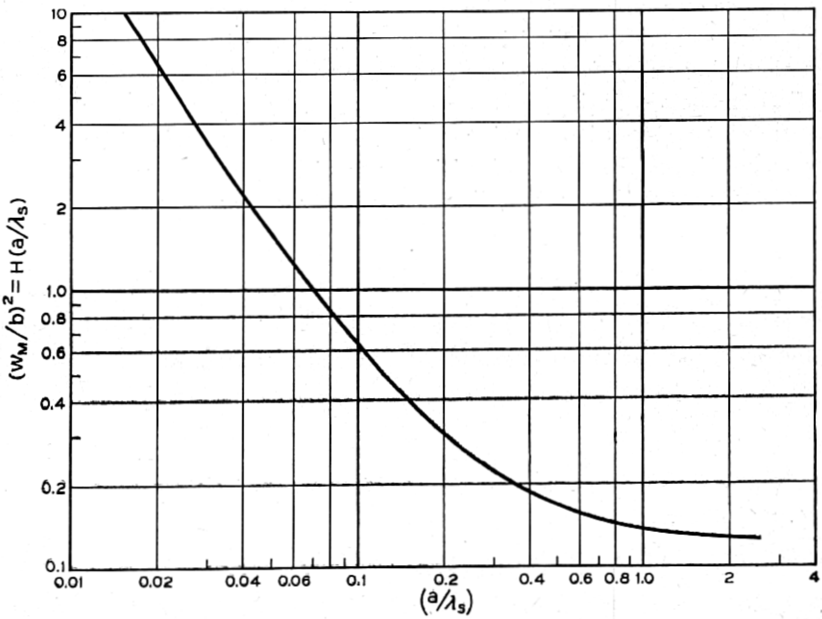

Fig. 4—As the ratio of beam radius a to wavelength in the beam, λ_s, is increased, the critical value of W, W_M, decreases and less current is needed in order to obtain gain. Here $(W_M/b)^2$, which is called $H(a/\lambda_s)$, is plotted vs. (a/λ_s).

ratio $(W/W_M)^2$ is the same as the parameter $(U/U_M)^2$ used there, although U and W are not the same.

The curve in Fig. 3 is useful in that it reduces the interdependence of a large number of parameters to a single curve. However, there are cases as, for example, when one is computing the bandwidth of an amplifier, in which it would be more convenient to have the curve in Fig. 3 broken up into a family of curves. We can do this by the following means:

We can write the gain in db/wavelength in the form

$$\text{db/wavelength} = bF(W/W_M)^2 \tag{29}$$

Here $F(W/W_M)^2$ is the function plotted in Fig. 3. If ℓ is the total length of the flow, the total gain in db, G, will thus be

$$G = \frac{\ell b}{\lambda_s} F(W/W_M)^2 \qquad (30)$$

We will now express $(W/W_M)^2$ in such a form as to indicate its dependence on wavelength in the beam, λ_s. We can write from (27)

$$W = \frac{\omega_e^2}{\omega^2} = \frac{\lambda_s^2}{\lambda_e^2} \qquad (31)$$

Here λ_e is a "plasma wavelength," defined by the relation

$$\lambda_e = \frac{u_0}{(\omega_e/2\pi)} \qquad (32)$$

We further have

$$W_M^2 = b^2 H(a/\lambda_s) \qquad (33)$$

Here $H(a/\lambda_s)$ is the function of (a/λ_s) which is plotted in Fig. 5.

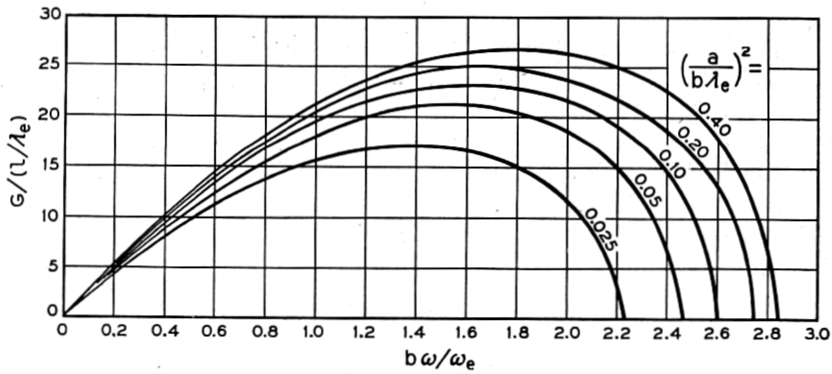

Fig. 5—In these curves the total gain in db, G, divided by the ratio of the length ℓ to the plasma wavelength λ_e, is plotted vs. $b\omega/\omega_e$, which is proportional to frequency, for several values of the parameter $(a/b\lambda_e)^2$. Changing b, the velocity seperation, changes both the parameter and the frequency scale.

Now, from (26), (27), and (29) we can write

$$G = \left(\frac{\ell}{\lambda_e}\right)\left(\frac{b\lambda_e}{a}\right)\left(\frac{a}{\lambda_s}\right) F\left[\left(\frac{\lambda_s}{a}\right)^2 \left(\frac{a}{b\lambda_e}\right)^2 \frac{1}{H(a/\lambda_s)}\right] \qquad (34)$$

For a given tube the parameters (ℓ/λ_e) and $(a/b\lambda_e)$ do not vary with frequency, while (a/λ_s) is proportional to frequency. Hence, we can construct universal frequency curves by plotting $G/(\ell/\lambda_e)$ vs. (a/λ_s) for various values of the parameter $(a/b\lambda_e)$. It is more convenient, however, to use as an abscissa $b\lambda_e/\lambda_s = b\omega/\omega_e$, and this has been done in Fig. 5.

In order to use these curves it is necessary to express the parameters $b\omega/\omega_e$, λ_e and $(a/b\lambda_e)^2$ in terms of convenient physical quantities. We obtain

$$
\begin{aligned}
b\omega/\omega_e &= .545 \times 10^{-10} bV^{1/4} \omega/J_0^{1/2} \\
\lambda_e &= 2.04 \times 10^{-2} V_0^{3/4}/J_0^{1/2} \qquad (35) \\
(a/b\lambda_e)^2 &= 767\ I_0/b^2V^{3/2}
\end{aligned}
$$

Here I_0 is current in amperes and J_0 is in amperes / cm.^{2}

The broadness of the frequency response curves of Fig. 5 is comparable to that of curves for helix-type traveling-wave tubes.

It is interesting to note that the maximum value of $G/(\ell/\lambda_e)$ varies little for a considerable range of the parameter $a/b\lambda_e$, approaching a constant for large values of the parameter. This means that, with a beam of given length, velocity and charge density, one can obtain almost the same optimum gain over a wide range of frequencies simply by adjusting the velocity-separation parameter b.

4. Concluding Remarks

There is a great deal of room for extension of the theory of double-stream amplifiers. This paper has not dealt with the setting up of the increasing wave, nor with other geometries than that of a cylindrical beam in a very remote tube, nor with the effect of physical separation of the electron streams of two velocities nor with streams of many velocities or streams with continuous velocity distributions.

This last is an interesting subject in that it may provide a means for dealing with problems of noise in multivelocity electron streams. Indeed, it was while attempting such a treatment that the writers were distracted by the idea of double-stream amplification.

APPENDIX

Derivation of Results Used in Section 3

Consider a double-stream electron beam whose axis coincides with the z-axis of a system of cylindrical coordinates (r, φ, z) and which is subject to an infinite, longitudinal, d-c. magnetic field. The radius of the beam is a and each of the streams is characterized by d-c. velocities, u_1 and u_2, which are vectors in the positive z direction, and d-c. space charge densities, ρ_{02} and ρ_{02}. All d-c. quantities are assumed to be independent of the coordinates and time, except, of course, for the discontinuities at the surface of the beam. Small a-c. disturbances are superimposed upon these d-c. quantities and they are small enough so that their cross products can be neglected compared with the products of d-c. quantities and a-c. quantities. It is further assumed that only those a-c. quantities are allowed which have no azimuthal variation, that is, $\frac{\partial}{\partial \varphi} = 0$. Fig. 6 shows the electron beam.

Outside the beam the appropriate Maxwell's equations are

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\varphi) = j\frac{k}{\eta_0}E_2 \tag{A-1}$$

$$\frac{\partial H_\varphi}{\partial z} = -j\frac{k}{\eta_0}E_r \tag{A-2}$$

$$\frac{\partial E_z}{\partial r} - \frac{\partial E_r}{\partial z} = jk\eta_0 H_\varphi \tag{A-3}$$

where

$$k = \frac{\omega}{c} \tag{A-4}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \text{ ohms} \tag{A-5}$$

Inside the beam, equations (A-2) and (A-3) remain the same, but instead of equation (A-1) we have

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\varphi) = j\frac{k}{\eta_0}E_z + q_1 + q_2 \tag{A-6}$$

where q_1 and q_2 are the first order a-c. convection current densities of the two streams. These quantities can be calculated from the force equation and the equation for the conservation of charge. Assuming that all a-c. quantities vary as $\exp j(\omega t - \beta z)$, the force equation is (for stream number one, say)

$$j\omega v_1 - j\beta u_1 v_1 = -(e/m)E_z \tag{A-7}$$

and the equation for the conservation of charge is

$$j\beta\rho_{01}v_1 + j\beta u_1\rho_1 = +j\omega\rho_1 \tag{A-8}$$

Equations (A-7) and (A-8) can be solved for v_1 and ρ_1 :

$$v_1 = \frac{-(e/m)E_z}{j\omega\left(1 - \frac{\beta}{\beta_1}\right)} \tag{A-7a}$$

$$\rho_1 = \frac{\beta\rho_{01}}{\omega\left(1 - \frac{\beta}{\beta_1}\right)}v_1 \tag{A-8a}$$

where

$$\beta_1 = \frac{\omega}{u_1}$$

Combining equations (A-7a) and (A-8a) one has

$$\rho_1 = \frac{j\beta\rho_{01}(e/m)E_z}{\omega^2\left(1 - \frac{\beta}{\beta_1}\right)^2} \tag{A-9}$$

The first order a-c. convection current density is given by

$$q_1 = \rho_{01}v_1 + \rho_1 u_1 \tag{A-10}$$

which, by combining with (A-7a) and (A-8b), becomes

$$q_1 = \frac{j(k/\eta_0)(\rho_{01}/m\epsilon_0)E_z}{\omega^2\left(1 - \frac{\beta}{\beta_1}\right)^2} \tag{A-11}$$

Similarly

$$q_2 = \frac{j\,\frac{k}{\eta_0}\,\rho_{01}\,\frac{e}{m\epsilon_0}\,E_z}{\omega^2\left(1 - \frac{\beta}{\beta_2}\right)^2} \tag{A-12}$$

If we now define

$$\beta_0 = \tfrac{1}{2}(\beta_1 + \beta_2) \tag{A-13}$$

$$B_1 = \frac{\beta_1}{\beta_0}; \qquad B_2 = \frac{\beta_2}{\beta_0} \tag{A-14}$$

and let

$$Z = \frac{\beta}{\beta_0} \tag{A-15}$$

$$W_1 = \frac{\omega_{e1}}{\omega}; \qquad W_2 = \frac{\omega_{e2}}{\omega} \tag{A-16}$$

where ω_e, the plasma-electron angular frequency given by

$$\omega_{e1}^2 = -\frac{e\rho_{01}}{m\epsilon_0}, \text{ etc.} \tag{A-17}$$

Equations (11) and (12) become

$$q_1 = \frac{-j(k/\eta_0)W_1^2 B_1^2}{(Z - B_1)^2}\,E_z \tag{A-18}$$

$$q_2 = \frac{-i(k/\eta_0)W_2^2 B_2^2}{(Z - B_2)^2} \tag{A-19}$$

Thus equation (A-6) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\varphi) = j\,\frac{k}{\eta_0}\,LE_z \tag{A-6a}$$

where

$$L = 1 - \frac{W_1^2 B_1^2}{(Z - B_1)^2} - \frac{W_2^2 B_2^2}{(Z - B_2)^2} \tag{A-20}$$

If we assume that the tube which surrounds the beam be taken as infinitely remote, the appropriate solutions outside the beam are

$$\hat{H}_{\varphi 0} = A_0 K_1(\gamma r) \tag{A-21}$$

$$\hat{E}_{z0} = j \frac{\eta_0 \gamma}{k} A_0 K_0(\gamma r) \tag{A-22}$$

and inside the beam

$$\hat{H}_{\varphi i} = A_i I_1(\xi r) \tag{A-23}$$

$$\hat{E}_{zl} = -j \frac{\eta_0 \gamma}{\sqrt{L}} A_i I_0(\xi r) \tag{A-24}$$

where

$$\begin{aligned} \gamma^2 &= \beta^2 - k^2 \approx \beta^2 \\ \xi^2 &= \gamma^2 L \end{aligned} \tag{A-25}$$

The I's and K's in equations (A-21)–(A-24) are modified Bessel functions.[3] At the surface of the beam ($r = a$), one has the following two independent boundary conditions

$$\hat{H}_{\varphi i} = \hat{H}_{\varphi 0} \tag{A-26}$$

$$\hat{E}_{zi} = \hat{E}_{z0} \tag{A-25a}$$

which, using equations (A-21)–(A-24), yield

$$\frac{I_0(\xi_a)}{\sqrt{L}\, I_1(\xi_a)} = -\frac{K_\eta(\gamma a)}{K_1(\gamma a)} \tag{A-27}$$

From equations (A-13), (A-14), (A-15) and (A-24) one has

$$\xi a = Z\beta_0 a \sqrt{L} \tag{A-28}$$

$$\gamma a = Z\beta_0 a \tag{A-29}$$

If we now define a beam wavelength, λ_s, by the relations

$$\beta_0 = \frac{2\pi}{\lambda_s} \tag{A-30}$$

and assume for the purpose of simplifying the calculation that in the expression for L in (A-20)

$$W_1^2 B_1^2 = W_2^2 B_2^2 = W^2 \tag{A-31}$$

[3] See A Treatise on the Theory of Bessel Functions, G. N. Watson, Chapter 3.

We easily see that

$$W^2 = (\omega_e/\omega)^2 \tag{A-32}$$

where

$$\omega_e = \frac{e}{m} J_0/\epsilon_0 u_0 \tag{A-33}$$

We obtain from (A-20), (A-28), (A-29) and (A-30)

$$\left[\frac{K_1\left(\frac{2\pi a Z}{\lambda_s}\right) I_0\left(\sqrt{L}\,\frac{2\pi a Z}{\lambda_s}\right)}{K_0\left(\frac{2\pi a Z}{\lambda_s}\right) I_1\left(\sqrt{L}\,\frac{2\pi a Z}{\lambda_s}\right)}\right]^2 = L$$

$$= 1 - \left[\frac{W^2}{(Z - B_1)^2} + \frac{W^2}{(Z - B_2)^2}\right] \tag{A-34}$$

Equation (A-31) is equivalent to Equation (12) of the text or to

$$\frac{J_1}{V_1^{3/2}} = \frac{J_2}{V_2^{3/2}} \tag{A-35}$$

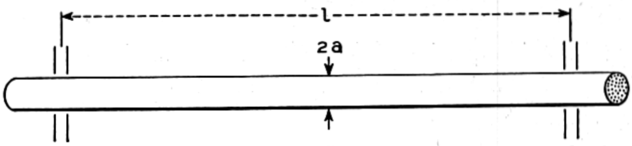

Fig. 6—The diameter of the electron flow considered is $2a$, and the length is ℓ.

Letting $Y = -L$ and making use of the following well known relations between the Bessel functions

$$I_0(jx) = J_0(x)$$
$$I_1(jx) = jJ_1(x) \tag{A-36}$$

Equation (A-34) becomes

$$Y = \left[\frac{K_1\left(\frac{2\pi a}{\lambda_s} Z\right) J_0\left(\sqrt{Y}\,\frac{2\pi a}{\lambda_s} Z\right)}{K_0\left(\frac{2\pi a}{\lambda_s} Z\right) J_1\left(\sqrt{Y}\,\frac{2\pi a}{\lambda_s} Z\right)}\right]^2$$

$$= \frac{W^2}{(Z - B_1)^2} + \frac{W^2}{(Z - B_2)^2} - 1 \tag{A-37}$$

Let the right-hand side of equation (A-37) be denoted by $F_1(Z)$ and the middle of $F_2(Z)$. In order to find the real roots of equation (A-37) one can plot F_1 and F_2 as functions of Z on the same chart. The abscissae of the intersections of the two curves will then be the real roots. In Fig. 7, F_1 is plotted as a function of Z for $B_1 = 0.9$ and $B_2 = 1.1$.

In view of the definitions in equations (A-13) and (A-14), both B_1 and B_2 are uniquely defined by a single parameter, namely, the fractional velocity separation, b. That is

$$b = 2(u_1 - u_2)/(u_1 + u_2) = 2(\beta_2 - \beta_1)/(\beta_2 + \beta_1) = B_2 - B_1 \quad \text{(A-38)}$$

and

$$B_1 = 1 - (b/2), \quad B_2 = 1 + (b/2) \quad \text{(A-39)}$$

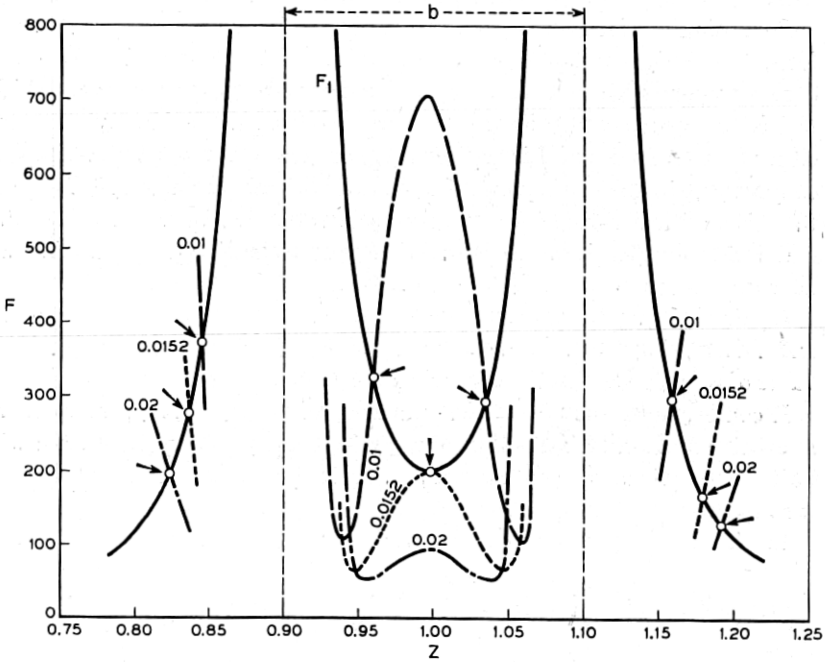

Fig. 7—A curve illustrating conditions giving rise to various types of roots.

A complete plot of F_2, for any value of the parameters W and (a/λ_s), would show that equation (A-37) has an infinite number of real solutions. A real solution of equation (A-37) means an unattenuated wave. Thus there are an infinite number of unattenuated waves possible. The waves which will actually be excited in any given case, however, depend upon the boundary conditions at the input and output of the tube. Ordinarily only those waves will be excited which do not have a reversal in phase of the longitudinal E vector, say, as r varies from 0 to a. Attention, therefore, will be given only to those waves for which E_z does not change sign over a cross-section of the beam. By inspection of equation (A-23), it is evident that this requirement is automatically satisfied if $L > 0$. On the other hand, if L is negative, one has

$$E_{zi} \sim J_0\left(\sqrt{Y}\,\frac{2\pi r}{\lambda_s}\,Z\right) \tag{A-23a}$$

Thus attention will be limited to those roots which satisfy

$$\sqrt{Y}\,\frac{2\pi a}{\lambda_s}\,Z < 2.405 \tag{A-40}$$

where 2.405 is the first zero of the Bessel function in equation (A-21a).

Returning to Fig. 7, portions of three different F_2 curves are plotted: one for $W^2 = 0.01$, one for $W^2 = 0.0152$ and one for $W^2 = 0.02$. All three curves are for $(a/\lambda_s) = 0.16$. The intersections which represent roots which satisfy the inequality (A-40) are marked with arrows. Evidently there are either four real roots of this type or there are two real roots and a complex conjugate pair, the distinction being determined by the value of W. Thus there is a critical value of W^2 (in this case it is 0.0152) for which two of the real roots are identical. This identical pair is indicated by two arrows near the minimum of the F_1 curve at $Z = 1$.

A pair of conjugate complex roots means that there are an increasing wave and a decreasing wave. Thus for each value of b and (a/λ_s) there is a least value of W^2 below which the tube will have no gain.

It can be shown that the critical tangency of the F_1 and F_2 curves occurs at a value of Z which is less than b^2 away from unity. Very little error will be incurred, then, by assuming that this critical point occurs at $Z = 1$ if b is small.

Letting $Z = 1$ in equation (A-37), and using equations (A-39) one has

$$8(W_M/b)^2 - 1 = \left(\frac{K_1(2\pi a/\lambda_s)J_0\left(\sqrt{8(W_M/b)^2 - 1}\;2\pi a/\lambda_s\right)}{K_0(2\pi a/\lambda_s)J_1\left(\sqrt{8(W_M/b)^2 - 1}\;2\pi a/\lambda_s\right)}\right) \tag{A-41}$$

where W_M is the critical value of W. Equation (A-41) determines $(W_M/b)^2$ as af unction of (a/λ_s). This relationship is plotted in Fig. 4.

We will find that there will be an increasing wave in the range $W_M \leq W < \infty$. The calculation of the gain in this interval would be very laborious since Bessel functions of complex argument would be involved. However, a good approximation can be made when b is small. The real part of Z will always be near unity and the imaginary part will be found to be less than $b/2$. Therefore one can let $Z = 1$ in equation (A-37) where it multiplies the factor $(2\pi a/\lambda_a)$ in the argument of the Bessel functions and let $Z - 1 = U$ in the right-hand side of Equation (A-37). With these assumptions Y can be determined as a function of (a/λ_s) and U can be determined as a function of Y. We have from Equation (A-37)

$$\frac{1}{(U + b/2)^2} + \frac{1}{(U - b/2)^2} = \frac{1 + Y}{W^2} \qquad \text{(A-37a)}$$

When $U = 0$, $W^2 = W_M^2 = W_M^2$, so that

$$1 + Y = 8(W_M/b)^2 \qquad \text{(A-42)}$$

and equation (A-37a) becomes

$$\frac{1}{(U + b/2)^2} + \frac{1}{(U - b/2)^2} = (8/b^2)(W_M/W)^2 \qquad \text{(A-37}b\text{)}$$

the solution of which, for the increasing wave, is

$$U = j(b/2)[(1/2)(W/W_M)^2(\sqrt{1 + 8(W_M/W)^2} - 1) - 1]^{\frac{1}{2}} \qquad \text{(A-43)}$$

and the gain will be given by

$$\text{Gain}/b = 27.3[(1/2)(W/W_M)^2(\sqrt{1 + 8(W_M/W)^2} - 1) - 1]^{\frac{1}{2}} \qquad \text{(A-44)}$$

$$\text{db/wavelength/unit } b$$

"Decibels gain/wavelength/unit b" is plotted against $(W/W_M)^2$ in Fig. 3.

As $(W/W_M)^2$ becomes very large, the gain per wavelength approaches 27.3 b db.