The Electrostatic Field in Vacuum Tubes With Arbitrarily Spaced Elements

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VACUUM tubes with close spacing between electrodes have become of increasing importance in recent years. The higher transconductances and lower electron transit times thus obtained combine with other features to raise both the frequency and band width at which the tube may operate satisfactorily as an amplifier. Specific designs have been discussed in papers by E. D. McArthur and E. F. Peterson[1], and by Fremlin, Hall and Shatford[2]. The important contributions to structural technique made by E. V. Neher have been described in the Radiation Laboratory Series[3]. Further important advances in the art have been recently announced by J. A. Morton and R. M. Ryder of the Bell Laboratories at the recent I.R.E. Electronics Conference held at Cornell University in June, 1948. The material of the present paper represents work done by the authors over a decade ago, and naturally there has been considerable publication on related topics in the intervening years. It has been suggested by our colleagues, however, that some of the results are not available in the technical literature and are of sufficient utility to warrant a belated publication. These results have to do with the variation of the electric intensity, amplification factor, and current density which takes place along the cathode surface because of the nearby grid wires.

We shall deal mainly with the approximate solution which neglects the effect of space charge. The correction required to take account of space charge is in general relatively small as shown by both qualitative argument and experimental data in an early paper by R. W. King[15]. More recent theoretical work[19] extending into the high frequency realm has confirmed the minor nature of the modification needed. The problem is thereby reduced to one of finding solutions of Laplace's equation which reduce to constant values on the cathode, grid, and anode surfaces. The original work on this problem was done by Maxwell[4] who calculated the electrostatic screening effect of a wire grating between conducting planes long before the vacuum tube was invented. All subsequent work has followed the methods outlined by Maxwell. In particular he suggested the replacement of the conducting planes by an infinite series of images of the grid wires and described an appropriate solution in series for the case of finite size wires. The useful approximation obtained when the diameter of the grid wires is assumed small compared to their spacing was discussed in detail only for the case of large distances between the grating and each of the conducting planes.

Figure 1 shows the assumed geometry of the grid, anode, and cathode. End effects are neglected. The origin is taken at the center of one of the grid wires which have radius c, and the X-axis is along the grid plane. The spacing of the wires between centers is a, the distance from grid to anode is d_2, and that from grid to cathode is d_1. No restrictions are placed on the sizes of a, d_2, and d_1. Above the anode and below the cathode is shown a doubly infinite set of images which may be inserted to replace the conducting planes of the anode and cathode. By symmetry the potential from the

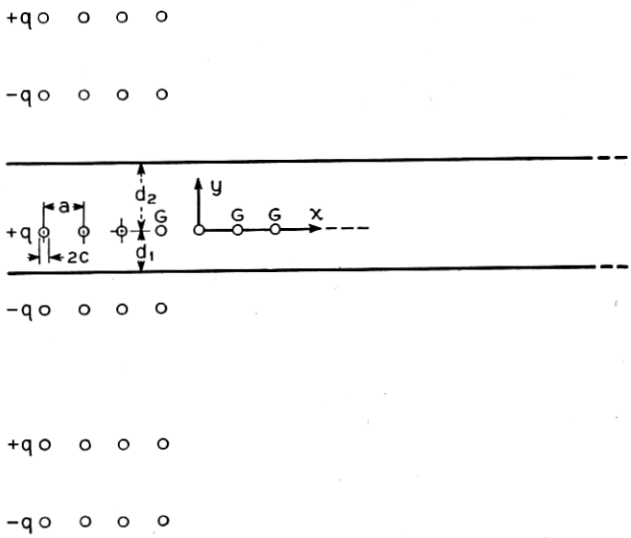

Fig. 1—Array of images for production of equipotential surfaces in planar triode.

array of charges there shown must be constant for all x when $y = d_2$ and also for all x when $y = -d_1$. The double periodicity of the array suggests immediately an application of elliptic functions. The solution of the symmetrical case was actually stated in terms of the elliptic function sn z by F. Noether[5]. The extension to the non-symmetrical case shown in Fig. 2 is fairly obvious. One of the authors worked out such a solution in terms of Jacobi's Theta functions in 1935, but abandoned any plans for publishing his analysis in view of the excellent treatment appearing shortly after that time in the Proceedings of the Royal Society by Rosenhead and Daymond[6], who applied Theta functions to both tetrodes and triodes, and both cylindrical and planar tube structures for the case of fine grid wires. Some of their formulas were later included in a book by Strutt[7]. Methods of calculating the case of thick grid wires in terms of expansions in series of elliptic functions were discussed by Knight, Howland and McMullen[8–10]. The problem of a finite number of grid wires was treated by Barkas[11]. More recently tubes with close spacing between grid and cathode, but with anode and grid assumed far apart, have been analyzed in terms of elementary functions by

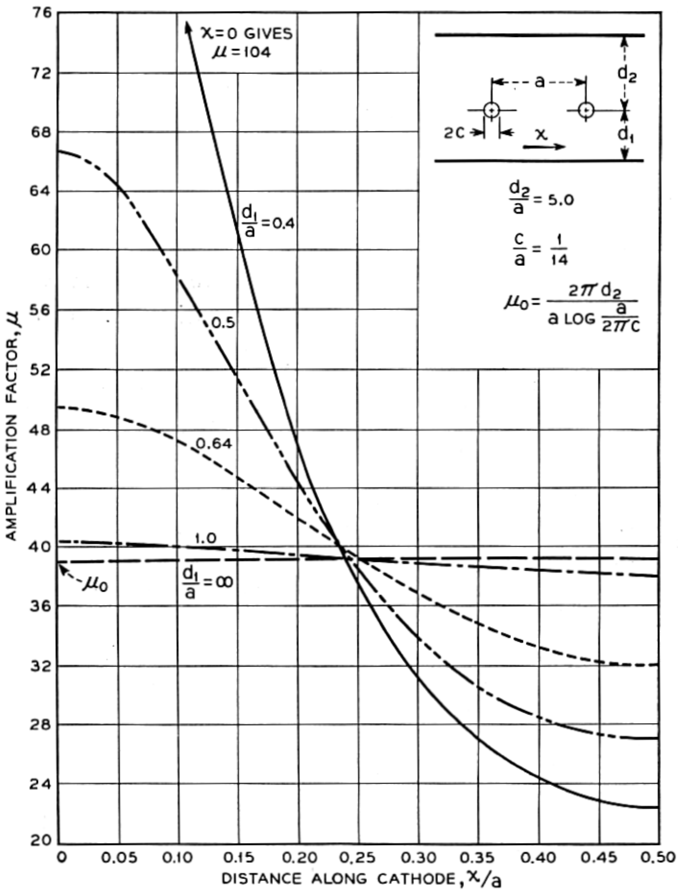

Fig. 2—Variation of amplification factor along the cathode surface of a triode.

Fremlin[12]. A solution based on the Schwartz-Christoffel transformation has been given by Herne[13] for the case of grid wires of finite size and approximately circular in shape.

Since the derivations have been adequately covered in the references cited, we merely state the final formula here and indicate how it may be verified as correct. Let $V(x, y)$ represent the potential function corresponding to

Fig. 1, the planar triode with fine grid wires. The potential of the cathode is set equal to zero. Then in the space between anode and cathode,

$$AV(x, y) = [2\pi\, d_2(y - d_2)/a + (d_1 + d_2)f(x, y)]V_g + [B(y + d_1) - 2\pi\, d_2\, y/a - d_1 f(x, y)]V_p, \quad (1)$$

where

$$f(x, y) = \ell n \left| \frac{\vartheta_1 [\pi(x + iy - 2i\, d_2)/a]}{\vartheta_1 [\pi(x + iy)/a]} \right| \quad (2)$$

$$A = (d_1 + d_2)B - 2\pi\, d_2^2/a \quad (3)$$

$$B = \ell n \left| \frac{a\vartheta_1(2\pi i\, d_2/a)}{\pi c\vartheta_1'(0)} \right| \quad (4)$$

Here we have used Jacobi's notation for the ϑ_1-function, as explained by Whittaker and Watson[14], rather than the Tannery-Molk notation used by Rosenhead and Daymond. We write $\vartheta_1(\pi z)$ for their $\vartheta_1(z)$. In our notation

$$\vartheta_1(z) = 2 \sum_{n=0}^{\infty} (-)^n e^{i(n+1/2)^2\tau} \sin (2n + 1)z \quad (5)$$

where the parameter τ in the above formulas is given by:

$$\tau = 2i(d_1 + d_2)/a \quad (6)$$

By $\vartheta_1'(z)$ is meant the derivative with respect to z:

$$\vartheta_1'(z) = 2 \sum_{n=0}^{\infty} (-)^n(2n + 1)e^{i(n+1/2)^2\tau} \cos (2n + 1)\, z \quad (7)$$

Verification of the solution is straightforward. The resulting $V(x, y)$ is seen to be the real part of a function which is analytic in the complex variable $x + iy$ except for logarithmic singularities at the points where the Theta functions vanish. Hence $V(x, y)$ satisfies Laplace's equation in two dimensions in the region excluding the singular points. Since the zeros of $\vartheta_1(z)$ occur at $z = m\pi + n\pi\tau$, where m and n take on all positive and negative values as well as zero, the singular points of the solution are at

$$\left. \begin{aligned} x + iy &= ma + 2in(d_1 + d_2) - 2id_2 \\ \text{and} \quad x + iy &= ma + 2in(d_1 + d_2) \end{aligned} \right\} \quad (8)$$

which coincide with the centers of the image circles of Fig. 1. The logarithmic singularities represent line charges with the first set arising from a ϑ_1-function in the numerator, yielding a positive charge, and the second set from the ϑ_1-function in the denominator giving a negative sign. The equipotential curves are approximately circular in the neighborhood of the charges and hence $V(x, y)$ gives a constant potential on the surface of each grid wire if the radius of the grid wire is small compared with the spacing.

We may show by direct substitution that $V(x, y)$ becomes equal to V_p at all points of the anode and equal to zero at all points of the cathode. On the anode we have $y = d_2$ which, when substituted in the expression for $f(x, y)$, gives the logarithm of the absolute value of the ratio of conjugate complex quantities, and hence

$$f(x, d_2) = 0$$

Substituting in (1), we then readily verify that $V(x, d_2) = V_p$. On the cathode we make use of the quasi-periodicity of the ϑ_1-function, as expressed by

$$\vartheta_1(z) = -e^{i(\pi\tau+2z)}\,\vartheta_1(z + \pi\tau), \tag{9}$$

to prove

$$f(x, -d_1) = \frac{2\pi\, d_2}{a}, \tag{10}$$

from which it follows that $V(x, -d_1) = 0$. To show that all grid wires are at the same potential, we make use of the other periodicity of the ϑ_1-function,

$$\vartheta_1(z + \pi) = -\vartheta_1(z), \tag{11}$$

which shows that

$$f(x \pm ma, y) = f(x, y),\ m = 0, 1, 2, \cdots \tag{12}$$

It remains to prove that V actually approaches the value V_g in the neighborhood of the typical wire, which may be taken at the origin since the solution repeats periodically with the wire spacing. We let

$$x + iy = ce^{i\theta} \tag{13}$$

and assume $c/a \ll 1$. Expanding in power series in c/a, we find that the first order terms are included in:

$$f(c\cos\theta, c\sin\theta) = \ell\mathrm{n}\left|\frac{\vartheta_1(-2\pi i\, d_2/a)}{\vartheta_1'(0)\pi c e^{i\theta}/a}\right| = B \tag{14}$$

The sign of the argument of the ϑ_1-function in the numerator is of no consequence since it does not affect the absolute value. Substituting back in (1), we then find

$$\lim_{c/a\to 0} V(c\cos\theta, c\sin\theta) = V_g \tag{15}$$

The solution is thus completely established.

The quantities in which we are specifically interested are electric field, amplification factor, and current density. The electric field is equal to the negative gradient of the potential function. The amplification factor is found by taking the ratio of partial derivatives of the electric field at the cathode with respect to grid and anode voltages. The current density may then be studied for any assumed operating values of grid and plate voltages.

To calculate the gradient we note that since $V(x, y)$ is the real part of an analytic function $W(z) = V + iU$, it follows from the Cauchy-Riemann equations,

$$W'(z) = \frac{\partial V}{\partial x} - i\frac{\partial V}{\partial y} = -E_x + iE_y \tag{16}$$

where E_x and E_y are the x- and y- components of the electric intensity. From (1),

$$\begin{aligned} AW(z) = {} & [(d_1 + d_2)\, F(z) - 2\pi d_2(iz + d_2)/a]V_g \\ & + [B(d_1 - iz) + 2\pi i\, d_2 z/a - d_1\, F(z)]V_p. \end{aligned} \tag{17}$$

where

$$F(z) = \ell n \frac{\vartheta_1[\pi(z - 2i\, d_2)/a]}{\vartheta_1(\pi z/a)} \tag{18}$$

Calculating the derivative and making use of the relation,

$$\frac{\vartheta_1'(z - \pi\tau)}{\vartheta_1(z - \pi\tau)} = \frac{\vartheta_1'(z)}{\vartheta_1(z)} + 2i, \tag{19}$$

we find at the cathode surface

$$F'(x - i\, d_1) = \frac{2\pi i}{a}[1 + C(x)] \tag{20}$$

where

$$C(x) = Im\frac{\vartheta_1'[\pi(x + i\, d_1)/a]}{\vartheta_1[\pi(x + i\, d_1)/a]} \tag{21}$$

It follows that when $y = -d_1$, we must have $E_x = 0$ and

$$\begin{aligned} aAE_y/2\pi = {} & [d_1 + (d_1 + d_2)C(x)]V_g \\ & + [d_2 - d_1 - aB/2\pi - d_1 C(x)]V_p \end{aligned} \tag{22}$$

The amplification factor is then given by

$$\mu = \frac{\partial E_y/\partial V_g}{\partial E_y/\partial V_p} = \frac{d_1 + (d_1 + d_2)C(x)}{d_2 - d_1 - aB/2\pi - d_1 C(x)} \tag{23}$$

Numerical calculation from these formulas can be made by means of (5) and (6). When d_1 and d_2 are both large compared with unity, Eq. (23)

reduces to the familiar approximate formula derived, for example, in an early paper by R. W. King[15],

$$\mu \doteq \frac{2\pi\, d_2}{a \ln \dfrac{a}{2\pi c}}, \tag{23a}$$

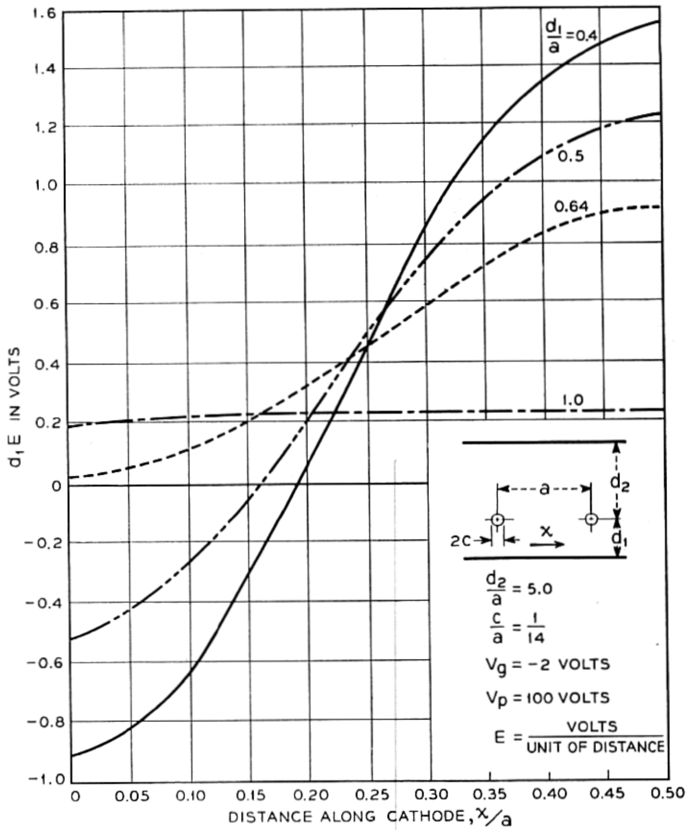

Fig. 3—Variation of cathode field strength in a triode.

Some calculated curves for μ and E_g are shown in Figs. 2 and 3. Figure 2 shows the amplification factor as a function of the distance along the cathode with the ratio of grid-cathode separation to grid wire spacing as a parameter. The ratio of grid-anode separation to grid wire spacing is held constant at five. Only half the grid spacing interval is included since the curves are symmetrical. The increase in μ-variation as the grid-cathode separation becomes small is clearly demonstrated. For negligible μ-variation we must select d_1/a of the order of 2 or greater.

Figure 3 shows the variation in field strength along the cathode for the typical operating point, $V_g = -2$ and $V_p = 100$ volts. It is to be noted that for d_1/a less than 0.6, the electric field actually changes sign as we move from a point immediately below a grid wire to the midpoint between two grid wires. In other words a part of the cathode will not emit at all in these cases while the remainder emits in a non-uniform manner. In the rather extreme case of $d_1/a = 0.4$ only about a quarter of the cathode is emitting. It is worth noting how relatively rapid the "shadow" or "island" formation increases between $d_1/a = 0.64$ and 0.5 as compared to the increase in the interval from 0.5 to 0.4.

If the equation for μ is solved for $C(x)$ and the result substituted back in the expression for E_y at the cathode we find:

$$-E_y = \frac{V_g + V_p/\mu}{d_1 + (d_1 + d_2)/\mu} \tag{24}$$

where here of course μ varies with x. This is identical with the expression derived by Benham[16] from Maxwell's approximate solution except that in the latter case μ was a constant. Our colleague, Mr. L. R. Walker, has pointed out that the equation follows directly from the assumption of small grid wires without explicit solution for the potential function. Since the charge density σ_c on the cathode is proportional to the field strength (the factor of proportionality in MKS units is the dielectric constant ϵ of vacuum or 9.854×10^{-12} farads/meter), Maxwell's capacity coefficients C_{gc} and C_{pc} may be calculated from

$$\sigma_c = \epsilon E_y = -(C_{gc}V_g + C_{pc}V_p) \tag{25}$$

The minus sign is used here because we are taking the ratio of charge to voltage at the negative plate of the condenser consisting of cathode, grid and anode surfaces. Hence

$$C_{gc} = \frac{\epsilon}{d_1 + (d_1 + d_2)/\mu} \tag{26}$$

$$C_{pc} = \frac{\epsilon/\mu}{d_1 + (d_1 + d_2)/\mu} \tag{27}$$

Since μ is variable, an integration is required to determine the total capacitance. From the periodicity of μ with grid spacing it is possible to express the result in terms of the average values of C_{gc} and C_{pc} over an interval of length a along a direction parallel to the grid plane and multiply these values by the total area of cathode surface.

Equation (24) may be interpreted in a number of different ways of which we shall mention the following two:

1. The "equivalent voltage" $V_g + V_p/\mu$ does not act at the grid but at a distance D from the cathode, where

$$D = d_1 + (d_1 + d_2)/\mu \tag{28}$$

Both the equivalent voltage and distance vary along the cathode surface.

2. The "equivalent voltage"

$$V_e = (V_g + V_p/\mu)/[1 + (1 + d_2/d_1)/\mu] \tag{29}$$

acts in the grid plane and varies with distance along the cathode surface.

As far as the cold tube is concerned the two formulas are equivalent at the cathode, but not at the grid. When the tube is heated and complete space charge is present, the two formulas also differ at the cathode. The current density in the presence of space charge is, according to (28) and Child's law:

$$I = K(V_g + V_p/\mu)^{3/2}/D^2 \tag{30}$$

while, from (29),

$$I = KV_e^{3/2}/d_1^2 \tag{31}$$

In both, $K^2 = 32\,\epsilon^2 e/81\, m$, where e/m is the ratio of electronic charge to mass. The value of current given by (31) is $[1 + (1 + d_2/d_1)/\mu]^{1/2}$ times as large as that given by (30). If $u >> 1 + d_2/d_1$ the two values are nearly the same. In tubes with close grid-to-cathode spacing the inequality may not be fulfilled. As to which viewpoint is more accurate, we note that Ferris and North in their papers [17, 18] on input loading adopted the latter, and that at high frequencies where electron transit time must be considered the second viewpoint is preferable because of the more accurate representation of effects at the grid. For a more complete discussion see Reference 19. Figure 4 shows curves of relative current density as a function of distance along the cathode as computed from Eq. (31). The transconductance for unit area of cathode surface as computed from the same equation is given by:

$$d_1^2 g_{mo} = d_1^2 \frac{\partial I}{\partial V_g} = \frac{2}{3}\epsilon\sqrt{\frac{2e}{m}\left(V_g + \frac{V_p}{\mu}\right)\left(\frac{d_1}{D}\right)^3}$$

$$= 3.512(V_g + V_p/\mu)^{1/2}(d_1/D)^{3/2} \text{ micromhos.} \tag{32}$$

The resulting variation with distance along the cathode is shown in Fig. 5.

Defining the figure of merit M at a point x along the cathode as the ratio between the transconductance $\partial I/\partial V_g$ and the sum of C_{gc} and C_{pc} at this point, we find from (30)

$$M = (4J/3)^{1/3}[d_1 + (d_1 + d_2)/\mu]^{-1/3}\mu/(\mu + 1) \tag{33}$$

where $J = eI/m\epsilon$, $e/m = 1.77 \times 10^{11}$ coulombs/kg. From (31), we find on the other hand

$$M = (4J/3\, d_1)^{1/3}\mu/(\mu + 1) \tag{34}$$

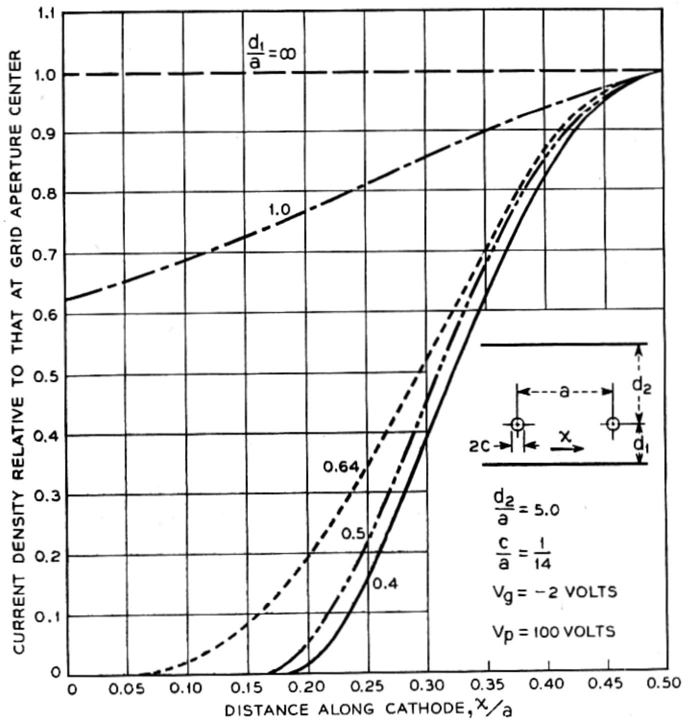

Fig. 4—Variation of current density in a triode.

Both formulas indicate that for a cathode capable of supplying a given current density the only means of improvement lies in decreasing the cathode-grid spacing. The improvement is extremely slow; doubling the figure of merit requires an eight-fold decrease in spacing.

We again emphasize that the calculated current densities and figures of merit are functions of x, the distance along the cathode. The total current between the two grid wires is found from (30) to be

$$I_T = 2K \int_{x_0}^{a/2} (V_g + V_p/\mu)^{3/2}\, dx/D^2 \tag{35}$$

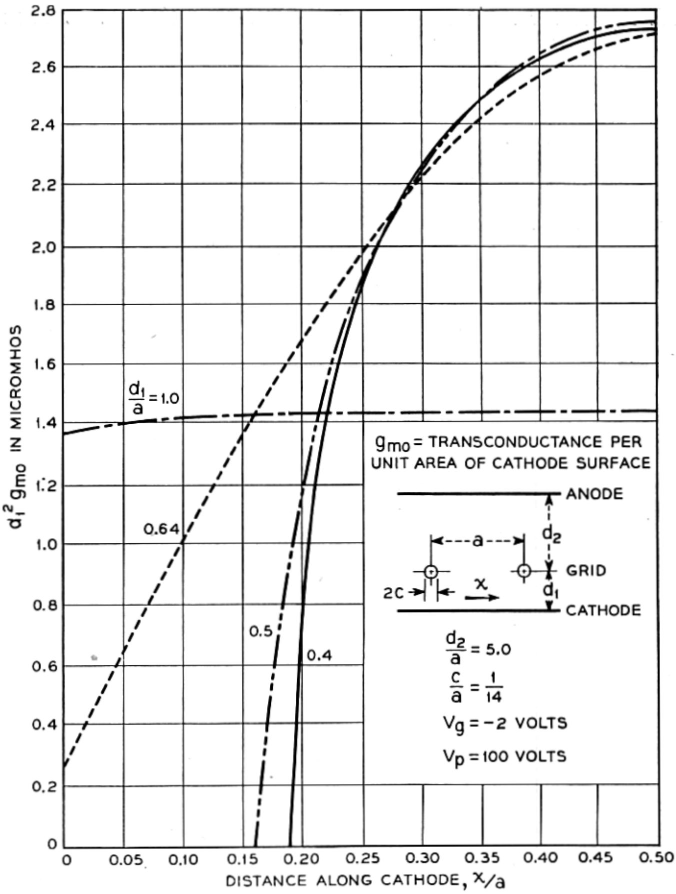

Fig. 5—Variation of transconductance along the cathode surface of a triode.

while, from (31),

$$I_T = \frac{2K}{d_1^2} \int_{x_0}^{a/2} [(\mu V_g + V_p)/(\mu + 1 + d_2/d_1)]^{3/2} \, dx \tag{36}$$

where x_0 is given by

$$V_g + V_p/\mu(x_0) = 0 \tag{37}$$

On the basis of several reasonable assumptions it may be shown that both (35) and (36) lead to an approximate 5/2 power law instead of 3/2 power law. Such a law has actually been observed in cases where shadow formation was suspected.

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