Theory of Transient Phenomena in the Transport of Holes in an Excess Semiconductor

By CONYERS HERRING

An analysis is given of the transient behavior of the density of holes n_h in an excess semiconductor as a function of time t and of position x with respect to the electrode from which they are being injected. When the geometry is one-dimensional, an exact solution for the function $n_h(x, t)$ can be constructed, provided certain simplifying assumptions are fulfilled, of which the most important are that there be no appreciable trapping of holes or electrons and that diffusion be negligible. An attempt is made to estimate the range of conditions over which the neglect of diffusion will be justified. A few applications of the theory to possible experiments are discussed.

A variety of experiments have been performed, and others are planned, which involve measurement of transient or steady-state phenomena due to the drift of positive holes along a specimen of n-type semiconductor after they have been introduced at an *injection electrode* or *emitter*.[1] These phenomena are presumably a result of the interplay of drift, space-charge, recombination, and diffusion effects. This paper seeks to relate these effects to the phenomena, and its principal contribution is an explicit calculation of the transient phenomena outside the range of small-signal theory, for cases where the geometry is one-dimensional and where certain simplifying assumptions, notably the neglect of diffusion, are justified. Removal of some of these simplifying assumptions and a more careful development of the theory will be necessary in certain applications.

Section 1 discusses the physical assumptions and boundary conditions involved in setting the problem up. Section 2 contains calculations of the distribution of holes along the length of the semiconductor at various times, for the mathematically simplest case where recombination and diffusion are ignored and all currents are held constant after the start of the injection. This simple case illustrates the method of attack to be used in the more general calculations of Section 4, and it is hoped that this sketching of basic ideas will enable the hasty reader to pass on to Section 6 without going

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[1] Experiments of this sort have been undertaken with the objective of testing and extending the theoretical interpretation of transistor action proposed by J. Bardeen and W. H. Brattain, *Phys. Rev.*, *75*, 1208 (1949), especially as regards the role of volume transport of holes, a role first suggested by J. N. Shive, *Phys. Rev.*, *75*, 689 (1949). Examples of the type of experiment discussed in the present paper have been described by: J. R. Haynes and W. Shockley, *Phys. Rev.*, *75*, 691 (1949) (transient effects); W. Shockley, G. L. Pearson, M. Sparks and W. H. Brattain, in a paper presented at the Cambridge Meeting of the American Physical Society, June 16–18, 1949 (steady-state transport); W. Shockley, G. L. Pearson, and J. R. Haynes, *Bell Sys. Tech. Jour.*, this issue (steady-state and transient effects).

through the mathematical details of Sections 3, 4, and 5. Section 3 contains the complete differential equations of the problem, including diffusion and recombination, and Section 4 gives the solution when only the diffusion terms are neglected. Section 5 contains some order-of-magnitude estimates regarding diffusion effects. Section 6 summarizes the capabilities of the theory so far developed, presents some obvious generalizations, and discusses an interesting *shock wave* phenomenon which occurs whenever the injected hole current is quickly decreased.

1. Basic Assumptions and Boundary Conditions

Consider the n-type semiconducting specimen shown in Fig. 1, having electrodes at its two ends, $x = -a$ and $x = b$, respectively, and an injection electrode system at $x = 0$ somewhere in between. Let a current of density j_a per unit area enter at the left-hand end, and let a current of density j_e be injected at $x = 0$. To make the problem strictly one-dimensional, it will be supposed that this injection takes place uniformly over the plane cross-section of the specimen at $x = 0$, instead of taking place at isolated points of the surface, as is usually the case in experiments. This idealization will presumably be justified if the thickness of the specimen is small compared with lengths in the x-direction which are significant in the experiment and if the injected positive holes are able to spread themselves uniformly over the cross-section before appreciable recombination has taken place.

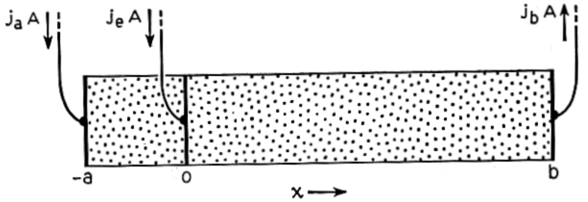

Fig. 1—Idealized experiment on hole transport in one dimension.

Unless otherwise stated, it will be supposed that j_e consists entirely of positive holes, i.e., that the number of electrons withdrawn from the specimen by the electrode at $x = 0$ is negligible compared with the number of holes injected. The currents j_a and j_e need not be constant in time, although most of the analysis to be given below will assume them constant after the time of initiation of j_e.

One can set up differential equations for the variation with x and time of the electron density, n_e, and the hole density, n_h. These equations will in the general case involve migration due to electrostatic fields, diffusion, recombination, trapping, and thermal release of electrons and holes from traps. It will be assumed, however, that trapping and thermal release from traps can be neglected, or, more precisely stated, that creation of mobile holes and electrons occurs only at the electrodes, and that the disappearance of mobile holes and electrons is caused only by mechanisms which cause holes and electrons to disappear in equal numbers at essentially the same time and place. If this assumption is valid, the charge density due to impurity centers will never differ from its equilibrium value by an amount comparable with the density due to free electrons. This assumption can be expected to be reasonably good for an n-type impurity semiconductor in which the number of donor levels is very much greater than the number of acceptor levels and for which, at the operating temperature, practically all the donor levels have been thermally ionized, while thermal excitation of electrons from the normally full band has not yet become appreciable.

As has just been mentioned, the differential equations for the behavior of the electron and hole densities involve migration under the influence of the local electric field $E(x, t)$. This field is in turn influenced by the space charge due to any inequality between the hole density n_h and the electron excess $(n_e - n_0)$, where n_0 is the normal electron density. If the difference $(n_h - n_e + n_0)$ were comparable with n_h or n_e, the problem would be very complicated. Fortunately, however, this difference cannot have an appreciable value over an appreciable range of x, on the scale of typical experiments. For example, if $(n_h - n_e + n_0)$ were 10^{-2} of n_0 for a range Δx of 1μ, and if n_0 is 10^{15} cm^{-3}, then the difference in field strength on the two sides of Δx would be about 2000v/cm, a field which would outweigh all other fields in the problem and rapidly neutralize the space charge. Moreover, the time required for the evening out of any such abnormally high space charge would be very short, of the order of magnitude of the resistivity of the specimen expressed in absolute electrostatic units (1 sec. $= 9 \times 10^{11}\ \Omega$ cm). Thus it will be quite legitimate to assume $(n_h - n_e + n_0) = 0$ in all equations of the problem except Poisson's equation which determines the field E, and so n_e can be eliminated from the conduction-diffusion equations for holes and electrons. These two equations can then be used, as is shown below, to determine the two unknown functions n_h and E, Poisson's equation being discarded as unnecessary.

The boundary conditions for these differential equations consist of two parts, the conditions at $t = 0$ and those at and to the left of $x = 0$. In most of the applications to be considered, the injection current j_e will be assumed to commence at $t = 0$. Thus, initially, the specimen will be free of holes and, at $t = 0^+$, will have a field $E_a = j_a/\sigma_0$ in the region $-a < x < 0$, and a field $E_0 = j_b/\sigma_0$ in the region $0 < x < b$, where σ_0 is the normal conductivity of the specimen and $j_b = j_a + j_e$ is the total current density to the right of $x = 0$. The boundary condition at $x = 0$ is determined by the magnitudes of the electronic and hole contributions to the injection current j_e. If no electrons are withdrawn by the electrode at $x = 0$, then the electron currents just to the left and just to the right of $x = 0$ must be equal, and the hole current densities on the two sides must differ by j_e ; if a part of j_e is due to withdrawal of electrons, then the electronic current will have a corresponding discontinuity. If j_a is positive, i.e., flows from left to right in the specimen, the current can be assumed to be practically entirely electronic over most of the range from $-a$ to 0; i.e., as x becomes negative the hole current must rapidly approach zero and the electron current must rapidly approach j_a . In fact, if diffusion is ignored the electron and hole currents must have these limiting values for any negative x.

The preceding discussion and the mathematics to follow have been couched in purely one-dimensional language, i.e., have been formulated as if the electron and hole densities were functions of x alone, independent of y and z, and as if the semiconductor extended to infinity in the y- and z-directions. However, it is easy to see at each stage that practically the same equations can be written for transport of holes along a narrow filament whose thickness is small compared with the linear scale of the phenomena along its length, even when the density of holes is not uniform over the cross-section of the filament. If the density of holes is uniform over the cross-section, all the equations will of course hold as written. However, recent work[2] has suggested that holes recombine with electrons so rapidly at the surface that the density of holes may be much smaller near the surface than in the center of the cross-section. In such case all the equations of this memorandum must be interpreted as applying to the mean value, $\bar{n}_h(x)$, of the density of holes, $n_h(x, y, z)$, averaged over the cross-section of the filament; also, the rate of recombination of holes and electrons must be set equal to some function of $\bar{n}_h$, as yet not reliably known, instead of to a constant times the product of electron and hole densities. This will of course alter most of the quantitative predictions of Section 4, but will not require any change in the method of calculation.

2. FORMULATION AND SOLUTION OF THE PROBLEM WITH NEGLECT OF DIFFUSION AND RECOMBINATION

For this case the electron and hole currents can each be equated to the product of field strength E by particle density n by mobility μ, and the continuity equations are

$$\frac{\partial n_h}{\partial t} = -\frac{\partial}{\partial x}(E\mu_h n_h) \tag{1}$$

$$\frac{\partial n_e}{\partial t} = \frac{\partial}{\partial x}(E\mu_e n_e). \tag{2}$$

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[2] H. Suhl and W. Shockley, paper Q11 presented at the Washington Meeting of the American Physical Society, April 29, 1949; see also Shockley, Pearson, Sparks and Brattain, reference 1.

Since the neutrality condition requires $\frac{\partial n_h}{\partial t} = \frac{\partial n_e}{\partial t}$, subtracting (1) and (2) and integrating gives the equation of conservation of total current:

$$E(\mu_e n_e + \mu_h n_h) = j(t)/e = \text{const. indep. of } x$$

where of course $j = j_b = (j_a + j_e)$ when $0 < x < b$ and when conditions are such that all currents flow from left to right. Putting the neutrality condition $n_e = n_h + n_0$, into the equation gives the following relation between E and n_h:

$$E[(\mu_e + \mu_h)n_h + \mu_e n_0] = j/e \tag{3}$$

This can be used to eliminate either E or n_h from (1). If E is eliminated we have

$$\frac{\partial n_h}{\partial t} = -\frac{\mu_e \mu_h n_0 j}{e[(\mu_e + \mu_h)n_h + \mu_e n_0]^2}\frac{\partial n_h}{\partial x} = -V(n_h)\frac{\partial n_h}{\partial x} \tag{4}$$

where $V(n_h)$ is an abbreviation for the coefficient shown. If, instead, n_h is eliminated from (1) a similar equation results:

$$\frac{\partial E}{\partial t} = \frac{E}{j}\frac{dj}{dt} - V(E)\frac{\partial E}{\partial x} \tag{5}$$

where

$$V(E) = eE^2\mu_h\mu_e n_0/j = E\mu_h(E/E_0) \tag{6}$$

where

$$E_0 = j/\sigma_0 \tag{7}$$

i.e., the field necessary to maintain the total current by electronic conduction in the normal state of the specimen. The velocity $V(E)$ is of course numerically the same as the $V(n_h)$ occurring in (4) when E and n_h are related by (3).

The solution can be based on either (4) or (5). We shall use (4), as n_h is the most interesting quantity for direct measurement, and as the differential equation to be given below for the case where diffusion terms are included is simpler when n_h is chosen as the dependent variable.

Equation (4) (or (5)) describes a wave propagated with the variable velocity V. If $j_e \ll j_a$, so that E is never greatly different from E_0, (4) (or (5)) and (6) indicate that n_h (or E) is propagated with the constant velocity $E_0 \mu_h$, as is of course to be expected. More interesting is the case where j_e and j_a are comparable, so that V departs significantly from constancy. It is tempting to suppose that, for this case also, the curve of n_h against x at any time t can be constructed by taking the graph of n_h against x at $t = 0$ and moving each point of the curve horizontally to the right a distance $V(n_h)t$. One can, in fact, easily verify that this construction gives a solution of (4), by writing (4) in the form

$$\left(\frac{\partial x}{\partial t}\right)_{n_h} = -\frac{\left(\dfrac{\partial n_h}{\partial t}\right)_x}{\left(\dfrac{\partial n_h}{\partial x}\right)_t} = V(n_h)$$

whence it is obvious that the function $n_h(x, t)$ defined implicitly by

$$x(n_h\,,\, t) = x(n_h\,,\, 0) + V(n_h)t$$

satisfies (4) for any form of the arbitrary function $x(n_h\,,\, 0)$, and that, conversely, any solution of (4) must be of this form.

Assuming, as in the preceding, that all currents flow from left to right, the boundary conditions at $t = 0^+$ are:

$$n_h = 0 \text{ for } x < 0 \text{ and } x > 0 \tag{8}$$

or, equivalently,

$$\left.\begin{aligned} E = E_a &= j_a/\sigma_0 &\quad \text{for} \quad x < 0 \\ E = E_0 &= (j_a + j_e)/\sigma_0 &\quad \text{for} \quad x > 0 \end{aligned}\right\}. \tag{9}$$

The boundary conditions at $x = 0$ are, for $t > 0$,

$$n_h = 0 \text{ or, equivalently, } E = E_a \text{ for } x = 0^- \tag{10}$$

and

$$n_h = n_{h1} \text{ or, equivalently, } E = E_1\,,\text{ for } x = 0^+ \tag{11}$$

where E_1 and n_{h1} are given by the requirement of continuity of electronic current, i.e.,

$$E_a n_0 \mu_e = E_1(n_0 + n_{h1})\mu_e$$

whence, using the relation (3) between E_1 and n_{h1} and expressing E_a as $j_a/n_0 e \mu_e$

$$n_{h1} = \frac{n_0}{\dfrac{j_a}{j_e}\dfrac{\mu_h}{\mu_e} - 1} \tag{12}$$

or, alternatively,

$$E_1 = E_0\left[1 - \frac{(\mu_e + \mu_h)}{\mu_h}\,\frac{j_e}{(j_a + j_e)}\right]. \tag{13}$$

According to (12), n_{h1} is small when j_e is small; and, by (13), E_1 is only slightly below E_0 for this case. As j_e increases, n_{h1} increases and E_1 decreases, and (12) and (13) would make n_{h1} infinite and E_1 zero when $j_e/j_a = \mu_h/\mu_e$. This merely means that the assumptions made in this section, in particular the neglect of diffusion and recombination or the assumption that no electrons are taken out by the injection electrode, must fail to be valid before j_e gets as large as $\mu_h j_a/\mu_e$. It will, in fact, be shown in Section 5 how the presence of enormous concentration gradients makes it essential to consider the effects of diffusion near $x = 0$ when j_e becomes large.

Putting the boundary conditions (8), (9), (10), and (11) into the wave-propagation construction described above gives the solution shown schematically in Fig. 2. An infinitesimal instant after $t = 0$, n_h is zero everywhere except in an infinitesimal interval at $x = 0$, where it rises to a maximum value n_{h1} given by (12). This is shown schematically in the upper left diagram of Fig. 2. The corresponding plot of E, shown in the upper right, dips down to E_1, which is less than either E_a or E_0, in this infinitesimal interval. After a finite time has elapsed, the curves of n_{h1} and E against x are simply those obtained by moving each point of the right-hand portions of these $t = 0^+$ curves a distance Vt horizontally to the right, as shown in the bottom two sketches. Here V depends on the ordinate in each diagram, taking on its maximum value $E_0\mu_h$ when $n_{h1} = 0$ or $E = E_0$. Since V is proportional to E^2, the curve in the lower right diagram is a parabola in the range be-

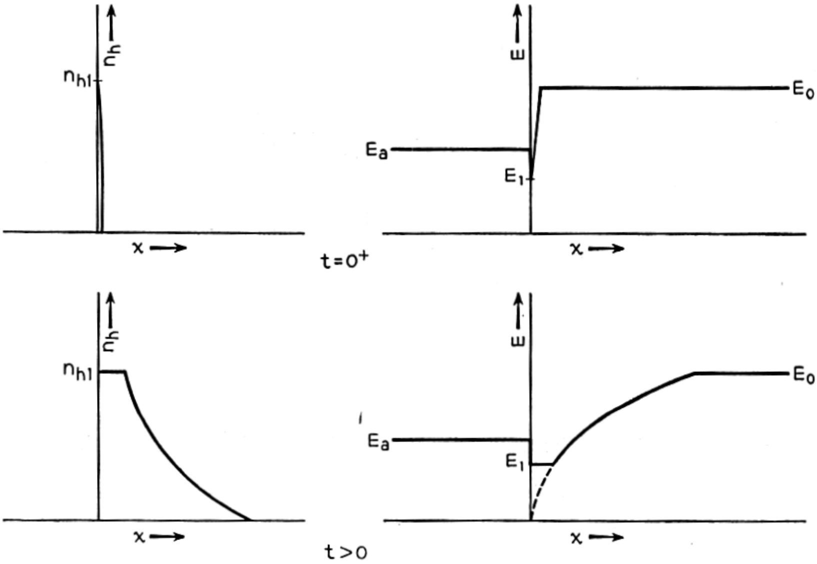

Fig. 2—Schematic variation of hole density n_h and electric field E with distance x from injection electrode and time t after the start of the injected current, in the approximation neglecting diffusion and recombination.

tween the front and the rear of the transient disturbance; this parabola, if continued, would have its vertex at the origin. After a sufficiently long time a steady state will be reached in which the field for positive x has the uniform value E_1 and the density of holes the uniform value n_{h1}.

It is possible to measure n_h as a function of t for fixed x by using a closely spaced pair of probes to measure the potential gradient E, and converting E to n_h by (3); alternatively, the current to a single negatively biased probe can be used as a measure of n_h, if calibrated by the two-probe method. The portion of this curve of n_h against t for which $0 < n_h < n_{h1}$ is given, in the present approximation, by

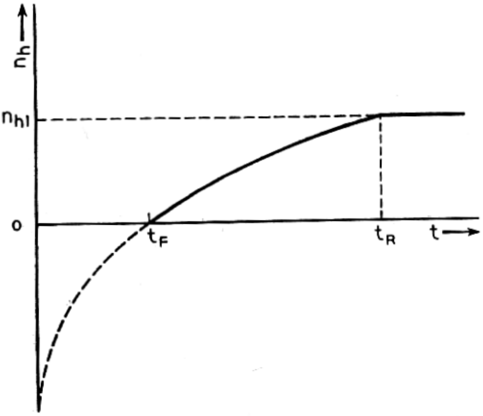

Fig. 3—Schematic variation of hole density n_h with time t after the start of the injected current, at some given distance downstream from the injection electrode, in the approximation neglecting diffusion and recombination.

$$t = x/V(n_h) = \frac{x[(\mu_e + \mu_h)n_h + \mu_e n_0]^2 e}{\mu_e \mu_h n_0 (j_a + j_e)} \tag{14}$$

$$= t_F[1 + (1 + \mu_h/\mu_e)n_h/n_0]^2$$

where

$$t_F = x/E_0\mu_h \tag{15}$$

is the time of arrival of the front of the disturbance. This curve is a parabola, as shown in Fig. 3; if continued, the parabola would have its vertex on the negative n_h axis, as shown. The rear of the disturbance, at which n_h becomes constant and equal to n_{h1}, arrives at a time t_R given by inserting n_{h1} from (12) into (14):

$$t_R = t_F/[1 - (1 + \mu_e/\mu_h)j_e/(j_a + j_e)]^2 \tag{16}$$

Note that the velocity of advance of the rear of the disturbance is less than that with which the holes drift in the steady-state field E_1. In other words, wave velocity and particle velocity must be distinguished in phenomena of this sort, although they happen to coincide at the front of the disturbance.

The discussion just given has been based on the assumption that j_a and j_e are independent of time, and that they both flow from left to right in Fig. 1. Time changes in the currents are easily taken into account in the n_h construction of Fig. 2: according to (4), it is merely necessary to move the various points of the curve of n_h against x to the right with the variable velocity $V(n_h, t)$ instead of the constant velocity $V(n_h)$; in addition, n_{h1} will in general not be a constant, so that the part of the curve for small x will no longer be a horizontal line. As for the restriction that the currents all flow from left to right, only a change of notation is needed to make all formulas apply to the case where all currents flow from right to left; and the case where part of j_e flows to the right and part to the left can, obviously, occur only under conditions where the assumptions of this section are not fulfilled, i.e., can occur only if electrons are removed at $x = 0$ or if both diffusion and recombination are important. For, if diffusion is negligible, the existence of a potential maximum at $x = 0$ implies a convergence of electrons from both sides onto the plane $x = 0$, and recombination alone cannot annihilate electrons at a finite rate in an infinitesimal volume.

Mention has already been made of the fact that equations such as (12) and (13) give an infinite density of holes when $j_e/j_a = \mu_h/\mu_e$, and are nonsensical for larger values of j_e/j_a. It is easy to see why any theory which neglects diffusion must break down for values of j_e/j_a of this size and larger if no electrons are removed by the injection electrode. If j_e/j_a is too large, any positive field just to the right of the injection plane $x = 0$ will cause more electrons to flow in the negative x-direction than can be carried off by the current j_a which flows in the region of negative x. This difficulty cannot be eliminated by making the field smaller in the region of small positive x, since making the field smaller requires a higher density of holes to carry the hole current j_e; and this in turn requires a higher density of electrons to preserve electrical neutrality. Thus, though it may be possible to realize experimental conditions under which the approximations of this section are valid for moderate values of j_e/j_a, increase of j_e/j_a above the critical value will always result in the building up of an enormously high density of holes and electrons near $x = 0$, and one must then consider diffusive transport and possibly other phenomena such as breakdown of the assumption that no electrons are removed by the injection electrode.

It will be shown below that the effect of recombination on the curves of n_h against x at various times t can be taken into account by using a geometrical construction similar to that of Fig. 2 except that, instead of moving the various points of the curve horizontally to the right with increasing time, one must move them along a family of decreasing curves (cf. Figs. 4, 5, and 6). The effect of diffusion can be described roughly as a migration of each point from one of these curves to another.

3. COMPLETE DIFFERENTIAL EQUATIONS OF THE PROBLEM

As was mentioned in Section 1, the transport of electrons and holes along a narrow filament can be described by one-dimensional equations even if recombination at the surface of the filament causes the distribution of electrons and holes to be non-uniform over its cross-section. In the equations to follow, n_h and n_e will be understood to refer to averages, over the cross-section, of the hole and electron densities, respectively; the electrostatic field E can always be assumed uniform over the cross-section of the filament, if the latter is thin. The as yet uncertain influence of the surface on the rate of recombination of electrons and holes can be allowed for by writing the recombination rate as $n_0 R(n_h/n_0)/\tau$ particles per unit volume per unit time, where R is a function which is asymptotically n_h/n_0 as its argument $\rightarrow 0$, and where τ is the recombination time for small hole densities. For pure volume recombination, $R = n_h n_e/n_0^2 = (n_h/n_0)(1 + n_h/n_0)$, while a conceivable extreme of surface recombination would be $R = n_h/n_0$.

Using this function, the continuity equations for electrons and holes can then be written, with inclusion of recombination and diffusion terms

$$\frac{\partial n_h}{\partial t} = -\frac{\partial}{\partial x}(E\mu_h n_h) - \frac{n_0}{\tau} R\left(\frac{n_h}{n_0}\right) + \frac{\partial}{\partial x}\left(D_h \frac{\partial n_h}{\partial x}\right) \tag{17}$$

$$\frac{\partial n_e}{\partial t} = \frac{\partial}{\partial x}(E\mu_e n_e) - \frac{n_0}{\tau} R\left(\frac{n_h}{n_0}\right) + \frac{\partial}{\partial x}\left(D_e \frac{\partial n_e}{\partial x}\right) \tag{18}$$

where the D's are the diffusion constants, related to the mobilities μ by the Einstein relation

$$D/\mu = kT/e \tag{19}$$

Using the neutrality condition $n_e = n_0 + n_h$, subtracting (17) from (18) and integrating gives the equation of constancy of current, the generalization of (3):

$$E[(\mu_e + \mu_h)n_h + \mu_e n_0] + \frac{kT}{e}(\mu_e - \mu_h)\frac{\partial n_h}{\partial x} = j(t)/e. \tag{20}$$

Solving for E gives

$$E = \frac{j - kT(\mu_e - \mu_h)\dfrac{\partial n_h}{\partial x}}{e[(\mu_e + \mu_h)n_h + \mu_e n_0]} \tag{21}$$

which can be substituted into (17) to give a differential equation for n_h alone:

$$\frac{\partial n_h}{\partial t} = -\frac{j}{e}\frac{\partial}{\partial x}\left[\frac{\mu_h n_h}{(\mu_e + \mu_h)n_h + \mu_e n_0}\right] - \frac{n_0}{\tau} R\left(\frac{n_h}{n_0}\right) + \frac{kT}{e}\mu_h \mu_e \frac{\partial}{\partial x}\left[\frac{(n_0 + 2n_h)\dfrac{\partial n_h}{\partial x}}{(\mu_e + \mu_h)n_h + \mu_e n_0}\right]. \tag{22}$$

The first term on the right represents drift, the second recombination, and the third diffusion. This holds whether j is constant in time or not. However, as the remainder of this memorandum will be devoted to the case where the currents involved are held constant after their initiation, it will be convenient to simplify the notation by introducing a current-dependent scale for x and writing the equation in terms of the dimensionless variables

$$\nu = n_h/n_0\,, \; s = t/\tau,\; \xi = x/E_0\mu_h\tau = xen_0\mu_e/j\mu_h\tau \tag{23}$$

In terms of these (22) becomes simply

$$\frac{\partial \nu}{\partial s} = -\frac{\partial}{\partial \xi}\left[\frac{\nu}{1 + (1 + \mu_h/\mu_e)\nu}\right] - R(\nu) + \left(\frac{J}{j}\right)^2 \frac{\partial}{\partial \xi}\left[\frac{(1 + 2\nu)\dfrac{\partial \nu}{\partial \xi}}{1 + (1 + \mu_h/\mu_e)\nu}\right] \tag{24}$$

where $R(\nu) = \nu(1 + \nu)$ for pure volume recombination, or $= \nu$ for a surface recombination uninfluenced by the electron density, and where

$$\begin{aligned} J &= (kTe\,\mu_e^2 n_0^2/\mu_h\tau)^{1/2} \\ &= \sigma_0(kT/e\,\mu_h\tau)^{1/2} \end{aligned} \tag{25}$$

Numerically the characteristic field is, at 300°K, with $\mu_h = 1700$ cm^2/v sec,[3]

$$(kT/e\mu_h\tau)^{1/2} = 3.90\,(\tau/1\mu s)^{-1/2} \text{ volts/cm} \tag{26}$$

Note that the importance of the diffusion term in (24) goes down inversely as the *square* of the current density used and inversely as the *square* of the recombination time; this is because an increase in the distance the holes travel decreases the distance they diffuse by decreasing the concentration gradient, and also makes a given diffusion distance less serious by comparison with the total distance traveled. Note also that, if $\mu_e = \mu_h$, the last term of (24) reduces simply to $\left(\frac{J}{j}\right)^2 \frac{\partial^2 \nu}{\partial \xi^2}$, but that, if $\mu_e \neq \mu_h$, the diffusion term is not a simple second derivative.

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[3] G. L. Pearson, paper Q9 presented at the Washington Meeting of the American Physical Society, April 29, 1949.

4. SOLUTION INCLUDING RECOMBINATION BUT NEGLECTING DIFFUSION

It is plausible to expect by analogy with Fig. 2 that (24) can be solved, neglecting the last term, by a similar construction in which the curve of n_h against x at time t is derived from that at time 0 by moving each point to the right along a descending curve, instead of along a horizontal line as before. To show that this is indeed the case, and at the same time to show that the diffusion term cannot so easily be taken into account, let (24) be written, omitting its last term, as

$$\frac{\partial \nu}{\partial s} = -\Phi(\nu)\,\frac{\partial \nu}{\partial \xi} - R(\nu)$$

where Φ is just the translation into dimensionless variables of the velocity V encountered in (4). This can be converted into a differential equation for ξ by writing

$$\left(\frac{\partial \nu}{\partial s}\right)_\xi = -\frac{\left(\frac{\partial \xi}{\partial s}\right)_\nu}{\left(\frac{\partial \xi}{\partial \nu}\right)_s}$$

and multiplying through by $\left(\frac{\partial \xi}{\partial \nu}\right)_s$:

$$\left(\frac{\partial \xi}{\partial s}\right)_\nu = R(\nu)\left(\frac{\partial \xi}{\partial \nu}\right)_s + \Phi(\nu) \tag{27}$$

or with $w = \int \frac{d\nu}{R(\nu)}$,

$$\frac{\partial \xi}{\partial s} - \frac{\partial \xi}{\partial w} = \Phi(w)$$

$$\frac{\partial\left(\xi + \int \Phi\, dw\right)}{\partial s} - \frac{\partial\left(\xi + \int \Phi\, dw\right)}{\partial w} = 0$$

whence the general solution is

$$\xi = -\int \Phi\, dw + f(s + w) \tag{28}$$

where f is an arbitrary function. If the same transformation is tried on (24) with the diffusion term retained, the equation corresponding to (27) has an additional term on the right containing a quotient of second and first derivatives of ξ with respect to ν, and the simple explicit solution fails.

To apply (28) to explicit calculation, or even to visualize it physically, it is necessary to determine the proper form of the arbitrary function f to fit the boundary conditions of the problem. This is most conveniently done by introducing a family of curves as suggested by the analogy with Fig. 2. The analogy suggests that we should try to find curves in the ν, ξ plane (the full curves of Fig. 4) such that a point can move along any one of them with velocity components

$$\frac{d\xi}{ds} = \Phi, \qquad \frac{d\nu}{ds} = -R.$$

The equation of any such curve is

$$\frac{d\xi}{d\nu} = -\Phi/R$$

or

$$\xi(\nu, \nu_0) = \int_{\nu}^{\nu_0} \frac{\Phi\, d\nu}{R} \tag{29}$$

where ν_0, the intercept of the curve on the ν-axis, is taken as a parameter distinguishing the curve in question from others of the family. A point which starts at height ν_0 on the ν-axis at time $s = 0$ will reach height ν at time

$$s(\nu, \nu_0) = \int_{\nu}^{\nu_0} \frac{d\nu}{R}. \tag{30}$$

Thus, after time s, the locus of all points which start at all the various heights ν_0 will be the curve obtained by eliminating ν_0 between (29) and (30) (shown dotted in Fig. 4). That this curve is, in fact, of the form (28) and therefore a solution of the differential equation is easily seen by writing (29) and (30) in terms of integrals taken from some arbitrary but fixed lower limit:

$$\xi(\nu, \nu_0) = -\int^{\nu} \frac{\Phi\, d\nu}{R} + \int^{\nu_0} \frac{\Phi\, d\nu}{R}$$

$$s(\nu, \nu_0) = -\int^{\nu} \frac{d\nu}{R} + \int^{\nu_0} \frac{d\nu}{R}.$$

As ν_0 is varied both the integrals with upper limit ν_0 will vary, and either can be expressed as a function of the other:

$$\int^{\nu_0} \frac{\Phi d\nu}{R} = f\left(\int^{\nu_0} \frac{d\nu}{R}\right)$$

whence

$$\xi = -\int^{\nu} \frac{\Phi\, d\nu}{R} + f\left(s + \int^{\nu} \frac{d\nu}{R}\right)$$

which is identical with (28).

The equations (29) and (30) of course apply only to the portion of the curve of ν against ξ which is derived from starting points ν_0 on the ν axis which are less than the maximum value ν_1 corresponding to the value n_{h1} given by (12): The points for $\nu_0 < \nu_1$ are merely initiated at time $s = 0$ and propagated by the differential equation from then on; the point $\nu = \nu_1$, $\xi = 0$, on the other hand, remains a source at all times from the initiation of the injection onward. Thus the complete curve of ν against ξ for any positive s follows the dotted construction of Fig. 4 from the ξ axis up to where it intersects the full curve corresponding to $\nu_0 = \nu_1$, after which it follows the latter curve, as indicated by the crosses in the figure. The steady-state distribution is thus simply the full curve for $\nu_0 = \nu_1$.

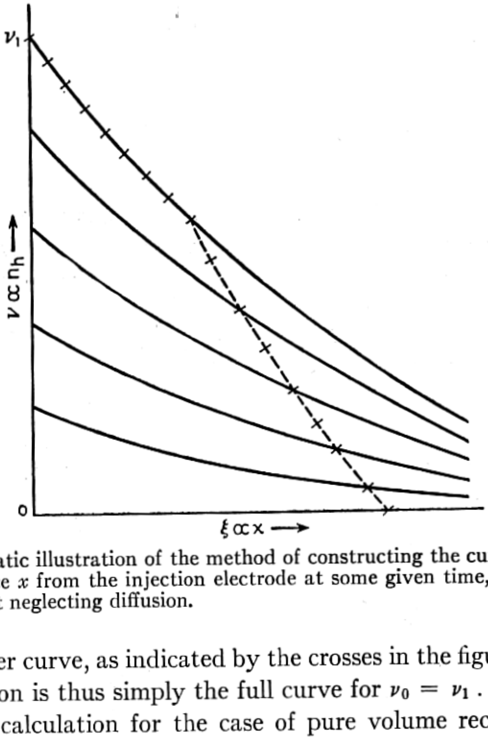

Fig. 4—Schematic illustration of the method of constructing the curve of hole density n_h against distance x from the injection electrode at some given time, taking account of recombination but neglecting diffusion.

For explicit calculation for the case of pure volume recombination one must insert $\Phi = 1/[1 + (1 + \mu_h/\mu_e)\nu]^2$, $R = \nu(1 + \nu)$ into (29) and (30). The integrations are easily carried out and give

$$\xi = \left[\frac{1 + \mu_e/\mu_h}{1 + (1 + \mu_h/\mu_e)\nu_0} + \ln \frac{\nu_0}{1 + (1 + \mu_h/\mu_e)\nu_0} + \frac{\mu_e^2}{\mu_h^2} \ln \frac{1 + (1 + \mu_h/\mu_e)\nu_0}{1 + \nu_0}\right] - [\text{same with } \nu \text{ instead of } \nu_0] \tag{31a}$$

$$s = \ln \frac{\nu_0}{1 + \nu_0} - \ln \frac{\nu}{1 + \nu} \tag{32a}$$

For the case of a surface recombination uninfluenced by electron concentration one obtains similarly, with $R = \nu$:

$$\xi = \left[\frac{1}{1 + (1 + \mu_h/\mu_e)\nu_0} - \ln \frac{\nu_0}{1 + (1 + \mu_h/\mu_e)\nu_0}\right] - [\text{same with } \nu \text{ instead of } \nu_0] \tag{31b}$$

$$s = \ln \frac{\nu_0}{\nu} \tag{32b}$$

When $\mu_e = 3\mu_h/2$, as for germanium, (31a) and (31b) become respectively

$$\xi = \left[\frac{5/2}{1 + 5\nu_0/3} + \ln \frac{\nu_0}{1 + 5\nu_0/3} + \frac{9}{4} \ln \frac{1 + 5\nu_0/3}{1 + \nu_0}\right] - [\text{same with } \nu \text{ instead of } \nu_0] \tag{33a}$$

and

$$\xi = \left[\frac{1}{1 + 5\nu_0/3} + \ln \frac{\nu_0}{1 + 5\nu_0/3}\right] - [\text{same with } \nu \text{ instead of } \nu_0] \tag{33b}$$

These can also be written, using (32a) and (32b),

$$\xi = s + \frac{5/2}{1 + 5\nu_0/3} - \frac{5/2}{1 + 5\nu/3} + \frac{5}{4} \ln \left[\frac{(1 + 5\nu_0/3)(1 + \nu)}{(1 + 5\nu/3)(1 + \nu_0)}\right] \tag{34a}$$

and

$$\xi = s + \ln \left(\frac{\nu + 3/5}{\nu_0 + 3/5}\right) - \frac{1}{1 + 5\nu/3} + \frac{1}{1 + 5\nu_0/3} \tag{34b}$$

Figures 5a and 5b show as a full curve the plot of eq. (33a) for the case $\nu_0 = \infty$, and the full curve in Fig. 6 shows in the same way the plot of (33b) for $\nu_0 = \infty$. Changing ν_0 of course merely shifts either curve horizontally. Note the very sharp increase of ν for small ξ, which shows up in pronounced manner on the expanded scale of Fig. 5b. The corresponding values of s, computed from (32a) or (34a), are marked on the curve of Fig. 5; the corresponding marks on the curve of Fig. 6 also represent values of s at intervals of 0.2, but are not labeled with absolute values because (32b) is infinite for $\nu_0 = \infty$.

For large ξ, ν becomes very small and it becomes legitimate to expand the logarithms. The first few terms of the resulting asymptotic expression for ξ are, for $\nu_0 = \infty$ and the recombination function leading to (31a),

$$\xi \sim \left(\frac{\mu_e^2}{\mu_h^2} - 1\right) \ln (1 + \mu_h/\mu_e) - (1 + \mu_e/\mu_h) - \ln \nu + (3 + 2\mu_h/\mu_e)\nu \tag{35a}$$

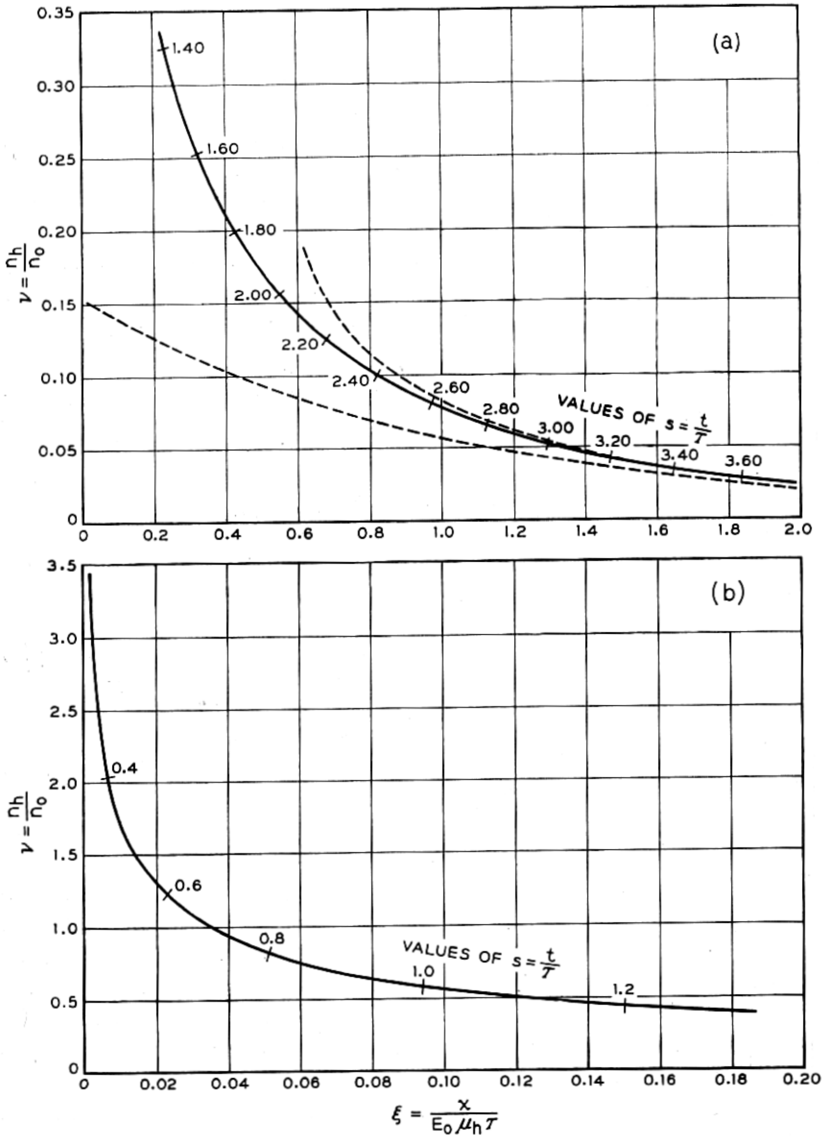

Fig. 5—Steady-state curve of hole density n_h against distance x, for the case of ideal volume recombination (recombination rate $= n_h n_e/\tau n_0$), and asymptotic approximations to this curve.

while, for the recombination function leading to (31b),

$$\xi \sim -\ln(1 + \mu_h/\mu_e) - 1 - \ln\nu + 2(1 + \mu_e/\mu_h)\nu \qquad (35b)$$

In Figs. 5a and 6 the lower dotted curve represents the sum of the terms of (35a) or (35b) respectively as far as the term in ln ν: in this approximation the dependence of ν on ξ is exponential. An exponential behavior of this sort is assumed in the small-signal theory of the modulation of the resistance of a filament of semiconductor by hole injection.[4] The upper dotted curve in Figs. 5a and 6 is a plot of (35a) or (35b), respectively, with the linear term included. It will be seen that in both figures the simple exponential approximation is already quite far off when $\nu = n_h/n_0 = 0.1$, though it improves rapidly for smaller ν.

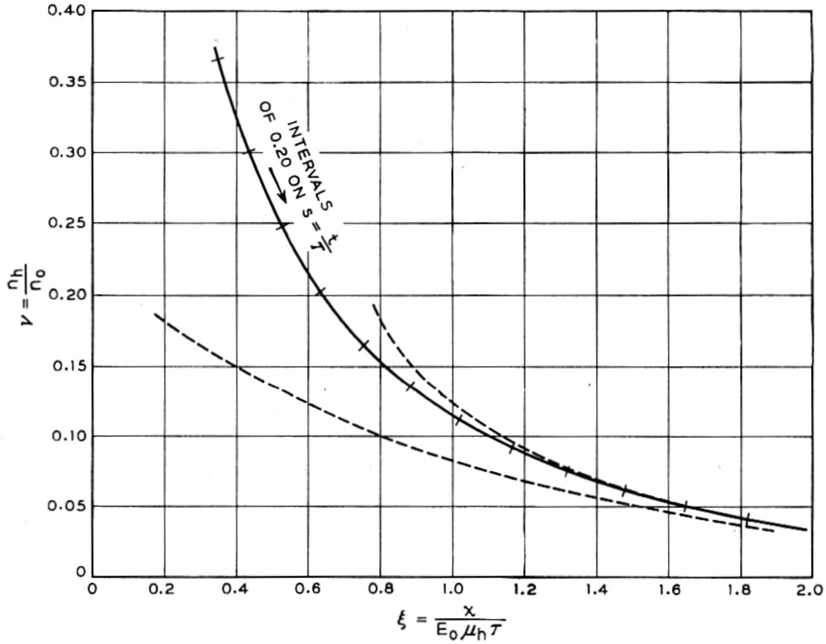

Fig. 6—Steady-state curve of hole density n_h against distance x, for the case of ideal surface recombination (recombination rate $= n_h/\tau$), and asymptotic approximations to this curve.

Figure 7 shows a sample plot of ν against ξ for the case of ideal volume recombination (eqs. (31a) etc.), for the numerical conditions $s = 1$, $\nu_1 = 0.3$ (cf Fig. 4). According to (12), whose validity at $\xi = 0$ is unimpaired by the occurrence of recombination, this value of ν_1 implies $j_a/j_c = 6.5$. The left-

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[4] W. Shockley, G. L. Pearson, and J. R. Haynes, *Bell Sys. Tech. Jour.*, this issue.

hand portion of this curve is simply traced from Fig. 5, with a horizontal shift sufficient to give an intercept at $\nu = 0.3$; the right-hand portion was constructed by placing the paper for Fig. 7 over that for Fig. 5, shifting horizontally until the point corresponding to one of the values of s marked on Fig. 5 lay on the axis of ordinates of Fig. 7, marking the position of the point labeled with one plus this value of s, and repeating.

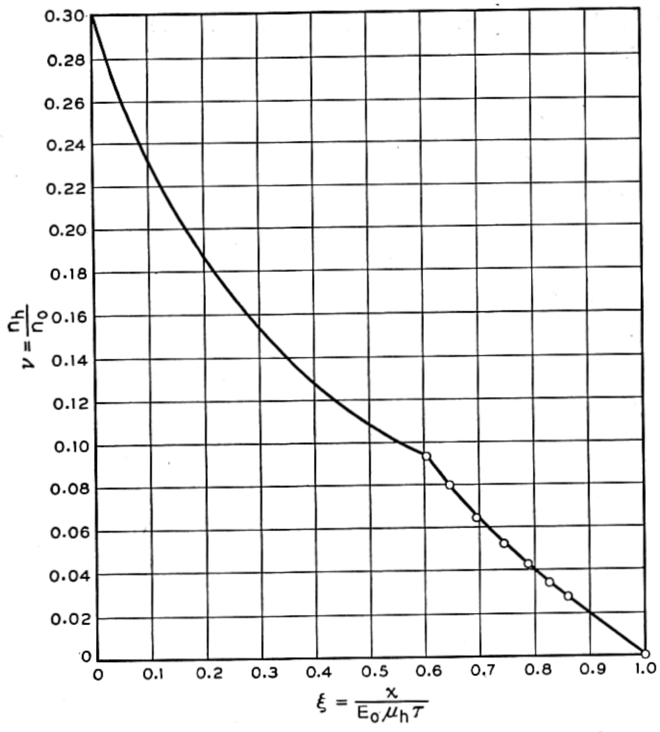

Fig. 7—Variation of hole density n_h with distance x at time $t = \tau$ assuming $n_{h1} = 0.3\, n_0$ recombination rate $= n_h n_e / \tau n_0$, and neglecting diffusion.

Figure 8 shows sample plots of ν against s for the same case of ideal volume recombination, with $\nu_1 = 0.3$, for $\xi = 0.5$ and $\xi = 1.0$. Curves for a different ν_1 would start out exactly the same, but rise higher. The rising portion of the curve for $\xi = 0.5$, for example, was constructed from the curve of Fig. 5a by locating various points (ξ, ν) on the latter curve and associating with the ν value of each such point a value of s equal to the difference of the s values marked on the curve of Fig. 5a for the two points abscissae ξ and $(\xi - 0.5)$. As Fig. 5a was prepared entirely by slide rule, the accuracy is not all that can be desired; the individual computed points are shown to give an idea of the magnitude of the computational errors.

For convenience in future calculations the equations will be appended which correspond to (31) to (34) when, instead of n_h, the field E is used as dependent variable in the differential equations. In terms of the dimensionless variable

$$\epsilon = E/E_0 = \frac{1}{1 + \nu(1 + \mu_h/\mu_e)} \tag{36}$$

and the parameter ϵ_0 corresponding to $\nu = \nu_0$, the equations are, for ideal volume recombination (eqs. (31a) etc.),

$$\xi = \left[(1 + \mu_e/\mu_h)\epsilon_0 - \frac{\mu_e^2}{\mu_h^2} \ln\left(1 + \frac{\mu_h}{\mu_e}\epsilon_0\right) + \ln(1 - \epsilon_0)\right] \tag{37a}$$
$$- \text{[same with } \epsilon \text{ instead of } \epsilon_0]$$

$$s = \ln \frac{(1 - \epsilon_0)}{\left(1 + \frac{\mu_h}{\mu_e}\epsilon_0\right)} - \ln \frac{(1 - \epsilon)}{\left(1 + \frac{\mu_h}{\mu_e}\epsilon\right)} \tag{38a}$$

while, for the recombination function leading to eqs. (31b) etc.,

$$\xi = \epsilon_0 - \epsilon + \ln \frac{1 - \epsilon_0}{1 - \epsilon} \tag{37b}$$

$$s = \ln \left[\frac{\frac{1}{\epsilon_0} - 1}{\frac{1}{\epsilon} - 1}\right]. \tag{38b}$$

The electrostatic potential U is

$$U = -\int E\,dx = -E_0^2 \mu_h \tau \int \epsilon\, d\xi.$$

In the steady state the relation between ϵ and ξ is given by (37) with ϵ_0 set equal to ϵ_1 which, by (13), is $1 - \frac{(\mu_e + \mu_h)}{\mu_h} \frac{j_e}{(j_a + j_e)}$. For this case

$$U = -E_0^2 \mu_h \tau \left[\epsilon\xi - \int \xi\, d\epsilon\right]$$
$$= -E_0^2 \mu_h \tau \left[\epsilon(\mu_e^2/\mu_h^2 - 1) - \epsilon^2(1 + \mu_e/\mu_h)/2 \right. \tag{39a}$$
$$\left. - \ln(1 - \epsilon) - \frac{\mu_e^2}{\mu_h^2} \ln\left(1 + \frac{\mu_h}{\mu_e}\epsilon\right)\right] + \text{const.}$$

for ideal volume recombination; while, for the assumptions leading to eqs. (31b), etc. the relation is

$$U = -E_0^2\mu_h\,\tau[\epsilon - \epsilon^2/2 - \ln(1 - \epsilon)] + \text{const.} \tag{39b}$$

Thus, in the steady state, the difference in potential between any two points to the right of $x = 0$ can be obtained by finding the values of ϵ for these two points by (37), and then using these to evaluate the difference in the values of (39) at the two points. To the left of $x = 0$, of course, E is constant and equal to j_a/σ.

5. Diffusion Effects

Diffusion will obviously be very important at small values of $\xi = x/E_0\mu_h\tau$ when n_{h1} is large, because of the tremendous concentration gradients which Figs. 5 and 6 predict for such cases. Also, of course, diffusion will round off the discontinuities in slope which appear at the front and rear of the transient disturbance as in Fig. 7 and Fig. 8. At other points the importance of diffusion effects can be roughly estimated either by comparing the diffusion current with the drift current or by comparing the divergences of these two contributions to the current, i.e., the last and first terms on the right of (24). Referring to these terms in (24) we have

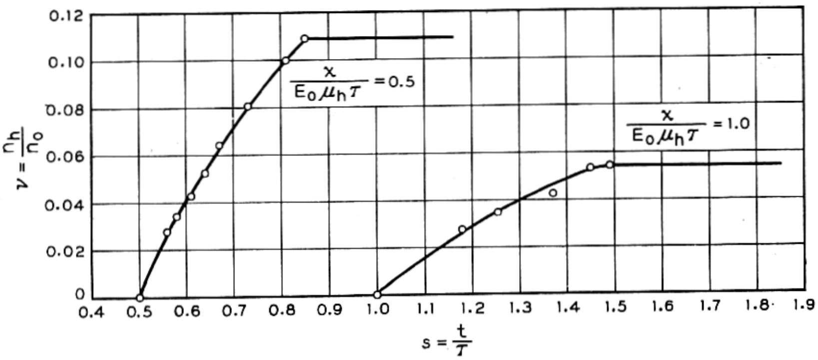

Fig. 8—Transient behavior of n_h with time at position $x/E_0\,\mu_h\tau = 0.5$ and 1.0, assuming $n_{h1} = 0.3\,n_0$ and recombination rate $= n_h n_e/\tau n_0$.

$$\left[\frac{\text{diffusion current}}{\text{drift current}}\right] = \left(\frac{J}{j}\right)^2 \frac{(1 + 2\nu)}{\nu}\left[\frac{\partial\nu}{\partial\xi}\right] \tag{40}$$

$$\left[\frac{\text{div. diffusion current}}{\text{div. drift current}}\right] = \left(\frac{J^2}{j}\right) \cdot \left[[1 + (1 + \mu_h/\mu_e)\nu][1 + 2\nu]\,\frac{\partial^2\nu}{\partial\xi^2} \bigg/ \frac{\partial\nu}{\partial\xi} + (1 - \mu_h/\mu_e)\,\frac{\partial\nu}{\partial\xi}\right]. \tag{41}$$

For the steady-state curve approximate values of the expressions (40) and (41) can be computed by evaluating the derivatives of ξ with respect to ν from (29) or (31). For the case of ideal volume recombination with $\mu_e/\mu_h = 3/2$ this gives, if the diffusion effects are not too large,

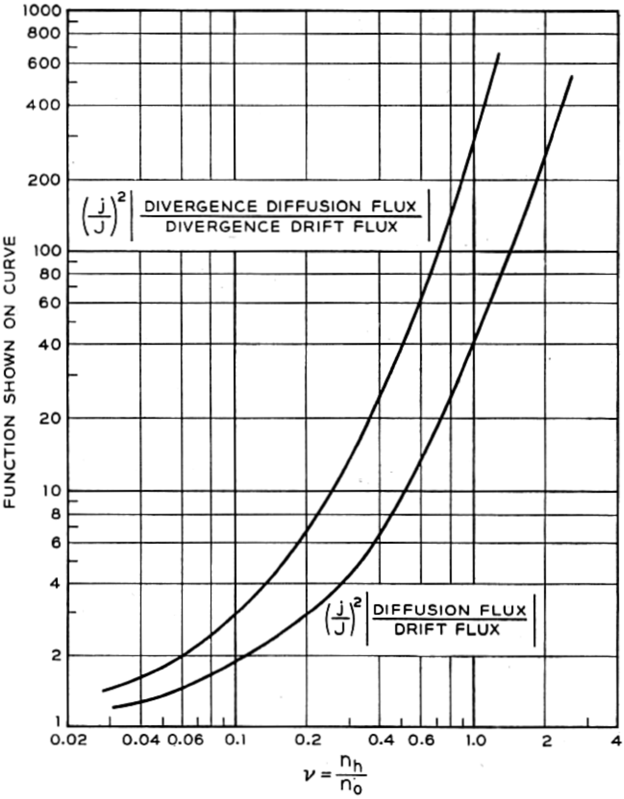

Fig. 9—Asymptotic magnitude of diffusion terms in the steady-state flux of holes, when j/J is large.

$$\left[\frac{\text{diffusion current}}{\text{drift current}}\right] \approx \left(\frac{J}{j}\right)^2 (1+\nu)(1+2\nu)(1+\tfrac{5}{3}\nu)^2 \tag{42}$$

$$\left[\frac{\text{div. diffusion current}}{\text{div. drift current}}\right] \approx \left(\frac{J}{j}\right)^2 (1+\tfrac{5}{3}\nu)^2(1+\tfrac{28}{3}\nu+21\nu^2+\tfrac{40}{3}\nu^3). \tag{43}$$

These functions are plotted in Fig. 9. From this figure one can estimate roughly when diffusion will begin to have serious effects other than a slight rounding of the leading and trailing ends of the transient. For example, if it is desired that the ratio (43) be less than about 0.1 in the steady state for values of ν as high as 0.3, the upper curve of Fig. 9 shows that the current density used must be large enough to make $\left(\frac{J}{j}\right)^2 \leq \frac{0.1}{13.6}$, i.e., $j \geq 11.7\ J$, where J is given by (25) and (26).

An approximate evaluation of (40) and (41) in the transient region can be performed by graphical or numerical differentiation of a curve such as that of Fig. 7. For example, a rough calculation based on Fig. 7 gives, in the middle of the transient portion ($\xi = 0.75$),

$$\left[\frac{\text{div. diffusion current}}{\text{div. drift current}}\right] \approx 3\left(\frac{J}{j}\right)^2.$$

More important and also more difficult to estimate is the effect of diffusion in rounding off the front and rear edges of the transient. Various ways can be devised to estimate a rough upper limit to the amount of rounding off to be expected. One such is to compute what the diffusive flux just behind the front of the advancing disturbance would be if the distribution of holes were the same as in the absence of diffusion. Under conditions where diffusion is not too serious the time integral of this diffusive flux between any two times can be equated to the increase in rounding of the front, as measured by the area between an ideal curve such as that of Fig. 7 and the actual curve of ν against ξ for the same time s. The integration cannot be extended back to time zero, however, since the integral of the flux diverges logarithmically. The fact that the diffusive flux is actually finite instead of infinite of course arises from the fact that at small times the concentration gradient a short distance behind the front can no longer be approximated by the gradient which would obtain in the absence of diffusion, but instead is very much less. This suggests that an upper limit to the total diffusive flux passing into the region of the front from time 0 to time s can be obtained by taking the flux computed as described above between the times s_0 and s, and adding to it the total number of holes which have left the injection electrode between time 0 and time s_0. Since this gives an upper limit for any s_0, one may use the minimum of the resulting sum as s_0 is varied.

The results of some sample calculations of this sort are shown in Fig. 10, which refers to the same time, currents, and recombination function as Fig. 7, viz., $s = t/\tau = 1.0$, $j_e/j_a = 2/13$, ideal volume recombination. The full curve is the transient portion of Fig. 7 replotted on a larger scale. The lower dotted curve is a curve drawn in by hand in such a way as to make the area between it and the full curve equal the upper limit computed in the manner just described, for the case $j = 100J$. The upper curve was drawn similarly for $j = 31.6J$. Since the true curve of ν against ξ must lie between the dotted curve and the full curve in each case, it can be concluded that for times and current ratios of this order the diffusionless theory of Section 4 gives a useful approximation to the transient when $j \gtrsim 100J$. At the other end, it seems likely that for $j \lesssim 10J$ the theory of Section 4 has no quantitative utility at all in the transient region.

When diffusion effects are sufficiently great, account must also be taken of the fact that the boundary conditions at the injection electrode ($x = 0$)

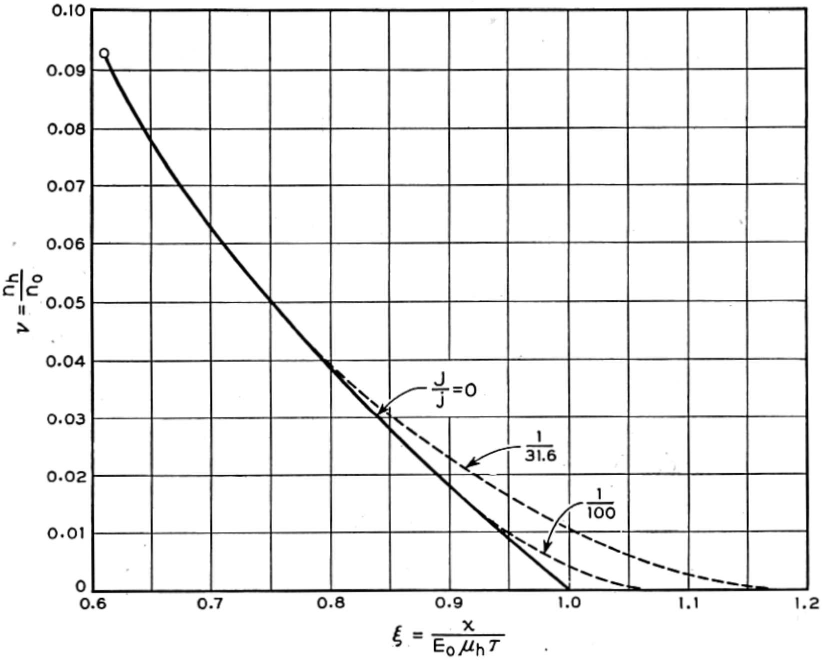

Fig. 10—Approximate magnitudes of the rounding of the front by diffusion for various values of j/J, for the case $t = \tau$, $j_e/j_a = 2/13$, ideal volume recombination. Ordinate is proportional to hole density, abscissa to distance from injection electrode.

take a different form from those in the absence of diffusion. In the absence of diffusion and with the assumption that only holes are injected at $x = 0$, the current just to the right of $x = 0$ must consist of a contribution j_e from holes and a contribution j_a from electrons, while the current just to the left of $x = 0$ is purely electronic and of magnitude j_a. This implies, as we have seen in Section 2, that the hole density be discontinuous, with the value n_{h1} given by (12) just to the right of $x = 0$, and the value zero just to the left. But if diffusion is allowed, the hole density must be continuous. For the idealized case where holes are injected on the plane $x = 0$ and no electrons are removed there, the equations to be satisfied are

$$D_e\left(\frac{\partial n_e}{\partial x}\right)_+ + n_e\mu_e E_+ = D_e\left(\frac{\partial n_e}{\partial x}\right)_- + n_e\mu_e E_- \tag{44}$$

$$-D_h\left(\frac{\partial n_h}{\partial x}\right)_+ + n_h\mu_h E_+ = \frac{j_e}{e} - D_h\left(\frac{\partial n_h}{\partial x}\right)_- + n_h\mu_h E_- \tag{45}$$

$$D_e\left(\frac{\partial n_e}{\partial x}\right)_- - D_h\left(\frac{\partial n_h}{\partial x}\right)_- + (n_e\mu_e + n_h\mu_h)E_- = j_a/e \tag{46}$$

where subscripts + and − refer to conditions just to the right of $x = 0$ and just to the left, respectively. Using the neutrality condition $n_e = n_0 + n_h$ these are three equations for the five unknowns $\left(\frac{\partial n_h}{\partial x}\right)_\pm$, $E_\pm$, n_h. To complete the determination of these quantities the differential equation (22) must be solved and the boundary conditions imposed that $n_h \to 0$ as $x \to \pm\infty$.

Actually the problem of estimating conditions at $x = 0$ may not be quite as formidable as the preceding paragraph suggests, at least if the diffusion parameter J/j is reasonably small and if j_e/j_a is also not too large. For such cases the "upstream diffusion" of holes into the region of negative x will probably reach a steady state in a very short time. Solutions of the steady state differential equation in this region have been obtained numerically by W. van Roosbroeck (unpublished). Such solutions will give one relation between n_h and $\left(\frac{\partial n_h}{\partial x}\right)_-$; another relation, in the form of a fairly narrow range of limits, is provided by the fact that $\left(\frac{\partial n_h}{\partial x}\right)_+$ will under these conditions be $\ll \left(\frac{\partial n_h}{\partial x}\right)_-$, being in fact probably somewhere between zero and the value for the diffusionless case with the same value of n_{h1}.

Of course, if the mathematical solution for this one-dimensional idealization is to be applied to a case where holes are injected into a filament by a pointed electrode on its boundary, little meaning can be attached to variations in the n_h of the mathematical solution within a range of x values smaller than the diameter of the filament.

6. Summary and Discussion

There are three principal factors which limit the range of conditions within which the present theory provides a useful approximation to the transient behavior of n_h as a function of t and x. These are diffusion, trapping, and departure from one-dimensional geometry. If the geometry is sufficiently nearly one-dimensional and trapping is negligible, the discussion of Section 5 shows that the theory of Section 4, with its neglect of diffusion, will give a useful approximation to the truth whenever the field in which the holes migrate is sufficiently strong—e.g., strong enough to make the current density $j \gtrsim 100\ J$, where J is given by (25) and (26). The obtaining of "sufficiently strong" fields without excessive heating or other undesirable effects is facilitated by the use of specimens with as long a recombination time τ as possible, and by the use of specimens of low conductivity. However, it is hard to say how low the conductivity can be made without danger that the "no trapping" assumption will break down, since for this assumption to be valid the density of hole traps must be $\ll$ the density of donors.

The numerical predictions of the theory depend upon the way in which the rate of recombination is assumed to depend upon the concentrations of electrons and holes, i.e., upon the form of the function $R(\nu)$ introduced in (17) and (18). The full curves of Figs. 5 and 6 give the steady-state dependence of n_h on x for two simple assumptions regarding $R(\nu)$, the dependence corresponding to any given boundary value n_{h1} at $x = 0$ being simply obtained by a suitable horizontal shift of the curve plotted. When the currents are held constant after their initiation, the auxiliary time scale in these figures can be used to construct the transient disturbance at any time, by the methods described in connection with the examples of Figs. 7 and 8.

These results should hold for a plane-parallel arrangement of electrodes or, to a good approximation, for electrodes placed along the length of a narrow filament, provided the n_h appearing in the equations is interpreted as a cross-sectional average of the hole density and provided the other assumptions given in Section 1 are fulfilled. It is easy to see, however, that practically the same equations apply to cases of cylindrical or spherical geometry, in the approximation where diffusion is neglected. For, in these cases, the original equations (17) and (18) merely have $\frac{\partial}{\partial x}(\cdots)$ replaced by $\frac{1}{r}\frac{\partial}{\partial r}(r\cdots)$ or $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdots)$; if the diffusion terms are neglected the solution is the same as before with x replaced by $r^2/2$ (cylindrical case) or $r^3/3$ (spherical case) and with j replaced by $I/2\pi d$ (cylindrical case, d = thickness of sample, I = total current) or by $I/4\pi$ (spherical case). However, it may be difficult to realize experimentally conditions approximating cylindrical or spherical geometry which satisfy the requirement that diffusion effects be small.

Another generalization which is easily made is the removal of the assumption that no electrons are withdrawn by the electrode at $x = 0$. As far as conditions to the right of $x = 0$ are concerned (Fig. 1), the only change required in the diffusionless theory is to interpret j_e as the current density leaving the emitter electrode in the form of holes, rather than as the total current from the emitter electrode, and to interpret j_a as the sum of the current leaving the emitter electrode in the form of electrons and any current to the left of $x = 0$.

It should also be clear that the entire analysis of this paper, though it has for definiteness been formulated for the case where holes are injected into an excess semiconductor, applies just as well to any case where electrons can be injected into a defect semiconductor. For the latter case it is merely necessary to interchange the subscripts e and h in the formulas. Though the types of experiments discussed in this paper have to date only been reported for n-type germanium, the occurrence of similar phenomena in p-type specimens is indicated by the successful use of such specimens in transistors.[5]

An interesting and possibly quite useful phenomenon should occur when, after establishment of a steady state, the current j_e is suddenly turned off. There will result a transient disturbance propagated in the direction of increasing x, which is very much like a shock wave in a gas. This, the most interesting feature of the phenomenon, will occur regardless of whether j_a remains constant when j_e is cut off; however, the simplest example for illustrative purposes is the case where j_a is increased by the amount j_e at the instant when the latter is cut off, so that j remains constant. For this case, illustrated in Fig. 11, the values of n_h ahead of the advancing front will remain the same at each point as in the previous steady state. Just behind the front, n_h must drop abruptly to zero. If j/J is large, where J is given by (25), the drop will be extremely sharp. For the change in the form of the front with time is compounded out of diffusion and propagation with variable velocity along descending curves, as shown schematically in Fig. 4. Since the latter propagation involves a more rapid motion to the right, the smaller n_h, it tends to steepen the front, and this steepening must continue until

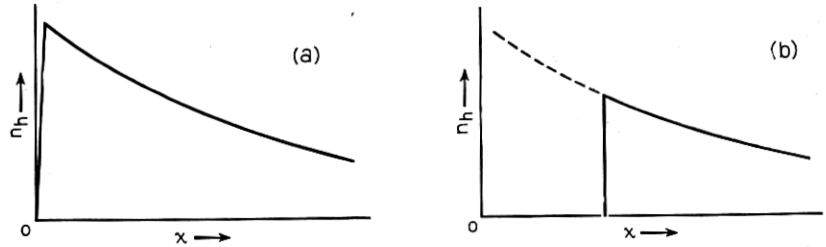

Fig. 11—Schematic variation of hole density n_h with distance x, illustrating formation of a *shock wave* by quickly decreasing j_e to zero, for the case where $j = j_e + j_a$ is kept constant.
(a) Immediately after reduction of j_e to zero.
(b) Later time.

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[5] W. G. Pfann and J. H. Scaff, paper presented at the Cambridge Meeting of the American Physical Society, June 16–18, 1949.

the diffusive spreading becomes sufficient to counterbalance it. It is not necessary, for the production of a steep front of this kind, that the decrease of j_e to zero be brought about with corresponding rapidity; even a gradual decrease of j_e will lead to a front which becomes steeper as it advances, and if the decrease of j_e is not too gradual a "shock front" will have developed after a short distance. The order of magnitude of the "shock front thickness" can be estimated by finding the value of the time Δt for which the diffusion distance $\Delta x_D = (2D \Delta t)^{1/2}$ equals the difference Δx_V between the drift distances of the holes at the top and bottom of the front, i.e., $\Delta x_V = [V(0) - V(n_h)]\Delta t$, where V is given by (4) and n_h is the height of the front. For this value of Δt,

$$\Delta x_D = 2D/[V(0) - V(n_h)] \tag{48}$$

and this is presumably of the order of magnitude of the thickness of the front. If D is interpreted as $D_h = kT\mu_h/e$, which is good enough for the present purpose, this gives

$$\Delta x_D = \frac{2kT}{eE_0} \cdot \frac{1}{\left[1 - \dfrac{1}{1 + \nu(1 + \mu_h/\mu_e)}\right]^2}. \tag{49}$$

Of course, this extremely sharp front can be realized only when the conditions of one-dimensional geometry are accurately fulfilled. When the geometry is made sufficiently ideal, observation of the thickness of the "shock front" can provide a valuable check on the validity of the basic assumptions of the theory such as the neglect of trapping.[6]

The author would like to express his indebtedness to many of his colleagues, and especially to J. Bardeen, J. R. Haynes, and W. van Roosbroeck, for many illuminating discussions of the topics covered in this paper.

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[6] The accompanying paper by W. Shockley, G. L. Pearson and J. R. Haynes describes some observations of this *shock wave* effect, though under conditions where $\nu \ll 1$, so that the thickness of the front as given by (49) is still fairly large.