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Traveling-Wave Tubes

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The following material on traveling-wave tubes is taken from a book which will be published by Van Nostrand in September, 1950. Substantially the entire contents of the book will be published in this and the three succeeding issues of the Bell System Technical Journal.

This material will cover in detail the theory of traveling-wave amplifiers. In addition, brief discussions of magnetron amplifiers and double-stream amplifiers are included. Experimental material is drawn on in a general way only, as indicating the range of validity of the theoretical treatments.

The material deals only with the high-frequency electronic and circuit behavior of tubes. Such matters as matching into circuits are not considered; neither are problems of beam formation and electron focusing, which have been dealt with elsewhere.[1]

The material opens with the presentation of a simplified theory of the traveling-wave tube. A discussion of circuits follows, including helix calculations, a treatment of filter-type circuits, some general circuit considerations which show that gain will be highest for low group velocities and low stored energies, and a justification of a simple transmission line treatment of circuits by means of an expansion in terms of the normal modes of propagation of a circuit. Then a detailed analysis of overall electronic and circuit behavior is made, including a discussion of various electronic and circuit waves, the fitting of boundary conditions to obtain overall gain, noise figure calculations, transverse motions of electrons and field solutions appropriate to broad electron streams. Short treatments of the magnetron amplifier and the double-stream amplifier follow.

[1] For instance, "Theory and Design of Electron Beams," J. R. Pierce, Van Nostrand, 1949.

CHAPTER I

INTRODUCTION

ASTRONOMERS are interested in stars and galaxies, physicists in atoms and crystals, and biologists in cells and tissues because these are natural objects which are always with us and which we must understand. The traveling-wave tube is a constructed complication, and it can be of interest only when and as long as it successfully competes with older and newer microwave devices. In this relative sense, it is successful and hence important.

This does not mean that the traveling-wave tube is better than other microwave tubes in all respects. As yet it is somewhat inefficient compared with most magnetrons and even with some klystrons, although efficiencies of over 10 per cent have been attained. It seems reasonable that the efficiency of traveling-wave tubes will improve with time, and a related device, the magnetron amplifier, promises high efficiencies. Still, efficiency is not the chief merit of the traveling-wave tube.

Nor is gain, although the traveling-wave tubes have been built with gains of over 30 db, gains which are rivaled only by the newer double-stream amplifier and perhaps by multi-resonator klystrons.

In noise figure the traveling-wave tube appears to be superior to other microwave devices, and noise figures of around 12 db have been reported. This is certainly a very important point in its favor.

Structurally, the traveling-wave tube is simple, and this too is important. Simplicity of structure has made it possible to build successful amplifiers for frequencies as high as 48,000 megacycles (6.25 mm). When we consider that successful traveling-wave tubes have been built for 200 mc, we realize that the traveling-wave amplifier covers an enormous range of frequencies.

The really vital feature of the traveling-wave tube, however, the new feature which makes it different from and superior to earlier devices, is its tremendous bandwidth.

It is comparatively easy to build tubes with a 20 per cent bandwidth at 4,000 mc, that is, with a bandwidth of 800 mc, and L. M. Field has reported a bandwidth of 3 to 1 extending from 350 mc to 1,050 mc. There seems no reason why even broader bandwidths should not be attained.

As it happens, there is a current need for more bandwidth in the general field of communication. For one thing, the rate of transmission of intelligence by telegraph, by telephone or by facsimile is directly proportional to bandwidth; and, with an increase in communication in all of these fields, more bandwidth is needed.

Further, new services require much more bandwidth than old services. A bandwidth of 4,000 cycles suffices for a telephone conversation. A bandwidth of 15,000 cycles is required for a very-high-fidelity program circuit. A single black-and-white television channel occupies a bandwidth of about 4 mc, or approximately a thousand times the bandwidth required for telephony.

Beyond these requirements for greater bandwidth to transmit greater amounts of intelligence and to provide new types of service, there is currently a third need for more bandwidth. In FM broadcasting, a radio frequency bandwidth of 150 kc is used in transmitting a 15 kc audio channel. This ten-fold increase in bandwidth does not represent a waste of frequency space, because by using the extra bandwidth a considerable immunity to noise and interference is achieved. Other attractive types of modulation, such as PCM (pulse code modulation) also make use of wide bandwidths in overcoming distortion, noise and interference.

At present, the media of communication which have been used in the past are becoming increasingly crowded. With a bandwidth of about 3 mc, approximately 600 telephone channels can be transmitted on a single coaxial cable. It is very hard to make amplifiers which have the high quality necessary for single sideband transmission with bandwidths more than a few times broader than this. In television there are a number of channels suitable for local broadcasting in the range around 100 mc, and amplifiers sufficiently broad and of sufficiently good quality to amplify a single television channel for a small number of times are available. It is clear, however, that at these lower frequencies it would be very difficult to provide a number of long-haul television channels and to increase telephone and other services substantially.

Fortunately, the microwave spectrum, which has been exploited increasingly since the war, provides a great deal of new frequency space. For instance, the entire broadcast band, which is about 1 mc wide, is not sufficient for one television signal. The small part of the microwave spectrum in the wavelength range from 6 to $7\frac{1}{2}$ cm has a frequency range of 1,000 mc, which is sufficient to transmit many simultaneous television channels, even when broad-band methods such as FM or PCM are used.

In order fully to exploit the microwave spectrum, it is desirable to have amplifiers with bandwidths commensurate with the frequency space available. This is partly because one wishes to send a great deal of information in the microwave range: a great many telephone channels and a substantial number of television channels. There is another reason why very broad bands are needed in the microwave range. In providing an integrated nation-wide communication service, it is necessary for the signals to be amplified by many repeaters. Amplification of the single-sideband type of signal used in coaxial systems, or even amplification of amplitude modulated signals, requires a freedom from distortion in amplifiers which it seems almost impossible to attain at microwave frequencies, and a freedom from interfering signals which it will be very difficult to attain. For these reasons, it seems almost essential to rely on methods of modulation which use a large bandwidth in order to overcome both amplifier distortion and also interference.

Many microwave amplifiers are inferior in bandwidth to amplifiers available at lower frequencies. Klystrons give perhaps a little less bandwidth than good low-frequency pentodes. The type 416A triode, recently developed at Bell Telephone Laboratories, gives bandwidths in the 4,000 mc range somewhat larger than those attainable at lower frequencies. Both the klystron and the triode have, however, the same fundamental limitation as do other conventional tubes. As the band is broadened at any frequency, the gain is necessarily decreased, and for a given tube there is a bandwidth beyond which no gain is available. This is so because the signal must be applied by means of some sort of resonant circuit across a capacitance at the input of the tube.

In the traveling-wave tube, this limitation is overcome completely. There is no input capacitance nor any resonant circuit. The tube is a smooth transmission line with a negative attenuation in the forward direction and a positive attenuation in the backward direction. The bandwidth can be limited by transducers connecting the circuit of the tube to the source and the load, but the bandwidth of such transducers can be made very great. The tube itself has a gradual change of gain with frequency, and we have seen that this allows a bandwidth of three times and perhaps more. This means that bandwidths of more than 1,000 mc are available in the microwave range. Such bandwidths are indeed so great that at present we have no means for fully exploiting them.

In all, the traveling-wave tube compares favorably with other microwave devices in gain, in noise figure, in simplicity of construction and in frequency range. While it is not as good as the magnetron in efficiency, reasonable efficiencies can be attained and greater efficiencies are to be expected. Finally, it does provide amplification over a bandwidth commensurate with the frequency space available at microwaves.

The purpose of this book is to collect and present theoretical material which will be useful to those who want to know about, to design or to do research on traveling-wave tubes. Some of this material has appeared in print. Other parts of the material are new. The old material and the new material have been given a common notation.

The material covers the radio-frequency aspects of the electronic behavior of the tube and its internal circuit behavior. Matters such as matching into and out of the slow-wave structures which are described are not considered. Neither are problems of producing and focusing electron beams, which have been discussed elsewhere,[1] nor are those of mechanical structure nor of heat dissipation.

In the field covered, an effort has been made to select material of practical value, and to present it as understandably as possible. References to various publications cover some of the finer points. The book refers to experimental data only incidentally in making general evaluations of theoretical results.

To try to present the theory of the traveling-wave tube is difficult without some reference to the overall picture which the theory is supposed to give. One feels in the position of lifting himself by his bootstraps. For this reason the following chapter gives a brief general description of the traveling-wave tube and a brief and specialized analysis of its operation. This chapter is intended to give the reader some insight into the nature of the problems which are to be met. In Chapters III through VI, slow-wave circuits are discussed to give a qualitative and quantitative idea of their nature and limitations. Then, simplified equations for the overall behavior of the tube are introduced and solved, and matters such as overall gain, insertion of loss, a-c space-charge effects, noise figure, field analysis of operation and transverse field operation are considered. A brief discussion of power output is given.

Two final chapters discuss briefly two closely related types of tube; the traveling-wave magnetron amplifier and the double-stream amplifier.

[1] loc. cit.

CHAPTER II

SIMPLE THEORY OF TRAVELING-WAVE TUBE GAIN

Synopsis of Chapter

IT IS difficult to describe general circuit or electronic features of traveling-wave tubes without some picture of a traveling-wave tube and traveling-wave gain. In this chapter a typical tube is described, and a simple theoretical treatment is carried far enough to describe traveling-wave gain in terms of an increasing electromagnetic and space-charge wave and to express the rate of increase in terms of electronic and circuit parameters.

In particular, Fig. 2.1 shows a typical traveling-wave tube. The parts of this (or of any other traveling-wave tube) which are discussed are the electron beam and the slow-wave circuit, represented in Fig. 2.2 by an electron beam and a helix.

In order to derive equations covering this portion of the tube, the properties of the helix are simulated by the simple delay line or network of Fig. 2.3, and ordinary network equations are applied. The electrons are assumed to flow very close to the line, so that all displacement current due to the presence of electrons flows directly into the line as an impressed current

For small signals a wave-type solution of the equations is known to exist, in which all a-c electronic and circuit quantities vary with time and distance as $\exp(j\omega t - \Gamma z)$. Thus, it is possible to assume this from the start.

On this basis the excitation of the circuit by a beam current of this form is evaluated (equation (2.10)). Conversely, the beam current due to a circuit voltage of this form is calculated (equation (2.22)). If these are to be consistent, the propagation constant Γ must satisfy a combined equation (2.23).

The equation for the propagation constant is of the fourth degree in Γ, so that any disturbance of the circuit and electron stream may be expressed as a sum of four waves.

Because some quantities are in practical cases small compared with others, it is possible to obtain good values of the roots by making an approximation. This reduces the equation to the third degree. The solutions are expressed in the form

$$-\Gamma = -j\beta_e + \beta_e C\delta$$

Here β_e is a phase constant corresponding to the electron velocity (2.16) and C is a gain parameter depending on circuit and beam impedance (2.43). A solution of the equation for the case of an electron speed equal to the speed of the undisturbed wave yields 3 values of δ which are shown in Fig. 2.4. These represent an increasing, a decreasing and an unattenuated wave. The increasing wave is of course responsible for the gain of the tube. A different approximation yields the missing backward unattenuated wave (2.32).

The characteristic impedance of the forward waves is expressed in terms of β_e, C, and δ (2.36) and is found to differ little from the impedance in the absence of electrons.

The gain of the increasing wave is expressed in terms of C and the length of the tube in wavelengths, N

$$G = 47.3\ CN \text{ db} \tag{2.37}$$

It will be shown later that the gain of the tube can be expressed approximately as the sum of the gain of the increasing wave plus a constant to take into account the setting up of the increasing wave, or the boundary conditions (2.39).

Finally, the important gain parameter C is discussed. The circuit part of this parameter is measured by the cube root of an impedance, $(E^2/\beta^2P)^{1/3}$, which relates the peak field E acting on the electrons, the phase constant $\beta = \omega/v$, and the power flow. $(E^2/\beta^2P)^{1/3}$ is a measure of circuit goodness as far as gain is concerned.

We should note also that a desirable circuit property is constancy of phase velocity with frequency, for the electron velocity must be near to the circuit phase velocity to produce gain.

Evaluation of the effects of attenuation, of varying the electron velocity and many other matters are treated in later chapters.

2.1 Description of a Traveling-Wave Tube

Figure 2.1 shows a typical traveling-wave tube such as may be used at frequencies around 4,000 megacycles. Such a tube may operate with a cathode current of around 10 ma and a beam voltage of around 1500 volts. There are two essential parts of a traveling-wave amplifier; one is the helix, which merely serves as a means for producing a slow electromagnetic wave with a longitudinal electric field; and the other is the electron flow. At the input the wave is transferred from a wave guide to the helix by means of a short antenna and similarly at the output the wave is transferred from the helix to a short antenna from which it is radiated into the output wave guide. The wave travels along the wire of the helix with approximately the speed of light. For operation at 1500 volts, corresponding to about $\frac{1}{13}$ the speed of light, the wire in the helix will be about thirteen times as long as the axial length of the helix, giving a wave velocity of about $\frac{1}{13}$ the speed of light along the axis of the helix. A longitudinal magnetic focusing field of a few hundred gauss may be used to confine the electron beam and enable it to pass completely through the helix, which for 4000 megacycle operation may be around a foot long.

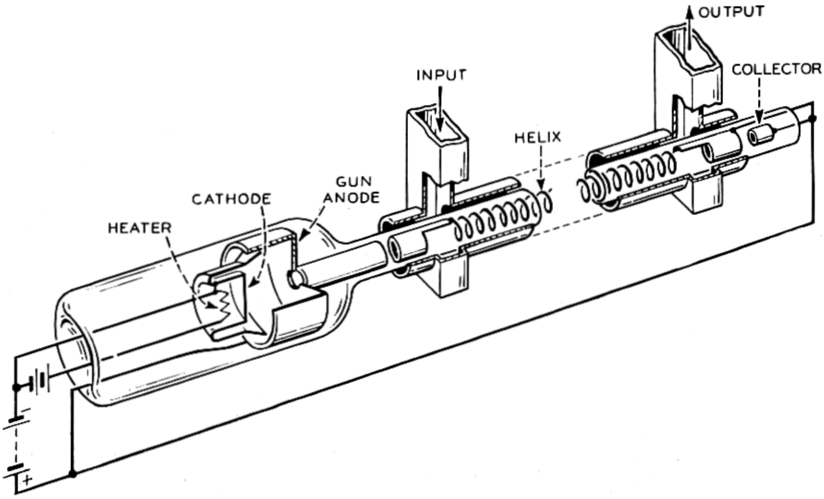

Fig. 2.1—Schematic of the traveling-wave amplifier.

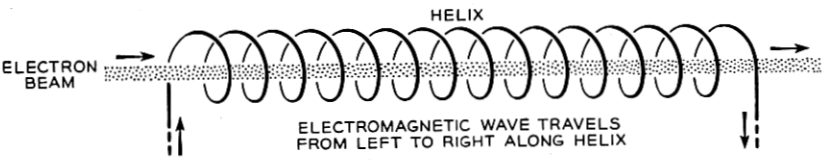

Fig. 2.2—Portion of the traveling-wave amplifier pertaining to electronic interaction with radio-frequency fields and radio-frequency gain.

In analyzing the operation of the traveling-wave tube, it is necessary to focus our attention merely on the two essential parts shown in Fig. 2.2, the circuit (helix) and the electron stream.

2.2 The Type of Analysis Used

A mathematical treatment of the traveling-wave tube is very important, not so much to give an exact numerical prediction of operation as to give a picture of the operation and to enable one to predict at least qualitatively the effect of various physical variations or features. It is unlikely that all of the phenomena in a traveling-wave tube can be satisfactorily described in a theory which is simple enough to yield useful results. Most analyses, for instance, deal only with the small-signal or linear theory of the traveling-wave tube. The distribution of current in the electron beam can have an important influence on operation, and yet in an experimental tube it is often difficult to tell just what this distribution is. Even the more elaborate analyses of linear behavior assume a constant current density across the beam. Similarly, in most practical traveling-wave tubes, a certain fraction of the current is lost on the helix and yet this is not taken into account in the usual theories.

It has been suggested that an absolutely complete theory of the traveling-wave tube is almost out of the question. The attack which seems likely to yield the best numerical results is that of writing the appropriate partial differential equations for the disturbance in the electron stream inside the helix and outside of the helix. This attack has been used by Chu and Jackson[2] and by Rydbeck.[3] While it enables one to evaluate certain quantities which can only be estimated in a simpler theory, the general results do not differ qualitatively and are in fair quantitative agreement with those which are derived here by a simpler theory.

In the analysis chosen here, a number of approximations are made at the very beginning. This not only simplifies the mathematics but it cuts down the number of parameters involved and gives to these parameters a simple physical meaning. In terms of the parameters of this simple theory, a great many interesting problems concerning noise, attenuation and various boundary conditions can be worked out. With a more complicated theory, the working out of each of these problems would constitute essentially a new problem rather than a mere application of various formulae.

There are certain consequences of a more general treatment of a traveling-wave tube which are not apparent in the simple theory presented here. Some of these matters will be discussed in Chapters XII, XIII and XIV.

The theory presented here is a small signal theory. This means that the equations governing electron flow have been linearized by neglecting certain quantities which become negligible when the signals are small. This results in a wave-type solution. Besides the small signal limitation of the analyses presented here, the chief simplifying assumption which has been made is that all the electrons in the electron flow are acted on by the same a-c field, or at least by known fields. The electrons will be acted on by essentially the same field when the diameter of the electron beam is small enough or when

[2] L. J. Chu and J. D. Jackson, "Field Theory of Traveling-Wave Tubes," *Proc. I. R. E.*, Vol. 36, pp. 853–863, July 1948.

[3] Olof E. H. Rydbeck, "The Theory of the Traveling-Wave Tube," *Ericsson Technics*, No. 46, 1948.

the electrons form a hollow cylindrical beam in an axially symmetrical circuit, a case of some practical importance.

Besides these assumptions, it is assumed in this section that the electrons are displaced by the a-c field in the axial direction only. This may be approximately true in many cases and is essentially so when a strong magnetic focusing field is used. The effects of transverse motion will be discussed in Chapter XIII.

In this chapter an approximate relation suitable for electron speeds small compared to the velocity of light is used in computing interaction between electrons and the circuit.

A more general relation between impressed current and circuit field, valid for faster waves, will be given in Chapter VI. Non-relativistic equations of motion will, however, be used throughout the book. With whatever speed the waves travel, it will be assumed that the electron speed is always small compared with the speed of light.

We consider here the interaction between an electric circuit capable of propagating a slow electromagnetic wave and a stream of electrons. We can consider that the signal current in the circuit is the result of the disturbed electron stream acting on the circuit and we can consider that the disturbance on the electron stream is the result of the fields of the circuit acting on the electrons. Thus the problem naturally divides itself into two parts.

2.3 The Field Caused by an Impressed Current

We will first consider the problem of the disturbance produced in the circuit by a bunched electron stream. In considering this problem in this section in a manner valid for slow waves and small electron velocities, we will use the picture in Fig. 2.3. Here we have a circuit or network with uniformly

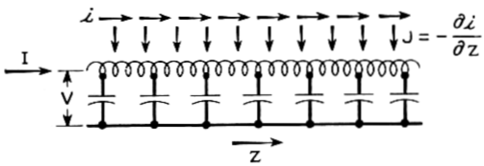

Fig. 2.3—Equivalent circuit of a traveling-wave tube. The distributed inductance and capacitance are chosen to match the phase velocity and field strength of the field acting on the electrons. The impressed current due to the electrons is $-\partial i/\partial z$, where i is the electron convection current.

distributed series inductance and shunt capacitance and with current I and voltage V. The circuit extends infinitely in the z direction. An electron convection current i flows along very close to the circuit. The sum of the displacement and convection current into any little volume of the electron beam must be zero. Because the convection current varies with distance in the direction of flow, there will be a displacement current J amperes per meter impressed on the transmission circuit. We will assume that the electron beam is very narrow and very close to the circuit, so that the displacement current along the stream is negligible compared with that from the stream to the circuit. In this case the displacement current to the circuit will be given by the rate of change of the convection current with distance.

If the convection current i and the impressed current J are sinusoidal with time, the equations for the network shown in Fig. 2.3 are

$$\frac{\partial I}{\partial z} = -jBV + J \tag{2.1}$$

$$\frac{\partial V}{\partial z} = -jXI \tag{2.2}$$

Here I and V are the current and the voltage in the line, B and X are the shunt susceptance and series reactance per unit length and J is the impressed current per unit length.

It may be objected that these "network" equations are not valid for a transmission circuit operating at high frequencies. Certainly, the electric field in such a circuit cannot be described by a scalar electric potential. We can, however, choose BX so that the phase velocity of the circuit of Fig. 2.3 is the same as that for a particular traveling-wave tube. We can further choose X/B so that, for unit power flow, the longitudinal field acting on the electrons according to Fig. 2.3, that is, $-\partial V/\partial z$, is equal to the true field for a particular circuit. This lends a plausibility to the use of (2.1) and (2.2). The fact that results based on these equations are actually a good approximation for phase velocities small compared with the velocity of light is established in Chapter VI.

We will be interested in cases in which all quantities vary with distance as $\exp(-\Gamma z)$. Under these circumstances, we can replace differentiation with respect to z by multiplication by $-\Gamma$. The impressed current per unit length is given by

$$J = -\frac{\partial i}{\partial z} = \Gamma i \tag{2.3}$$

Equations (2.1) and (2.2) become

$$-\Gamma I = -jBV + \Gamma i \tag{2.4}$$

$$-\Gamma V = -jXI \tag{2.5}$$

If we eliminate I, we obtain

$$V(\Gamma^2 + BX) = -j\Gamma Xi \tag{2.6}$$

Now, if there were no impressed current, the righthand side of (2.6) would be zero and (2.6) would be the usual transmission-line equation. In this case, Γ assumes a value Γ_1, the natural propagation constant of the line, which is given by

$$\Gamma_1 = j\sqrt{BX} \tag{2.7}$$

The forward wave on the line varies with distance as $\exp(-\Gamma_1 z)$ and the backward wave as $\exp(+\Gamma_1 z)$.

Another important property of the line itself is the characteristic impedance K, which is given by

$$K = \sqrt{X/B} \tag{2.8}$$

We can express the series reactance X in terms of Γ_1 and K

$$X = -jK\Gamma_1 \tag{2.9}$$

Here the sign has been chosen to assure that X is positive with the sign given in (2.7). In terms of Γ_1 and K, (2.6) may be written

$$V = \frac{-\Gamma\Gamma_1 Ki}{(\Gamma^2 - \Gamma_1^2)} \tag{2.10}$$

In (2.10), the convection current i is assumed to vary sinusoidally with time and as $\exp(-\Gamma z)$ with distance. This current will produce the voltage V in the line. The voltage of the line given by (2.10) also varies sinusoidally with time and as $\exp(-\Gamma z)$ with distance.

2.4 Convection Current Produced by the Field

The other part of the problem is to find the disturbance produced on the electron stream by the fields of the line. In this analysis we will use the quantities listed below, all expressed in M.K.S. units.[4]

η—charge-to-mass ratio of electrons
  $\eta = 1.759 \times 10^{11}$ coulomb/kg
u_0—average velocity of electrons
V_0—voltage by which electrons are accelerated to give them the velocity
  u_0. $u_0 = \sqrt{2\eta V_0}$
I_0—average electron convection current
ρ_0—average charge per unit length
  $\rho_0 = -I_0/u_0$
v—a-c component of velocity
ρ—a-c component of linear charge density
i—a-c component of electron convection current

[4] Various physical constants are listed in Appendix I.

The quantities v, ρ, and i are assumed to vary with time and distance as $\exp(j\omega t - \Gamma z)$.

One equation we have concerning the motion of the electrons is that the time rate of change of velocity is equal to the charge-to-mass ratio times the electric gradient.

$$\frac{d(u_0 + v)}{dt} = \eta \frac{\partial V}{\partial z} \tag{2.11}$$

In (2.11) the derivative represents the change of velocity observed in following an individual electron. There is, of course, no change in the average velocity u_0. The change in the a-c component of velocity may be expressed in terms of partial derivatives, $\frac{\partial v}{\partial t}$, which is the rate of change with time of the velocity of electrons passing a given point, and $\frac{\partial v}{\partial z}$, which is variation of electron velocity with distance at a fixed time.

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial z}\frac{dz}{dt} = \eta \frac{\partial V}{\partial z} \tag{2.12}$$

Equation (2.12) may be rewritten

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial z}(u_0 + v) = \eta \frac{\partial V}{\partial z} \tag{2.13}$$

Now it will be assumed that the a-c velocity v is very small compared with the average velocity u_0, and v will be neglected in the parentheses. The reason for doing this is to obtain differential equations which are linear, that is, in which products of a-c terms do not appear. Such linear equations necessarily give a wave type of variation with time and distance, such as we have assumed. The justification for neglecting products of a-c terms is that we are interested in the behavior of traveling-wave tubes at small signal levels, and that it is very difficult to handle the non-linear equations. When we have linearized (2.13) we may replace the differentiation with a respect to time by multiplication by $j\omega$ and differentiation with respect to distance by multiplication by $-\Gamma$ and obtain

$$(j\omega - u_0\Gamma)v = -\eta\Gamma V \tag{2.14}$$

We can solve (2.14) for the a-c velocity and obtain

$$v = \frac{-\eta\Gamma V}{u_0(j\beta_e - \Gamma)} \tag{2.15}$$

Where

$$\beta_e = \omega/u_0 \tag{2.16}$$

We may think of β_e as the phase constant of a disturbance traveling with the electron velocity.

We have another equation to work with, a relation which is sometimes called the equation of continuity and sometimes the equation of conservation of charge. If the convection current changes with distance, charge must accumulate or decrease in any small elementary distance, and we see that in one dimension the relation obeyed must be

$$\frac{\partial i}{\partial z} = -\frac{\partial \rho}{\partial t} \tag{2.17}$$

Again we may proceed as before and solve for the a-c charge density ρ

$$-\Gamma i = -j\omega\rho$$

$$\rho = \frac{-j\Gamma i}{\omega} \tag{2.18}$$

The total convection current is the total velocity times the total charge density

$$-I_0 + i = (u_0 + v)(\rho_0 + \rho) \tag{2.19}$$

Again we will linearize this equation by neglecting products of a-c quantities in comparison with products of a-c quantities and a d-c quantity. This gives us

$$i = \rho_0 v + u_0 \rho \tag{2.20}$$

We can now substitute the value ρ obtained from (2.18) into (2.20) and solve for the convection current in terms of the velocity, obtaining

$$i = \frac{j\beta_e \rho_0 v}{(j\beta_e - \Gamma)} \tag{2.21}$$

Using (2.15) which gives the velocity in terms of the voltage, we obtain the convection current in terms of the voltage

$$i = \frac{jI_0 \beta_e \Gamma V}{2V_0(j\beta_e - \Gamma)^2} \tag{2.22}$$

2.5 Overall Circuit and Electronic Equation

In (2.22) we have the convection current in terms of the voltage. In (2.10) we have the voltage in terms of the convection current. Any value of Γ for which both of these equations are satisfied represents a natural mode of propagation along the circuit and the electron stream. When we combine (2.22) and (2.10) we obtain as the equation which Γ must satisfy:

$$1 = \frac{jKI_0\beta_e\Gamma^2\Gamma_1}{2V_0(\Gamma_1^2 - \Gamma^2)(j\beta_e - \Gamma)^2} \tag{2.23}$$

Equation (2.23) applies for any electron velocity, specified by β_e, and any wave velocity and attenuation, specified by the imaginary and real parts of the circuit propagation constant Γ_1. Equation (2.23) is of the fourth degree. This means that it will yield four values of Γ which represent four natural modes of propagation along the electron stream and the circuit. The circuit alone would have two modes of propagation, and this is consistent with the fact that the voltages at the two ends can be specified independently, and hence two boundary conditions must be satisfied. Four boundary conditions must be satisfied with the combination of circuit and electron stream. These may be taken as the voltages at the two ends of the helix and the a-c velocity and a-c convection current of the electron stream at the point where the electrons are injected. The four modes of propagation or the waves given by (2.23) enable us to satisfy these boundary conditions.

We are particularly interested in a wave in the direction of electron flow which has about the electron speed and which will account for the observed gain of the traveling-wave tube. Let us assume that the electron speed is made equal to the speed of the wave in the absence of electrons, so that

$$-\Gamma_1 = -j\beta_e \tag{2.24}$$

As we are looking for a wave with about the electron speed, we will assume that the propagation constant differs from β_e by a small amount ξ, so that

$$\begin{aligned} -\Gamma &= -j\beta_e + \xi \\ &= -\Gamma_1 + \xi \end{aligned} \tag{2.25}$$

Using (2.24) and (2.25) we will rewrite (2.23) as

$$1 = \frac{-KI_0\beta_e^2(-\beta_e^2 - 2j\beta_e\xi + \xi^2)}{2V_0(2j\beta_e\xi - \xi^2)(\xi^2)} \tag{2.26}$$

Now we will find that, for typical traveling-wave tubes, ξ is much smaller than β_e; hence we will neglect the terms involving $\beta_e\xi$ and ξ^2 in the numerator in comparison with β_e^2 and we will neglect the term ξ^2 in the denominator in comparison with the term involving $\beta_e\xi$. This gives us

$$\xi^3 = -j\beta_e^3\frac{KI_0}{4V_0} \tag{2.27}$$

While (2.27) may seem simple enough, it will later be found very convenient to rewrite it in terms of other parameters, and we will introduce them now. Let

$$KI_0/4V_0 = C^3 \tag{2.28}$$

C is usually quite small and is typically often around .02. Instead of ξ we will use a quantity or a parameter δ

$$\xi = \beta_e C\delta \tag{2.29}$$

In terms of δ and C, (2.27) becomes

$$\delta = (-j)^{1/3} = (e^{j(2n-1/2)\pi})^{1/3} \tag{2.30}$$

This has three roots which will be called δ_1, δ_2 and δ_3, and these represent three forward waves. They are

$$\begin{aligned} \delta_1 &= e^{-j\pi/6} = \sqrt{3}/2 - j/2 \\ \delta_2 &= e^{-j5\pi/6} = -\sqrt{3}/2 - j/2 \\ \delta_3 &= e^{j\pi/2} = j \end{aligned} \tag{2.31}$$

Figure 2.4 shows the three values of δ. Equation (2.23) was of the fourth degree, and we see that a wave is missing. The missing root was eliminated

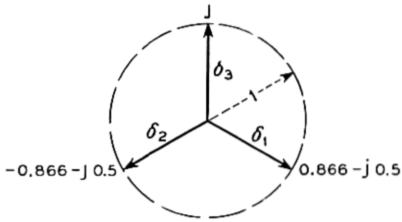

Fig. 2.4—There are three forward waves, with fields which vary with distance as $\exp(-j\beta_e + \beta_e C\delta)z$. The three values of δ for the case discussed, in which the circuit is lossless and the electrons move with the phase velocity of the unperturbed circuit wave, are shown in the figure.

by the approximations made above, which are valid for forward waves only. The other wave is a backward wave and its propagation constant is found to be

$$-\Gamma = j\beta_e\left(1 - \frac{C^3}{4}\right) \tag{2.32}$$

As C is a small quantity, C^3 is even smaller, and indeed the backward wave given by (2.32) is practically the same as the backward wave in the absence of electrons. This is to be expected. In the forward direction, there is a cumulative interaction between wave and the electrons because both are moving at about the same speed. In the backward direction there is no cumulative action, because the wave and the electrons are moving in the opposite directions.

The variation in the z direction for three forward waves is as

$$\exp -\Gamma z = \exp -j\beta_e z \exp \delta C \beta_e z \tag{2.33}$$

We see that the first wave is an increasing wave which travels a little more slowly than the electrons. The second wave is a decreasing wave which travels a little more slowly than the electrons. The third wave is an unattenuated wave which travels faster than the electrons. It can be shown generally that when a stream of electrons interacts with a wave, the electrons must go faster than the wave in order to give energy to it.

It is interesting to know the ratio of line voltage to line current, or the characteristic impedance, for the three forward waves. This may be obtained from (2.5). We see that the characteristic impedance K_n for the nth wave is given in terms for the propagation constant for the nth wave, Γ_n, by

$$K_n = V/I = jX/\Gamma_n \tag{2.34}$$

In terms of δ_n this becomes

$$K_n = K(1 - \beta_e C \delta_n / \Gamma_1) \tag{2.35}$$

$$K_n = K(1 - jC\delta_n) \tag{2.36}$$

We see that the characteristic impedance for the forward waves differs from the characteristic impedance in the absence of electrons by a small amount proportional to C, and that the characteristic impedance has a small reactive component.

We are particularly interested in the rate at which the increasing wave increases. In a number of wave lengths N, the total increase in db is given by

$$\begin{aligned} &20 \log_{10} \exp [(\sqrt{3}/2)(C)(2\pi N)] \text{ db} \\ &= 47.3\, CN \text{ db} \end{aligned} \tag{2.37}$$

We will see later that the overall gain of the traveling-wave tube with a uniform helix can be expressed in the form

$$G = A + BCN \text{ db} \tag{2.38}$$

Here A is a loss relating voltage associated with the increasing wave to the total applied voltage. This loss may be evaluated and will be evaluated later by a proper examination of the boundary conditions at the input of the tube. It turns out that for the case we have considered

$$G = -9.54 + 47.3\, CN \text{ db} \tag{2.39}$$

In considering circuits for traveling-wave tubes, and in reformulating the theory in more general terms later on, it is valuable to express C in terms of parameters other than the characteristic impedance. Two physically significant parameters are the power flow in the circuit and the electric field associated with it which acts on the electron stream. The ratio of the square of the electric field to the power can be evaluated by physical measurement even when it cannot be calculated. For instance, Cutler[5] did this by allowing the power from a wave guide to flow into a terminated helix, so that the power in the helix was the same as the power in the wave guide. He then compared the field in the helix with the field in the wave guide by probe measurements. The field strength in the wave guide could be calculated in terms of the power flow, and hence Cutler's measurements enabled him to evaluate the field in the helix for a given power flow.

The magnitude of the field is given in terms of the magnitude of the voltage by

$$E = |\Gamma V| \tag{2.40}$$

Here E is taken as the magnitude of the field. The power flow in the circuit is given in terms of the circuit voltage by

$$P = |V|^2/2K \tag{2.41}$$

A quantity which we will use as a circuit parameter is

$$E^2/\beta^2 P = 2K \tag{2.42}$$

Here it has been assumed that we are concerned with low-loss circuits, so that Γ_1^2 can be replaced by the phase constant β^2. Usually, β can be taken as equal to β_e, the electron phase constant, with small error, and in the preceding work this has been assumed to be exactly true in (2.23).

In terms of this new quantity, C is given by

$$C^3 = (2K)(I_0/8V_0) = (E^2/\beta^2 P)(I_0/8V_0) \tag{2.43}$$

If we call V_0/I_0 the beam impedance, C^3 is $\frac{1}{4}$ the circuit impedance divided by the beam impedance. It would have been more sensible to use $E^2/2\beta^2 P$ instead of $E^2/\beta^2 P$. Unfortunately the writer feels stuck with his benighted first choice because of the number of curves and published equations which make use of it.

Besides the circuit impedance, another important circuit parameter is the phase velocity. As the electron velocity is made to deviate from the phase velocity of the circuit, the gain falls off. An analysis to be given later

[5] C. C. Cutler, "Experimental Determination of Helical-Wave Properties," *Proc. IRE*, Vol. 36, pp. 230–233, February 1948.

discloses that the allowable range of velocity Δv is of the order of

$$\Delta v \approx \pm\, C u_0 \tag{2.44}$$

Thus, the allowable difference between the phase velocity of the circuit and the velocity of the electrons increases as circuit impedance and beam current are increased and decreases as voltage is increased.

We have illustrated the general method of attack to be used and have introduced some of the important parameters concerned with the circuit and with the overall behavior of the tube. In later chapters, the properties of various circuits suitable for traveling-wave tubes will be discussed in terms of impedance and phase velocity and various cases of interest will be worked out by the methods presented.

CHAPTER III

THE HELIX

Synopsis of Chapter

ANY circuit capable of propagating a slow electromagnetic wave can be used in a traveling-wave tube. The circuit most often used is the helix. The helix is easy to construct. In addition, it is a very good circuit. It has a high impedance and a phase velocity that is almost constant over a wide frequency range.

In this chapter various properties of helices are discussed. An approximate expression for helix properties can be obtained by calculating the properties, not of a helix, but of a helically conducting cylindrical sheet of the same radius and pitch as the helix. An analysis of such a sheet is carried out in Appendix II and the results are discussed in the text.

Parameters which enter into the expressions are the free-space phase constant $\beta_0 = \omega/c$, the axial phase constant $\beta = \omega/v$, where v is the phase velocity of the wave, and the radial phase constant γ. The arguments of various Bessel functions are, for instance, γr and γa, where r is the radial coordinate and a is radius of the helix. The parameters β_0, β and γ are related by

$$\beta^2 = \beta_0^2 + \gamma^2$$

For tightly wound helices in which the phase velocity v is small compared with the velocity of light, γ is very nearly equal to β. For instance, at a velocity corresponding to that of 1,000 volt electrons, γ and β differ by only 0.4%.

Figure 3.1 illustrates two parameters of the helically conducting sheet, the radius a and pitch angle ψ. For an actual helix, a will be taken to mean the mean radius, the radius to the center of the wire.

Figure 3.2 shows a single curve which enables one to obtain γ, and hence β, for any value of the parameter

$$\beta_0 a \cot \psi = \frac{\omega a \cot \psi}{c}.$$

This parameter is proportional to frequency. The curve is an approximate representation of velocity vs. frequency. At high frequencies γ approaches

$\beta_0 \cot \psi$ and β thus approaches $\beta_0/\sin \psi$; this means that the wave travels with the velocity of light around the sheet in the direction of conduction. In the case of an actual helix, the wave travels along the wire with the velocity of light.

The gain parameter C is given by

$$C = (I_0/8V_0)^{1/3}(E^2/\beta^2 P)^{1/3}$$

Values of $(E^2/\beta^2 P)^{1/3}$ on the axis may be obtained through the use of Fig. 3.4, where an impedance parameter $F(\gamma a)$ is plotted vs. γa, and by use of (3.9). For a given helix, $(E^2/\beta^2 P)^{1/3}$ is approximately proportional to $F(\gamma a)$. $F(\gamma a)$ falls as frequency increases. This is partly because at high frequencies and short wavelengths, for which the sign of the field alternates rapidly with distance, the field is strong near the helix but falls off rapidly away from the helix and so the field is weak near the axis. At very high frequencies the field falls off away from the helix approximately as $\exp(-\gamma \Delta r)$, where Δr is distance from the helix, and we remember that γ is very nearly proportional to frequency. $(E^2/\beta^2 P)^{1/3}$ measured at the helix also falls with increasing frequency.

In many cases, a hollow beam of radius r (the dashed lines of Fig. 3.5 refer to such a beam) or a solid beam of radius r (the solid lines of Fig. 3.5 refer to such a beam) is used. For a hollow beam we should evaluate E^2 in $(E^2/\beta^2 P)^{1/3}$ at the beam radius, and for a solid beam we should use the mean square value of E averaged over the beam.

The ordinate in Fig. 3.5 is a factor by which $(E^2/\beta^2 P)^{1/3}$ as obtained from Fig. 3.4 and (3.9) should be multiplied to give $(E^2/\beta^2 P)^{1/3}$ for a hollow or solid beam.

The gain of the increasing wave is proportional to $F(\gamma a)$ times a factor from Fig. 3.5, and times the length of the tube in wavelengths, N. N is very nearly proportional to frequency. Also γ, and hence γa, are nearly proportional to frequency. Thus, $F(\gamma a)$ from Fig. 3.4 times the appropriate factor from Fig. 3.5 times γa gives approximately the gain vs. frequency, (if we assume that the electron speed matches the phase velocity over the frequency range). This product is plotted in Fig. 3.6. We see that for a given helix size the maximum gain occurs at a higher frequency and the bandwidth is broader as r/a, the ratio of the beam radius to the helix radius, is made larger.

It is usually desirable, especially at very short wavelengths, to make the helix as large as possible. If we wish to design the tube so that gain is a maximum at the operating frequency, we will choose a so that the appropriate curve of Fig. 3.6 has its maximum at the value of γa corresponding to the operating frequency. We see that this value of a will be larger the larger is r/a. In an actual helix, the maximum possible value of r/a is less than unity, since the inside diameter of the helix is less than a by the radius of the wire. Further, focusing difficulties preclude attaining a beam radius equal even to the inside radius of the helix.

Experience indicates that at very short wavelengths (around 6 millimeters, say) it is extremely important to have a well-focused electron beam with as large a value of r/a as is attainable.

A characteristic impedance K_t may be defined in terms of a "transverse" voltage V_t, obtained by integrating the peak radial field from a to ∞, and from the power flow. In Fig. 3.7, $(v/c)\ K_t$ is plotted vs. γa. A "longitudinal" characteristic impedance K_ℓ is related to K_t (3.13). For slow waves K_ℓ is nearly equal to K_t. The impedance parameter $E^2/\beta^2 P$ evaluated at the surface of the cylinder is twice K_ℓ. We see that K_ℓ falls with increasing frequency.

A simplified approach in analysis of the helically conducting sheet is that of "developing" the sheet; that is, slitting it normal to the direction of conduction and flattening it out as in Fig. 3.8. The field equations for such a flattened sheet are then solved. For large values of γa the field is concentrated near the helically conducting sheet, and the fields near the developed sheet are similar to the fields near the cylindrical sheet. Thus the dashed line in Fig. 3.7 is for the developed sheet and the solid line is for a cylindrical sheet.

For the developed sheet, the wave always propagates with the speed of light in the direction of conduction. In a plane normal to the direction of conduction, the field may be specified by a potential satisfying Laplace's equation, as in the case, for instance, of a two-wire or coaxial line. Thus, the fields can be obtained by the solution of an electrostatic problem.

One can develop not only a helically conducting sheet, but an actual helix, giving a series of straight wires, shown in cross-section in Fig. 3.9. In Case I, corresponding to approximately two turns per wavelength, successive wires are $-$, $+$, $-$, $+$ etc.; in case II, corresponding to approximately four turns per wavelength, successive wires are $+$, 0, $-$, 0, $+$, 0 etc.

Figures 3.10 and 3.11 illustrate voltages along a developed sheet and a developed helix.

Figure 3.13 shows the ratio, $R^{1/3}$, of $(E^2/\beta^2 P)^{1/3}$ on the axis to that for a developed helically conducting sheet, plotted vs. d/p. We see that, for a large wire diameter d, $(E^2/\beta^2 P)^{1/3}$ may be larger on the axis than for a helically conducting sheet with the same mean radius and hence the same pitch angle and phase velocity. This is merely because the thick wires extend nearer to the axis than does the sheet. The actual helix is really inferior to the sheet.

We see this by noting that the highest value of $(E^2/\beta^2 P)^{1/3}$ for a helically conducting sheet is that at the sheet ($r = a$). With a finite wire size, the largest value r can have is the mean helix radius a minus the wire radius. In Fig. 3.14, the ratio of $(E^2/\beta^2P)^{1/3}$ for this largest allowable radius to $(E^2/\beta^2P)^{1/3}$ at the surface of the developed sheet is plotted vs. d/p. We see that, in terms of maximum available field, $(E^2/\beta^2P)^{1/3}$ is no more than 0.83 as high as for the sheet for four turns per wavelength and 0.67 as high as for the sheet for two turns per wavelength. We further see that there is an optimum ratio of wire diameter to pitch; about 0.175 for four turns per wavelength and about 0.125 for two turns per wavelength. Because the maxima are so broad, it is probably better in practice to use larger wire, and in most tubes which have been built, d/p has been around 0.5.

In designing tubes it is perhaps best to do so in terms of field on the axis (Fig. 3.13), the allowable value of r/a and the curves of Fig. 3.6.

Figure 3.15 compares the impedance of the developed helix with that of the developed sheet as given by the straight line of Fig. 3.7.

There are factors other than wire size which can cause the value of E^2/β^2P for an actual helix to be less than the value for the helically conducting sheet. An important cause of impedance reduction is the influence of dielectric supporting members. Even small ceramic or glass supporting rods can cause some reduction in helix impedance. In some tubes the helix is supported inside a glass tube, and this can cause a considerable reduction in helix impedance.

When a field analysis seems too involved, it may be possible to obtain some information by considering the behavior of transmission lines having parameters adjusted to make the phase constant and the characteristic impedance equal to those of the helix. For instance, suppose that the presence of dielectric material results in an actual phase constant β_d as opposed to a computed phase constant β. Equation (3.64) gives an estimate of the consequent reduction of $(E^2/\beta^2P)^{1/3}$ on the axis.

This method is of use in studying the behavior of coupled helices. For instance, concentric helices may be useful in producing radial fields in tubes in which transverse fields predominate in the region of electron flow (see Chapter XIII). A concentric helix structure might be investigated by means of a field analysis, but some interesting properties can be deduced more simply by considering two transmission lines with uniformly distributed self and mutual capacitances and inductances, or susceptance and reactances. The modes of propagation on such lines are affected by coupling in a manner similar to that in which the modes of two resonant circuits are affected by coupling.

If two lines are coupled, their two independent modes of propagation are mixed up to form two modes of propagation in which both lines participate. If the original phase velocities differ greatly, or if the coupling between the lines is weak, the fields and velocity of one of these modes will be almost like the original fields and velocity of one line, and the fields and velocity of the other mode will be almost like the original fields and velocity of the other line. However, if the coupling is strong enough compared with the original separation of phase velocities, both lines will participate almost equally in each mode. One mode will be a "longitudinal mode" for which the excitations on the two lines are substantially equal, and the other mode will be a "transverse" mode for which the excitations are substantially equal and opposite.

The ratios of the voltages on the lines for the two modes are given by (3.75). Here it is assumed that the series reactances X and shunt susceptances B of the lines are almost equal, differing only enough to make a difference $\Delta\Gamma_0$ in the propagation constants. B_{12} and X_{12} are the mutual susceptance and reactance. We see that to make the voltages on the two lines nearly equal or equal and opposite, B_{12} and X_{12} should have the same sign, so that capacitive and inductive couplings add.

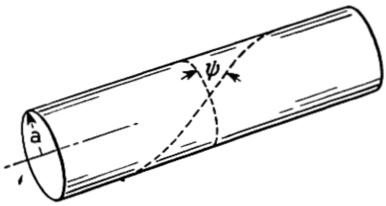

Fig. 3.1—A helically conducting sheet of radius a. The sheet is conducting along helical paths making an angle ψ with a plane normal to the axis.

Increasing the coupling increases the velocity separation between the two modes, and this is desirable. When there is a substantial difference in velocity, operation in the desired mode can be secured by making the electron velocity equal to the phase velocity of the desired mode.

To make the capacitive and inductive couplings add in the case of concentric helices (Fig. 3.17), the helices should be wound in opposite directions.

3.1 The Helically Conducting Sheet

In computing the properties of a helix, the actual helix is usually replaced by a helically conducting cylindrical sheet of the same mean radius. Such a sheet is illustrated in Fig. 3.1. This sheet is perfectly conducting in a helical direction making an angle ψ, the pitch angle, with a plane normal to the axis (the direction of propagation), and is non-conducting in a helical direction normal to this ψ direction, the direction of conduction. Appropriate solutions of Maxwell's equations are chosen inside and outside of the cylindrical sheet. At the sheet, the components of the electric field in the ψ direction are made zero, and those normal to the ψ direction are made equal inside and outside. Since there can be no current in the sheet normal to the ψ direction, the components of magnetic field in the ψ direction must be the same inside and outside of the sheet. When these boundary conditions are imposed, one can solve for the propagation constant and $E^2/\beta^2 P$ can then be obtained by integrating the Poynting vector.

The helically conducting sheet is treated mathematically in Appendix II. The results of this analysis will be presented here.

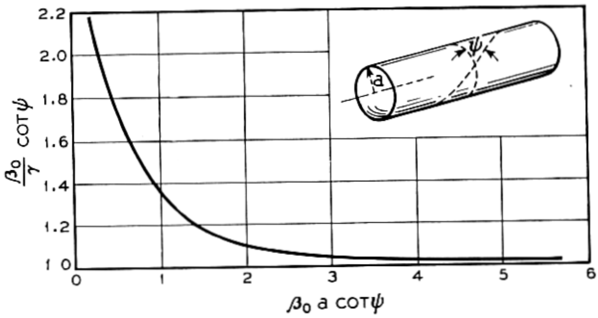

Fig. 3.2—The radial propagation constant is $\gamma^2 = (\beta^2 - \beta_0^2)^{1/2}$. Here $(\beta_0/\gamma)\cot\psi$ is plotted vs $\beta_0 a \cot\psi$, a quantity proportional to frequency. For slow waves the ordinate is roughly the ratio of the wave velocity to the velocity the wave would have if it traveled along the helically conducting sheet with the speed of light in the direction of conduction.

3.1a *The Phase Velocity*

The results for the helically conducting sheet are expressed in terms of three phase or propagation constants. These are

$$\beta_0 = \omega/c, \qquad \beta = \omega/v \tag{3.1}$$

$$\gamma = \sqrt{\beta^2 - \beta_0^2} \tag{3.2}$$

$$\gamma = \beta\sqrt{1 - (v/c)^2} \tag{3.3}$$

Here c is the velocity of light and v is the phase velocity of the wave. β_0 is the phase constant of a wave traveling with the speed of light, which would vary with distance in the z direction as $\exp(-j\beta_0 z)$. The actual axial phase constant is β, and the fields vary with distance as $\exp(-j\beta z)$.

γ is the radial propagation constant. Various field components vary as modified Bessel functions of argument γr, where r is the radius. Particularly, the longitudinal electric field, which interacts with the electrons, varies as $I_0(\gamma r)$.

For the phase velocities usually used, γ is very nearly equal to β, as may be seen from the following table of accelerating voltages V_0 (to give an electron the velocity v), v/c and γ/β.

V	v/c	γ/β
100	.0198	1.000
1,000	.0625	.998
10,000	.1980	.980

Figure 3.2 gives information concerning the phase velocity of the wave in the form of a plot of $(\beta_0/\gamma) \cot \psi$ as a function of $\beta_0 a \cot \psi$.

The ratio of the phase velocity v to the velocity of light c may be expressed

$$v/c = \beta_0/\beta = (\gamma/\beta)(\beta_0/\gamma) \cot \psi \tan \psi \tag{3.4}$$

$$v/c = (\gamma/\beta) \tan \psi \, [(\beta_0/\gamma) \cot \psi\,]$$

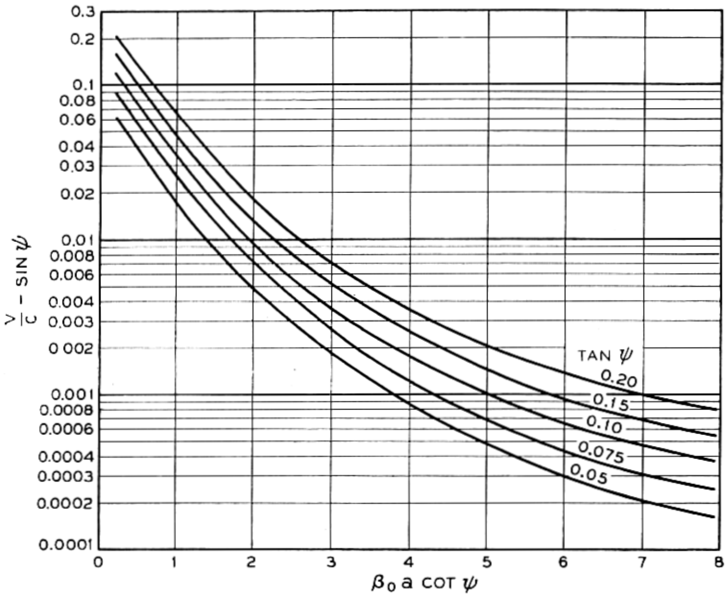

Fig. 3.3—From these curves one can obtain v/c, the ratio of the phase velocity of the wave to the velocity of light, for various values of $\tan \psi$ and $\beta_0 a \cot \psi$.

From Fig. 3.2 we see that, for large values of $\beta_0 a \cot \psi$, $(\beta_0/\gamma) \cot \psi$ approaches unity. For slow waves γ/β approaches unity. Under these circumstances, very nearly

$$v/c = \tan \psi \tag{3.5}$$

If the wave traveled in the direction of conduction with the speed of light we would have

$$v/c = \sin \psi$$

This is essentially the same as (3.5) for small pitch angles ψ. Thus, for large values of the abscissa in Fig. 3.2, the phase velocity is just about that corresponding to propagation along the sheet in the direction of conduction with the speed of light and hence in the axial direction at a much reduced speed. For helices of smaller radius compared with the wavelength, the speed is greater.

The bandwidth of a traveling-wave tube is in part determined by the range over which the electrons keep in step with the wave. The abscissa of Fig. 3.2 is proportional to frequency, but the ordinate is not strictly proportional to phase velocity. Hence, it seems desirable to have a plot which does show velocity directly. To obtain this we can assign various values to cot ψ.

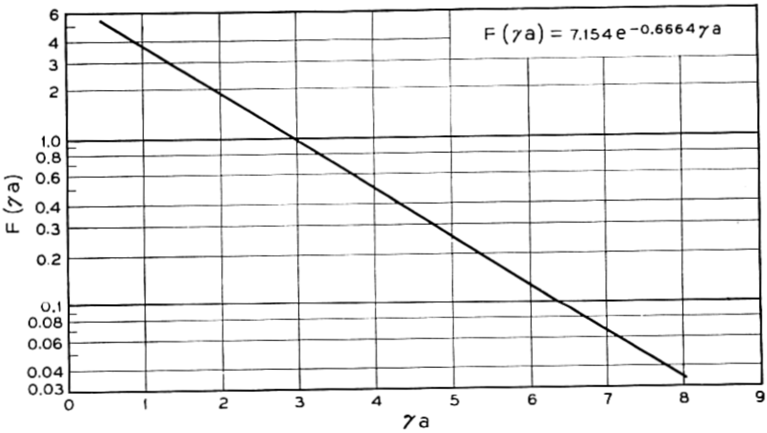

Fig. 3.4—A curve giving the impedance function $F(\gamma a)$ vs. γa. On the axis, $(E^2/\beta^2P)^{1/3} = (\beta/\beta_0)^{1/3}(\gamma/\beta)^{4/3}F(\gamma a)$.

The ordinate (β_0/γ) cot ψ then gives us γ/β_0 and from (3.2) we see that

$$v/c = \beta_0/\beta = (1 + (\gamma/\beta_0)^2)^{-1/2} \tag{3.6}$$

We have seen that, for large values of $\beta_0 a$ cot ψ, (β_0/γ) cot ψ approaches unity, and v/c approaches a value

$$v/c = (1 + \cot^2 \psi)^{-1/2} = \sin \psi \tag{3.7}$$

To emphasize the change in velocity with frequency it seems best to plot the difference between the actual velocity ratio and this asymptotic velocity ratio on a semi-log scale. Accordingly, Fig. 3.3 shows $(v/c) - \sin \psi$ vs. $\beta_0 a$ cot γ for tan ψ = .05, .075, .1, .15, .2.

For large values of the abscissa the velocities are those corresponding to about 640 volts ($\tan\psi = .05$), 1,400 volts (.075), 2,500 volts (.1), 5,600 volts (.15), 9,800 volts (.2).

3.1b *The Impedance Parameter* $(E^2/\beta^2 P)$

Figure 3.4 shows a plot of a quantity $F(\gamma a)$ vs. γa. This quantity is computed from a very complicated expression (Appendix II), but it is accurately given over the range shown by the empirical relation

$$F(\gamma a) = 7.154\, e^{-.6664\gamma a} \tag{3.8}$$

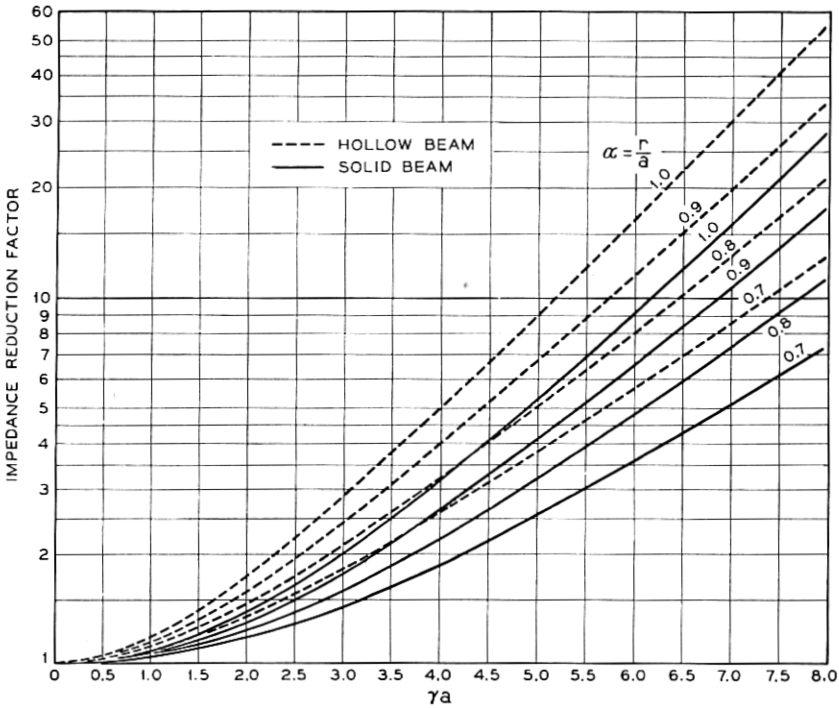

Fig. 3.5—Factors by which $(E^2\beta/^2P)^{1/3}$ on the axis should be multiplied to give the correct value for hollow and solid beams of radius r.

For the field on the axis of the helix,

$$(E^2/\beta^2 P)^{1/3} = (\beta/\beta_0)^{1/3}(\gamma/\beta)^{4/3}F(\gamma a) \tag{3.9}$$

We should remember that $\beta/\beta_0 = c/v$ and that γ/β is nearly unity for velocities small compared with the velocity of light.

In the expression for the gain parameter C, the square of the field E is multiplied by the current I_0 (2.28). If we were to assume that two electron streams of different currents, I_1 and I_2, were coupled to the circuit through transformers, so as to be acted on by fields E_1 and E_2, but that the streams did not interact directly with one another, we would find the effective value of C^3 to be given by

$$C^3 = (E_1^2/\beta^2 P)(I_1/8V_0) + (E_2^2/\beta^2 P)(I_2/8V_0)$$

Thus, if we neglect the direct interaction of electron streams through fields due to local space charge, we can obtain an effective value of C^3 by integrating $E^2 dI_0$ over the beam. If we assume a constant current density, we can merely use the mean square value of E over the area occupied by electron flow.

The axial component of electric field at a distance r from the axis is $I_0(\gamma r)$ times the field on the axis. Hence, if we used a tubular beam of radius r, we should multiply $(E^2/\beta^2 P)^{1/3}$ as obtained from Fig. 3.4 by $[I_0(\gamma r)]^{2/3}$. The quantity $[I_0(\gamma r)]^{2/3}$ is plotted vs. γa for several values of r/a as the dashed lines in Fig. 3.5.

Suppose the current density is uniform out to a radius r and zero beyond this radius. The average value of E^2 is greater than the value on the axis by a factor $[I_0^2(\gamma r) - I_1^2(\gamma r)]$ and $(E^2/\beta^2 P)^{1/3}$ from Fig. 3.4 should in this case be multiplied by this factor to the $\frac{1}{3}$ power. The appropriate factor is plotted vs. γa as the solid lines of Fig. 3.5.

We note from (2.39) that the gain contains a term proportional to CN, where N is the number of wavelengths. For slow waves and usual values of γa, very nearly, N will be proportional to the frequency and hence to γ, while C is proportional to $(E^2/\beta^2 P)^{1/3}$. We can obtain $(E^2/\beta^2 P)^{1/3}$ from Figs. 3.4 and 3.5. The gain of the increasing wave as a function of frequency will thus be very nearly proportional to this value of $(E^2/\beta^2 P)^{1/3}$ times γ, or, times γa if we prefer.

In Fig. 3.6, $\gamma a F(\gamma a)$ is plotted vs. γa for hollow beams of radius r for various values of r/a (dashed lines) and for uniform density beams of radius r for various values of r/a (solid lines). If we assume that the electron speed is adjusted to equal the phase velocity of the wave, we can take the ordinate as proportional to gain and the abscissa as proportional to frequency.

We see that the larger is r/a, the larger is the value of γa for maximum gain. For one typical 7.5 cm wavelength traveling-wave tube, γa was about 2.8. For this tube, the ratio of the inside radius of the helix to the mean radius of the helix was 0.87. We see from Fig. 3.6 that, if a solid beam just filled this helix, the maximum gain should occur at about the operating wavelength. As a matter of fact, the beam was somewhat smaller than the inside diameter of the helix, and there was an observed increase of gain with an increase in wavelength (a higher gain at a lower frequency). In a particular tube for 0.625 cm wavelength, it was felt desirable to use a relatively large helix diameter. Accordingly, a value of γa of 6.7 was chosen. We see that, unless r/a is 0.9 or larger, this must result in an appreciable increase in gain at some frequency lower than operating frequency. It was only by use of great care in focusing the beam that gain was attained at 0.625 cm wavelength, and there was a tendency toward oscillation, presumably at longer wavelengths. This discussion of course neglects the effect of transmission

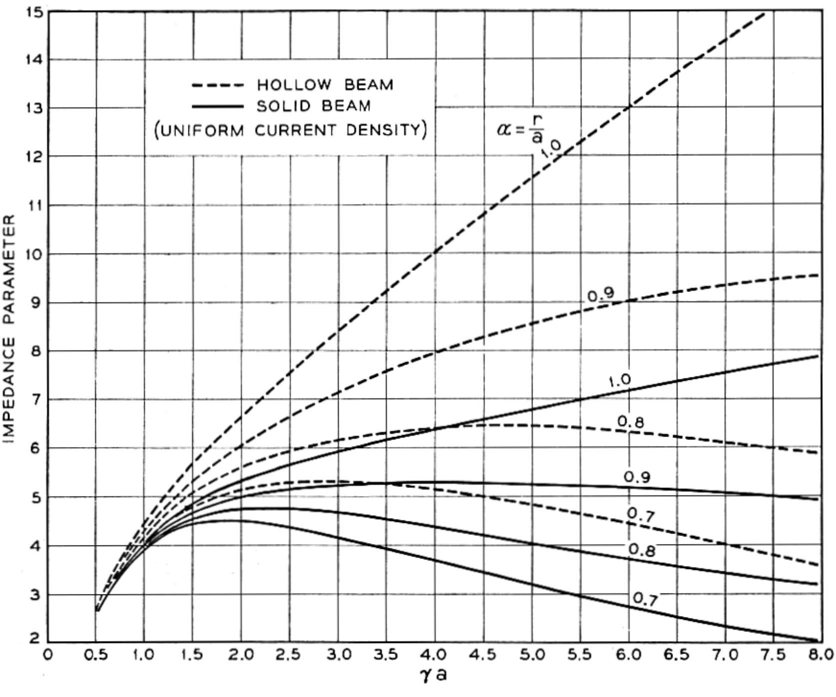

Fig. 3.6—The ordinate is $\gamma a F(\gamma a)$ times the parameters from Fig. 3.5. For a fixed current and voltage it is nearly proportional to gain per unit length, and hence the curves give roughly the variation of gain with frequency.

loss or gain. Usually the loss decreases when the frequency is decreased, and this favors oscillation at low frequencies.

3.1c *Impedance of the Helix*

No impedance which can be assigned to the helically conducting sheet can give full information for matching a helix to a waveguide or transmission line. As in the case of transducers between a coaxial line and a waveguide or between waveguides of different cross-section, the impedance is important, but discontinuity effects are also important. However, a suitably defined helix impedance is of some interest.

Figure 3.7 presents the impedance as defined on a voltage-power basis. The peak "transverse" voltage V_t is obtained by integrating the radial electric field from the radius a of the helically conducting sheet to ∞. The "transverse" characteristic impedance K_t is defined by the relation

$$P = (\tfrac{1}{2})(V_t^2/K_t)$$

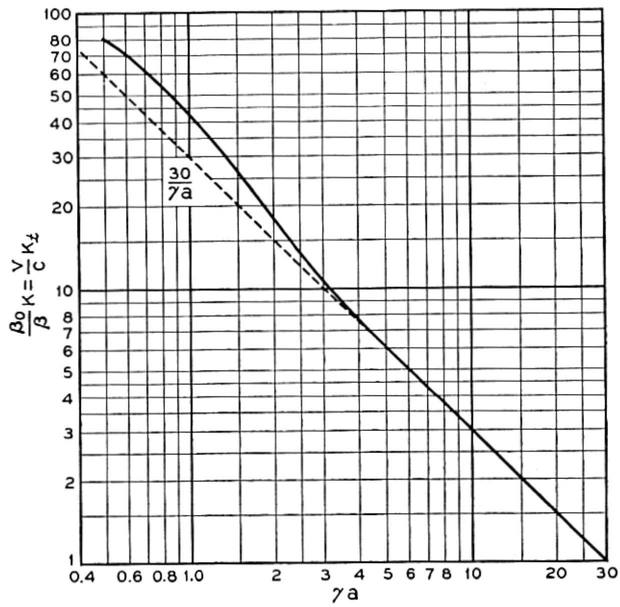

Fig. 3.7—Curves giving the variation of transverse impedance, K_t, with γa.

The impedance is found to be given by

$$\left(\frac{\beta}{\gamma}\right)^2\left(\frac{\beta_0}{\beta}\right)K_t = \frac{120 I_0^2}{(\gamma a)^2}\left[\left(1+\frac{I_0 K_1}{I_1 K_0}\right)(I_1^2 - I_0 I_2) + \left(\frac{I_0}{K_0}\right)^2\left(1+\frac{I_1 K_0}{I_0 K_1}\right)(K_0 K_2 - K_1^2)\right]^{-1} \tag{3.10}$$

The I's and K's are modified Bessel functions of argument γa.

The dashed line on Fig. 3.7 is a plot of $30/\gamma a$ vs. γa. It may be seen that, for large values of γa, very nearly

$$K_t = (\beta/\beta_0)(\gamma/\beta)^2(30/\gamma a) \tag{3.11}$$

and in the whole range shown the impedance differs from this value by a factor less than 1.5.

We might have defined a "longitudinal" voltage V_ℓ as half of the integral of the longitudinal component of electric field at the surface of the helically conducting sheet for a half wavelength (between successive points of zero field). We find that

$$V_\ell = \sqrt{1 - (v/c)^2}\, V_t = (\gamma/\beta)V_t \tag{3.12}$$

and, accordingly, the "longitudinal impedance" K_ℓ will be

$$K_\ell = [1 - (v/c)^2]K_t = (\gamma/\beta)^2 K_t \tag{3.13}$$

Our impedance parameter, $E^2/\beta^2 P$, is just twice this "longitudinal impedance."

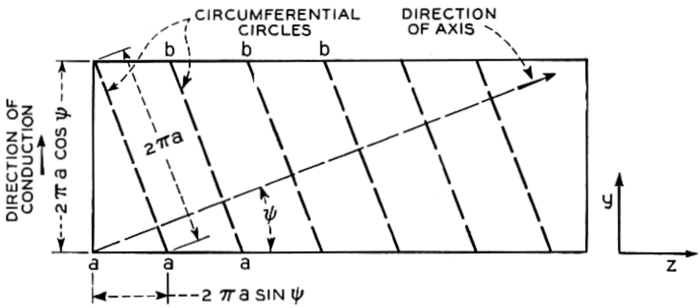

Fig. 3.8—A "developed" helically conducting sheet. The sheet has been slit along a line normal to the direction of conduction and flattened out.

The transverse voltage V_t is greater than the longitudinal voltage V_ℓ because of the circumferential magnetic flux outside of the helix. For slow waves V_ℓ is nearly equal to V_t and the fields are nearly curl-free solutions of Laplace's equation. In this case the circumferential magnetic flux is small compared with the longitudinal flux inside of the helix.

For the circuit of Fig. 2.3 the transverse and longitudinal voltages are equal, and it is interesting to note that this is approximately true for slow waves on a helix. For very fast waves, the longitudinal voltage becomes small compared with the transverse voltage.

For a typical 4,000-megacycle tube, for which $\gamma a = 2.8$, Fig. 5 indicates a value of K_t of about 150 ohms.

3.2 The Developed Helix

For large helices, i.e., for large values of γa, the fields fall off very rapidly away from the wire. Under these circumstances we can obtain quite accurate results by slitting the helically conducting sheet along a spiral line normal to the direction of conduction and flattening it out. This gives us the plane conducting sheet shown in Fig. 3.8. The indicated coordinates are z to the right and y upward: x is positive into the paper. The fields about the developed sheet approximate those about the helically conducting sheet for distances always small compared with the original radius of curvature.

The straight dashed line shown on the helix impedance curve of Fig. 3.7 can be obtained as a solution for the "developed helix." We see that it is within 10% of the true curve for values of γa greater than 2.8. We might note that a 10% error in impedance means only a $3\frac{1}{3}$% error in the gain parameter C.

In solving for the fields around the sheet, the developed surface can be extended indefinitely in the plus and minus y directions. In order that the fields may match when the sheet is rolled up, they must be the same at $y = 0$, $z = 2\pi a \sin\psi$ and $y = 2\pi a \cos\psi$, $z = 0$. The appropriate solutions are plane electromagnetic waves traveling in the y direction with the speed of light.

For positive values of x, the appropriate electric and magnetic fields are

$$\begin{aligned} E_x &= E_0 e^{-\gamma x}\, e^{-j\gamma z}\, e^{-j\beta_0 y} \\ E_z &= jE_0 e^{-\gamma x}\, e^{-j\gamma z}\, e^{-j\beta_0 y} \\ E_y &= 0 \end{aligned} \tag{3.14}$$

We should note that the x and z components of the field can be obtained as gradients of a function

$$\Phi = -(E_0/\gamma) e^{-\gamma x}\, e^{-j\gamma z}\, e^{-j\beta_0 y} \tag{3.15}$$

where

$$\begin{aligned} E_x &= -\partial\Phi/\partial z \\ E_z &= -\partial\Phi/\partial y \end{aligned} \tag{3.16}$$

$$\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial z^2 = 0 \tag{3.17}$$

Thus, in the xz plane, Φ satisfies Laplace's equation.

The magnetic field is given by the curl[6] of the electric field times $j/\omega\mu$. Its components are:

$$\begin{aligned} H_x &= \frac{-j}{\mu c} E_0 e^{-\gamma x}\, e^{-j\gamma z}\, e^{-j\beta_0 y} \\ H_z &= \frac{-1}{\mu c} E_0 e^{-\gamma x}\, e^{-j\gamma z}\, e^{-j\beta_0 y} \\ H_y &= 0 \end{aligned} \tag{3.18}$$

[6] Maxwell's equations are given in Appendix I.

The fields in the $-x$ direction may be obtained by substituting $\exp(\gamma x)$ for $\exp(-\gamma x)$.

If the sheet is to roll up properly, the points a on the bottom coinciding with the points b on the top, we have

$$2\pi\gamma a \sin\psi - 2\pi\beta_0 a \cos\psi = 2n\pi \tag{3.19}$$

where n is an integer.

The solution corresponding most nearly to the wave on a singly-wound helix is that for $n = 0$. The others lead to a variation of field by n cycles along a circumferential line. These can be combined with the $n = 0$ solution to give a solution for a developed helix of thin tape, for instance. Or, appropriate combinations of them can represent modes of helices wound of several parallel wires. For instance, we can imagine winding a balanced transmission line up helically. One of the modes of propagation will be that in which the current in one wire is 180° out of phase with the current in the other. This can be approximated by a combination of the $n = +1$ and $n = -1$ solutions. This mode should not be confused with a fast wave, a perturbation of a transverse electromagnetic wave, which can exist around an unshielded helix.

Usually, we are interested in the slow wave on a singly-wound helix, and in this case we take $n = 0$ in (3.19), giving

$$\begin{aligned} \gamma \sin\psi - \beta_0 \cos\psi &= 0 \\ \tan\psi &= \beta_0/\gamma \end{aligned} \tag{3.20}$$

$$\sin\psi = \frac{\beta_0}{(\gamma^2 + \beta_0^2)^{1/2}} \tag{3.21}$$

$$\cos\psi = \frac{\gamma}{(\gamma^2 + \beta_0^2)^{1/2}} \tag{3.22}$$

Let us evaluate the propagation constant in the axial direction. From Fig. 3.8 we see that, in advancing unit distance in the axial direction, we proceed a distance $\cos\psi$ in the z direction and $\sin\psi$ in the y direction. Hence, the phase constant β in the axial direction must be

$$\beta = \beta_0 \sin\psi + \gamma \cos\psi \tag{3.23}$$

Using (3.18) and (3.19), we obtain

$$\beta = (\beta_0^2 + \gamma^2)^{1/2} \tag{3.24}$$

$$\gamma = (\beta^2 - \beta_0^2)^{1/2} \tag{3.25}$$

These are just relations (3.2, 3.3).

The power flow along the axis is that crossing a circumferential circle, represented by lines *a-b* in Fig. 3.8. As the power flows in the y direction, this is the power associated with a distance $2\pi a \sin\psi$ in z direction. Also, the power flow in the $+x$ region will be equal to the power flow in the $-x$ region. Hence, the power flow in the helix will be twice that in the region $x = 0$ to $x = +\infty$, $z = 0$ to $z = 2\pi a \sin\psi$.

$$P = 2\int_{z=0}^{2\pi a \sin\psi}\int_{x=0}^{\infty} (\tfrac{1}{2})(E_z H_x^* - E_x H_z^*)\, dx\, dz \qquad (3.26)$$

This is easily integrated to give

$$P = \frac{2\pi a \sin\psi E_0^2}{\gamma\mu c} \qquad (3.27)$$

The magnitude E of the axial component of field is

$$E = E_0 \cos\psi \qquad (3.28)$$

Using (3.21), (3.22), (3.24) and (3.28) in connection with (3.27) we obtain

$$(E^2/\beta^2 P) = (\gamma/\beta)^4(\beta/\beta_0)(\mu c/2\pi\gamma a) \qquad (3.29)$$

We have

$$\mu c = \mu/\sqrt{\mu\epsilon} = \sqrt{\mu/\epsilon} = 377 \text{ ohms}$$

Thus

$$E^2/\beta^2 P = (\gamma/\beta)^4(\beta/\beta_0)(60/\gamma a) \qquad (3.30)$$

The longitudinal impedance is half this, and the transverse impedance is $(\beta/\gamma)^2$ times the longitudinal impedance.

3.3 Effect of Wire Size

An actual helix of round wire, as used in traveling-wave tubes, will of course differ somewhat in properties from the helically conducting sheet for which the foregoing material applies.

One might expect a small difference if there were many turns per wavelength, but actual tubes often have only a few turns per wavelength. For instance, a typical 4,000 mc tube has about 4.8 turns per wavelength, while a tube designed for 6 mm operation has 2.4 turns per wavelength.

If the wire is made very small there will be much electric and magnetic energy very close to the wire, which is not associated with the desired field component (that which varies as $\exp(-j\beta z)$ in the z direction). If the wire is very large the internal diameter of the helix becomes considerably less than the mean diameter, and the space available for electron flow is reduced. As the field for the helically conducting sheet is greatest at the sheet, this means that the maximum available field is reduced. Too, the impedance will depend on wire size.

It thus seems desirable to compare in some manner an actual helix and the helically conducting sheet. It would be very difficult to solve the problem of an actual helix. However, we can make an approximate comparison by a method suggested by R. S. Julian.

In doing this we will develop the helix of wires just as the helically con-

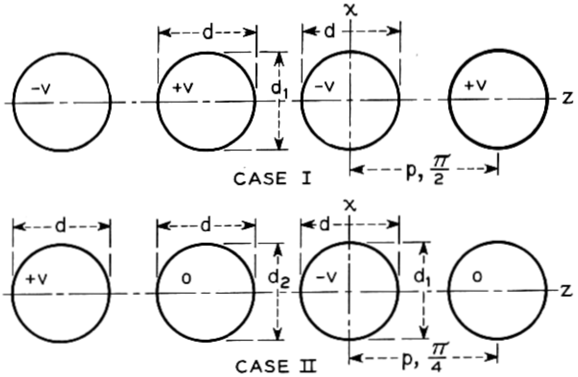

Fig. 3.9—The wires of a developed helix with about two turns per wavelength (case I) and about four turns per wavelength (case II). In the analysis used, the wires are not quite round.

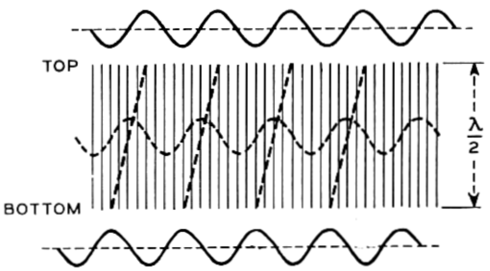

Fig. 3.10—Voltages on a developed helically conducting sheet for two turns per wavelength.

ducting sheet was developed, by slitting it along a helical line normal to the wires. We will then consider two special cases, one in which the wires of the developed helix are one half wavelength long and the other in which the wires are one quarter wavelength long.

The waves propagated on the developed helix are transverse electromagnetic waves propagated in the direction of the wires, and the electric fields normal to the direction of propagation can be obtained from a solution of Laplace's equation in two dimensions (as in (3.15)–(3.17)).

It is easy to make up two-dimensional solutions of Laplace's equation with equipotentials or conductors of approximately circular form, as shown in Fig. 3.9. In case I, the conductors are alternately at potentials $-V, +V, -V$, etc.; and in case II, the potentials are $-V$, 0, $+V$, 0, $-V$, 0, $+V$, etc. Far away in the x direction from such a series of conductors, the field will vary sinusoidally in the z direction and will vary in the same manner with x as in the developed helically conducting sheet. Hence, we can make the distant fields of the conductors of cases I and II of Fig. 3.9 equal to the distant fields of developed helically conducting sheets, and compare the $E^2/\beta^2 P$ and the impedance for the different systems. Case I would correspond to a helix of approximately two turns per wavelength and case II to four turns per wavelength.

3.3a *Two Turns per Wavelength*

Figure 3.10 is intended to illustrate the developed helically conducting sheet. The vertical lines indicate the direction of conduction. The dashed slanting lines are intersections of the original surface with planes normal to the axis. That is, on the original cylindrical surface they were circles about the surface, and they connect positions along the top and bottom which should be brought together in rolling up the flattened surface to reconstitute the helically conducting sheet.

Waves propagate on the developed sheet of Fig. 3.10 vertically with the speed of light. The vertical dimension of the sheet is in this case taken as $\lambda/2$, where λ is the free-space wavelength.[7] The sine waves above and below Fig. 3.10 indicate voltages at the top and the bottom and are, of course, 180° out of phase. As is necessary, the voltages at the ends of the dashed slanting lines, (really, the voltages at the same point before the sheet was slit) are equal.

A wave sinusoidal at the bottom of the sheet, zero half way up and 180° out of phase with the bottom at the top would constitute along any horizontal line a standing wave,not a traveling wave. Actually, this is only one component of the field. The other is a wave 90° out of phase in both the horizontal and vertical directions. Its maximum voltage is half-way up, and it is indicated by the dotted sine wave in Fig. 3.10. The voltage of this component is zero at top and bottom. It may be seen that these two components propagating upward together constitute a wave traveling to the right. The two components are orthogonal spatially, and the total power is twice the power of either component taken separately.

Figure 3.11 indicates an array of wires obtained by developing an actual

[7] Section 3.3a is referred to as "two turns per wavelength." This is not quite accurate; it is in error by the difference between the lengths of the vertical and the slanting lines in Fig. 3.10.

helix which has been slit along a helical line normal to the wire of which the helix is wound. The dashed slanting lines again connect points which were the same point before the helix was slit and developed. Again we assume a height of a half wavelength. Thus, if the polarities are maximum +, −, +, − etc. as shown at the bottom, they will be maximum −, +, −, +, −, + etc. as shown at the top, and zero half-way up. In this case the field is a standing wave along any horizontal line, and no other component can be introduced to make it a traveling wave. Half of the field strength can be regarded as constituting a component traveling to the right and half as a component traveling to the left.

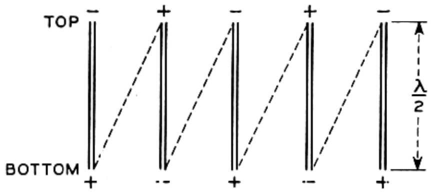

Fig. 3.11—Voltages on a developed helix for two turns per wavelength.

The equipotentials used to represent the field about the wires of Fig. 3.9, Case I and Fig. 3.10 belong to the field

$$V + j\psi = \ln \tan (z + jx) \tag{3.31}$$

Here V is potential and ψ is a stream function. There are negative equipotentials about $z = x = 0$ and positive equipotentials about $x = 0$, $z = \pm\pi/2$. For an equipotential coinciding with the surface of a wire of z-diameter, $2\, z_{\text{wire}}$, d/p is thus

$$d/p = \frac{z_{\text{wire}}}{\pi/4} \tag{3.32}$$

at $x = 0$, $z < \pi/4$

$$V = \ln \tan z \tag{3.33}$$

at $z = 0$

$$V = \ln \tanh x \tag{3.34}$$

Hence, for an equipotential on the wire with an z-diameter $2z$, the x-diameter $2x$ can be obtained from (3.33) and (3.34) as

$$2x = 2 \tanh^{-1} \tan z \tag{3.35}$$

Of course, the ratio of the x-diameter d_1 to the pitch is given by

$$d_1/p = \frac{x}{\pi/4} \tag{3.36}$$

where x is obtained from (3.35).

In Fig. 3.12, d_1/d is plotted vs. d/p by means of (3.35) and (3.36). This shows that for wire diameters up to $d/p = .5$ (open space equal to wire diameter) the equipotentials representing the wire are very nearly round.

The total electric flux from each wire is $2\pi\epsilon$ and the potential of a wire of z-diameter $2z$ is $V = -\ln \tan z$. Hence, the stored energy W_1 per unit length per wire, half the product of the charge and the voltage, is

$$W_1 = -\pi\epsilon \ln \tan z \tag{3.37}$$

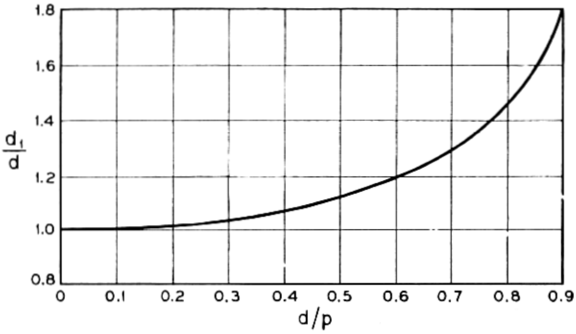

Fig. 3.12—Ratio of the two diameters of the wire of a helix for two turns per wavelength (see Fig. 3.9) vs. the ratio of one of the diameters to the pitch.

The total distant field and the useful field component are given by expanding (3.31) in Fourier series and taking the fundamental component, giving

$$V = -2 \cos 2z e^{\mp 2x} \tag{3.38}$$

The $-$ sign applies for $x > 0$ and the $+$ sign for $x < 0$. Half of this can be regarded as belonging to a field moving to the right and half to a field moving to the left.

For a field equal to half that specified by (3.38), which might be part of the field of a developed helically conducting sheet, the stored energy W_2 per unit depth can be obtained by integrating $(E_z^2 + E_x^2)\ \epsilon/2$ from $x = -\infty$ to $x = +\infty$ and from $z = -\pi/4$ to $+\pi/4$, and it turns out to be

$$W_2 = \tfrac{1}{2}\ \pi\epsilon \tag{3.39}$$

If we add another field component similar to half of (3.38), but in quadrature with respect to z and t, we will have the traveling wave of a helically conducting sheet with the same distant traveling field component as given by (3.31). Hence, the ratio R of the stored energy for the developed sheet to the stored energy for the developed helix is

$$R = 2W_2/W_1 = -\frac{1}{\ln \tan z} \tag{3.40}$$

R is the ratio of the stored energies, and hence of the power flows (since the waves both propagate with the speed of light) of a developed helically conducting sheet and a developed helix with the same distant traveling fundamental field components. Hence, at a given distance $(E^2/\beta^2 P)^{1/3}$ for the helix is $R^{1/3}$ times as great as for the helically conducting sheet. In Fig. 3.13, $R^{1/3}$ is plotted vs. d/p.

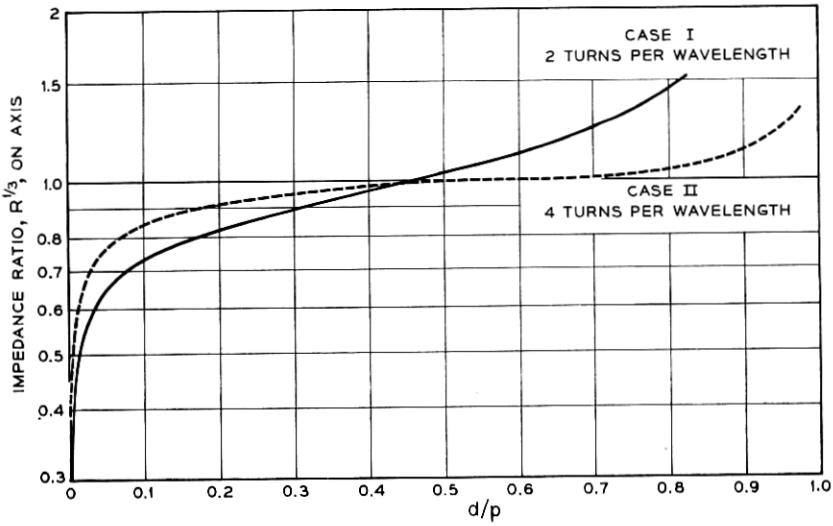

Fig. 3.13—Ratio $R^{1/3}$ of $(E^2/\beta^2 P)^{1/3}$ for a helix to the value for a helically conducting sheet for the distant field.

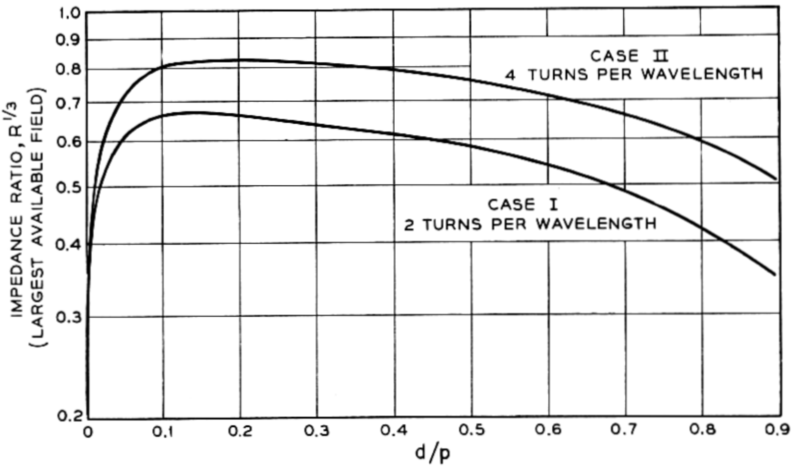

Fig. 3.14—Ratio $R^{1/3}$ of $(E^2/\beta^2 P)^{1/3}$ for a helix to the value for a helically conducting sheet, field at the inside diameter of the helix or sheet.

The maximum available field for the developed helically conducting sheet (equation (3.38)) is that for $x = 0$. The maximum available field for the developed helix (equation (3.31)) is that for an electron grazing the helix inner or outer diameter, that is, an electron at a value of x given by (3.35). The fundamental sinusoidal component of the field varies as $\exp(-2x)$ for both the sheet and the helix, and hence there is a loss in E^2 by a factor $\exp(-4x)$ because of this. We wish to make a comparison on the basis of E^2 and power or energy. Hence, on basis of maximum available field squared we would obtain from (3.40)

$$R = -\frac{1}{\ln \tan z} e^{-4x} \tag{3.41}$$

where x is obtained from (3.35). Figure 3.14 was obtained from (3.32), (3.35) and (3.41).

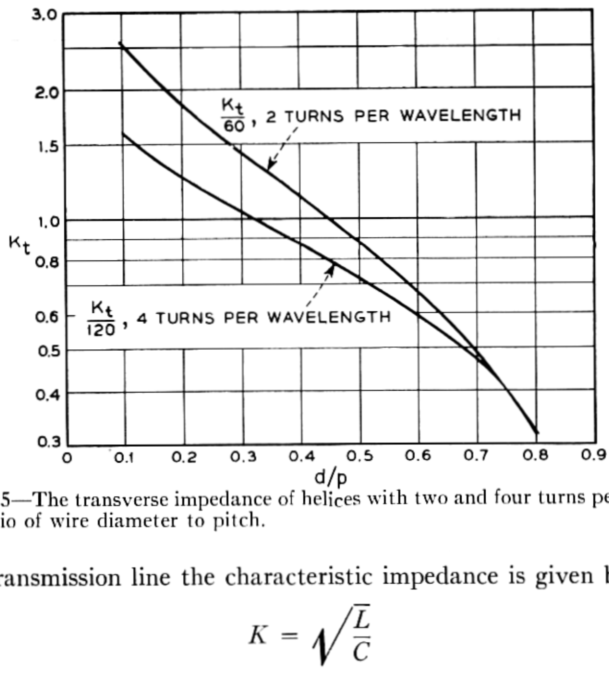

Fig. 3.15—The transverse impedance of helices with two and four turns per wavelength vs. the ratio of wire diameter to pitch.

In a transmission line the characteristic impedance is given by

$$K = \sqrt{\frac{L}{C}} \tag{3.42}$$

Here L and C are the inductance and capacitance per unit length. This impedance should be identified with the transverse impedance of the helix. We also have for the velocity of propagation, which will be the velocity of light, c,

$$c = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\epsilon}} \tag{3.43}$$

From (3.42) and (3.43) we obtain

$$K_t = \sqrt{\mu\epsilon}/C = \sqrt{\mu/\epsilon}(\epsilon/C)$$
$$= 377\ \epsilon/C \tag{3.44}$$

Now C is the charge Q divided by the voltage V. Hence

$$K_t = 377\ \epsilon V/Q \tag{3.45}$$

In this case we have

$$K_t = \frac{337\epsilon \ln \tan z}{2\pi\epsilon}$$
$$K_t = -60 \ln \tan_z \tag{3.46}$$

To obtain the impedance of the corresponding helically conducting sheet we assume, following (3.30)

$$K_t = (\gamma/\beta)\ (\gamma/\beta_0)\ (30/\gamma a) \tag{3.47}$$

and assuming a slow wave, let $\gamma = \beta$, so that

$$K_t = 30/\beta_0 a \tag{3.48}$$

If we are to have n turns per wavelength, and the speed of light in the direction of conduction, then we must have

$$\beta_0 a = 1/n \tag{3.49}$$

whence

$$K_t = 30n \tag{3.50}$$

For $n = 2$ (two turns per wavelength), $K = 60$. In Fig. 3.15, the characteristic impedance K_t as obtained from (3.46) divided by 60 (from (3.50)) is plotted vs. d/p.

3.3b *Four Turns per Wavelength*

In this case there are enough wires so that we can add a quadrature component as in Fig. 3.10 and thus produce a traveling wave rather than a standing wave. Thus, we can make a more direct comparison between the developed sheet and the developed helix.

For the developed helix we have

$$V + j\psi = \ln \tan (z + jx) + \frac{A}{\cos 2\,(z + jx)} \tag{3.15}$$

If we transform this to new coordinates z_1, x_1 about an origin at $z = 0$, $x = \pi/4$ we obtain

$$V + j\psi = \ln\left(\frac{1 + \tan(z_1 + jx_1)}{1 - \tan(z_1 + jx_1)}\right) - \left(\frac{A}{\sin 2(z_1 + jx_1)}\right) \tag{3.52}$$

We can now adjust A to give a zero equipotential of diameter $2z_1$ about $x = x_1 = 0$, $z_1 = 0$ ($z = \pi/4$) by letting

$$A = (\sin 2z_1)\ln\left(\frac{1 + \tan z_1}{1 - \tan z_1}\right) \tag{3.53}$$

If A is so chosen, there will be roughly circular equipotentials of z-diameter $2z_1$ about $z = \pm\pi/4$, etc. There will also be roughly circular equipotentials of the same z-diameter about $z = 0, \pm\pi/2$, etc., of potential $\pm V$. That about $z = 0$ has a potential

$$V = \ln\left(\frac{1 + \tan z_1}{1 - \tan z_1}\right)\frac{A}{\cos 2z_1} \tag{3.54}$$

where A is taken from (3.53).

The distance between centers of equipotentials is $p = \pi/4$, so that the ratio of z-diameter of the equipotentials to pitch is

$$d/p = 2z_1/(\pi/4) = z_1/(\pi/8) \tag{3.55}$$

The x-diameter of the equipotential about $z = 0$ (and of those about $z = \pm\frac{\pi}{2}$ etc.) can be obtained as $2x$ by letting V have the value given by (3.54) and setting $z = 0$ in (3.51), giving

$$V = \ln\tanh x + \frac{A}{\cosh 2x} \tag{3.56}$$

The ratio of this x-diameter to the pitch, d_1/p, is

$$d_1/p = y/(\pi/8), \tag{3.57}$$

x is obtained from (3.56).

To obtain the x-diameter of the 0 potential electrodes we take the derivative (3.52) with respect to z_1, giving the gradient in the z direction

$$\frac{\partial V}{\partial z_1} + j\frac{\partial\psi}{\partial z_1} = \frac{\sec^2(z_1 + jx_1)}{1 + \tan(z_1 + jx_1)} + \frac{\sec^2(z_1 + jx_1)}{1 - \tan(z_1 + jx_1)} - \frac{2A\cos 2(z_1 + jx_1)}{\sin 2(z_1 + jx_1)} \tag{3.58}$$

We then let $z_1 = 0$ and find the value of x_1 for which $\partial V/\partial z_1 = 0$. When $z_1 = 0$, (3.58) becomes

$$A = \sinh 2x_1 \tanh 2x_1 \frac{(1 - \tanh^2 x_1)}{(1 + \tanh^2 x_1)} \tag{3.59}$$

As A is given by (3.53), we can obtain x, from (3.57), and the ratio of the x-diameter d_2 to the pitch is

$$d_2/p = x_1/(\pi/8) \tag{3.60}$$

Figure 3.16 shows d_1/d and d_2/d vs. d/p.

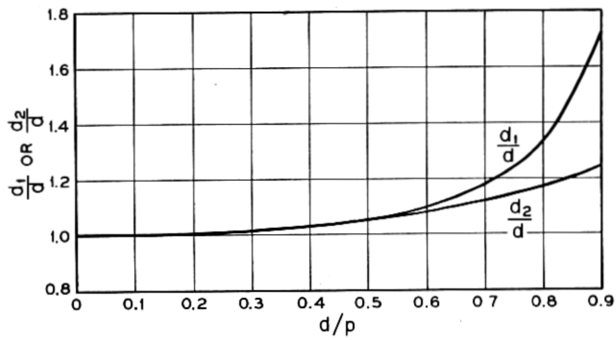

Fig. 3.16—Ratios of the wire diameters for the four turns per wavelength analysis.

The ratios R and the impedance are obtained merely by comparing the power flow for the developed sheet with a single sinusoidally distributed component with the power flow for case II for the same distant field. In a comparison with the helically conducting sheet, $n = 2$ is used in (3.50). The results are shown in Figs. 3.13, 3.14, 3.15. We see that on the basis of the largest available field, the best wire size is $d/p = .19$.

3.4 Transmission Line Equations and Helices

It is of course possible at any frequency to construct a transmission line with a distributed shunt susceptance B per unit length and a distributed shunt reactance X per unit length and, by adjusting B and X to make the phase velocity and $E^2/\beta^2 P$ the same for the artificial line as for the helix. In simulating the helix with the line, B and X must be changed as frequency is changed. Indeed, it may be necessary to change B and X somewhat in simulating a helix with a forced wave on it, as, the wave forced by an electron stream. Nevertheless, a qualitative insight into some problems can be obtained by use of this type of circuit analogue.

3.4a *Effect of Dielectric on Helix Impedance Parameter*

One possible application of the transmission line equivalent is in estimating the lowering of the helix impedance parameter $(E^2/\beta^2P)^{1/3}$.

In the case of a transmission line of susceptance B and reactance X per unit length, we have for the phase constant β and the characteristic impedance K

$$\beta = \sqrt{BX} \tag{3.61}$$

$$K = \sqrt{X/B} \tag{3.62}$$

Now, suppose that B is increased by capacitive loading so that β has a larger value β_d. Then we see that K will have a value K_d

$$K_d = (\beta/\beta_d)K \tag{3.63}$$

Where should K be measured? It is reasonable to take the field at the surface of the helix or the helically conducting sheet as the point at which the field should be evaluated. The field at the axis will, then, be changed by a different amount, for the field at the surface of the helix is $I_0(\gamma a)$ times the field at the axis.

Suppose, then, we design a helix to have a phase constant β (a phase velocity ω/β) and, in building it, find that the dielectric supports increase the phase constant to a value β_d giving a smaller phase velocity ω/β_d. Suppose β/β_0 is large, so that γ is nearly equal to β. How will we estimate the actual axial value of $(E^2/\beta^2P)^{1/3}$? We make the following estimate:

$$(E^2/\beta^2 P)_d^{1/3} = \left(\frac{\beta}{\beta_d}\right)^{1/3} \left(\frac{I_0(\beta a)}{I_0(\beta_d a)}\right)^{2/3} (E^2/\beta^2 P)^{1/3} \tag{3.64}$$

Here the factor $(\beta/\beta_d)^{1/3}$ is concerned with the reduction of impedance measured at the helix surface, and the other factor is concerned with the greater falling-off of the field toward the center of the helix because of the larger value of γ (taken equal to β and β_d in the two cases).

The writer does not know how good this estimate may be.

3.4b *Coupled Helices*

Another case in which the equivalent transmission line approach is particularly useful is in considering the problem of concentric helices. Such configurations have been particularly suggested for producing slow transverse fields. They can be analyzed in terms of helically conducting cylinders or in terms of developed cylinders. A certain insight can be gained very quickly, however, by the approach indicated above.

We will simulate the helices by two transmission lines of series impedances jX_1 and jX_2, of shunt admittances jB_1 and jB_2 coupled by series mutual impedance and shunt mutual admittance jX_{12} and jB_{12}. If we consider a wave which varies as $\exp(-j\Gamma z)$ in the z direction we have

$$\Gamma I_1 - jB_1V_1 - jB_{12}V_2 = 0 \tag{3.65}$$

$$\Gamma V_1 - jX_1I_1 - jX_{12}I_2 = 0 \tag{3.66}$$

$$\Gamma I_2 - jB_2V_2 - jB_{12}V_1 = 0 \tag{3.67}$$

$$\Gamma V_2 - jX_2I_2 - jX_{12}I_1 = 0 \tag{3.68}$$

If we solve (3.65) and (3.67) for I_1 and I_2 and eliminate these, we obtain

$$\frac{V_2}{V_1} = \frac{-(\Gamma^2 + X_1 B_1 + X_{12} B_{12})}{X_1 B_{12} + B_2 X_{12}} \tag{3.69}$$

$$\frac{V_1}{V_2} = \frac{-(\Gamma^2 + X_2 B_2 + X_{12} B_{12})}{X_2 B_{12} + B_1 X_{12}} \tag{3.70}$$

Multiplying these together we obtain

$$\begin{aligned} \Gamma^4 &+ (X_1 B_1 + X_2 B_2 + 2X_{12} B_{12})\Gamma^2 \\ &+ (X_1 X_2 - X_{12}^2)(B_1 B_2 - B_{12}^2) = 0 \end{aligned} \tag{3.71}$$

We can solve this for the two values of Γ^2

$$\begin{aligned} \Gamma^2 = &-\tfrac{1}{2}(X_1 B_1 + X_2 B_2 + 2X_{12} B_{12}) \\ &\pm \tfrac{1}{2} [(X_1 B_1 - X_2 B_2)^2 + 4 (X_1 B_1 + X_2 B_2)(X_{12} B_{12}) \\ &+ 4 (X_1 X_2 B_{12}^2 + B_1 B_2 X_{12}^2)]^{1/2} \end{aligned} \tag{3.72}$$

Each value of Γ^2 represents a normal mode of propagation involving both transmission lines. The two square roots of each Γ^2 of course indicate waves going in the positive and negative directions.

Suppose we substitute (3.72) into (3.69). We obtain

$$\frac{V_2}{V_1} = \frac{\begin{aligned}-(X_1 B_1 - X_2 B_2) \pm [(X_1 B_1 - X_2 B_2)^2 \\ + 4(X_1 B_1 + X_2 B_2)(X_{12} B_{12}) + 4(X_1 X_2 B_{12}^2 + B_1 B_2 X_{12}^2)]^{1/2}\end{aligned}}{2(X_1 B_{12} + B_2 X_{12})} \tag{3.73}$$

We will be interested in cases in which X_1B_1 is very nearly equal to X_2B_2. Let

$$\Delta\Gamma_0^2 = X_1B_1 - X_2B_2 \tag{3.74}$$

and in the parts of (3.73) where the difference of (3.74) does not occur use

$$\begin{aligned} X_1 &= X_2 = X \\ B_1 &= B_2 = B \end{aligned} \tag{3.75}$$

Then, approximately

$$\frac{V_2}{V_1} = \frac{-\Delta\Gamma_0^2 \pm [(\Delta\Gamma_0^2)^2 + 4(XB_{12} + BX_{12})^2]^{1/2}}{2(XB_{12} + BX_{12})} \tag{3.76}$$

Let us assume that $\Delta\Gamma^2$ is very small and retains terms up to the first power of $\Delta\Gamma^2$

$$\frac{V_2}{V_1} = \pm 1 + \frac{\Delta\Gamma_0^2}{2(XB_{12} + BX_{12})} \tag{3.77}$$

Let

$$\Gamma_0^2 = -XB \tag{3.78}$$

$$\frac{V_2}{V_1} = \pm 1 - \frac{\Delta\Gamma_0^2/\Gamma_0^2}{2(B_{12}/B + X_{12}/X)} \tag{3.79}$$

Let us now interpret (3.79). This says that if $\Delta\Gamma_0^2$ is zero, that is, if $X_1B_1 = X_2B_2$ exactly, there will be two modes of transmission, a ***longitudinal*** mode in which $V_2/V_1 = +1$ and a *transverse* mode in which $V_2/V_1 = -1$. If we excite the transverse mode it will persist. However, if $\Delta\Gamma_0^2 \neq 0$, there will be two modes, one for which $V_2 > V_1$ and the other for which $V_2 < V_1$; in other words, as $\Delta\Gamma_0^2$ is increased, we approach a condition in which one mode is nearly propagated on one helix only and the other mode nearly propagated on the other helix only. Then if we drive the pair with a transverse field we will excite both modes, and they will travel with different speeds down the system.

We see that to get a good transverse field we must make

$$\frac{\Delta\Gamma_0^2}{\Gamma_0^2} << 2(B_{12}/B + X_{12}/X) \tag{3.80}$$

In other words, the stronger the coupling (B_{12}, X_{12}) the more the helices can afford to differ (perhaps accidentally) in propagation constant and the pair still give a distinct transverse wave.

Thus, it seems desirable to couple the helices together as tightly as possible and especially to see that B_{12} and X_{12} have the same signs.

Let us consider two concentric helices wound in opposite directions, as in Fig. 3.17. A positive voltage V_1 will put a positive charge on helix 1 while a positive voltage V_2 will put a negative charge on helix 1. Thus, B_{12}/B is negative. It is also clear that the positive current I_2 will produce flux linking helix 1 in the opposite direction from the positive current I_1, thus making X_{12}/X negative. This makes it clear that to get a good transverse field between concentric helices, the helices should be wound in opposite directions. If the helices were wound in the same direction, the "transverse" and "longitudinal" modes would cease to be clearly transverse and longitudinal should the phase velocities of the two helices by accident differ a little. Further, even if the phase velocities were the same, the transverse and longitudinal modes would have almost the same phase velocity, which in itself may be undesirable.

Field analyses of coupled helices confirm these general conclusions.

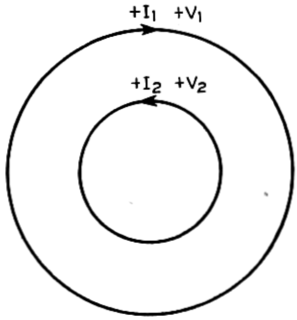

Fig. 3.17—Currents and voltages of concentric helices.

3.5 About Loss in Helices

The loss of helices is not calculated in this book. Some matters concerning deliberately added loss will be considered, however.

Loss is added to helices so that the backward loss of the tube (loss for a wave traveling from output to input) will be greater than the forward gain. If the forward gain is greater than the backward loss, the tube may oscillate if it is not terminated at each end in a good broad-band match.

In some early tubes, loss was added by making the helix out of lossy wire, such as nichrome or even iron, which is much lossier at microwave frequencies because of its ferromagnetism. Most substances are in many cases not lossy enough. Iron is very lossy, but its presence upsets magnetic focusing.

When the helix is supported by a surrounding glass tube or by parallel ceramic or glass rods, loss may be added by spraying aquadag on the inside or outside of the glass tube or on the supporting rods. This is advantageous in that the distribution of loss with distance can be controlled.

It is obvious that for lossy material a finite distance from the helix there is a resistivity which gives maximum attenuation. A perfect conductor would introduce no dissipation and neither would a perfect insulator.

If lossy material is placed a little away from the helix, loss can be made greater at lower frequencies (at which the field of the helix extends out into the lossy material) than at higher frequencies (at which the fields of the helix are crowded near the helix and do not give rise to much current in the lossy material. This construction may be useful in preventing high-frequency tubes from oscillating at low frequencies.

Loss may be added by means of tubes or collars of lossy ceramic which fit around the helix.

APPENDIX I

MISCELLANEOUS INFORMATION

This appendix presents an assortment of material which may be useful to the reader.

Constants

Electronic charge-to-mass ratio:

$\eta = e/m = 1.759 \times 10^{11}$ Coulomb/kilogram

Electronic charge: $e = 1.602 \times 10^{-19}$ Coulomb

Dielectric constant of vacuum: $\epsilon = 8.854 \times 10^{-12}$ Coulomb/meter

Permitivity of vacuum: $\mu = 1.257 \times 10^{-6}$ Henry/meter

Boltzman's constant: $k = 1.380 \times 10^{-23}$ Joule/degree

Cross Products

$$(A' \times A'')_x = A_y' A_z'' - A_z' A_y''$$
$$(A' \times A'')_y = A_z' A_x'' - A_x' A_z''$$
$$(A' \times A'')_z = A_x' A_y'' - A_y' A_x''$$

Maxwell's Equations: Rectangular Coordinates

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\epsilon E_x + J_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \qquad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y + J_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z + J_z$$

Maxwell's Equations: Axially Symmetrical

$$\frac{\partial E_\varphi}{\partial z} = -j\omega\mu H_\rho \qquad \frac{\partial H_\varphi}{\partial z} = -(j\omega\epsilon E_\rho + J_\rho)$$

$$\frac{\partial E\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = -j\omega\mu H_\varphi \qquad \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = j\omega\epsilon E_\varphi + J_\varphi$$

$$\frac{\partial}{\partial \rho}(\rho E_\varphi) = -j\omega\mu\rho H_z \qquad \frac{\partial}{\partial \rho}(\rho H_\varphi) = \rho(j\omega\epsilon E_z + J_z)$$

Miscellaneous Formulae Involving $I_n(x)$ and $K_n(x)$

1. $I_{\nu-1}(Z) - I_{\nu+1}(Z) = \frac{2\nu}{Z} I_\nu(Z), \qquad K_{\nu-1}(Z) - K_{\nu+1}(Z) = -\frac{2\nu}{Z} K_\nu(Z)$

2. $I_{\nu-1}(Z) + I_{\nu+1}(Z) = 2I'_\nu(Z), \qquad K_{\nu-1}(Z) + K_{\nu+1}(Z) = -2K'_\nu(Z)$

3. $ZI'_\nu(Z) + \nu I_\nu(Z) = ZI_{\nu-1}(Z), \qquad ZK'_\nu(Z) + \nu K_\nu(Z) = -ZK_{\nu-1}(Z)$

4. $ZI'_\nu(Z) - \nu I_\nu(Z) = ZI_{\nu+1}(Z), \qquad ZK'_\nu(Z) - \nu K_\nu(Z) = -ZK_{\nu+1}(Z)$

5. $\left(\frac{d}{ZDZ}\right)^m \{Z^\nu I_\nu(Z)\} = Z^{\nu-m} I_{\nu-m}(Z), \left(\frac{d}{ZdZ}\right)^m \{Z^\nu K_\nu(Z)\}$

$$= (-)^m Z^{\nu-m} K_{\nu-m}(Z)$$

6. $\left(\frac{d}{ZdZ}\right)^m \left\{\frac{I_\nu(Z)}{Z^\nu}\right\} = \frac{I_{\nu+m}(Z)}{Z^{\nu+m}}, \qquad \left(\frac{d}{ZdZ}\right)^m \left\{\frac{K_\nu(Z)}{Z^\nu}\right\} = (-)^m \frac{K_{\nu+m}(Z)}{Z^{\nu+m}}$

7. $I'_0(Z) = I_1(Z), \qquad K'_0(Z) = -K_1(Z)$

8. $I_{-\nu}(Z) = I_\nu(Z), \qquad K_{-\nu}(Z) = K_\nu(Z)$

9. $K_{1/2}(Z) = \left(\frac{\pi}{2Z}\right)^{1/2} e^{-Z}$

10. $I_\nu(Ze^{m\pi i}) = e^{m\nu\pi i} I_\nu(Z)$

11. $K_\nu(Ze^{m\pi i}) = e^{-m\nu\pi i} K_\nu(Z) - i \frac{\sin m\nu\pi}{\sin \nu\pi} I_\nu(Z)$

12. $I_\nu(Z)\, K_{\nu+1}(Z) + I_{\nu+1}(Z)\, K_\nu(Z) = 1/Z$

For small values of X:

13. $I_0(X) = 1 + .25\, X^2 + .015625\, X^4 + \cdots$

14. $I_1(X) = .5X + .0625\, X^3 + .002604\, X^5 + \cdots$

15. $K_0(X) = -\left\{\gamma + \ln\left(\frac{X}{2}\right)\right\} I_0(X) + \frac{1}{4} X^2 + \frac{3}{128} X^4 + \cdots\}$

16. $K_1(X) = \left\{\gamma + \ln\left(\frac{X}{2}\right)\right\} I_1(X) + \frac{1}{X} - \frac{1}{4} X - \frac{5}{64} X^3 + \cdots\}$

$$\gamma = .5772 \ldots \text{ (Euler's constant)}$$

For large values of X:

17. $I_0(X) \sim \frac{e^X}{(2\pi X)^{1/2}} \left\{1 + \frac{.125}{X} + \frac{.0703125}{X^2} + \frac{.073242}{X^3} + \cdots\right\}$

$$18.\ I_1(X) \sim \frac{e^X}{(2\pi X)^{1/2}} \left\{1 - \frac{.375}{X} - \frac{.1171875}{X^2} - \frac{.102539}{X^3} - \cdots\right\}$$

$$19.\ K_0(X) \sim \left(\frac{\pi}{2X}\right)^{1/2} e^{-X} \left\{1 - \frac{.125}{X} + \frac{.0703125}{X^2} - \frac{.073242}{X^3} + \cdots\right\}$$

$$20.\ K_1(X) \sim \left(\frac{\pi}{2X}\right)^{1/2} e^{-X} \left\{1 + \frac{.375}{X} - \frac{.1171875}{X^2} + \frac{.102539}{X^3} - \cdots\right\}.$$

Fig. A1.1 shows $I_0(X)$ (solid line) and the first two terms of 13 and the first term of 17 (dashed lines).

Fig. A1.2 shows $I_1(X)$ (solid line) and the first term of 14 and the first term of 18 (dashed lines).

Fig. A1.3 shows $K_0(X)$ (solid line) and $-\left\{\gamma + \ln\left(\frac{X}{2}\right)\right\} I_0(X)$ and the first term of 19 (dashed lines).

Fig. A1.4 shows $K_1(X)$ (solid line) and $\left\{\gamma + \ln\left(\frac{X}{2}\right)\right\} I_1(X) + 1/X$ and the first term of 20 (dashed lines).

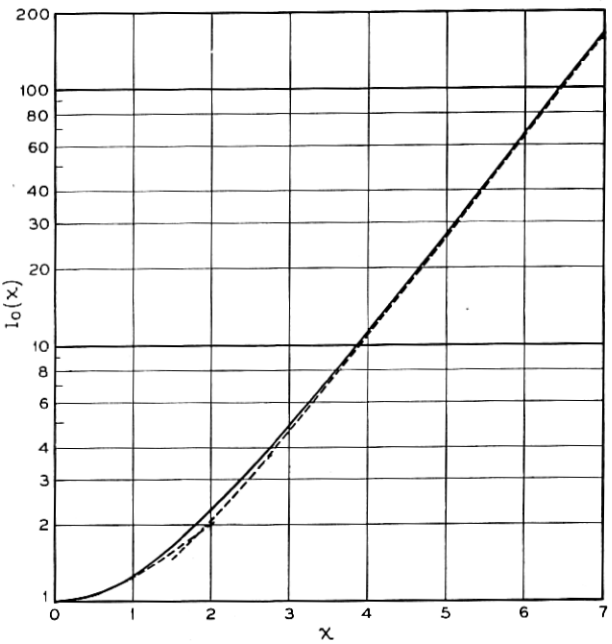

Fig. A1.1—The correct value of $I_0(X)$ (solid line), the first two terms of the series expansion 13 (dashed line from origin), and the first term of the asymptotic series 17 (dashed line to right).

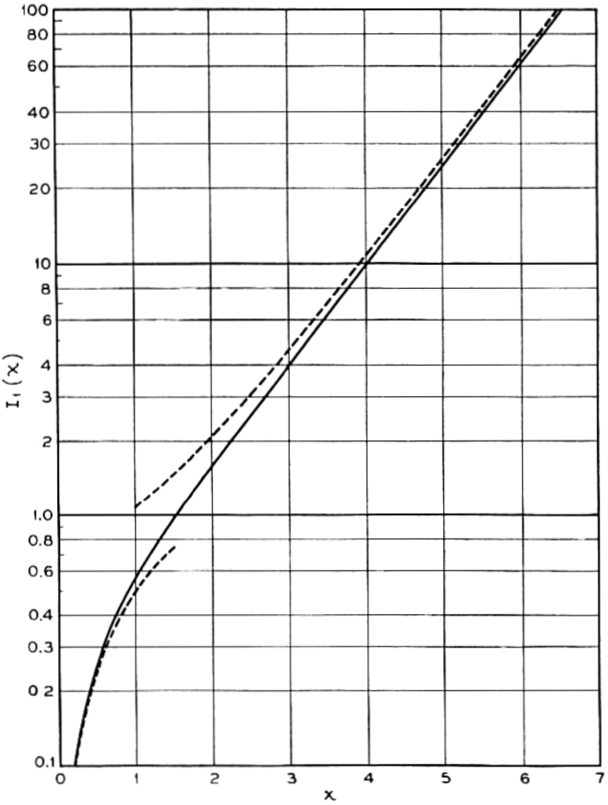

Fig. A1.2—The correct value of $I_1(X)$ (solid line), the first term of the series expansion 14'(lower dashed line), and the first term of the asymptotic series 18 (upper dashed line).

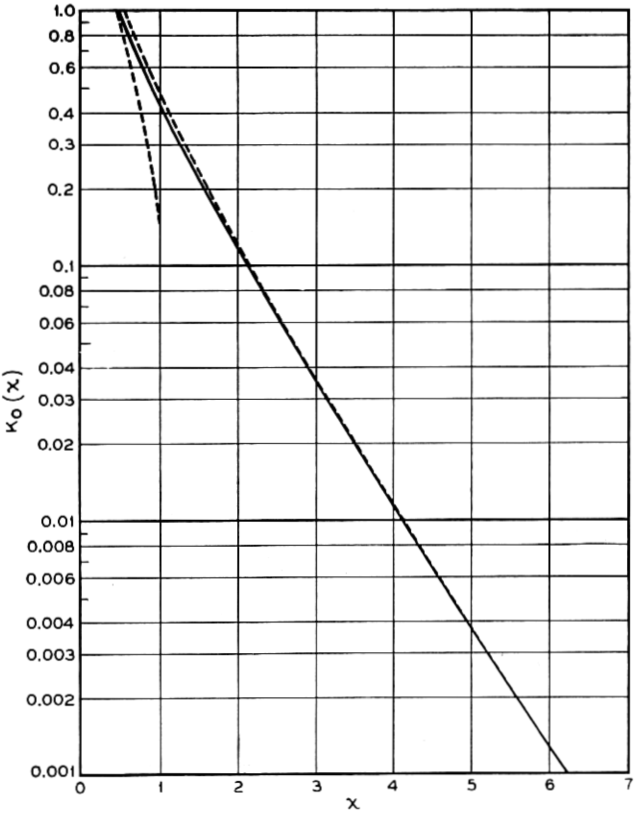

Fig. A1.3—The correct value of $K_0(X)$ (solid line), $-\left\{\gamma + \ln\left(\frac{X}{2}\right)\right\} I_0(X)$ from the series expansion 15 (left dashed line), and the first term of the asymptotic series 19 (right dashed line).

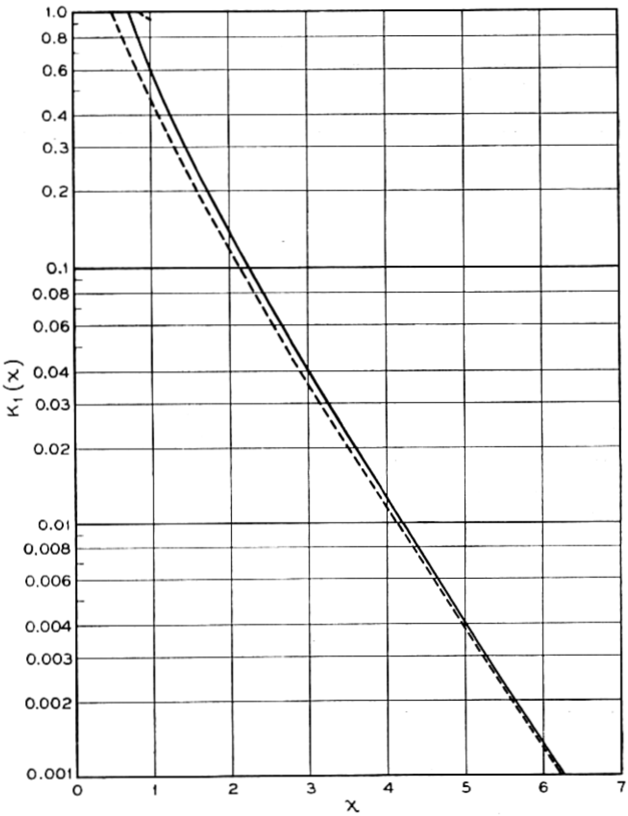

Fig. A1.4—The correct value of $K_1(X)$ (solid line), $\left\{\gamma + \ln\left(\frac{X}{2}\right)\right\} I_1(X)$ from the series expansion 16 (upper dashed line), and the first term of the asymptotic series 20 (lower dashed line).

APPENDIX II

PROPAGATION ON A HELICALLY CONDUCTING CYLINDER

The circuit parameter important in the operation of traveling-wave tubes is:

$$(E_z^2/\beta^2 P)^{1/3} \tag{1}$$

$$\beta = \omega/v. \tag{2}$$

Here E_z is the peak electric field in the direction of propagation, P is the power flow along the helix, and v is the phase velocity of the wave. The quantity $E_z^2/\beta^2 P$ has the dimensions of impedance.

While the problem of propagation along a helix has not been solved, what appears to be a very good approximation has been obtained by replacing the helix with a cylinder of the same mean radius α which is conducting only in a helical direction making an angle Ψ with the circumference, and nonconducting in the helical direction normal to this.

An appropriate solution of the wave equation in cylindrical co-ordinates for a plane wave having circular symmetry and propagating in the z direction with velocity

$$v = \frac{\omega}{\beta}, \tag{3}$$

less than the speed of light c, is

$$E_z = [AI_0(\gamma r) + BK_0(\gamma r)]e^{j(\omega t - \beta_z)} \tag{4}$$

where I_0 and K_0 are the modified Bessel functions, and

$$\gamma^2 = \beta^2 - \left(\frac{\omega}{c}\right)^2 = \beta^2 - \beta_0^2. \tag{5}$$

The form of the z (longitudinal) components of an electromagnetic field varying as $e^{j(\omega t - \beta z)}$ and remaining everywhere finite might therefore be

$$H_{z1} = B_1 I_0(\gamma r)e^{j(\omega t - \beta z)} \tag{6}$$

$$E_{z3} = B_3 I_0(\gamma r)e^{j(\omega t - \beta z)} \tag{7}$$

inside radius α, and

$$H_{z2} = B_2 K_0(\gamma r)e^{j(\omega t - \beta z)} \tag{8}$$

$$E_{z4} = B_4 K_0(\gamma r) e^{j(\omega t - \beta z)} \tag{9}$$

outside radius α. Omitting the factor $e^{j(\omega t - \beta z)}$ the radial and circumferential components associated with these, obtained by applying the curl equation, are, inside radius α,

$$H_{\phi 3} = B_3 \frac{j\omega\epsilon}{\gamma} I_1(\gamma r) \tag{10}$$

$$H_{r1} = B_1 \frac{j\beta}{\gamma} I_1(\gamma r) \tag{11}$$

$$E_{\phi 1} = -B_1 \frac{j\omega\mu}{\gamma} I_1(\gamma r) \tag{12}$$

$$E_{r3} = B_3 \frac{j\beta}{\gamma} I_1(\gamma r) \tag{13}$$

and outside radius α

$$H_{\phi 4} = -B_4 \frac{j\omega\epsilon}{\gamma} K_1(\gamma r) \tag{14}$$

$$H_{r2} = -B_2 \frac{j\beta}{\gamma} K_1(\gamma r) \tag{15}$$

$$E_{\phi 3} = B_2 \frac{j\omega\mu}{\gamma} K_1(\gamma r) \tag{16}$$

$$E_{r4} = -B_4 \frac{j\beta}{\gamma} K_1(\gamma r). \tag{17}$$

The boundary conditions which must be satisfied at the cylinder of radius α are that the tangential electric field must be perpendicular to the helix direction

$$E_{z3} \sin \Psi + E_{\phi 1} \cos \Psi = 0 \tag{18}$$

$$E_{z4} \sin \Psi + E_{\phi 2} \cos \Psi = 0, \tag{19}$$

the tangential electric field must be continuous across the cylinder

$$E_{z3} = E_{z4} \text{ (and } E_{\phi 1} = E_{\phi 2}), \tag{20}$$

and the tangential component of magnetic field parallel to the helix direction must be continuous across the cylinder, since there can be no current in the surface perpendicular to this direction.

$$\begin{aligned} H_{z1} \sin \Psi + H_{\phi 3} \cos \Psi = H_{z2} \sin \Psi \\ + H_{\phi 4} \cos \Psi. \end{aligned} \tag{21}$$

These equations serve to determine the ratios of the B's and to determine γ through

$$(\gamma \alpha)^2 \frac{I_0(\gamma\alpha)K_0(\gamma\alpha)}{I_1(\gamma\alpha)K_1(\gamma\alpha)} = (\beta_0 \alpha \cot \Psi)^2. \tag{22}$$

We can easily express the various field components listed in (6) through (17) in terms of a common amplitude factor. As such expressions are useful in understanding the nature of the field, it seems desirable to list them in an orderly fashion.

INSIDE THE HELIX:

$$E_z = BI_0(\gamma r)e^{j(\omega t - \beta z)} \tag{23}$$

$$E_r = j\beta \frac{\beta}{\gamma} I_1(\gamma r)e^{j(\omega t - \beta z)} \tag{24}$$

$$E_\Phi = -B \frac{I_0(\gamma a)}{I_1(\gamma a)} \frac{1}{\cot \psi} I_1(\gamma r)e^{j(\omega t - \beta z)} \tag{25}$$

$$H_z = -j \frac{B}{k} \frac{\gamma}{\beta_0} \frac{I(\gamma a)}{I_1(\gamma a)} \frac{1}{\cot \psi} I_0(\gamma r)e^{j(\omega t - \beta z)} \tag{26}$$

$$H_r = \frac{B}{k} \frac{\beta}{\beta_0} \frac{I_0(\gamma a)}{I_1(\gamma a)} \frac{1}{\cot \psi} I_1(\gamma r)e^{j(\omega t - \beta z)} \tag{27}$$

$$H_\Phi = j \frac{B}{k} \frac{\beta_0}{\gamma} I(\gamma r)e^{j(\omega t - \beta z)}. \tag{28}$$

OUTSIDE THE HELIX:

$$E_z = B \frac{I_0(\gamma a)}{K_0(\gamma a)} K_0(\gamma r)e^{j(\omega t - \beta z)} \tag{29}$$

$$E_r = -jB \frac{\beta}{\gamma} \frac{I_0(\gamma a)}{K_0(\gamma a)} K_1(\gamma r)e^{j(\omega t - \beta z)} \tag{30}$$

$$E_\Phi = -B \frac{I_0(\gamma a)}{K_1(\gamma a)} \frac{1}{\cot \psi} K_1(\gamma r)e^{j(\omega t - \beta z)} \tag{31}$$

$$H_z = j \frac{B}{k} \frac{\gamma}{\beta_0} \frac{I_0(\gamma a)}{K_1(\gamma a)} \frac{1}{\cot \psi} K_0(\gamma r)e^{j(\omega t - \beta z)} \tag{32}$$

$$H_r = \frac{B}{k} \frac{I_0(\gamma a)}{K_1(\gamma a)} \frac{1}{\cot \psi} K_1(\gamma r)e^{j(\omega t - \beta z)} \tag{33}$$

$$H_\Phi = -j \frac{B}{k} \frac{\beta_0}{\gamma} \frac{I_0(\gamma a)}{K_0(\gamma a)} K_1(\gamma r)e^{j(\omega t - \beta z)} \tag{34}$$

Here

$$k = \sqrt{\mu/\epsilon} = 120\,\pi \text{ ohms} \tag{35}$$

The power associated with the propagation is given by

$$P = \tfrac{1}{2}\,\mathrm{Re}\int E \times H^*\, d\tau \tag{36}$$

taken over a plane normal to the axis of propagation. This is

$$P = \pi\,\mathrm{Re}\left[\int_0^a (E_r H_\Phi^* - E_\Phi H_r^*) r\, dr + \int_a^\infty (E_r H_\Phi^* - E_\Phi H_r^*) r\, dr\right] \tag{37}$$

or

$$\begin{aligned} P &= \pi E_z^2(0)\,\frac{\beta\beta_0^2}{\gamma^2\omega\mu}\left[\left(1 + c\,\frac{I_0 K_1}{I_1 K_0}\right)\int_0^\alpha I_1^2(\gamma r) r\, dr \right. \\ &\qquad \left. + \left(\frac{I_0}{K_0}\right)^2\left(1 + \frac{I_1 K_0}{I_0 K_1}\right)\int_\alpha^\infty K_1^2(\gamma r) r\, dr\right] \\ &= E_z^2(0)\,\frac{\pi}{2k}\,\frac{\beta\beta_0\alpha^2}{\gamma^2}\left[\left(1 + \frac{I_0 K_1}{I_1 K_0}\right)(I_1^2 - I_0 I_2) \right. \\ &\qquad \left. + \left(\frac{I_0}{K_0}\right)^2\left(1 + \frac{I_1 K_0}{I_0 K_1}\right)(K_0 K_2 - K_1^2\right]. \end{aligned} \tag{38}$$

where $k = 120\,\pi$ ohms.

Let us now write

$$(E_z^2/\beta^2 P)^{1/3} = (\beta/\beta_0)^{1/3}(\gamma/\beta)^{4/3} F(\gamma a) \tag{39}$$

where

$$\begin{aligned} F(\gamma\alpha) = \left\{\left(\frac{(\gamma\alpha)^2}{240}\right)\left[(I_1^2 - I_0 I_2)\left(1 + \frac{I_0 K_1}{I_1 K_0}\right)\right.\right. \\ \left.\left. + \left(\frac{I_0}{K_0}\right)^2 (K_0 K_2 - K_1^2)\left(1 + \frac{I_1 K_0}{K_1 I_0}\right)\right]\right\}^{-1/3}. \end{aligned} \tag{40}$$

We can rewrite the expression for $F(\gamma\alpha)$ by using relations, Appendix I:

$$F(\gamma\alpha) = \left(\frac{\gamma\alpha}{240}\,\frac{I_0}{K_0}\left[\left(\frac{I_1}{I_0} - \frac{I_0}{I_1}\right) + \left(\frac{K_0}{K_1} - \frac{K_1}{K_0}\right) + \frac{4}{\gamma\alpha}\right]\right)^{-1/3}. \tag{41}$$