Communication in the Presence of Noise—Probability of Error for Two Encoding Schemes

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Recent work by C. E. Shannon and others has led to an expression for the maximum rate at which information can be transmitted in the presence of random noise. Here two encoding schemes are described in which the ideal rate is approached when the signal length is increased. Both schemes are based upon drawing random numbers from a normal universe, an idea suggested by Shannon's observation that in an efficient encoding system the typical signal will resemble random noise. In choosing these schemes two requirements were kept in mind: (1) the ideal rate must be approached, and (2) the problem of computing the probability of error must be tractable. Although both schemes meet both requirements, considerable work has been required to put the expression for the probability of error into manageable form.

1. Introduction

In recent work concerning the theory of communication it has been shown that the maximum or ideal rate of signaling which may be achieved in the presence of noise is (1, 2, 3, 4, 5)

$$R_I = F \log_2 (1 + W_S/W_N) \text{ bits/sec.} \tag{1-1}$$

In this expression F is the width of the frequency band used for signaling (which we suppose to extend from 0 to F cps), W_S is the average signaling power and W_N the average power of the noise. The noise is assumed to be random and to have a constant power spectrum of W_N/F watts per cps over the frequency band $(0, F)$.

This ideal rate is achieved only by the most efficient encoding schemes in which, as Shannon (1, 2) states, the typical signal has many of the properties of random noise. Here we shall study two different encoding schemes, both of them referring to a bandwidth F and a time interval T. By making the product FT large enough the ideal rate of signaling may be approached in either case* and we are interested in the probability of error for rates of signaling a little below the rate (1-1). The work given here is closely associated with Section 7 of Shannon's second paper (2).

In the first encoding scheme the signal corresponding to a given message lasts exactly T seconds, but (because the signal is zero outside this assigned interval of duration) the power spectrum of the signal is not exactly zero for frequencies exceeding F. In the second encoding scheme, the signal

* A recent analysis by M. J. E. Golay (*Proc. I. R. E.*, Sept. 1949, p. 1031) indicates that the ideal rate of signaling may also be approached by quantized PPM under suitable conditions.

power spectrum is limited to the band $(0, F)$ but the signal, regarded as a function of time, is not exactly zero outside its allotted interval of length T.

It turns out that both schemes lead to the same mathematical problem which may be stated as follows: Given two universes of random numbers both distributed normally about zero with standard deviations σ and ν, respectively. Let the first universe be called the σ (signal) universe and the second the ν (noise) universe. Draw $2N + 1$ numbers $A_{-N}^{(0)}, A_{-N+1}^{(0)}, \cdots, A_0^{(0)}, \cdots, A_N^{(0)}$ at random from the σ universe. These $2N + 1$ numbers may be regarded as the rectangular coordinates of a point P_0 in $2N + 1$-dimensional space. Draw $2N + 1$ numbers $B_{-N}, \cdots, B_0, \cdots, B_N$ at random from the ν universe and imagine a (hyper-) sphere S of radius $x_0^{1/2} = P_0Q$, where

$$x_0 = \sum_{n=-N}^{N} B_n^2 = \overline{P_0Q^2}, \tag{1-2}$$

centered on the point Q whose coordinates are $A_n^{(0)} + B_n$, $n = -N, \cdots, 0, \cdots, N$. Return to the σ universe, draw out K sets of $2N + 1$ numbers each, denote the kth set by $A_{-N}^{(k)}, \cdots, A_0^{(k)}, \cdots, A_N^{(k)}$ and the associated point by P_k.

What is the probability that none of the K points $P_1, \cdots, P_K$ lie within the sphere S? In other words what is the probability, which will be denoted by "Prob. $(P_1Q, \cdots, P_KQ > P_0Q)$," that the K distances $P_1Q, \cdots, P_KQ$ will all exceed the radius P_0Q? In terms of the A_n's and B_n's we ask for the probability that all K of the numbers $x_1, x_2, \cdots, x_K$ exceed x_0 where

$$x_k = \sum_{n=-N}^{N} (A_n^{(k)} - A_n^{(0)} - B_n)^2 = \overline{P_kQ^2} \tag{1-3}$$

Expression (1-2) for x_0 is seen to be a special case of (1-3). The relationship between the points $P_0, Q, P_1, P_2, \cdots, P_k, \cdots, P_K$ is indicated in Fig. 1.

The answer to this problem is given by the rather complicated expression (4-12) which, when written out, involves Bessel functions of imaginary argument and of order $N - 1/2$. When N and K become very large the work of Section 5 shows that the probability in question is given by

$$\text{Prob. } (P_1Q, \cdots, P_KQ > P_0Q) = (1 + \operatorname{erf} H)/2 + 0(1/K) + 0(N^{-1/2} \log^{3/2} N) \tag{1-4}$$

where, with $r = \nu^2/\sigma^2$,

$$H = \left(\frac{1+r}{4N}\right)^{1/2} \left[(N + 1/2) \log_e (1 + 1/r) - \log_e (K + 1) + \frac{1}{2} \log_e \frac{2\pi N(1 + 2r)}{(1 + r)^2} \right] \tag{1-5}$$

The symbol $0(N^{-1/2} \log^{3/2} N)$ stands for a term of order $N^{-1/2} \log^{3/2} N$, i.e., a positive constant C and a value N_0 can be found such that the absolute value of the term in question is less than $CN^{-1/2} \log^{3/2} N$ when $N > N_0$. In order to obtain actual numerical values for C and N_0, considerably more work than is given here would be required. The term $0(1/K)$ is of the same nature. The "order of" terms have been carried along in the work of Section 5 in order to guard against error in the many approximations which are made in the derivation of (1-4).

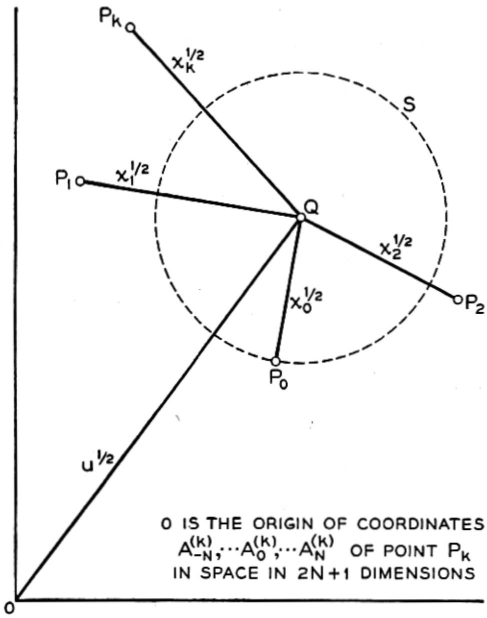

Fig. 1—Diagram indicating relationship between points P_0, Q, and P_k corresponding to signal, signal plus noise, and k^{th} signal not sent ($k > 0$), respectively.

The last term within the bracket in (1-5) has been retained even though it gives terms of order $N^{-1/2} \log N$ when (1-5) is put in (1-4) and could thus be included in $0(N^{-1/2} \log^{3/2} N)$. As shown by the table in the next paragraph, inclusion of this term considerably improves the agreement between (1-4) and values of Prob. $(P_1Q, \cdots, P_KQ > P_0Q)$ obtained by integrating the exact expression (4-12) numerically. This suggests that the term $0(N^{-1/2} \log^{3/2} N)$ in (1-4) is unnecessarily large.

Although the "order of" terms in (1-4) give us some idea of the accuracy of the approximation expressed by (1-4) and (1-5), a better one is desirable. With this in mind the lengthy task of computing the exact expression (4-12) for Prob. $(P_1Q, \cdots, P_KQ > P_0Q)$ by numerical integration was undertaken.

The values obtained in this way are listed in the second column of the following table. The values of Prob. $(P_1Q, \cdots, P_KQ > P_0Q)$ obtained from (1-4) (in which the "order of" terms are ignored) and (1-5) are given in the third column. Column IV lists values obtained from (1-4) and a simplified form of (1-5) obtained by omitting the last term in (1-5). These values are less accurate than those in the third column. The values in Column V are computed from (1-5) and a modified form of (1-4) obtained by adding the correction term shown in equation (5-53) (with $B = H$). The values in Column V are presumably the best that can be done with the approximations made in Section V of this paper, although the first entry renders this a little doubtful.

Prob. $(P_1Q, \cdots, P_KQ > P_0Q)$ for $N = 99.5$ & $r = 1$

$K + 1$	*Numerical Integration*	(1-4) & (1-5)	*Col. IV*	*Col. V*
$2^{100}e^{-30}$	.994	.9995	.9987	1.0001
$2^{100}e^{-15}$	.962	.9650	.9337	.9710
2^{100}	.603	.621	.5000	.605
$2^{100}e^{15}$	.1196	.1159	.0663	.1176
$2^{100}e^{30}$	.0065	.00347	.0013	.00586

It will become apparent later that the value $K + 1 = 2^{100}$ corresponds to the ideal rate of signaling. The non-integer value of 99.5 for N is explained by the fact that the calculations were started before the present version of the theory was worked out. It will be noticed that for $K + 1 = 2^{100}e^{-30}$ all of the approximate values exceed the .994 obtained by numerical integration. I am in doubt as to whether the major part of the discrepancy is due to errors in numerical integration (due to the considerable difficulty encountered) or to errors in the approximations.

In both encoding schemes, the point P_0 corresponds to the transmitted signal, Q to the transmitted signal plus noise, and $P_1, P_2, \cdots P_K$ to K other possible signals. The average signal power turns out to be $(N + 1/2)\sigma^2$ and the average noise power to be $(N + 1/2)\nu^2$. Furthermore,

x_0 = twice the average power in the noise.
x_k = " " " " " " " plus the kth signal.

Prob. $(P_1Q, \cdots P_KQ > P_0Q)$ = Probability that none of the K other signals will be mistaken for the signal sent, i.e., the probability of no error.

The random numbers $A_n^{(k)}$ are taken to be distributed normally instead of some other way because this choice makes the encoding signals (in our two schemes) resemble random noise, a condition which seems to be necessary for efficient encoding (1, 2).

Both of the encoding schemes are concerned with sending, in an interval of duration T, one of $K + 1$ different messages. According to communication theory (1, 2, 3) this corresponds to sending at the rate of $T^{-1} \log_2 (K + 1)$ bits per second. However, instead of discussing the rate of transmission, it is more convenient, from the standpoint of (1-4), to deal with the total number of bits of information sent in time T. Thus, selecting and sending one of the $K + 1$ possible messages is equivalent to sending

$$M = \log_2(K + 1) \tag{1-6}$$

bits of information. M, or one of the adjacent integers if M is not an integer, is the number of "yes or no" questions required to select the sent message from the $K + 1$ possible messages (divide the $K + 1$ messages into two equal, or nearly equal, groups; select the group containing the sent message by asking the person who knows, "Is the sent message in the first group?"; proceed in this way until the last subgroup consists of only the sent message). The amount of information which would be sent in time T at the ideal rate R_I defined by (1-1) is

$$M_I = TR_I = FT \log_2 (1 + 1/r) = (N + 1/2) \log_2 (1 + 1/r) \tag{1-7}$$

where use has been made of $W_N/W_S = \nu^2/\sigma^2 = r$, and the relation $N < FT < N + 1$ (which turns out to be common to both encoding schemes) has been approximated by $N + 1/2 = FT$.

When (1-6) and (1-7) are used to eliminate N and K from (1-5) the result is an expression for the actual amount M of information sent (in time T) in terms of (1) the amount M_I which is sent by transmitting at the ideal rate (1-1) for a time T, (2) the ratio r of the noise power to the signal power, and (3) the probability of no error in sending M bits of information in time T, this probability being given as $(1 + \text{erf } H)/2$:

$$M = M_I - aM_I^{1/2}H + b \tag{1-8}$$

where

$$a = 2\left[\frac{\log_2 e}{(1 + r)\log_e (1 + 1/r)}\right]^{1/2}, \qquad b = \frac{1}{2}\log_2\left[\frac{2\pi(1 + 2r)M_I}{(1 + r)^2 \log_2 (1 + 1/r)}\right] \tag{1-9}$$

Here the "order of" terms in (1-4) have been neglected together with similar terms which arise when $N + 1/2$ is used for N in computing a and b. The term b is usually small compared to $aM_I^{1/2}H$.

The more slowly we send, the less chance there is of error. The relationship between M, M_I and the probability of no error, as computed from

(1–8), is shown in the following table. The probability of no error is denoted by p and the terms are given in the same order as on the right of (1–8) in order to show their relative importance. The ratio $M/M_I(=R/R_I)$ for $r = 0.1$ is shown as a function of M in Fig. 2.

For $r = W_N/W_S = 0.1$

M_I bits	M for $p = .5$	M for $p = .99$	M for $p = .99999$
10^2	$M_I - 0 + 3.75$	$M_I - 24.3 + 3.75$	$M_I - 44.6 + 3.75$
10^4	" " $+ 7.07$	" $- 243 + 7.07$	" $- 446 + 7.07$
10^6	" " $+ 10.38$	" $- 2430 + 10.38$	" $- 4460 + 10.38$

For $r = W_N/W_S = 1$

M_I bits	M for $p = .5$	M for $p = .99$	M for $p = .99999$
10^2	$M_I - 0 + 4.44$	$M_I - 33.4 + 4.44$	$M_I - 61.2 + 4.44$
10^4	" " $+ 7.76$	" $- 334 + 7.76$	" $- 612 + 7.76$
10^6	" " $+ 11.08$	" $- 3340 + 11.08$	" $- 6120 + 11.08$

There may be some question as to the accuracy of the values for $p = .99999$, especially for $M_I = 100$, since this corresponds to points on the tail of the probability distribution where the "order of" terms in (1-4) become relatively important.

Of course, for a given bandwidth, the ideal rate of signaling R_I (given by (1-1)) for $r = .1$ exceeds that for $r = 1$ in the ratio $(\log_2 11)/(\log_2 2) = 3.46$.

The above results agree with the statement that, by efficient encoding, the rate of signaling R can be made to approach the ideal rate $R_I = M_I/T$ given by (1-1). As applied to our two schemes, the term "efficient encoding" means using a very large value of FT or N. To see this, divide both sides of (1-8) by M_I and rearrange the terms:

$$1 - M/M_I = aH\, M_I^{-1/2} + 0(M_I^{-1} \log M_I) \qquad \text{(1-10)}$$

When M_I is replaced by $R_I T$ in M/M_I, the fraction M/T occurs. We shall set $R = M/T$ and call R the rate of signaling corresponding to some fixed probability of error (which determines H). Thus, when (1-7) and the definition (1-9) for a are used, (1-10) goes into

$$\frac{(R_I - R)}{R_I} = \frac{2H}{[(1 + r)FT]^{1/2} \log_e (1 + 1/r)} + 0((\log FT)/FT) \qquad \text{(1–11)}$$

Equation (1-11) shows that when r and H are fixed (i.e. when the noise power/signal power and the probability of error are fixed) R/R_I approaches unity as $FT \to \infty$. This is shown in Fig. 2 for the case $r = 0.1$. Since $R/R_I = M/M_I$, M/M_I must approach unity and consequently M as well as M_I increases linearly with FT. Thus, for efficient encoding M is large and, from (1-6), so is K.

It should be remembered that equation (1-8) has been established only for the two encoding schemes of this article. The question of how much faster M/T approaches R_I for the more efficient encoding schemes mentioned at the end of Section 2 still remains unanswered.

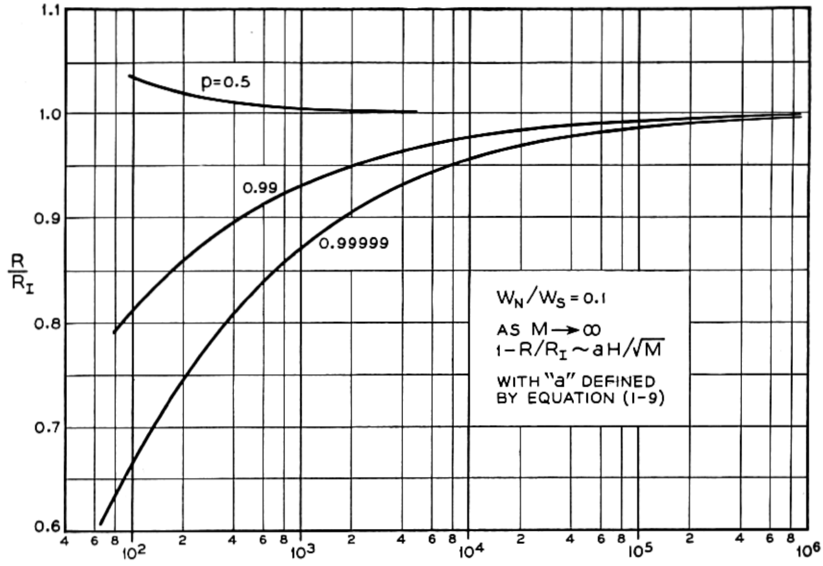

Fig. 2—Curves showing the approach of R/R_I ($= M/M_I$) to unity as the message length increases and the probability of no error remains fixed. R is the rate of signaling at which the probability of no error is p and R_I is the ideal rate.

It gives me pleasure to acknowledge the help I have received in the preparation of this memorandum from conversations with Messrs. H. Nyquist, John Riordan, C. E. Shannon, and M. K. Zinn. I am also indebted to Miss M. Darville for computing the tables shown above and for checking a number of the equations numerically.

2. The First Encoding Scheme

Suppose that we have $K + 1$ different messages any one of which is to be transmitted over a uniform frequency band extending from zero to the nominal cut-off frequency F in a time interval of length T. The adjective "nominal" is used because the sudden starting and stopping of the signals given by the first encoding scheme produces frequency components higher than F. A shortcoming of this nature must be accepted since it is impossible to have a signal possessing both finite duration and finite bandwidth.

The first step of the encoding process is to compute the integer N given by

$$N < FT < N + 1 \tag{2-1}$$

We assume that FT is not an integer in order to avoid borderline cases. Let W_S be the average signal power available for transmission and define the standard deviation σ of the σ universe introduced in Section 1 by $(N + 1/2)\sigma^2 = W_S$. To encode the first message, draw $2N + 1$ numbers $A_{-N}^{(0)}, \cdots, A_0^{(0)}, \cdots A_N^{(0)}$ at random from the σ universe. The signal corresponding to the first message is then taken to be

$$I_0(t) = 2^{-1/2}A_0^{(0)} + \sum_{n=1}^{N} (A_n^{(0)} \cos 2\pi nt/T + A_{-n}^{(0)} \sin 2\pi nt/T) \tag{2-2}$$

The remaining K messages are encoded in the same way, the signal representing the kth message being

$$I_k(t) = 2^{-1/2}A_0^{(k)} + \sum_{n=1}^{N} (A_n^{(k)} \cos 2\pi nt/T + A_{-n}^{(k)} \sin 2\pi nt/T). \tag{2-3}$$

It is apparent that each signal consists of a d-c term plus terms corresponding to N discrete frequencies, the highest being $N/T < F$, and that the average power (assuming $I_k(t)$ to flow through a unit resistance) in the kth signal is

$$T^{-1} \int_{-T/2}^{T/2} I_k^2(t)\, dt = 2^{-1}(A_0^{(k)})^2 + \sum_{n=1}^{N} 2^{-1}[(A_n^{(k)})^2 + (A_{-n}^{(k)})^2] \tag{2-4}$$

Since the A's were drawn from a universe of standard deviation σ, the expected value of the right hand side is $(2N + 1)\sigma^2/2$ which is equal to the average signal power W_S, as required.

We pick one of the $K + 1$ messages at random and send the corresponding signal over a transmission system subject to noise. We choose our notation so that the sent signal is represented by $I_0(t)$ as given by (2-2). Let the noise be given by

$$J(t) = 2^{-1/2}B_0 + \sum_{n=1}^{N} (B_n \cos 2\pi nt/T + B_{-n} \sin 2\pi nt/T) \tag{2-5}$$

where $B_{-N}, \cdots, B_0, \cdots, B_N$ are $(2N + 1)$ numbers drawn at random from the normally distributed ν universe mentioned in the introduction. The standard deviation ν of the universe is given by $(N + 1/2)\nu^2 = W_N$, W_N being the average noise power. We call $J(t)$ simply "noise" rather than "random noise" to emphasize that (2-5) does not represent a random noise current unless N and T approach infinity.

The input to the receiver is $I_0(t) + J(t)$. Let the process of reception consist of computing the $K + 1$ integrals

$$x_k = 2T^{-1} \int_{-T/2}^{T/2} [I_k(t) - I_0(t) - J(t)]^2 \, dt, \qquad k = 0, 1, \cdots, K \quad (2\text{-}6)$$

and selecting the smallest one (all of the $K + 1$ encodings have been carried to the receiver beforehand). If the value of k corresponding to the smallest integral happens to be 0, as it will be if the noise $J(t)$ is small, no error is made. In any other case the receiver picks out the wrong message.

When the representations (2-2), (2-3), and (2-5) are put in (2-6) and the integrations performed, it is found that

$$x_k = \sum_{n=-N}^{N} (A_n^{(k)} - A_n^{(0)} - B_n)^2, \qquad x_0 = \sum_{n=-N}^{N} B_n^2 \quad (2\text{-}7)$$

which have already appeared in equations (1-2) and (1-3). If, as in Section 1, P_k is interpreted as a point in $2N + 1$ — dimensional Euclidean space with coordinates $A_{-N}^{(k)}, \cdots, A_0^{(k)}, \cdots, A_N^{(k)}$ and Q is the point $A_{-N}^{(0)} + B_{-N}, \cdots, A_0^{(0)} + B_0, \ldots, A_N^{(0)} + B_N$, then x_k is the square of the distance between points P_k and Q. Point P_0 corresponds to the signal actually sent, points $P_1, \cdots, P_K$ to the remaining signals, and point Q to the signal plus noise at the receiver. The expected distance between the origin and P_k is $\sigma(2N + 1)^{1/2} = (2W_S)^{1/2}$, that between P_0 and Q is $\nu(2N + 1)^{1/2} = (2W_N)^{1/2}$, and that between the origin and Q is

$$(\sigma^2 + \nu^2)^{1/2}(2N + 1)^{1/2} = (2W_N + 2W_S)^{1/2}$$

No error is made when x_0 is less than every one of $x_1, x_2, \cdots, x_K$, i.e., when none of the points $P_1, \cdots, P_K$ lies within the sphere S of radius $x_0^{1/2}$ centered on Q and passing through P_0. Therefore the probability of obtaining no error when the first encoding scheme is used is equal to the probability denoted by Prob. $(P_1Q, \cdots, P_KQ > P_0Q)$ in the mathematical problem of Section 1.

One might wonder why probability theory has played such a prominent part in the encoding scheme just described. It is used because we do not know the best method of encoding. In fact, it would not be used if we knew how to solve the following problem:* Arrange $K + 1$ points $P_0, \cdots P_K$ on the hyper-surface of the $2N + 1$ — dimensional sphere of radius $(2W_S)^{1/2}$

* C. E. Shannon has commented that although the solution of this problem leads to a good code, it may not be the best possible, i.e., it is not obvious that the code obtained in this way is the same as the one obtained by choosing a set of points so as to minimize the probability of error (calculated from the given set of points and some given W_N) averaged over all $K + 1$ points.

in such a way that the smallest of the $K(K+1)/2$ distances $P_k P_\ell$, $k, \ell = 0, 1, \cdots, K$, $k \neq l$, has the largest possible value. This would maximize the difference (as measured by the distance between their representative points) between the two (or more) most similar encoding signals.†

In this paper we have been forced to rely on the randomness of probability theory to secure a more or less uniform scattering of the points $P_0, \cdots, P_K$. In our work they do not lie exactly on a sphere of radius $(2W_S)^{1/2}$ but this causes us no trouble.

3. The Second Encoding Scheme

The second of the two encoding schemes is suggested by one of Shannon's (2) proofs of the fundamental result (1-1). In this scheme the $K + 1$ messages are to be sent over a transmission system having a frequency band extending from zero to F cycles per second, and are to be sent during a time interval of nominal length T.

The first few steps in the encoding process are just the same as in the first scheme. N is still given by (2-1) and σ by $(N + 1/2)\sigma^2 = W_S$. After drawing $K + 1$ sets of A's, with $2N + 1$ in each set, the $K + 1$ messages are encoded so that the signal corresponding to the kth message, $k = 0, 1, \cdots, K$, is

$$I_k(t) = (FT)^{1/2} \sum_{n=-N}^{N} A_n^{(k)} \frac{\sin \pi(2Ft - n)}{\pi(2Ft - n)} \tag{3-1}$$

From (3-1), the value of $I_k(t)$ at $t = n/(2F)$ is zero if the integer n exceeds N in absolute value. If the integer n is such that $|n| \leq N$, the corresponding value of $I_k(t)$ is $(FT)^{1/2}A_n^{(k)}$. The energy in the kth signal is obtained by squaring both sides of (3-1) and integrating with respect to t. Thus

$$\int_{-\infty}^{\infty} I_k^2(t)\, dt = 2^{-1}T \sum_{n=-N}^{N} A_n^{(k)2} \tag{3-2}$$

which has the expected value $(N + 1/2)\sigma^2 T$. The average power developed when this amount of energy is expended during the nominal signal length T is $(N + 1/2)\sigma^2$ which is equal to W_S, as it should be.

The noise introduced by the transmission system is taken to be

$$J(t) = (FT)^{1/2} \sum_{n=-N}^{N} B_n \frac{\sin \pi(2Ft - n)}{\pi(2Ft - n)} \tag{3-3}$$

† Possibly if $K + 1$ discrete unit charges of electricity were allowed to move freely on the sphere, their mutual repulsion would separate them in the required manner. In $2N + 1$ dimensions this leads to the problem of minimizing the mutual potential energy

$$\tfrac{1}{2}\Sigma(\overline{P_k P_\ell})^{-2N+1}$$

where $N \geq 1$ and the summation extends over $k, \ell = 0, 1, \ldots K$ with $k \neq \ell$. However, this problem also appears to be difficult.

where the ν universe from which the B's are drawn has, as before, standard deviation ν given by $(N + 1/2)\nu^2 = W_N$. When the signal $I_0(t)$ is sent, the input to the receiver is $I_0(t) + J(t)$ and the process of reception consists of selecting the smallest of the $K + 1$ x_k's

$$x_k = 2T^{-1} \int_{-\infty}^{\infty} [I_k(t) - I_0(t) - J(t)]^2 \, dt \tag{3-4}$$

$$= \sum_{n=-N}^{N} (A_n^{(k)} - A_n^{(o)} - B_n)^2$$

The second expression for x_k is the same as the one given by (2-7) for the first encoding scheme, and the discussion in Section 2 following (2-7) may also be applied to the second encoding scheme. In particular, the probability of obtaining no error in transmitting a signal through noise is the same in both systems of encoding, and is given by the Prob. $(P_1Q, \cdots, P_KQ > P_0Q)$ of the mathematical problem of Section 1.

4. Solution of the Mathematical Problem

We shall simplify the work of solving the mathematical problem stated in Section 1 by taking $\sigma = 1$ and $\nu^2/\sigma^2 = r$. First regard the $4N + 2$ numbers $A_n^{(0)}$, B_n, $n = -N, \cdots, N$ as fixed or given beforehand. Geometrically, this corresponds to having the points P_0 and Q given. Select a typical set of random variables $A_n^{(k)}$, $n = -N, \cdots, N$, $k > 0$ and consider the associated set of variables

$$y_n = A_n^{(k)} - A_n^{(0)} - B_n = A_n^{(k)} + \bar{y}_n. \tag{4-1}$$

y_n is a random variable distributed normally about its average value

$$\bar{y}_n = -A_n^{(0)} - B_n \tag{4-2}$$

with standard deviation $\sigma = 1$. The quantity x_k, defined by (1-3) and representing the square of the distance between P_k and Q, may be written as

$$x_k = \sum_{n=-N}^{N} y_n^2 \tag{4-3}$$

Thus x_k is the sum of the squares of $2N + 1$ independent and normally distributed variates, having the same standard deviation but different average values. The probability density of such a sum is remarkable in that it does not depend upon the $\bar{y}_n$'s individually but only on the smu of their squares which we denote by

$$u = \sum_{n=-N}^{n} \bar{y}_n^2 = \sum_{n=-N}^{n} (A_n^{(0)} + B_n)^2$$

$$= 2T^{-1} \left[\begin{array}{l} \text{Energy in sent signal + Energy} \\ \text{in noise} \end{array} \right] \tag{4-4}$$

This behavior follows from the fact that the probability density of P_k has spherical symmetry about the origin (because all the $A_n^{(k)}$'s have the same σ). For the probability that x_k is less than some given value x is the probability that P_k lies within a sphere of radius $x^{1/2}$ centered on Q, and this, because of the symmetry, depends only on x and the distance $u^{1/2}$ of Q from the origin. Accordingly, we write $p(x, u)dx$ for the probability that $x < x_k < x + dx$ when the $\bar{y}_n'$'s (and hence u) are fixed.

The probability density $p(x, u)$ may be obtained from its characteristic function:

$$p(x, u) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-izx}[\text{ave. } e^{izx}]\, dz$$

$$\text{ave. } e^{izx} = \text{ave. } \exp\left[iz \sum_{n=-N}^{N} y_n^2 \right] \tag{4-5}$$

$$= \prod_{n=-N}^{N} \text{ave. } \exp\,[izy_n^2] = (1 - 2iz)^{-N-1/2} \exp\,[iuz(1 - 2iz)^{-1}]$$

where we have used (4-3) and, since y_n is distributed normally about $\bar{y}_n$,

$$\text{ave. } \exp\,[izy_n^2] = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{izy_n^2-(y_n-\bar{y}_n)^2/2}\, dy_n$$

$$= (1 - 2iz)^{-1/2} \exp\,[\bar{y}_n^2\, iz(1 - 2iz)^{-1}]$$

Hence

$$p(x, u) = (2\pi)^{-1} \int_{-\infty}^{\infty} (1 - 2iz)^{-N-1/2} \exp\,[izu(1 - 2iz)^{-1} - izx]\, dz$$
$$= 2^{-1}(x/u)^{N/2-1/4}\, I_{N-1/2}\,[(ux)^{1/2}]\, e^{-(u+x)/2} \tag{4-6}$$

where it is to be understood that x is never negative. The Bessel function of imaginary argument appears when we change the variable of integration from z to t by means of $1 - 2iz = 2t/x$, and bend the path of integration to the left in the t plane (6). This expression for the probability density of the sum of the squares of a number of normal variates having the same standard deviation but different averages has been given by R. A. Fisher (7).

We are now in a position to solve the following problem which is somewhat simpler than the one stated in Section 1: Given the $2N + 1$ coordinates $A_n^{(0)}$ of the point P_0 and the $2N + 1$ numbers B_n so that the coordinates $A_n^{(0)} + B_n$ of the point Q are given. What is the probability that none of the K points $P_1, P_2, \cdots, P_K$, whose coordinates $A_n^{(k)}$ are drawn at random from a universe distributed normally about zero with standard deviation $\sigma = 1$, be inside the sphere centered on the given point Q and passing through the other given point P_0? In other words, what is the probability that all K of the independent random variables $x_1, x_2, \cdots, x_K$ will exceed the given value x_0 when u has the value defined by (4-4) together with the given values of the $A_n^{(0)}$'s and B_n's? The variables $x_1, x_2, \cdots, x_K$ have the probability density $p(x, u)$ shown in (4-6) and x_0 is defined by (1-2) and the given values of the B_n's.

The answer to the above problem follows at once when we note that the probability of any one of $x_1, \cdots, x_K$, say x_1 for example, being less than x_0 is

$$P(x_0, u) = \int_0^{x_0} p(x, u)\, dx. \tag{4-7}$$

The probability of x_1 exceeding x_0 is then $1 - P(x_0, u)$ and the probability of all K of $x_1, \cdots, x_K$ exceeding x_0 is

$$[1 - P(x_0, u)]^K \tag{4-8}$$

Instead of being assigned quantities, x_0 and u are actually random variables when we consider the problem of Section 1. Now we take up the problem of finding the probability density of u when x_0 is fixed. Thus, from (4-4), we wish to find the probability density of

$$u = \sum_{n=-N}^{N} (A_n^{(0)} + B_n)^2 \tag{4-9}$$

in which the $2N + 1$ numbers $A_n^{(0)}$ are drawn at random from a universe distributed normally about zero with standard deviation $\sigma = 1$ and the numbers $B_{-N}, \cdots, B_0, \cdots, B_N$ are given. It is seen that u is the sum of the squares of $2N + 1$ normal variates all having the standard deviation $\sigma = 1$. The nth variate, $A_n^{(0)} + B_n$, has the average value B_n. This is just the problem which was encountered at the beginning of this section. Equation (4-9) is of the same form as (4-3) and we have the following correspondence:

Equation (4-3)	*Equation (4-9)*
x_k	u
y_n	$A_n^{(0)} + B_n$
$\bar{y}_n$	B_n
$u = \sum \bar{y}_n^2$	$x_0 = \sum B_n^2$

The probability that u lies in the interval $u, u + du$ when x_0 is given is therefore $p(u, x_0)\, du$ where $p(u, x_0)$ is obtained by putting u for x and x_0 for u in the probability density $p(x, u)$.

Until now x_0 has been fixed. At this stage we regard $B_{-N}, \cdots, B_0, \cdots, B_N$ as random variables drawn from a normal universe of average zero and standard deviation $\nu = \sigma r^{1/2} = r^{1/2}$. If the standard deviation were unity, the probability density of x_0 could be obtained directly from $p(x, u)$ by letting $u \to 0$ in (4-6). As it is, the x's appearing in the resulting expression must be divided by r to obtain the correct expression. Thus, the probability of finding x_0 between x_0 and $x_0 + dx_0$ is

$$p_0(x_0) = \frac{[x_0/2r]^{N-1/2}}{2r\Gamma(N + 1/2)} \cdot e^{-x_0/2r} \tag{4-10}$$

which is of the χ^2 type frequently encountered in statistical theory.

It follows that the probability of finding u in $(u, u + du)$ and x_0 in $(x_0, x_0 + dx_0)$ at the same time is $p_0(u, x_0)\, du\, dx_0$ where

$$\begin{aligned} p_0(u, x_0) &= p(u, x_0)p_0(x_0) \\ &= \frac{1}{4r\Gamma(N + 1/2)} \left(\frac{ux_0}{4r^2}\right)^{N/2-1/4} I_{N-1/2}[(x_0 u)^{1/2}]\, e^{-[u+x_0(1+1/r)]/2} \end{aligned} \tag{4-11}$$

The replacement of (x, u) in (4-6) by (u, x_0) should be noted.

Now that we have the probability density of u and x_0 we may combine it with the probability (4-8) that all K of $x_1, \cdots, x_K$ exceed x_0 when x_0 and u are fixed. The result is the answer to the problem stated in Section 1:

$$\begin{aligned} &\text{Prob. } (P_1Q, \cdots, P_KQ > P_0Q) \\ &\quad = \int_0^\infty du \int_0^\infty dx_0\, p_0(u, x_0)[1 - P(x_0, u)]^K \end{aligned} \tag{4-12}$$

This result is more complicated than it seems, for $p_0(u, x_0)$ is given by (4-11) and $P(x_0, u)$ is obtained by integrating $p(x, u)$ of (4-6) from $x = 0$ to $x = x_0$ in accordance with (4-7). The remaining portion of the paper is concerned with obtaining an approximation to (4-12) which holds when N and K are very large numbers.

5. Behavior of Prob. $(P_1Q, \cdots, P_KQ > P_0Q)$ as N and K Become Large

In this section we introduce a number of approximations which lead to a manageable expression for Prob. $(P_1Q, \cdots, P_KQ > P_0Q)$ when N and K become large.

Since u and x_0 are sums of independent random variables, namely

$$\begin{aligned} u &= \sum_{n=-N}^{N} (A_n^{(0)} + B_n)^2 \\ x_0 &= \sum_{n=-N}^{N} B_n^2, \end{aligned} \tag{5-1}$$

the central limit theorem tells us that the probability density $p_0(u, x_0)$ approaches a two-dimensional normal distribution centered on the average values

$$\bar{u} \doteq \sum_{n=-N}^{N} \text{ave.}\,[A_n^{(0)2} + B_n^2] = (2N + 1)(1 + r)$$
$$\bar{x}_0 = \sum_{n=-N}^{N} \text{ave.}\; B_n^2 = (2N + 1)r \tag{5-2}$$

Here we keep the convention $\sigma = 1$, $\nu^2/\sigma^2 = r$ used in Section 4. The same sort of reasoning as used to establish (5-2) shows that the spread about these average values is given by

$$\text{ave.}\;(u - \bar{u})^2 = (4N + 2)(1 + r)^2$$
$$\text{ave.}\;(x_0 - \bar{x}_0)^2 = (4N + 2)r^2 \tag{5-3}$$
$$\text{ave.}\;(u - \bar{u})(x_0 - \bar{x}_0) = (4N + 2)r^2$$

If the parameters N, K, and r in the integral (4-12) are such that its value is appreciably different from zero, most of the contribution comes from the region around $\bar{u}$ and $\bar{x}_0$ where $p_0(u, x_0)$ is appreciably different from zero. However, instead of taking $\bar{u}$ and $\bar{x}_0$ as reference values, we take the nearby values

$$u_2 = \bar{u} - 2 - 2r = (2N - 1)(1 + r) = 2q(1 + r)$$
$$x_2 = \bar{x}_0 - 2r = (2N - 1)r = 2qr \tag{5-4}$$

as these turn out to be better representatives of the center of the distribution. We have introduced the number

$$q = N - 1/2 \tag{5-5}$$

in order to simplify the writing of later equations. We assume $q > 1$.

First, we shall show that

$$\text{Prob.}\;(P_1Q, \cdots, P_KQ > P_0Q)$$
$$= \int_{u_2-a}^{u_2+a} du \int_{x_2-b}^{x_2+b} dx_0\, p_0(u, x_0)[1 - P(x_0\,, u)]^K + R_1 \tag{5-6}$$

where $a = 2(1 + r)(2q \log q)^{1/2}$, $b = 2r(2q \log q)^{1/2}$ and R_1 is of order $1/q$ (denoted by $0(1/q)$), i.e. a constant C and a value q_0 can be found such that $|R_1| < C/q$ when $q > q_0$. From (4-12) it is seen that R_1 is positive and less than

$$\left[\int_0^{u_2-a} du + \int_{u_2+a}^{\infty} du\right] \int_0^{\infty} dx_0\, p_0(u, x_0)$$
$$+ \left[\int_0^{x_2-b} dx_0 + \int_{x_2+b}^{\infty} dx_0\right] \int_0^{\infty} du\, p_0(u, x_0) \tag{5-7}$$

Since $p_0(u, x_0)$ is the joint probability density of u and x_0, the integration with respect to x_0 in the first part of (5-7) yields the probability density of u, and the integration with respect to u in the second part gives the probability density $p_0(x_0)$ (stated in (4-10)) of x_0. Thus (8)

$$\begin{aligned}\int_0^\infty dx_0\, p_0(u, x_0) &= \frac{[u/2(1+r)]^q e^{-u/2(1+r)}}{2(1+r)\Gamma(q+1)} \\ \int_0^\infty du\, p_0(u, x_0) &= \frac{[x_0/2r]^q e^{-x_0/2r}}{2r\Gamma(q+1)}\end{aligned} \tag{5-8}$$

Setting (5-8) in (5-7) and putting $u = 2(1+r)y$ and $x_0 = 2ry$ in the two parts of (5-7) reduces them to the same form. Thus (5-7) is equal to

$$2 - \frac{2}{\Gamma(q+1)} \int_{q-\ell}^{q+\ell} y^q e^{-y}\, dy \tag{5-9}$$

with $\ell = (2q \log q)^{1/2}$. In order to show that (5-9) is $0(1/q)$ we use the expansion

$$-y + q \log y = -q + q \log q - (y-q)^2/(2q) + (y-q)^3/(3q^2) - (y-q)^4 q[q + (y-q)\theta]^{-4}/4$$

where $0 \leqslant \theta \leqslant 1$. Let v represent the sum of the $(y-q)^3$ and $(y-q)^4$ terms, and expand $\exp v$ as $1 + v$ plus a remainder term. The integral of $\exp - (y-q)^2/(2q)$, taken between the limits $q \pm \ell$, can be shown to be of the form $1 - 0(1/q)$ by integrating by parts as in obtaining the asymptotic expansion for the error function. The term in $(y-q)^3$ vanishes upon integration and the remainder terms may be shown to be of $0(1/q)$. In all of this work a square root of q comes in through the fact that

$$1 > (2\pi q)^{1/2} q^q e^{-q}/\Gamma(q+1) > \exp[-1/(12q)] \tag{5-10}$$

We have just shown that the error introduced by restricting the region of integration as indicated by (5-6) introduces an error of order $1/q$ which vanishes as $q \to \infty$. The normal law approximation to $p_0(u, x_0)$ predicted by the central limit theorem holds over this restricted region. However, instead of appealing to the central limit theorem to determine the accuracy of the approximation, we prefer to deal directly with the functions involved.

Consideration of (5-4) and the behavior of $p_0(u, x_0)$ suggests the substitution

$$\begin{aligned}x_0 &= 2r(q + \alpha) \\ u &= 2(1+r)(q + \beta)\end{aligned} \tag{5-11}$$

where α and β are new variables whose absolute values never exceed $(2\,q \log q)^{1/2}$ in the restricted region of integration of (5-6). From (4-11)

$$p_0(u, x_0)\, du\, dx_0 = \frac{(1+r)}{\Gamma(q+1)}\left(\frac{z}{4r^2}\right)^{q/2} I_q(z^{1/2}) e^{-(1+r)(2q+\alpha+\beta)}\, d\alpha\, d\beta \qquad (5\text{-}12)$$

in which

$$z = ux_0 = 4r(1+r)(q+\alpha)(q+\beta) \qquad (5\text{-}13)$$

In Appendix II it is shown that

$$I_q(z^{1/2}) = \frac{q^{q+1/2}\, e^{-q}\, z^{q/2} \exp\,[(q^2+z)^{1/2} + V]}{\Gamma(q+1)(q^2+z)^{1/4}[q+(q^2+z)^{1/2}]^q} \qquad (5\text{-}14)$$

where $|V| < 1/(2q-1)$ when $q > 1$. Upon using (5-10) and (5-14) the right hand side of (5-12) may be written as

$$d\alpha\, d\beta (2\pi)^{-1/2}(1+r)(2r)^{-q}(q^2+z)^{-1/4} \exp\,[-(1+r)(2q+\alpha+\beta) + f(z) - \log \Gamma(q+1) + 0(1/q)] \qquad (5\text{-}15)$$

with

$$f(z) = q \log z - q \log\,[q + (q^2+z)^{1/2}] + (q^2+z)^{1/2} \qquad (5\text{-}16)$$

The value z_2 of z corresponding to the central point (u_2, x_2) of $p_0(u, x_0)$ is obtained by putting $\alpha = \beta = 0$ in (5-13):

$$\begin{aligned} z_2 &= 4r(1+r)q^2 \\ z - z_2 &= 4r(1+r)[q(\alpha+\beta) + \alpha\beta]. \end{aligned} \qquad (5\text{-}17)$$

Since we are interested in the form of $p_0(u, x_0)$ in the restricted region of integration of (5-6) we expand $f(z)$ about $z = z_2$ in a Taylor's series plus a remainder term.

$$\begin{aligned} f(z) &= q \log 2rq + q(1+2r) + (z - z_2)/(4rq) \\ &\quad - \frac{(z-z_2)^2}{32r^2q^3(1+2r)} + \frac{(z-z_2)^3}{3!}\left[\frac{(\xi_3+q)^3(3\xi_3-q)}{8\, z_3^3\, \xi_3^3}\right] \end{aligned} \qquad (5\text{-}18)$$

In the last term $z_3 = z_2 + (z - z_2)\theta$, $0 \leqslant \theta \leqslant 1$, $\xi_3^2 = q^2 + z_3$. The work of obtaining this expansion is simplified if $(q^2+z)^{1/2}$ is replaced by ξ in (5-16) before differentiating. For example, by using $2\xi'\xi = 1$, it can be shown that $f'(z)$ is simply $(q+\xi)/(2z)$. When the extreme values of α and β are put in (5-17), it is seen that $z - z_2$ does not exceed $0(q^{3/2} \log^{1/2} q)$ in the restricted region of integration. In the last term of (5-18) z_3 is $0(q^2)$, ξ_3 is $0(q)$ and consequently the last term itself is $0(q^{-1/2} \log^{3/2} q)$.

When the expression (5-17) for $(z - z_2)$ is put in (5-18) an expression for $f(z)$ is obtained. This expression, together with

$$\log \Gamma(q+1) = (q+1/2)\log q - q + (1/2)\log 2\pi + 0(1/q),$$

enables us to write the argument of the exponential function in (5-15) as $q \log 2r - (1/2) \log 2\pi q - Q(\alpha, \beta) + 0(q^{-1/2} \log^{3/2} q)$ where $Q(\alpha, \beta)$ denotes the quadratic function

$$\begin{aligned} Q(\alpha, \beta) &= [(1+r)^2(\alpha^2+\beta^2) - 2r(1+r)\alpha\beta]D \\ D &= 1/[2q(1+2r)] \end{aligned} \tag{5-19}$$

Similar considerations show that

$$(q^2 + z)^{-1/4} = q^{-1/2}(1+2r)^{-1/2}[1 + 0(q^{-1/2}\log^{1/2} q)] \tag{5-20}$$

When the above results are gathered together it is found that (5-12) may be written as

$$p_0(u, x_0)\, du\, dx_0 = D_1 \exp\,[-Q(\alpha, \beta) + 0(q^{-1/2}\log^{3/2} q)]\, d\alpha\, d\beta \tag{5-21}$$

where

$$D_1 = \frac{1+r}{2\pi q(1+2r)^{1/2}} \tag{5-22}$$

Expression (5-21) is valid as long as $|\alpha|$ and $|\beta|$ do not exceed

$$(2q \log q)^{1/2}.$$

Expression (5-21) differs from the one predicted by the central limit theorem (and (5-2) and (5-3)) in that it is not quite centered on the average values $\bar{x}_0$, $\bar{u}$, which correspond to $\alpha = 1$, $\beta = 1$, respectively. Also, q enters in place of $q + 1$. However, these differences amount to $0(q^{1/2}\log^{1/2} q)$ at most, as may be seen by putting $\alpha - 1$ and $\beta - 1$ for α and β in (5-19).

By using relations (5-6) and (5-21), it may be shown that

$$\begin{aligned} &\text{Prob. } (P_1Q, \cdots, P_KQ > P_0Q) \\ &= \int_{-q}^{\infty} d\alpha \int_{-\varsigma}^{\infty} d\beta\, D_1\, e^{-Q(\alpha,\beta)}[1 - P(x_0, u)]^K + 0(q^{-1/2}\log^{3/2} q) \end{aligned} \tag{5-23}$$

where it is understood that x_0 and u in $P(x_0, u)$ depend on α and β through (5-11). The term $0(q^{-1/2}\log^{3/2} q)$ in (5-23) represents the sum of three contributions. The first is R_1 in (5-6) which is $0(1/q)$. The second arises from the fact that when the factor $\exp\,[0(q^{-1/2}\log^{3/2} q)]$ in (5-21) is neglected in integrating (5-21) over $-\ell < \alpha < \ell$, $-\ell < \beta < \ell$, where $\ell = (2q \log q)^{1/2}$, the resulting integral is in error by $0(q^{-1/2}\log^{3/2} q)$. The third is due to the contributions of the integral from the region $|\alpha| > \ell$, $|\beta| > \ell$.

By introducing polar coordinates $\alpha = \rho \cos \theta$, $\beta = \rho \sin \theta$ it can be shown that the region $\rho > \ell$ more than covers the region in question and that

$$Q(\alpha, \beta) \geqslant (1 + r)\rho^2 D \tag{5-24}$$

Upon integrating with respect to ρ and setting in the lower limit ℓ, it is seen that the third contribution is $0(q^{-1/2})$.

We now assume K to be large. Since $0 \leqslant P(x_0, u) \leqslant 1$ we have

$$0 \leqslant e^{-KP} - (1 - P)^K \leqslant KP^2 e^{-KP} < 1/K \tag{5-25}$$

The last inequality follows from $x^2 \exp(-x) < 1$ for $x \geqslant 0$. A proof of the remaining portions will be found in "Modern Analysis" by Whittaker and Watson, Cambridge University Press, Fourth Edition (1927), page 242. When we observe that replacing $[1 - P(x_0, u)]^K$ by $1/K$ in the right hand side of (5-23) gives an integral whose value is less than $1/K$, we see that

$$\text{Prob.}\ (P_1Q, \cdots, P_KQ > P_0Q) \tag{5-26}$$
$$= \int_{-q}^{\infty} d\alpha \int_{-q}^{\infty} d\beta\, D_1\, e^{-Q(\alpha,\beta) - KP(x_0,u)} + 0(1/K) + 0(q^{-1/2} \log^{3/2} q)$$

We now take up the problem of expressing the cumulative probability density $P(x_0, u)$ in terms of α and β. When x_0 and u lie in the restricted region of integration shown in (5-6) they are near their average values $\bar{x}_0 = (2N + 1)r$ and $\bar{u} = (2N + 1)(1 + r)$. On the other hand the average value $\bar{x}$ of x and the mean square value σ_x^2 of $(x - \bar{x})^2$ as computed from (4-6), or directly, are $2N + 1 + u$ and $4N + 2 + 4u$, respectively. Thus we see that $\bar{x} - x_0$ is of the same magnitude as $4N$ and becomes much larger than σ_x as $N \rightarrow \infty$. The asymptotic development of Appendix I may therefore be used. In Appendix *I* (equations (A1-27) and (A1-29)) it is shown that when $M(= 2m = 2N + 1)$ is a large number and $1 << (\bar{x} - x_0)/\sigma_x$

$$P(x_0, u) = (4\pi m b_2)^{-1/2} (1 + 0(1/m)) \exp [m\mathrm{F}(v_1)] \tag{5-27}$$

where we have introduced the number $m = N + 1/2 = q + 1$ to save writing $N + 1/2$ or $q + 1$ repeatedly and where

$$\begin{aligned} 2b_2 &= (1 - 1/v_1)^2(1 + 4st)^{1/2} \\ v_1 &= [1 + (1 + 4st)^{1/2}]/2s \\ F(v_1) &= (1 + 4st)^{1/2} - s - t - \log v_1 \\ x_0 &= 2ms = (2N + 1)s, \qquad u = 2mt = (2N + 1)t \end{aligned} \tag{5-28}$$

Comparison of the last line in (5-28) with (5-11) shows that ms and mt are equal to $r(q + \alpha) = r(m + \alpha - 1)$ and

$$(1 + r)(q + \beta) = (1 + r)(m + \beta - 1),$$

respectively. It is convenient to introduce the notation

$$\gamma = \alpha - 1, \qquad \delta = \beta - 1$$
$$s = r(1 + \gamma/m), \qquad t = (1 + r)(1 + \delta/m). \tag{5-29}$$

It is seen that for the restricted region in which $|\alpha|$ and $|\beta|$ are less than $\ell = (2q \log q)^{1/2}$, $|\gamma|$ and $|\delta|$ are at most

$$0(q^{1/2} \log^{1/2} q) = 0(m^{1/2} \log^{1/2} m).$$

Hence s, t, $(1 + 4st)^{1/2}$, v_1 differ at most from r, $1 + r$, $1 + 2r$, $1 + 1/r$, respectively, by terms of order $m^{-1/2} \log^{1/2} m$. Similar considerations show that

$$(4\pi m b_2)^{-1/2} = (2\pi q)^{1/2} D_1[1 + 0(m^{-1/2} \log^{1/2} m)] \tag{5-30}$$

The argument of the exponential function in (5-27) must be expanded in powers of γ and δ. It turns out that when γ and δ lie in the restricted region, powers above the second may be neglected. For the sake of convenience we rewrite (5-13) and introduce z_1:

$$z = x_0 u = 4m^2 st = 4r(1 + r)(m + \gamma)(m + \delta)$$
$$z_1 = 4r(1 + r)m^2 \tag{5-31}$$
$$z - z_1 = 4r(1 + r)[m(\gamma + \delta) + \gamma\delta]$$

so that $z - z_1$ is $0(m^{3/2} \log^{1/2} m)$. Then

$$\begin{aligned}(1 + 4st)^{1/2} &= (1 + z/m^2)^{1/2} \\ &= (1 + z_1/m^2)^{1/2} + (z - z_1)(1 + z_1/m^2)^{-1/2}/(2m^2) \\ &\quad - (z - z_1)^2(1 + z_1/m^2)^{-3/2}/(8m^4) + R_2\end{aligned} \tag{5-32}$$

where R_2 is of the same order as $(z - z_1)^3/m^6$, or $m^{-3/2} \log^{3/2} m$. It follows that

$$\begin{aligned}(1 + 4st)^{1/2} &= 1 + 2r + \frac{2r(1 + r)}{1 + 2r}\left[\frac{\gamma + \delta}{m} + \frac{\gamma\delta}{m^2}\right] \\ &\quad - \frac{2r^2(1 + r)^2}{(1 + 2r)^3}\frac{(\gamma + \delta)^2}{m^2} + 0(m^{-3/2} \log^{3/2} m) \\ v_1 &= \frac{(1 + r)}{r(1 + \gamma/m)}\left\{1 + \frac{r}{1 + 2r}\left[\frac{\gamma + \delta}{m} + \frac{\gamma\delta}{m^2}\right]\right. \\ &\quad \left. - \frac{r^2(1 + r)(\gamma + \delta)^2}{m^2(1 + 2r)^2} + 0(m^{-3/2} \log^{3/2} m)\right\}.\end{aligned} \tag{5-33}$$

Combining these and a similar expression for log v_1 leads to

$$\begin{aligned} mF(v_1) &= -m \log (1 + 1/r) + \gamma - \delta \\ &\quad -[(1 + r)\gamma - r\delta]^2/[2m(1 + 2r)] + 0(m^{-1/2} \log^{3/2} m) \\ &= -(q + 1) \log (1 + 1/r) + \alpha - \beta - [(1 + r)\alpha - r\beta]^2 D \\ &\qquad + 0(q^{-1/2} \log^{3/2} q) \end{aligned} \tag{5-34}$$

Substitution of (5-30) and (5-34) in (5-27) gives the result we seek:

$$\begin{aligned} P(x_0, u) &= (1 + 1/r)^{-q-1}(2\pi q)^{1/2} D_1 \\ &\quad \exp (\alpha - \beta - [(1 + r)\alpha - r\beta]^2 D + 0(q^{-1/2} \log^{3/2} q)) \end{aligned} \tag{5-35}$$

Since $P(x_0, u)$ occurs only in the product $KP(x_0, u)$ in (5-26) we set, in view of (5-35),

$$KP(x_0, u) = A\lambda(\alpha, \beta) \exp S(\alpha, \beta) \tag{5-36}$$

where $\lambda(\alpha, \beta)$ stands for the terms denoted by $\exp [0(q^{-1/2} \log^{3/2} q)]$ in (5-35) and

$$\begin{aligned} A &= K(1 + 1/r)^{-q-1}(2\pi q)^{1/2} D_1 \\ S(\alpha, \beta) &= \alpha - \beta - [(1 + r)\alpha - r\beta]^2 D \end{aligned} \tag{5-37}$$

As long as $|\alpha| < \ell$ and $|\beta| < \ell$, $\lambda(\alpha, \beta)$ is nearly unity and we write

$$\begin{aligned} &\lambda_1 < \lambda(\alpha, \beta) < \lambda_2 \\ &\lambda_1 = 1 - \epsilon, \lambda_2 = 1 + \epsilon, \epsilon = Cq^{-1/2} \log^{3/2} q \end{aligned} \tag{5-38}$$

where C is a positive constant large enough to make ϵ dominate the terms of order $q^{-1/2} \log^{3/2} q$ in (5-35). q is supposed to be so large that ϵ is very small in comparison with unity.

Setting (5-36) in (5-26) gives

$$\text{Prob. } (P_1Q, \cdots, P_KQ > P_0Q) = I + 0(1/K) + 0(q^{-1/2} \log^{3/2} q) \tag{5-39}$$

where the contribution of the region outside $|\alpha| < \ell$, $|\beta| < \ell$ has been returned to the terms denoted by $0(q^{-1/2} \log^{3/2} q)$ (we could have stayed in the region $|\alpha| < \ell$, $|\beta| < \ell$ from (5-23) onward, but didn't do so because we wanted to show that the results coming from (5-25) were not restricted to this region) and

$$I = \int_{-\ell}^{\ell} d\alpha \int_{-\ell}^{\ell} d\beta\, D_1 \exp [-Q(\alpha, \beta) - A\lambda(\alpha, \beta) e^{S(\alpha,\beta)}] \tag{5-40}$$

Let $L(\lambda)$ denote the integral obtained by replacing the function $\lambda(\alpha, \beta)$ in I by the positive constant λ (which we shall take to be either λ_1 or λ_2 defined by (5-38)). Then, since $A \exp S(\alpha, \beta)$ is positive, it follows from (5-40) that

$$L(\lambda_1) > I > L(\lambda_2) \tag{5-41}$$

Also since $\exp [-A\lambda \exp S(\alpha, \beta)]$ lies between 0 and 1 for all real values of α and β it may be shown from (5-24) that $L(\lambda)$ is equal to $J(\lambda) + 0(q^{-1/2})$ where

$$J(\lambda) = \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta\, D_1 \exp [- Q(\alpha, \beta) - A\lambda e^{S(\alpha,\beta)}] \tag{5-42}$$

Here λ is a constant and $Q(\alpha, \beta)$, A, $S(\alpha, \beta)$ are defined by (5-19) and (5-37). From (5-39) and (5-41) we obtain

$$\begin{aligned}\text{Prob. } (P_1Q, \cdots, P_KQ > P_0Q) &= J(1) + \theta[J(\lambda_1) - J(1)] \\ &+ (1 - \theta)[J(\lambda_2) - J(1)] + 0(1/K) + 0(q^{-1/2} \log^{3/2} q)\end{aligned} \tag{5-43}$$

where $0 < \theta < 1$. It will be shown later that $J(\lambda_1)$ and $J(\lambda_2)$ differ from $J(1)$ by terms which are certainly not larger than $0(q^{-1/2})$.

The problem now is to evaluate the integral (5-42) for $J(\lambda)$. It turns out that $\exp [-A\lambda \exp S(\alpha, \beta)]$ acts somewhat like a discontinuous factor which is unity when $S(\alpha, \beta) + \log A\lambda$ is negative and zero when it is positive. In order to investigate this behavior we make the change of variable

$$\begin{aligned} \alpha - \beta &= w & \alpha &= y - rw \\ (1 + r)\alpha - r\beta &= y & \beta &= y - (1 + r)w \\ & & d\alpha\, d\beta &= dw\, dy \end{aligned} \tag{5-44}$$

From (5-19), (5-37), and (5-42)

$$\begin{aligned} Q(\alpha, \beta) &= [y^2 + (1 + 2r)\beta^2]D = y^2D + \beta^2/2q \\ S(\alpha, \beta) &= w - y^2D \\ J(\lambda) &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dw\, D_1 \exp [- y^2D - \beta^2/2q - A\lambda e^{w-y^2D}] \end{aligned} \tag{5-45}$$

Here and in the following work β is to be regarded as a function of w and y.

Split the interval of integration with respect to w into the two subintervals $(-\infty, w_0)$ and (w_0, ∞) where

$$w_0 = y^2D - \log A\lambda \tag{5-46}$$

and y is temporarily regarded as constant. In the first interval

$$\begin{aligned} &\int_{-\infty}^{w_0} \exp [- \beta^2/2q - e^{w-w_0}]\, dw \\ &= \int_{-\infty}^{w_0} e^{-\beta^2/2q}\, dw - \int_{-\infty}^{w_0} (1 - \exp [- e^{w-w_0}])e^{-\beta^2/2q}\, dw \end{aligned} \tag{5-47}$$

Splitting the interval of integration $(-\infty, w_0)$ into $(-\infty, -\log A\lambda)$ and $(-\log A\lambda, w_0)$ in the first integral on the right of (5-47) shows that its contribution to $J(\lambda)$ is

$$D_1 \int_{-\infty}^{\infty} dy \int_{-\infty}^{-\log A\lambda} dw\, e^{-y^2 D - \beta^2/2q} + D_1 \int_{-\infty}^{\infty} dy \int_{-\log A\lambda}^{w_0} dw\, e^{-y^2 D - \beta^2/2q} \quad (5\text{-}48)$$

Integrating with respect to y, after inverting the order of integration, shows that the value of the first integral is

$$\pi^{-1/2} \int_{-\infty}^{B} e^{-t^2}\, dt = (1 + \operatorname{erf} B)/2 \quad (5\text{-}49)$$

where, from (5-37) and the definition (5-22) of D_1,

$$\begin{aligned} B &= -\tfrac{1}{2}(1 + r)^{1/2} q^{-1/2} \log A\lambda \\ &= -\tfrac{1}{2}(1 + r)^{1/2} q^{-1/2} \log \frac{\lambda K r(1 + 1/r)^{-q}}{[2\pi q(1 + 2r)]^{1/2}} \end{aligned} \quad (5\text{-}50)$$

That the value of $J(\lambda)$ differs from (5-49) by $0(q^{-1/2})$ may be seen as follows. Since $0 < \exp[-\beta^2/2q] < 1$, the integral over (w_0, ∞) (mentioned just above (5-46) and obtained by taking the limits of integration to be w_0 and ∞ in the left side of (5-47)) is positive and less than

$$\int_{w_0}^{\infty} \exp[-e^{w-w_0}]\, dw = \int_{1}^{\infty} e^{-x}\, dx/x = .219\ldots \quad (5\text{-}51)$$

Likewise, the second integral on the right side of (5-47) is less than

$$\int_{-\infty}^{w_0} (1 - \exp[-e^{w-w_0}])\, dw = \int_{0}^{1} (1 - e^{-x})\, dx/x = .796\ldots \quad (5\text{-}52)$$

Therefore the contribution of the first integral on the right of (5-47) differs from $J(\lambda)$ by a quantity less than

$$\int_{-\infty}^{\infty} D_1 e^{-y^2 D}(.219 + .796)\, dy = 0(q^{-1/2})$$

in absolute value. The contribution of the first integral on the right of (5-47) differs from (5-49) by the second integral in (5-48) which is $0(q^{-1/2})$ because it is less than

$$\int_{-\infty}^{\infty} D_1(y^2 D) e^{-y^2 D}\, dy$$

The factor $(y^2 D)$ arises from $w_0 - (-\log A\lambda)$ when the mean value theorem is applied to the integral in w. Hence $J(\lambda)$ differs from (5-49) by $0(q^{-1/2})$.

Although (5-49) is a sufficiently accurate expression of $J(\lambda)$ for our purposes, it seems worthwhile to set down approximate expressions for the terms which have been dismissed as $0(q^{-1/2})$. From the above work,

$$J(\lambda) = (1 + \text{erf } B)/2 + D_1 \int_{-\infty}^{\infty} dy\, e^{-y^2 D} \left\{ \int_{w_0}^{\infty} e^{-\beta^2/2q} \exp[- e^{w-w_0}]\, dw \right.$$

$$- \int_{-\infty}^{w_0} e^{-\beta^2/2q} (1 - \exp[- e^{w-w_0}])\, dw$$

$$\left. + \int_{-\log A\lambda}^{w_0} e^{-\beta^2/2q}\, dw \right\} \tag{5-53}$$

$$\approx (1 + \text{erf } B)/2 + D_1 \int_{-\infty}^{\infty} dy\, e^{-y^2 D} \{-.577.. + y^2 D\} e^{-\beta_1^2/2q}$$

$$= (1 + \text{erf } B)/2 + \left(\frac{1+r}{4\pi q}\right)^{1/2} [- .577 \cdots +$$

$$4^{-1}(1 + r)^{-1}\{1 + (2 + 4r)B^2\}]e^{-B^2}$$

where $\beta_1 = y + (1 + r) \log A\lambda$ and we have made use of the fact that $\beta^2/2q$ changes relatively slowly in comparison with w when q is large.

Since $J(\lambda)$ differs from $(1 + \text{erf } B)/2$ by $0(q^{-1/2})$, and since the three B's for λ equal to λ_1, 1, and λ_2 differ by not more than $0(q^{-1/2} \log (\lambda_2/\lambda_1)) = 0(q^{-1} \log^{3/2} q)$, from (5-50) and (5-38), it follows that the terms involving $J(\lambda_1)$ and $J(\lambda_2)$ in (5-43) may be included in the term $0(q^{-1/2} \log^{3/2} q)$. In using our result it is more convenient to deal with N and $K + 1$ instead of $q = N - 1/2$ and K. Hence instead of B we deal with H defined by

$$H = -\frac{1}{2}\frac{(1+r)^{1/2}}{(q+1/2)^{1/2}} \log \frac{(K+1)(1+1/r)^{-q-1}(1+r)}{[2\pi(q+1/2)(1+2r)]^{1/2}}. \tag{5-54}$$

The difference $B - H$, with $\lambda = 1$ and H finite, may be shown to be (with considerable margin) $0(1/K) + 0(q^{-1/2})$. From (5-43), as amended by the first sentence in this paragraph, it follows that

$$\text{Prob. } (P_1Q, \cdots, P_KQ > P_0Q) = (1 + \text{erf } H)/2 + 0(1/K) + 0(q^{-1/2} \log^{3/2} q) \tag{1-4}$$

where the difference between erf B and erf H has been absorbed by the "order of" terms. When $q + 1/2$ is replaced by N in (5-54) the result is expression (1-5) for H.

APPENDIX I

Cumulative Distribution Function for a Sum of Squares of Normal Variates

Let x be a random variable defined by

$$x = \sum_{n=1}^{M} y_n^2 \tag{A1-1}$$

where y_n is a random variable distributed normally about its average value $\bar{y}_n$ with unit standard deviation. In writing $(A1 - 1)$ we have been guided by (4-3), where $M = 2N + 1$, but here we shall let M be any positive integer. In much of the following work $M/2$ occurs and for convenience we put

$$m = M/2 \tag{A1-2}$$

From the work of Section 4 it follows that the probability density $p(x, u)$ of x is given by Fisher's expression

$$p(x, u) = 2^{-1}(x/u)^{m/2-1/2} I_{m-1}[(ux)^{1/2}]e^{-(u+x)/2} \tag{A1-3}$$

where u is the constant

$$u = \sum_{n=1}^{M} \bar{y}_n^2 \tag{A1-4}$$

Here we are interested in the cumulative distribution function, i.e., the probability that x is less than some given value x_0,

$$P(x_0, u) = \int_0^{x_0} p(x, u)\, dx \tag{A1-5}$$

as M becomes large. In this case the central limit theorem tells us that $p(x, u)$ approaches a normal law with average $\bar{x} = M + u$ and variance $=$ ave. $(x - \bar{x})^2 = 2M + 4u$. The function $P(x_0, u)$ has been studied by J. I. Marcum in some unpublished work, and by P. K. Bose(9). In particular, Marcum has used the Gram-Charlier series to obtain values for $P(x_0, u)$ in the vicinity of $\bar{x}$ for large values of M. However, since I have not been able to find any previous work covering the case of interest here, namely values of $P(x_0, u)$ when x_0 is appreciably less than $\bar{x}$, a separate investigation is necessary and will be given here.

Integrating the general expression (4-5) with respect to x between $-X$ and x_0, letting $X \to \infty$, and discarding the portions of the integrand which oscillate with infinite rapidity gives

$$P(x_0, u) = -\frac{1}{2\pi i}\int_{-\infty,\,\text{above}\,0}^{\infty} z^{-1}e^{-izx_0}\,[\text{ave. } e^{izx}]\,dz$$
$$= 1 - \frac{1}{2\pi i}\int_{-\infty,\,\text{below}\,0}^{\infty} z^{-1}e^{-izx_0}\,[\text{ave. } e^{izx}]\,dz \tag{A1-6}$$

where the subscripts "above 0" and "below 0" indicate that the path of integration is indented so as to pass above or below, respectively, the pole at $z = 0$. The value of ave. exp (izx) may be obtained by setting $N + 1/2 = m$ in (4-5). The new notation

$$x_0 = Ms = 2ms, \qquad u = 2mt, \qquad 2z = \zeta \tag{A1-7}$$

enables us to write

$$P(x_0, u) = -\frac{1}{2\pi i}\int_{-\infty,\,\text{above}\,0}^{\infty} \zeta^{-1}\exp m[-is\zeta - \log(1 - i\zeta) - t + t(1 - i\zeta)^{-1}]\,d\zeta. \tag{A1-8}$$

The further change of variable

$$1 - i\zeta = v \tag{A1-9}$$

carries (A1-8) into

$$P(x_0, u) = \frac{1}{2\pi i}\int_K (1 - v)^{-1}\exp[mF(v)]\,dv \tag{A1-10}$$

where the path of integration K is the straight line in the complex v plane running from $1 + i\infty$ to $1 - i\infty$ with an indentation to the right of $v = 1$, and

$$F(v) = sv - \log v + t/v - s - t. \tag{A1-11}$$

The K used here should not be confused with the K denoting the number of messages in the body of the paper. We have run out of suitable symbols.

An asymptotic expression for (A1-10) will now be obtained by the method of "steepest descents." The saddle points are obtained by setting the derivative

$$F'(v) = s - 1/v - t/v^2 \tag{A1-12}$$

to zero and are at

$$v_1 = [1 + (1 + 4st)^{1/2}]/2s$$
$$v_2 = [1 - (1 + 4st)^{1/2}]/2s \tag{A1-13}$$

As x_0 and s increase from 0 to ∞, u and t of course being fixed, we have the following behavior:

$$\begin{array}{llll} x_0 = 0 & \bar{x} & \infty \\ s = 0 & 1+t & \infty \\ v_1 = \infty & 1 & 0 \\ v_2 = -t & -t/(1+t) & 0 \end{array} \tag{A1-14}$$

It is seen that $v_1 \geqslant 0$ and $v_2 \leqslant 0$.

Putting aside for the moment the factor $(1 - v)^{-1}$ in (A1-10), the path of steepest descent through the saddle point v_1 is one of the two curves specified by equating the imaginary part of $F(v)$ to zero. Introducing polar coordinates gives

$$v = \rho e^{i\theta}$$

$$\begin{aligned} \text{Real } F(v) &= (s\rho + t/\rho) \cos\theta - \log\rho - s - t \\ \text{Imag. } F(v) &= (s\rho - t/\rho) \sin\theta - \theta \end{aligned} \tag{A1-15}$$

At v_1, $\theta = 0$, $\rho = v_1$. Imag. $F(v_1) = 0$ and, from (A1-12),

$$\begin{aligned} \text{Real } F(v_1) &= (2sv_1 - 1) - \log v_1 - s - t \\ &= (1 + 4st)^{1/2} - \log v_1 - s - t \end{aligned} \tag{A1-16}$$

The path of steepest descent through v_1 may be obtained in polar form by solving

$$(s\rho - t/\rho) = \theta/\sin\theta \tag{A1-17}$$

for ρ as a function of θ. Setting $\varphi = \theta \csc\theta$ and taking the positive value of ρ leads to

$$\rho = [\varphi + (\varphi^2 + 4st)^{1/2}]/2s \tag{A1-18}$$

As θ increases from 0 to π, φ increases from 1 to ∞, and ρ starts from v_1 (as it should) and ends at ∞. Thus, the path of steepest descent through v_1 comes in from $v = -\infty + i\pi/s$ (when θ is nearly π, $\rho \approx \varphi/s$, $\varphi \approx \pi/(\pi - \theta)$ and $\rho(\pi - \theta) \approx \pi/s$), crosses the positive imaginary v axis and bends down to cut the real positive v axis (at right angles) at v_1, and then goes out to $v = -\infty - i\pi/s$ along a similar path in the lower part of the plane. It thus avoids the branch cut (which we take to run from $-\infty$ to 0) in the v plane necessitated by the term $\log v$ in $F(v)$. Since m and s are positive the path of integration K in (A1-10) may be made to coincide with the path of steepest descent when $v_1 > 1$. This corresponds to the case in which $x_0 < \bar{x}$ as (A1-14) shows. When $0 < v_1 < 1$, i.e., $\infty > x_0 > \bar{x}$, the two paths may still be made to coincide but it is necessary to add the contribution of the pole at $v = 1$ as K is pulled over it. This is equivalent to passing from the first to the second of equations (A1-6). The path $\theta = 0$ which makes Imag. $F(v)$ of (A1-15) zero turns out to be the curve of "steepest ascent" and hence need not be considered. As (A1-13) shows, the saddle point v_2 does not enter into our considerations because it lies on the negative real v axis and the path of integration K in (A1-10) cannot be made to pass through it without trouble from the singularity of $F(v)$ at $v = 0$.

We now suppose $x_0 < \bar{x}$ so that s and t are such as to make $v_1 > 1$. In order to remove the factor $(1 - v)$ from the denominator of the integrand in (A1-10), we change the variable of integration from v to w:

$$v - 1 = e^w, \qquad (1 - v)^{-1} dv = -dw$$

$$P(x_0, u) = -\frac{1}{2\pi i} \int_L \exp\,[mF(1 + e^w)]\, dw \tag{A1-19}$$

As v comes in along the path of steepest descent, the path of integration L for w comes in from $w = \infty + i\pi$ and dips down towards the real w axis as arg v decreases from π. L crosses the real w axis perpendicularly at the point

$$w_1 = \log\,(v_1 - 1) \tag{A1-20}$$

and then runs out to $w = \infty - i\pi$ along a curve which tends to become parallel to the real w axis. w_1 may be either positive or negative. When x_0 is almost as large as $\bar{x}$, w_1 is large and negative.

Since $F(v)$ is real along the path of steepest descent, $F(1 + e^w)$ is real along L. This real value is $-\infty$ at the ends of L and attains its maximum value $F(v_1)$, given by (A1-16), at $w = w_1$. w_1 is a saddle point in the complex w plane because

$$\frac{d}{dw} F(1 + e^w) = F'(1 + e^w)e^w = F'(v)e^w \tag{A1-21}$$

vanishes at $w = w_1$.

Instead of $F(1 + e^w)$ itself we shall be concerned with

$$\tau = F(1 + e^{w_1}) - F(1 + e^w) \tag{A1-22}$$

so that (A1-19) may be written as

$$P(x_0, u) = -\frac{\exp\,[mF(1 + e^{w_1})]}{2\pi i} \int_L e^{-m\tau}\, dw. \tag{A1-23}$$

The variable τ is real on the path of integration L, is zero at w_1, and increases to $+\infty$ as we follow L out to $w = \infty \pm i\pi$. It is convenient to split

K into two parts (10). The first part connects $\infty + i\pi$ to w_1 and the second part connects w_1 to $\infty - i\pi$. The values of w on these two parts will be denoted by w_I and w_{II}, respectively. Corresponding to each value of τ there is a value w_I and a value w_{II} (in fact it turns out that w_{II} is the conjugate complex of w_I). Changing the variable of integration in (A1-23) from w to τ, and remembering that K starts at $\infty + i\pi$, gives

$$P(x_0, u) = \frac{\exp[mF(1 + e^{w_1})]}{2\pi i} \int_0^\infty e^{-m\tau} \left[\frac{d}{d\tau} w_I - \frac{d}{d\tau} w_{II}\right] d\tau \quad \text{(A1-24)}$$

Since m is large, most of the contribution to the value of the integral comes from around $\tau = 0$ or $w = w_1$. In order to obtain an expression for the integrand in this region we note that, because $F'(v_1) = 0$, the Taylor series for (A1-22) is of the form

$$\tau = -b_2(w - w_1)^2 - b_3(w - w_1)^3 - b_4(w - w_1)^4 - \cdots \quad \text{(A1-25)}$$

The circle of convergence of this series is centered on w_1 and extends out to $w = \pm i\pi$, these points being the nearest singularities of $F(1 + e^w)$ as may be seen by setting $v = 1 + e^w$ in (A1-11) and observing that the singularities of $\log v - t/v$ in the finite portion of the w plane occur at odd multiples of $\pm i\pi$. We imagine the branch cuts associated with $\log v$ to run out to the right from these points along lines parallel to the real w axis. Since (A1-25) has a non-zero radius of convergence, the same is true of the two series obtained from it by inversion, namely

$$\begin{aligned} w_I - w_1 = {} & ib_2^{-1/2}\tau^{1/2} + b_3\tau/2b_2^2 \\ & + i[b_2^{-2}b_4 - 5b_2^{-3}b_3^2/4]\tau^{3/2}/2b_2^{1/2} + \cdots \end{aligned} \quad \text{(A1-26)}$$

and the series for $w_{II} - w_1$ obtained from (A1-26) by changing the sign of i. Differentiation of these two series gives a series for $d(w_I - w_{II})/d\tau$ which also converges for sufficiently small $|\tau|$ (putting aside the term in $\tau^{-1/2}$), and which, when put in (A1-24), leads to

$$P(x_0, u) \sim \frac{e^{mF(v_1)}}{(4\pi m b_2)^{1/2}} \left\{1 + \frac{3}{4m} [b_2^{-2}b_4 - 5b_2^{-3}b_3^2/4] + \cdots\right\} \quad \text{(A1-27)}$$

That this is an asymptotic expansion holding for large values of m follows from a lemma given by Watson (11). The conditions of the lemma hold since we have already shown that the series for $d(w_I - w_{II})/d\tau$ converges for $|\tau|$ small enough. Furthermore, $d(w_I - w_{II})/d\tau$ is bounded for $a \leqslant \tau$ where τ is real and $0 < a \leqslant$ the radius of convergence of (A1-26). This follows the fact that

$$\frac{dw}{d\tau} = \left[\frac{d\tau}{dw}\right]^{-1} = [-F'(1 + e^w)e^w]^{-1}$$

is bounded except near $w = w_1$ (i.e., $\tau = 0$) and, indeed, decreases to zero like $-e^{-w}/s$ as $w \to \infty \pm i\pi$ (i.e., $\tau \to \infty$).

The values of b_2, b_3, b_4 obtained by expanding (A1-22) and comparing the result with (A1-25) are

$$\begin{aligned} b_2 &= F''(v_1)e^{2w_1}/2 \\ b_3 &= [F'''(v_1)e^{3w_1} + 3F''(v_1)e^{2w_1}]/6 \\ b_4 &= [F''''(v_1)e^{4w_1} + 6F'''(v_1)e^{3w_1} + 7F''(v_1)e^{2w_1}]/24 \end{aligned} \tag{A1-28}$$

$$F''(v) = v^{-2} + 2tv^{-3}, \quad F'''(v) = -2v^{-3} - 6tv^{-4}, \quad F''''(v) = 6v^{-4} + 24tv^{-5}$$

Our asymptotic expression for $P(x_0, u)$, when $x_0 < \bar{x}$, is given by (A1-28) and (A1-27). Only the leading term of (A1-27) is used in the paper. Sometimes the following expressions are more convenient than the ones which have already been given.

$$\begin{aligned} b_2 &= v_1^{-3}(v_1 + 2t)e^{2w_1}/2 = v_1^{-3}(v_1 + 2t)(v_1 - 1)^2/2 \\ &= (1 - 1/v_1)^2(1 + 4st)^{1/2}/2 \\ F(v_1) &= (1 + 4st)^{1/2} - s - t - \log v_1. \end{aligned} \tag{A1-29}$$

In all of these formulas v_1 is given in terms of s and t by (A1-13) and s and t in terms of x_0 and u by (A1-7).

When $x_0 > \bar{x}$, the saddle point v_1 lies between 0 and 1 in the v plane. As v follows the path of steepest descent (discussed just below equation (A1-18)) arg $(v - 1)$ now stays close to π. From (A1-19) Imag. w stays close to π on the new path of steepest descent in the w plane, and the saddle point w_1 now lies on the negative real portion of the line Imag. $w = \pi$. The new path starts at $w = \infty + i\pi$, swings down a little as it comes in, swerves up to pass through w_1 and then goes out to $w = \infty + i\pi$ above the branch cut joining $w = i\pi$ to $w = \infty + i\pi$. The analysis goes along much as for $v_1 > 1$ except that instead of being 0 the imaginary part of w_1 is $i\pi$. This causes the terms in b_3 and b_4 containing exp $(3w_1)$ to change sign. The numerical values of b_2 and $F(v_1)$ are computed by the formulas (A1-29) as before. The fact that b_2 contains the factor exp $(i2\pi)$ shows up only in changing the sign of $b_2^{1/2}$ to give the minus sign in the leading term:

$$P(x_0, u) \sim 1 - (4\pi m|\,b_2\,|)^{-1/2} \exp\,[mF(v_1)]$$

which holds for $x_0 > \bar{x}$. The one arises from the pole at $v = 1$ and is the same as the one in the second of equations (A1-6).

In order to see how (A1-27) breaks down near $x_0 = \bar{x}$, we set $x_0 - \bar{x} = 2m(s - 1 - t) = -2m\epsilon$ or $s = 1 + t - \epsilon$ where ϵ is a small positive number

Using $\sigma_x^2 \equiv$ ave. $(x - \bar{x})^2 = 4(m + u) = 4m(1 + 2t)$ it is found that

$$v_1 = 1 + \epsilon/(1 + 2t) = 1 - 2(x_0 - \bar{x})\sigma_x^2$$

$$mF(v_1) = -m\epsilon^2/(2 + 4t) = -(x_0 - \bar{x})^2/2\sigma_x^2$$

$$2mb_2 = m(v_1 - 1)^2(1 + 2t) = (x_0 - \bar{x})^2/\sigma_x^2$$

and that, since $w_1 \to -\infty$, $b_3 \to b_2$ and $b_4 \to 7b_2/12$. When these values are put in (A1-27) the leading term becomes

$$P(x_0, u) \sim (2\pi)^{-1/2}(\sigma_x/z) \exp[-z^2/2\sigma_x^2]$$

and the term within the braces in (A1-27) reduces to $1 - \sigma_x^2/z^2$ where $z = \bar{x} - x_0 > 0$. Since the asymptotic expansion is useful only in the region where the second term within the braces is small in comparison with the first term, which is unity, $\bar{x} - x_0$ must be several times as large as σ_x before we can use (A1-27). It will be noticed that the above expression for $P(x_0, u)$ is closely related to the asymptotic expansion of the error function.

APPENDIX II

An Approximation for $I_N(x)$

When z in the Bessel function $J_q(qz)$ is imaginary a formula given by Meissel (12) becomes

$$I_q(qy) = \frac{(qy)^q \exp(qw + V)}{e^q\Gamma(q + 1)w^{1/2}(1 + w)^q} \tag{A2-1}$$

where $w = (1 + y^2)^{1/2}$ and V is a function of y and q which, when q is large, has the formal expansion

$$V = \frac{1}{24q}\left\{2 - \frac{2 - 3y^2}{w^3}\right\} + \frac{y^4 - 4y^2}{16q^2w^6} - \frac{1}{5760q^3}\left\{16 - \frac{16 + 1512y^2 - 3654y^4 + 375y^6}{w^9}\right\} + \cdots \tag{A2-2}$$

Here we shall show that for $y \geqslant 0$ and $q > 1$

$$| V | < 1/(2q - 1) \tag{A2-3}$$

Consideration of (A2-2) and also of the method used to establish (A2-3) indicates that the inequality is very rough. It doubtlessly can be greatly improved (but not beyond the $1/(12q)$ obtained by letting y and $q \to \infty$ in (A2-2)). Incidentally, it may be shown that the constant terms which remain in (A2-2) when $y = \infty$ are associated with the asymptotic expansion of $\log \Gamma(q + 1)$.

When (A2-1) is substituted in Bessel's differential equation, which we write as

$$y^2 \frac{d^2}{dy^2} I_q(qy) + y \frac{d}{dy} I_q(qy) - q^2(1 + y^2) I_q(qy) = 0,$$

we obtain a differential equation for V:

$$V'' = (4 - y^2)w^{-4}/4 - (2qw + w^{-2})y^{-1}V' - V'^2 \qquad \text{(A2-4)}$$

Here the primes denote differentiation with respect to y. The constants of integration associated with (A2-4) are to be chosen so that

$$V \rightarrow y^2/(4q + 4) \text{ as } y \rightarrow 0. \qquad \text{(A2-5)}$$

This condition is obtained by comparing the limiting form of (A2-1), in which $w \rightarrow 1 + y^2/2$, with

$$I_q(qy) \rightarrow \frac{(qy/2)^q}{\Gamma(q+1)} \left[1 + \frac{(qy/2)^2}{q+1}\right] \rightarrow \frac{(qy/2)^q}{\Gamma(q+1)} \exp\left[\frac{q^2y^2}{4(q+1)}\right]$$

Condition (A2-5) completely determines V since substitution of the assumed solution

$$V = 4^{-1}(q + 1)^{-1}y^2 + c_1y^4 + c_2y^6 + \cdots$$

in (A2-4) leads to relations which determine $c_1, c_2, \cdots$ successively.

Let $V' = v$. Then (A2-4) becomes

$$v' = c - 2bv - v^2 \qquad \text{(A2-6)}$$

where c and b are known functions of y defined by

$$c = (4 - y^2)w^{-4}/4, \qquad b = (qw + w^{-2}/2)y^{-1} \qquad \text{(A2-7)}$$

From (A2-5), $v \rightarrow y/(2q + 2)$ as $y \rightarrow 0$ and therefore

$$V = \int_0^y v\, dy \qquad \text{(A2-8)}$$

We first show that $|v| < 1/(2q - 1)$ when $q > 1$. The (y, v) plane may be divided into regions according to the sign of v'. The equations of the dividing lines between these regions are obtained by setting $v' = 0$ in (A2-6). Thus, for a given value of y, v' is positive if $v_2 < v < v_1$ and negative if $v > v_1$ or $v < v_2$ where

$$v_1 = -b + (b^2 + c)^{1/2} = c/[b + (b^2 + c)^{1/2}]$$
$$v_2 = -b + (b^2 + c)^{1/2} \qquad \text{(A2-9)}$$

When $y > 0$ we have $b \geqslant q$. A plot of c versus y shows that $|c| \leqslant 1$. Hence, when $q > 1$,

$$
\begin{aligned}
b^2 + c &\geqslant q^2 - 1 > (q-1)^2 \\
|v_1| &< 1/(2q-1) \\
v_2 &< -2q+1
\end{aligned}
\tag{A2-10}
$$

The curve obtained by plotting v_1 as a function of y plays an important role because, as we shall show, the maxima and minima of the curve for v lie on it. Therefore, the maximum value of $|v|$ cannot exceed the maximum value of $|v_1|$. The maxima and minima must lie on either the v_1 or the v_2 curve since v' vanishes only on these curves. In order to show that it is the v_1 curve we note from (A2-9) that, near $y = 0$, v_1 behaves like $y/(2q+1)$. Consequently both the v_1 and v curves start from $v = 0$ at $y = 0$ but for a while v_1 lies above v which behaves like $y/(2q+2)$. Here v lies in a $v' > 0$ region and continues to increase until it intersects v_1 (as it must do before y reaches 2 because $v_1 = 0$ at $y = 2$) at which point $v' = 0$, $v_1' \leqslant 0$, and v has a maximum which is less than the maximum of $|v_1|$ so $v < 1/(2q-1)$ when $q > 1$. Upon passing through v_1, v enters a $v' < 0$ region and decreases steadily until it either again intersects the v_1 curve or else approaches some limit as $y \to \infty$. In either case $|v|$ does not exceed $1/(2q-1)$, since, in the first case v would have a minimum at the intersection and in the second $v_1 \to 0$ as $y \to \infty$. The same reasoning may be applied to the remaining points of intersection, if any, of the v and v_1 curves.

In order to obtain an inequality for V itself we rewrite (A2-6) as

$$v' = c - (2b + v)v \tag{A2-11}$$

The solution of this equation which behaves like $y/(2q+2)$ as $y \to 0$ also satisfies the relation

$$v(y) = \int_0^y c(x) \exp\left[-\int_x^y [2b(\xi) + v(\xi)]\, d\xi\right] dx.$$

as may be verified by making use of the relations $c(x) \to 1$ as $x \to 0$ and $2b(\xi) \to (2q+1)/\xi$, $v(\xi) \to \xi/(2q+2)$ as $\xi \to 0$. For then

$$-\int_x^y [2b(\xi) + v(\xi)]\, d\xi \to (2q+1)\log x/y$$

$$v(y) \to \int_0^y (x/y)^{2q+1}\, dx = y/(2q+2)$$

Hence, from (A2-8)

$$V(y_1) = \int_0^{y_1} dy \int_0^y c(x) \exp\left[-\int_x^y [2b(\xi) + v(\xi)]\, d\xi\right] dx$$

and

$$| V(y_1) | \leq \int_0^{y_1} dy \int_0^{y} | c(x) | \exp \left[-\int_x^y [2b(\xi) - | v(\xi) |] \, d\xi \right] dx.$$

From $b \geqslant q$ and $| v | < 1/(2q - 1)$ it follows that $2b(\xi) - | v(\xi) | > 2q - 1$ when $q > 1$. This and $| c(x) | \leqslant (4 + x^2)(1 + x^2)^{-2}/4$ gives

$$| V(y_1) | < \int_0^{\infty} dy \int_0^{y} (4 + x^2)(1 + x^2)^{-2} 4^{-1} \exp [-(2q - 1)(y - x)] \, dx$$

$$= \frac{5\pi}{16(2q - 1)} < \frac{1}{2q - 1}$$

which is the result we set out to establish. The double integral may be reduced to a single integral by inverting the order of integration and integrating with respect to y. Incidentally, most of the roughness of our result is due to the use of the inequality for $| c(x) |$.

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