Relay Armature Rebound Analysis

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Rebound of mechanical structures subsequent to impinging on stops generally has deleterious effects on their performance and should, therefore, be minimized. A considerable reduction in rebound can often be obtained by introducing additional degrees of freedom to the structure.

A mathematical treatise of the dynamics of rebound motion of systems representing idealized relay armatures is presented. Normalized differential equations of motion and their solutions for the "free" and "impact" intervals are derived for systems having one, two, and three degrees of freedom, allowing the rebound behavior of a specific system to be calculated. The equations of series of rebounds, and possible combinations of such series are considered next for systems having one and two degrees of freedom. The field of possible rebound maxima is mapped for a practical range of mass distribution constants, coefficients of restitution, and force ratios. A sufficiently broad optimum design region is indicated.

The results of this analysis have been checked closely on a model and have led to appreciable reduction of armature rebound in relay designs.

I. INTRODUCTION

In numerous types of mechanisms it is desirable to arrest the motion of a member at a particular point and to maintain it in this position. One of the simplest means of accomplishing this is to allow the moving member to impinge on a fixed member (stop) and to provide forces to tension it against this stop. Because the member to be arrested possesses kinetic energy and because the stop cannot generally absorb all of this energy, the moving member will rebound from the stop. The rebound motion generally deteriorates the performance of the mechanism and should be minimized.

Investigation of this phenomenon has been stimulated by the armature rebound problem in relay operation, where rebound from the front stop* tends to reclose contacts and must therefore be compensated for by additional (waste) travel, resulting in deleterious effects on speed and

* Among relay designers the front stop has been generally referred to as "backstop". In this paper the terms front stop and heel stop have been used throughout for easier identification.

magnetic characteristics. Analysis in this paper will be directed towards relay armature systems, but it is also applicable to rebound in similar mechanisms.

II. STATEMENT OF PROBLEM

Analysis will be restricted to planar motion of armature systems having one, two, and three degrees of freedom as depicted in Figs. 1, 2, and 3. Generally one stop must be provided for each degree of freedom, although in the three-degree-of-freedom system of Fig. 3, two of the stops have been combined.

Applied forces F_1, F_2, F_3, have been chosen so as to be most easily correlated with actual relay designs.

The initial condition in all cases will be a pure rotation about the "heel" just prior to a "zero" impact at the "front" of the armature. The "zero" impact will be followed by rebound motion and impacts at the various stops eventually bringing the armature to rest. The object

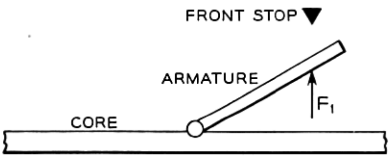

Fig. 1—Solidly hinged armature—one degree of freedom.

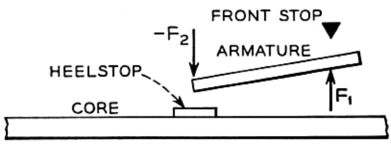

Fig. 2—Loosely hinged armature—two degrees of freedom.

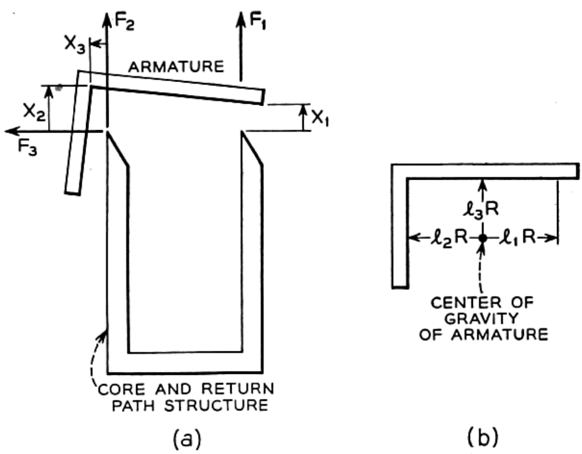

Fig. 3—Armature system—three degrees of freedom.

will be to minimize rebound motion at the front, since this is usually near the point actuating the relay contacts.

The basic problem is then to find the response of the armature subject to aperiodic but well defined impulses, which are functions of the positions and velocities of the system.

III. ASSUMPTIONS

In order to facilitate the solution of this problem, the following modifying assumptions are made:

(1) As mentioned in the previous section, analysis is restricted to planar motion.

(2) The armature is assumed to be a rigid body.

(3) Stops are assumed to be very stiff, massless springs capable of energy absorption during impact with the armature. The associated coefficient of restitution is assumed constant. Core and stop vibration are neglected.

(4) The tensioning forces F_1, F_2, F_3 are assumed to be constant forces. (This is fairly closely true for moderate rebound amplitudes of practical relay structures.)

(5) All displacements are small relative to the dimensions of the system and in particular the angular displacement θ is sufficiently small so that

$$\cos \theta \doteq 1$$

$$\sin \theta \doteq \theta$$

IV. DERIVATION OF EQUATIONS OF MOTION

The derivation of the equations of motion resolves itself into the solution of two different types of intervals:

(1) *Free Interval:* This is the period during which the armature is not in contact with any of its stops and only the tensioning forces are acting.

(2) *Impact Interval:* During such intervals the armature is in contact with at least one of the stops. The stiffness of the latter is assumed so high that the tensioning forces during this interval may be neglected.

The three-degree-of-freedom case will be considered first and the others subsequently deduced from it by allowing some of the constants to approach zero.

A. Free Interval

The motion of the armature will be described by the displacement at the stop points: x_1, x_3, x_3.* Let m be the mass and R the radius of gyration of the armature about the center of gravity. The latter is located by the dimensions $\ell_1 R$, $\ell_2 R$, and $\ell_3 R$ relative to the stop points, i.e., the points on the armature which contact the stops in the rest position (Fig. 3).

The equations of motion are derived in Appendix I and are put into dimensionless form:

$$\left.\begin{aligned} y_1 &= \frac{1}{2}\left[C_{11} + C_{12}\frac{(F_2)}{(F_1)} + C_{13}\frac{(F_3)}{(F_1)}\right]\left(\frac{t}{\tau}\right)^2 + \dot{y}_{10}\left(\frac{t}{\tau}\right) + \dot{y}_{10} \\ y_2 &= \frac{1}{2}\left[C_{21} + C_{22}\frac{(F_2)}{(F_1)} + C_{23}\frac{(F_3)}{(F_1)}\right]\left(\frac{t}{\tau}\right)^2 + \dot{y}_{20}\left(\frac{t}{\tau}\right) + \dot{y}_{20} \\ y_3 &= \frac{1}{2}\left[C_{31} + C_{32}\frac{(F_2)}{(F_1)} + C_{33}\frac{(F_3)}{(F_1)}\right]\left(\frac{t}{\tau}\right)^2 + \dot{y}_{30}\left(\frac{t}{\tau}\right) + \dot{y}_{30} \end{aligned}\right\} \quad (1)$$

where:

$$\left.\begin{aligned} y_i &= \frac{x_i}{\dot{x}_a\tau} = \frac{F_1}{\dot{x}_a^2 m}x_i \qquad \dot{y}_i = \frac{d}{d\left(\frac{t}{\tau}\right)}y_i = \frac{\dot{x}_i}{\dot{x}_a} \\ \tau &= \frac{\dot{x}_a m}{F_1} \end{aligned}\right\} \quad (2)$$

$\dot{x}_a$ is the front velocity $\dot{x}_1$, just prior to the "zero" impact, and

$$\left.\begin{aligned} C_{11} &= (\ell_1^2 + 1) \quad & C_{13} &= C_{31} = \ell_1\ell_3 \\ C_{22} &= (\ell_2^2 + 1) \quad & C_{12} &= C_{21} = (1 - \ell_1\ell_2) \\ C_{23} &= (\ell_3^2 + 1) \quad & C_{23} &= C_{32} = -\ell_2\ell_3 \end{aligned}\right\} \quad (3)$$

$\dot{y}_{10}$, $\dot{y}_{20}$, $\dot{y}_{30}$, are the initial velocities and y_{10}, y_{20}, y_{30} the initial displacements for the free interval in question.

The equations of motion for a two-degree-of-freedom system are obtained, if $F_3 = 0$. Then for the two coordinates of interest:

$$\left.\begin{aligned} y_1 &= \frac{1}{2}\left[C_{11} + C_{12}\left(\frac{F_2}{F_1}\right)\right]\left(\frac{t}{\tau}\right)^2 + \dot{y}_{10}\left(\frac{t}{\tau}\right) + y_{10} \\ y_2 &= \frac{1}{2}\left[C_{21} + C_{22}\left(\frac{F_2}{F_1}\right)\right]\left(\frac{t}{\tau}\right)^2 + \dot{y}_{20}\left(\frac{t}{\tau}\right) + y_{20} \end{aligned}\right\} \quad (4)$$

* A summary of all notations used in this paper is given in Appendix IV.

For a one-degree-of-freedom system $\ddot{y}_2 = C_{21} + C_{22}\left(\frac{F_2}{F_1}\right) = 0$, whence

$$y_1 = \frac{1}{2}\left[C_{11} - \frac{C_{12}^2}{C_{22}}\right]\left(\frac{t}{\tau}\right)^2 + \dot{y}_{10}\left(\frac{t}{\tau}\right) + y_{10}\bigg\} \tag{5}$$

B. Impact Interval

The change of velocity at point "i" due to an impact at "i" is, by definition of the coefficient of restitution "k",

$$\Delta\dot{x}_1 = -(1 + k_i)\dot{x}_i$$

It is assumed here that the action of the stops are true impacts, i.e., the changes in velocity take place while there is negligible motion of the body. The velocity changes then occur as instantaneous rotation about the conjugate axis, leading to the general relation for an impact at point "i":

$$\dot{y}_{j0n} = \dot{y}_{je(n-1)} + K_{ji}\dot{y}_{ie(n-1)} \tag{6}$$

The first subscript indicates the coordinate, the second subscript indicates the beginning (0) or the end (e) of the free interval described by the third subscript. The impact transfer coefficient K_{ji} relating a velocity change at point "j" to an impact at point "i":

$$K_{ji} = -\frac{C_{ji}}{C_{ii}}(1 + k_i) \tag{7}$$

Equations (1) through (7) allow any one specific case to be mapped, if the mass distribution and force ratio are known. A sample of such mapping of rebound motion for a rectangular two-degree-of-freedom armature appears in Fig. 4.

V. ANALYSIS OF REBOUND PATTERN—ONE-DEGREE-OF-FREEDOM SYSTEM

The rebound pattern for the one-degree-of-freedom system—as derived in Appendix II—consists of an infinite series of parabolic arcs of diminishing amplitudes. The structure comes to rest after a finite time interval. The maximum rebound occurs during the first bounce and equals

$$Y = -\frac{k^2}{2C} \tag{8}$$

where

$$C = C_{11} - \frac{C_{12}^2}{C_{22}} \tag{9}$$

The system returns to rest at

$$\frac{t}{\tau} = \frac{2}{C(1 - k)} \tag{10}$$

VI. ANALYSIS OF REBOUND PATTERN—TWO-DEGREE-OF-FREEDOM SYSTEM

The reason for choosing a two-degree-of-freedom system over a one-degree-of-freedom system would be, in keeping with the philosophy of

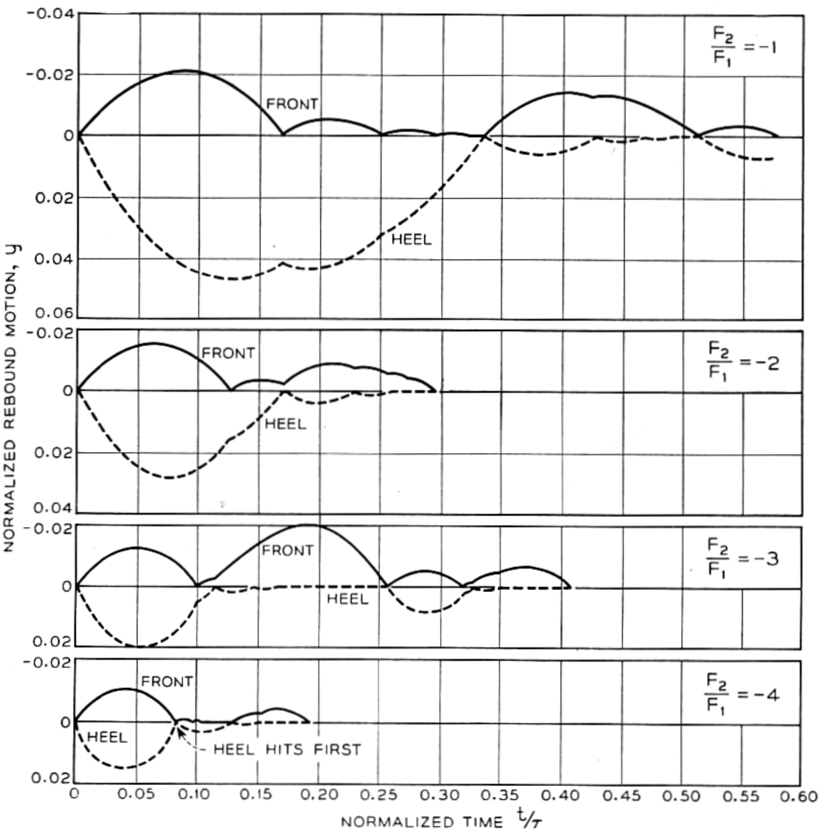

Fig. 4—Front and heel motion of plate type armature.

this treatment, to reduce Y_1, the greatest excursion at the front. In order to simplify mapping, this maximum excursion will be expressed as $2CY_1$, the ratio of Y_1 to Y as given by Equation (8) for the case of $k = 1$. Thus $2CY_1$ is the ratio of the greatest excursion of the two-degree-of-freedom system under consideration to the greatest excursion of the corresponding perfectly elastic one-degree-of freedom system.

We first introduce two basic constants which are functions of the mass distribution relative to the stop locations:

$$M_{ij} = \frac{C_{ij}^2}{C_{ii}C_{jj}} \tag{11}$$

This constant represents a mechanical coupling coefficient. As $M_{ij} = M_{ji}$, the two-degree-of-freedom system under consideration here has only one such non-trivial constant M_{12}.

The second constant represents a force transformation factor from the "j" coordinate to the "i" coordinate:

$$P_{ij} = \frac{C_{ij}}{C_{ii}} \tag{12}$$

In the analysis of the two-degree-of-freedom system only P_{12} is important.

If there is to be any heel motion, the "zero" impact at the front must impart a positive velocity to the heel. By Equations (6), (7), and (12), this requires that P_{12} be negative, which in turn implies that $\ell_1\ell_2 > 1$. For the limiting case of $\ell_1\ell_2 = 1$, $P_{12} = M_{12} = 0$ and no coupling exists between the heel and the front. Physically this means that the two stops are the centers of percussion of each other and the system will act as a simple hinge.

With the above foundation, it is possible to analyze the patterns of motion and maximum rebound amplitudes.

A. Motion Immediately Following "Zero" Impact

After the "zero" impact at the front, both front and heel will lift off in accordance with impact Equation (6) and continue to move in accordance with the free interval Equations (4). Whether the next impact occurs at the front or the heel depends on their respective periods, t_1 and t_2:

$$\frac{t_1}{t_2} = \frac{1 + \dfrac{P_{12}}{M_{12}} f}{1 + P_{12} f} \; \frac{k_1}{1 + k_1} \tag{13}$$

where:

$$f = \frac{F_2}{F_1}$$

A large value of t_1/t_2 will result in a series of heel impacts and the heel will come to rest while the front is still displaced from the stop. This will be called a complete heel series. A small value of t_1/t_2 results in a similar complete front series. If t_1/t_2 is near unity, a limited number of impacts on one end are followed by an impact on the other end, etc. An analysis of front and heel series follows:

B. Front Series

If $t_1/t_2 < 1$ a series of front impacts occurs. The impact velocities at the front are

$$\dot{y}_{1n} = 1, k, k^2, \cdots, k^n \tag{14}$$

The corresponding time intervals are

$$T_{1n} = \frac{2k_1}{A}, \frac{2k_1^2}{A}, \cdots, \frac{2k^n}{A} \tag{15}$$

where

$$A = (C_{11} + C_{12}f)$$

During this time, the heel velocity and displacement are given by

$$\left.\begin{aligned} \dot{y}_{20(n+1)} &= \dot{y}_{20n} + \left[\frac{2B}{A} - P_{12}(1 + k_1)\right] y_{i0n} \\ y_{20(n+1)} &= y_{20n} + \left[\frac{2B}{A}\dot{y}_{1n} + \dot{y}_{20n}\right]\dot{y}_{10n} \end{aligned}\right\} \tag{16}$$

where

$$B = C_{12} + C_{22}f$$

The velocity and displacement at the heel after a given number of front impacts are obtained by a summation of Equations (16). For a complete front series $n \to \infty$, and

$$\left.\begin{aligned} y_{2e\infty} &= \frac{2k_1}{A(1 + k_1)^2}\left[\frac{Bk_1}{A} - P_{12}\right] \\ \dot{y}_{2e\infty} &= \frac{1}{1 - k_1}\left[\frac{2Bk_1}{A} - P_{12}(1 + k_1)\right] \end{aligned}\right\} \tag{17}$$

In addition it is useful to set down energy equations in order to simplify evaluation of greatest rebound for the various groups of rebound patterns. The kinetic energy function T is evaluated in Appendix I. A potential energy term V—the work done against F_1 and F_2 from the equilibrium position—is introduced. If T_0 is the total energy of the system prior to the "zero" impact, then

$$\frac{T + V}{T_0} = \dot{y}_1^2 + \frac{M_{12}}{P_{12}^2} \dot{y}_2^2 - \frac{2M_{12}}{P_{12}} \dot{y}_1 \dot{y}_2 - 2C(y_1 + f y_2) \tag{18}$$

The energy loss due to n front impacts is

$$-\Delta \left(\frac{T + V}{T_0} \right) = (1 - k_1^{2n})(1 - M_{12}) \dot{y}_{1e0}^2 \tag{19}$$

For a complete front series $n \rightarrow \infty$, and

$$-\Delta \left(\frac{T + V}{T_0} \right) = (1 - M_{12}) \dot{y}_{1e0}^2 \tag{20}$$

If a complete front series follows the "zero" impact, $\dot{y}_{1e0} = 1$ and

$$-\Delta \left(\frac{T + V}{T_0} \right) = (1 - M_{12}) \tag{21}$$

After completion of this "initial" front series, the system maintains only one degree of freedom (rotation about the front) until a heel impact occurs. By setting $\dot{y}_1 = y_1 = y_2 = 0$ in (21) we obtain the heel impact approach velocity $\dot{y}_2 = P_{12}$.

Apparently energy loss due to n front impacts is a function of M_{12}, k_1, and the approach velocity of the first impact.

C. Heel Series

An analysis similar to the above can be made for partial and complete heel series following the "zero" impact. This is demonstrated in Appendix III, yielding, for $k_1 = k_2$*

$$\left. \begin{aligned} y_{1e\infty} &= \frac{AP_{12}(1 + k)^2}{B(1 - k)^2} \left[\frac{AP_{12}}{B} - \frac{k(1 - k)}{1 + k} - M_{12}k \right] \\ \dot{y}_{1e\infty} &= \frac{1 + k}{1 - k} \left[\frac{2AP_{12}}{B} - \frac{k(1 - k)}{(1 + k)} - M_{12}(1 + k) \right] \end{aligned} \right\} \tag{22}$$

* The more general form $k_1 \neq k_2$ can be obtained as indicated in Appendix III.

The energy relationships for heel series are

$$-\Delta\left(\frac{T+V}{T_0}\right) = (1 - k_2^{2n})\,\frac{M_{12}(1 - M_{12})}{P_{12}^2}\,\dot{y}_{2e0}^2 \tag{23}$$

For a complete series $n \rightarrow \infty$, and

$$-\Delta\left(\frac{T+V}{T_0}\right) = \frac{M_{12}(1 - M_{12})}{P_{12}^2}\,\dot{y}_{2e0}^2 \tag{24}$$

If a complete heel series follows the "zero" impact, $\dot{y}_{2e0} = P_{12}(1 + k_1)$, and

$$-\Delta\left(\frac{T+V}{T_0}\right) = M_{12}(1 - M_{12})(1 + k_1)^2 \tag{25}$$

Finally, for the special case where a complete heel series follows an initial complete front series $\dot{y}_{2e0} = P_{12}$, and

$$-\Delta\left(\frac{T+V}{T_0}\right) = M_{12}(M_{12} - 1) \tag{26}$$

It is to be noted that the energy loss due to a partial heel series is a function of M_{12}, P_{12}, k_2, and the approach velocity of the first impact, but that the equation for a complete heel series does not contain k_2. Finally, a complete initial heel series is a function of only M_{12} and k_1.

D. Complete Mapping of Problem

Equations (1) through (26) make it possible to completely map the two-degree-of-freedom rebound problem. The relative maximum amplitude $2CY_1$ and the rebound pattern will be determined.

Examination of the necessary equations, show that $2CY_1$ is in all cases a function of four parameters: k_1, k_2, M_{12} and $P_{12}f$. Of these, k_2 enters only if a partial heel series occurs prior to the time of maximum rebound. If it is assumed that for this limited group of cases $k_2 = k_1 = k$, the number of parameters is reduced to three: k, M_{12}, $P_{12}f$.

In Figs. 5 to 10, $2CY_1$ is plotted against $P_{12}f$ for the most useful range of $1/8 < M_{12} < 1/2.5$, $0.3 < k < 0.6$ and $0 < P_{12}f < 10$.

As $P_{12}f$ is increased from zero to infinity (corresponding to an increase in the heel tension F_2), the rebound pattern goes through some or all of five regions. The criterion for location in any one region is based upon the parameter

$$Q = \frac{1 + \dfrac{P_{12}}{M_{12}}f}{1 + P_{12}f} = \frac{t_1}{t_2}\,\frac{(1 + k)}{k} \tag{27}$$

Region I—Complete initial front series for $1 < Q < 1/k$. Within this region, if $M_{12}^2 > \frac{(1 - M_{12})k^2}{1 + P_{12}f}$, the maximum rebound occurs during the first bounce and

$$2CY_1 = \frac{(1 - M_{12})k^2}{1 + P_{12}f} \tag{28}$$

If the maximum rebound occurs later, it must occur during a complete heel series which follows the initial complete front series. From Equations (21) and (26)

$$2CY_1 = M_{12}^2 \tag{29}$$

By comparing Equations (28) and (29), the critical requirement

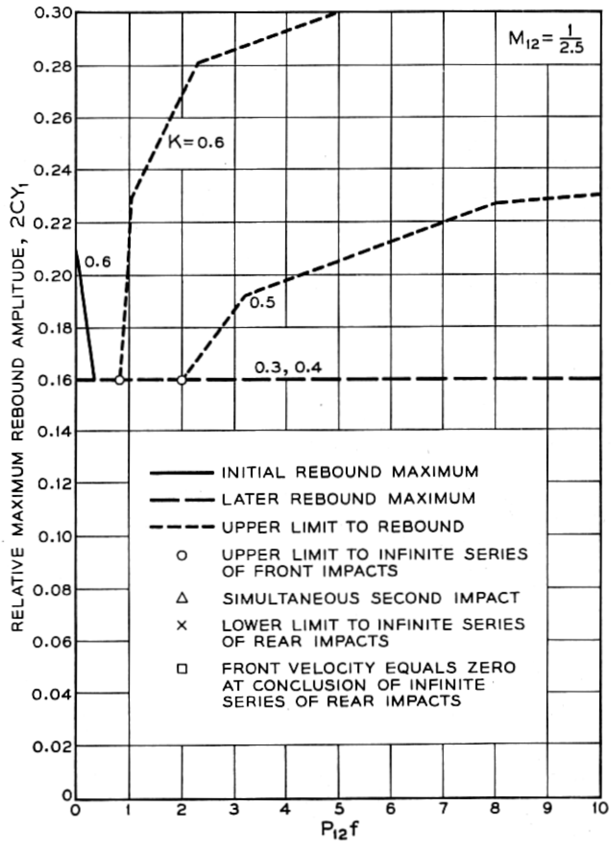

Fig. 5—Relative maximum rebound amplitude for $M_{12} = 1/2.5$.

for the latter case is that $P_{12}f > \frac{(1 - M_{12})}{M_{12}^2} k^2 - 1$. It should be noted that while the first rebound maximum, shown in solid lines on Figs. 5 to 10, is always realized, the later rebound given by (29) is an upper limit—shown in dashed lines—and is not always realized. In the dashed regions, phasing is extremely critical and small variations in the parameters may cause large variations in maximum rebound. From an engineering standpoint these regions are essentially undesirable.

Region II—Partial initial front series for

$$\frac{1}{k} < Q < \frac{1 + k}{k}.$$

This region is one of critical phasing, and attention is limited to special cases leading to maximum rebound. These cases occur when a

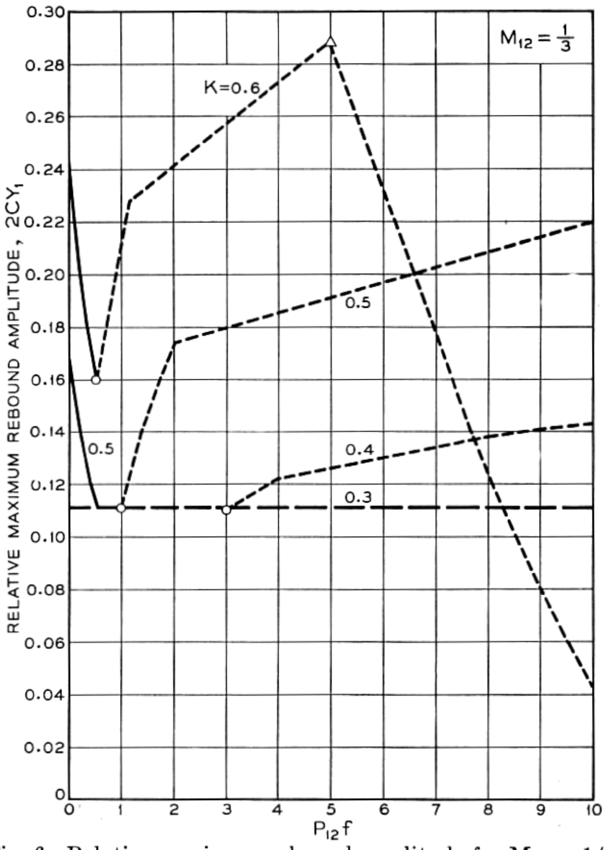

Fig. 6—Relative maximum rebound amplitude for $M_{12} = 1/3$.

heel impact immediately follows the last front impact of the series. These cases occur at

$$Q = \frac{1 - k^n}{k - k^n} \tag{30}$$

and lead to rebound amplitudes

$$2CY_1 = M_{12} + (1 - M_{12})[k^{2n} - M_{12}(1 - k^n)^2] \tag{31}$$

In Figs. 5 to 10, these special points are plotted and connected with straight dotted lines, which therefore indicate upper limits to rebound.

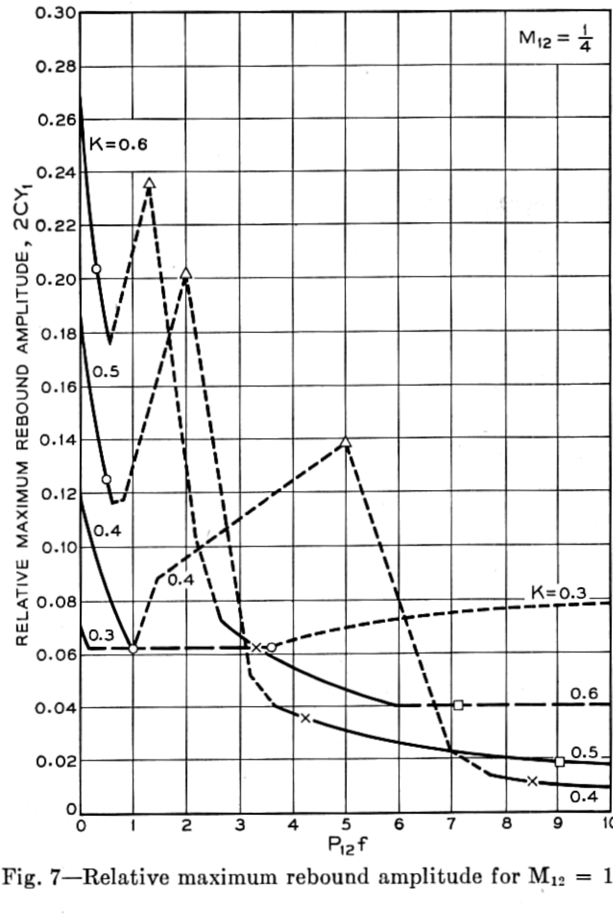

Fig. 7—Relative maximum rebound amplitude for $M_{12} = 1/4$.

Region III—Partial initial heel series for

$$\frac{1+k}{k} < Q < \frac{1+k}{k(1-k) + M_{12}k(1+k)}.$$

This is a region of critical phasing, and values were determined only for the maximum cases, where a front impact just precedes the last impact of the partial initial heel series. Here:

$$Q = \frac{1 - k^{n+1}}{\dfrac{k(1-k)}{1+k} + k(1-k^n)M_{12}} \tag{32}$$

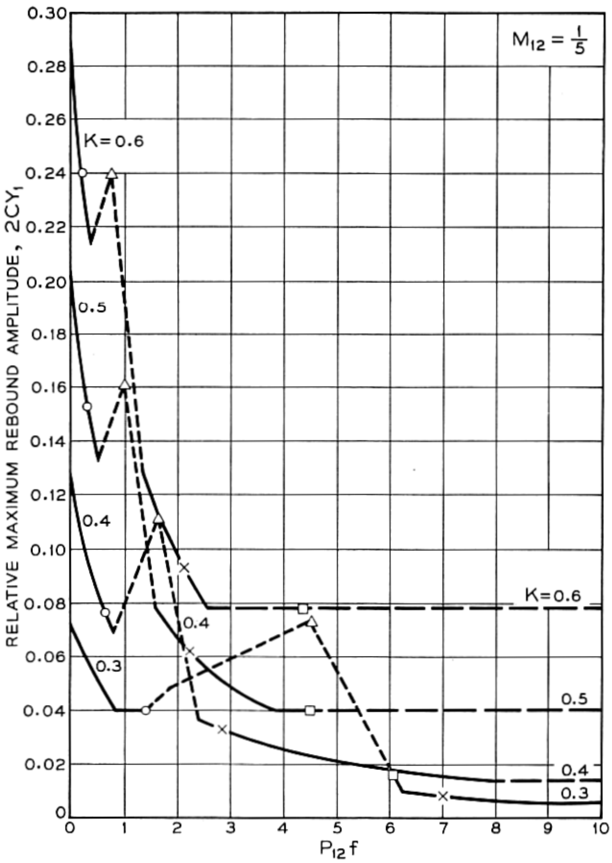

Fig. 8—Relative maximum rebound amplitude for $M_{12} = 1/5$.

and:

$$
\begin{aligned}
2CY_1 = 1 - (1 - M_{12})(1 - k^2)\{1 + [k - M_{12}(1 + k)(1 - k^n)]^2\} \\
- M_{12}(1 - M_{12})(1 + k)^2\{1 - k^{2n} \\
+ [k - k^n - M_{12}(1 + k)(1 - k^n)]^2\}
\end{aligned}
\tag{33}
$$

Region IV—Complete initial heel series. A complete initial heel series implies that when the heel has come to rest, the front is still away from the stop. When the front finally meets its stop, the situation is identical with that just prior to the zero impact except that the energy content of the system is lower. The pattern must then repeat with diminished amplitudes. For this region we recognize two groups of cases. The first

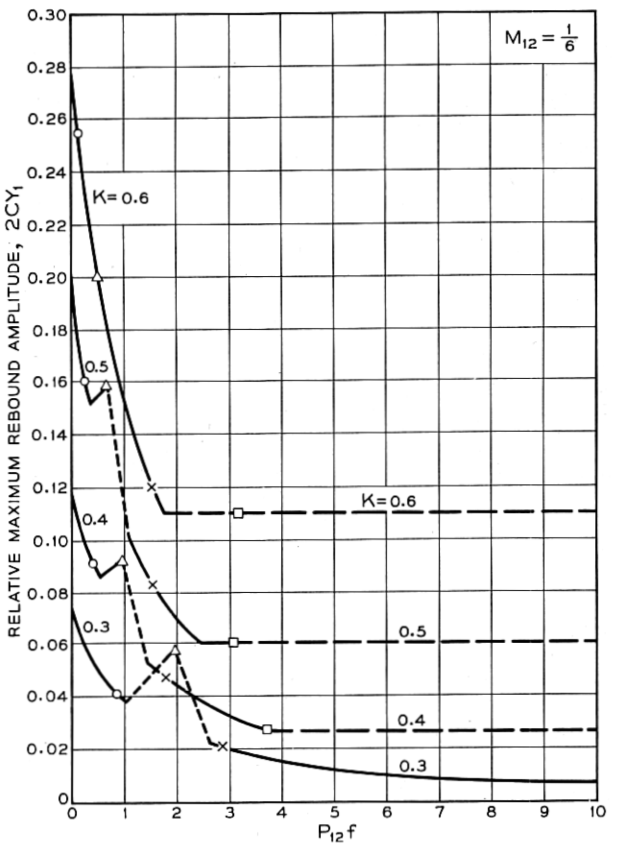

Fig. 9—Relative maximum rebound amplitude for $M_{12} = 1/6$.

group is that in which the front velocity is positive at the completion of the heel series. In that case

$$\frac{2(1+k)}{k(1-k)+M_{12}(1+k)^2} > Q > \frac{(1+k)}{k(1-k)+M_{12}k(1+k)^2}$$

and

$$2CY_1 = \frac{(1-M_{12})k^2}{1+P_{12}f} \tag{34}$$

For the second group the front velocity is still negative when the heel comes to rest from which point on the system acts as a one-degree-of-

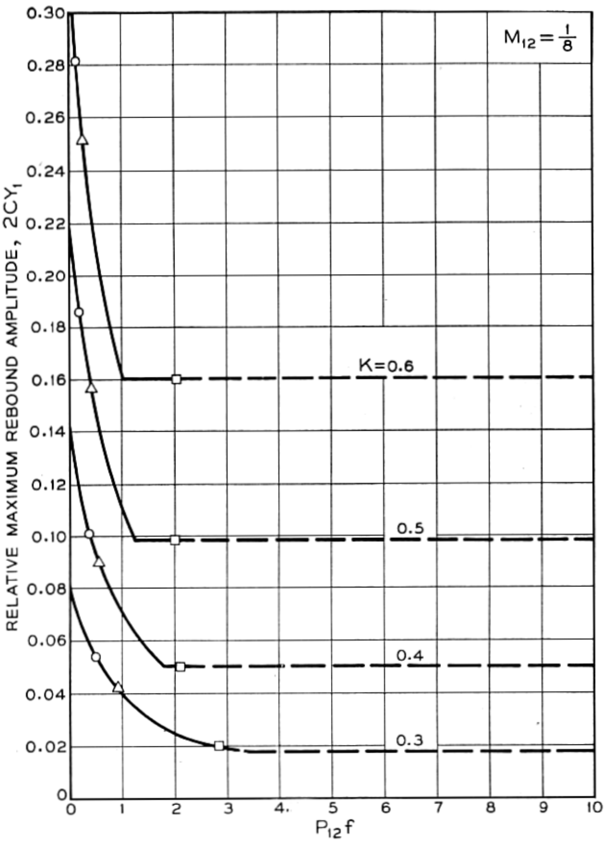

Fig. 10—Relative maximum rebound amplitude for $M_{12} = 1/8$.

freedom system until the next front impact. The requirement for this group is that

$$Q > \frac{2(1 + k)}{k(1 - k) + M_{12}(1 + k)^2}$$

and the maximum rebound is given by

$$2CY_1 = M_{12} - (1 - M_{12})[(1 - k^2) + M_{12}(1 + k)^2] \qquad (35)$$

It is to be noted that in the upper part of Group 1 the amplitude increases with successive heel impacts. This can be explored through the use of Equation (22). For simplicity of mapping, however, the limit given by Equation (35) has been extended back from the lower boundary of Group 2 until it intersects the line marking the first rebound amplitude of Group 1.

In Figs. 5 to 10 the respective regions have been identified by means of the symbols indicated below:

Region I	from $P_{12}f = 0$	to ○
Region II	from ○	to △
Region III	from △	to ×
Region IV, Group 1	from ×	to □
Region IV, Group 2	from □	to $P_{12}f \rightarrow \infty$

E. *Discussion of Rebound Charts*

Aside from quantitative data contained in Figs. 5 to 10, the following general trends are of interest:

For values of $M_{12} > \frac{1}{4}$, and the values of k under consideration, most of the useful range of $P_{12}f$ involves critical phasing and the rebound maxima are relatively high.

For values of $\frac{1}{6} \leq M_{12} \leq \frac{1}{4}$, consistently controllable rebound amplitude may be obtained.

For values of $M_{12} < \frac{1}{6}$ rebound increases again and the structure approaches the one-degree-of-freedom case.

VII. ANALYSIS OF REBOUND PATTERNS—THREE-DEGREES-OF-FREEDOM SYSTEM

Rebound pattern analysis as in Parts V and VI has so far not been performed for the three-degree-of-freedom system, partly because of complexity, and partly because for the system of Fig. 3 friction at the hinging stop will greatly influence the motion.

However, it is felt that the approach and notation of the analysis presented here is sufficiently general to allow extension of the rebound pattern analysis to the three-degree-of-freedom case. At any rate, with the assumption of the magnitudes of frictional forces, the basic equations of Part IV may be used to plot any particular case.

VIII. ARMATURE REBOUND MODEL

In order to verify the formal analysis presented in Parts III, IV and V, a large model of a two-degree-of-freedom system was constructed. It consisted essentially of a large bar constrained to move in a plane, biased against two stops, and to the ends of which writing pens were attached. As rebound conditions were simulated by releasing the bar against its stops, chart paper moved at right angles to the bar motion and thus produced a record of end displacement versus time.

By changing spring members and attaching masses to the bar, it was possible to vary the mass distribution and the biasing forces.

The results obtained closely agreed with those suggested by the analysis. The maximum rebound amplitudes were generally somewhat lower probably due to frictional effects.

IX. RELAY DESIGN CRITERIA RESULTING FROM ARMATURE REBOUND ANALYSIS

A. *Limitation of Analysis*

The assumptions which this analysis is subject to have been described under Part II. As applied to relays and probably the majority of mechanical structures, one assumption is most frequently and severely violated. The stops, which have been assumed to be stiff springs associated with a definite coefficient of restitution are, in practice, massive bodies which dissipate energy through excitation of high frequency modes of vibration. Accordingly, the assumption that the stops are at rest is violated, particularly if the mechanism is subject to repetitive (pulsing) impacts and the stop vibration does not decay greatly in the repetition period.

However, mechanisms designed in accordance with this analysis have performed well even under moderate pulsing conditions if the sensitive phasing region was avoided. In addition, every effort should be made to reduce the amount and duration of stop and mounting structure vibration by making them stiff, massive, and dissipative, if possible.

B. *Design Criteria*

1. Type of Armature Structure.

The selection of the number of degrees-of-freedom for an armature structure depends on the expected coefficient of rebound as well as practical considerations.

It can be shown without great difficulty that for very low coefficients of rebound the one-degree-of-freedom system is preferable. This is quite obvious when one considers the limiting value of $k = 0$. In this case the one-degree-of-freedom system will have no rebound whatsoever, while the two-degree-of-freedom system has a heel bounce followed by rebound at the front. The value of k below which the one degree system is preferable varies with the mass distribution relative to the stop points, being 0.18 for a rectangular plate armature with stops located at its edges.

Experience indicates that k in most practical relays and similar mechanical structures varies from 0.3 to 0.6. Hence the two-degree-of-freedom system is superior in all practical cases to the solidly hinged armature.

As far as three and higher degree-of-freedom systems are concerned, it may be said that generally the greater the number of modes resulting in impacts, the quicker the rebound energy can be diverted and dissipated and the lower theoretical rebound values can be obtained. This consideration would favor systems containing many degrees of freedom. However, while multi-degree-of-freedom systems can reach very low rebound values, their motion (phasing) must be very closely controlled or they may prove to be inferior to simpler systems particularly under vibratory (pulsing) operation. It is this difficulty which makes it appear that the two-degree-of-freedom system offers the best promise with the three-degree system also quite promising. By the same reasoning, additional spurious rocking modes should be minimized.

2. Armature Mass.

The armature mass should be as low as possible. This will minimize stop and structure vibration. In addition, in relay applications light armatures tend to increase magnetic "drag" losses of the armature during the release motion.

3. Stops and Mounting Structure.

As discussed before, it is desirable to reduce the amount and duration of stop and mounting structure vibration.

4. Coefficient of Restitution.

The coefficient of restitution should be kept low. Stops having low stiffness should, therefore, be avoided.

5. Biasing Forces.

F_1 should be kept as high as practicable.

For proper energy loss during impacts, the motion between impacts must occur outside the region of the compression, i.e., the armature and stop must separate. Therefore, because all practical stops have a finite stiffness, the biasing forces (F_1, F_2, etc.) should produce a static deflection less than say, arbitrarily, 5 per cent of the maximum expected rebound amplitude.

6. Design Parameters for Two-Degree-of-Freedom Systems.

As clearly indicated in Figs. 5 to 10 for the practical range of coefficients of restitution, most consistently good results are obtained with a coupling factor $M_{12} = \frac{1}{16}$ to $\frac{1}{4}$. This factor is most easily adjusted by correct placement of the front stop.

For the above range of M_{12} the force ratio F_2/F_1 should be such as to make the product

$$
\begin{aligned}
P_{12}\frac{F_2}{F_1} &> 4 \qquad M_{12} = \tfrac{1}{4} \\
&> 3 \qquad M_{12} = \tfrac{1}{5} \\
&> 3 \qquad M_{12} = \tfrac{1}{6}
\end{aligned}
$$

(Note: For a rectangular armature structure with the stops placed at its edges $M_{12} = \frac{1}{4}$, $P_{12} = \frac{1}{2}$ and force ratios in the neighborhood of 8 are desirable.)

X. ACKNOWLEDGMENT

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Appendix I

DERIVATION OF BASIC EQUATIONS OF MOTION THREE-DEGREE-OF-FREEDOM SYSTEM

(1) *Free Interval*

The motion of the armature will be described by the displacement at the stop points, x_1, x_2, x_3. Let m be the mass and R the radius of gyration of the armature about the center of gravity. The latter is located by the dimensions $\ell_1 R$, $\ell_2 R$, and $\ell_3 R$ relative to the stop points (Fig. 3).

The rotation and displacement of the center of gravity is then

$$\left.\begin{aligned} x_h &= (x_2 - x_1)\frac{\ell_3}{\ell_1 + \ell_2} + x_3 \\ x_v &= x_1 + (x_2 - x_1)\frac{\ell_1}{\ell_1 + \ell_2} \\ \theta &= \frac{x_2 - x_1}{R(\ell_1 + \ell_2)} \end{aligned}\right\} \quad \text{(a)}$$

From this the kinetic energy may be computed

$$\left.\begin{aligned} T &= \tfrac{1}{2}m(\dot{x}_h^2 + \dot{x}_v^2) + \tfrac{1}{2}mR^2\dot{\theta}^2 \\ &= \frac{\dot{x}_1^2(1 + \ell_2^2 + \ell_3^2) + \dot{x}_2^2(\ell_1^2 + \ell_3^2 + 1) + \dot{x}_3^2(\ell_1 + \ell_2)^2}{2(\ell_1 + l_2)^2} \\ &+ \frac{2\dot{x}_1\dot{x}_2(\ell_1\ell_2 - \ell_3^2 - 1) - 2\dot{x}_3\ell_3(\ell_1 + \ell_2)(\dot{x}_1 - \dot{x}_2)}{2(\ell_1 + \ell_2)^2} \end{aligned}\right\} \quad \text{(b)}$$

Applying LaGrange's Equation to the above, the equations of motion are obtained:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = P_r$$

$$\left.\begin{aligned} \frac{F_1}{m} &= \frac{\ddot{x}_1[\ell_2^2 + \ell_3^2 + 1] + \ddot{x}_2[\ell_1\ell_2 - \ell_3^2 - 1] - \ddot{x}_3\ell_3(\ell_1 + \ell_2)}{(\ell_1 + \ell_2)^2} \\ \frac{F_2}{m} &= \frac{\ddot{x}_1[\ell_1\ell_2 - \ell_3^2 - 1] + \ddot{x}_2[\ell_1^2 + \ell_3^2 + 1] + \ddot{x}_3\ell_3(\ell_1 + \ell_2)}{(\ell_1 + \ell_2)^2} \\ \frac{F_3}{m} &= \frac{-\ddot{x}_1\ell_3(\ell_1 + \ell_2) + \ddot{x}_2\ell_3(\ell_1 + \ell_2) + \ddot{x}_3(\ell_1 + \ell_2)^2}{(\ell_1 + \ell_2)^2} \end{aligned}\right\} \quad \text{(c)}$$

The Equations (3) may be solved for $\ddot{x}_1$, $\ddot{x}_2$, $\ddot{x}_3$ and the results integrated, yielding

$$\left.\begin{aligned} x_1 &= \frac{1}{2m}[C_{11}F_1 + C_{12}F_2 + C_{13}F_3]t^2 + \dot{x}_{10}t + x_{10} \\ x_2 &= \frac{1}{2m}[C_{21}F_1 + C_{22}F_2 + C_{23}F_3]t^2 + \dot{x}_{20}t + x_{20} \\ x_3 &= \frac{1}{2m}[C_{31}F_1 + C_{32}F_2 + C_{33}F_3]t^2 + \dot{x}_{30}t + x_{30} \end{aligned}\right\} \quad \text{(d)}$$

where

$$\left.\begin{array}{ll} C_{11} = (\ell_1^2 + 1) & C_{13} = C_{31} = \ell_1\ell_3 \\ C_{22} = (\ell_2^2 + 1) & C_{12} = C_{21} = (1 - \ell_1\ell_2) \\ C_{33} = (\ell_3^2 + 1) & C_{23} = C_{32} = -\ell_2\ell_3 \end{array}\right\} \quad (3)$$

$\dot{x}_{10}$, $\dot{x}_{20}$, $\dot{x}_{30}$ are the initial velocities, x_{10}, x_{20}, x_{30} the initial displacements for the free interval in question. Interpretation of the analytic results is simplified by the introduction of normalization. Let $\dot{x}_a$ be $\dot{x}_1$ just before the "zero" impact and define

$$\left.\begin{array}{ll} y_i = \dfrac{x_i}{\dot{x}_a\tau} = \dfrac{F_1}{\dot{x}_a^2 m} x_i & \dot{y}_i = \dfrac{d}{d\left(\dfrac{t}{\tau}\right)} y_i = \dfrac{\dot{x}_i}{\dot{x}_a} \\ \tau = \dfrac{\dot{x}_a m}{F_1} & \end{array}\right\} \quad (2)$$

Dividing Equations (d) by $\dot{x}_a\tau$ yields the normalized equations of motion:

$$\left.\begin{array}{l} y_1 = \dfrac{1}{2}\left[C_{11} + C_{12}\dfrac{(F_2)}{(F_1)} + C_{13}\dfrac{(F_3)}{(F_1)}\right]\left(\dfrac{t}{\tau}\right)^2 + \dot{y}_{10}\left(\dfrac{t}{\tau}\right) + y_{10} \\ y_2 = \dfrac{1}{2}\left[C_{21} + C_{22}\dfrac{(F_2)}{(F_1)} + C_{23}\dfrac{(F_3)}{(F_1)}\right]\left(\dfrac{t}{\tau}\right)^2 + \dot{y}_{20}\left(\dfrac{t}{\tau}\right) + y_{20} \\ y_3 = \dfrac{1}{2}\left[C_{31} + C_{32}\dfrac{(F_2)}{(F_1)} + C_{33}\dfrac{(F_3)}{(F_1)}\right]\left(\dfrac{t}{\tau}\right)^2 + \dot{y}_{30}\left(\dfrac{t}{\tau}\right) + y_{30} \end{array}\right\} \quad (1)$$

(2) *Impact Interval*

The change of velocity at point "i" due to an impact at "i" is, by definition of the coefficient of restitution "k":

$$\Delta\dot{x}_i = -(1 + k_i)\dot{x}_i \quad (e)$$

Since this velocity change occurs as rotation about the conjugate point as an instant center of rotation, the impact relationships may be written, for an impact at point "1",

$$\Delta\dot{x}_1 = -(1 + k_1)\dot{x}_1$$

$$\Delta\dot{x}_2 = -(1 + k_1)\dot{x}_1 \frac{\left(\dfrac{R}{\ell_1} - \ell_2 R\right)}{\left(\ell_1 R + \dfrac{R}{\ell_1}\right)}$$

$$= (1 + k_1)\dot{x}_1 \frac{(\ell_1\ell_2 - 1)}{(\ell_1^2 + 1)} = -\frac{C_{12}}{C_{11}}(1 + k_1)\dot{x}_1$$

$$\Delta\dot{x}_3 = -(1 - k_1)\dot{x}_1 \frac{\ell_3 R}{\ell_1 R + \frac{R}{\ell_1}}$$

$$= -(1 + k_1)\dot{x}_1 \frac{\ell_1\ell_3}{\ell_1^2 + 1)} = -\frac{C_{13}}{C_{11}}(1 + k_1)\dot{x}_1$$

Similarly it can be shown that impacts at points (2) and (3) follow the same pattern. The general impact relations for impact at point "i" are then

$$\dot{y}_{j0n} = \dot{y}_{je(n-1)} + K_{ji}\dot{y}_{ie(n-1)} \tag{6}$$

The first subscript indicates the coordinate, and the second subscript indicates the beginning (0) or end (e) of the free interval denoted by the third subscript.

The impact transfer coefficient K_{ji} relating a velocity change at point "j" to an impact at point "i":

$$K_{ji} = -\frac{C_{ji}}{C_{ii}}(1 + k_i) \tag{7}$$

Appendix II

ANALYSIS OF REBOUND PATTERNS—ONE-DEGREE-OF-FREEDOM SYSTEM

The equation of motion of this system is

$$y_{1n} = \tfrac{1}{2}Ct'^2 + \dot{y}_{10n}t' + y_{10n} \tag{f}$$

where

$$C = C_{11} - \frac{C_{12}^2}{C_{22}} \tag{9}$$

$$t' = \frac{t}{\tau}$$

and is measured from the start of the particular interval of free motion in question. The impact relationship is

$$\dot{y}_{10n} = -k_1\dot{y}_{1e(n-1)}$$

The motion consists of a series of parabolic arcs having periods of $2\dot{y}_{10}/C$ in general, or $2/C$, $2k/C$, $2k^2/C$, $\cdots$, $2k^{n-1}/C$. The time elapsed is a convergent series and approaches, for a complete series:

$$\lim_{n\to\infty} \frac{2}{C}[1 + k + k^2 + \cdots k^n] = \frac{2}{C(1-k)} \tag{10}$$

The maximum rebound amplitude in any interval is $-\dot{y}_{10n}^2/2C$. The maximum excursion occurs during the first bounce at $t' = 1/C$ and equals $-k^2/2C$.

Appendix III

ANALYSIS OF REBOUND PATTERNS—TWO-DEGREE-OF-FREEDOM SYSTEM

The equations of motion of this system are

$$\left.\begin{aligned} y_1 &= \tfrac{1}{2}At'^2 + \dot{y}_{10n}t' + y_{10n} \\ y_2 &= \tfrac{1}{2}Bt'^2 + \dot{y}_{20n}t' + y_{20n} \end{aligned}\right\} \tag{g}$$

$$\left.\begin{aligned} \text{where } A &= C_{11} + C_{12}f \\ B &= C_{12} + C_{22}f \\ f &= \frac{F_2}{F_1} \\ t' &= \frac{t}{\tau} \text{ measured from the start of the particular free interval in question.} \end{aligned}\right\} \tag{h}$$

A. Complete Front Series

At the beginning of a front series

$$\left.\begin{aligned} y_1 &= 0 \\ \dot{y}_1 &= \dot{y}_{1e0} \\ y_2 &= y_{2e0} \\ \dot{y}_2 &= \dot{y}_{2e0} \end{aligned}\right\} \tag{i}$$

In a manner analogous to that for the one-degree-of-freedom system each front impact reduces $\dot{y}_1$ to $-k_1\dot{y}_1$. Therefore, after the n^{th} impact,

$$\dot{y}_{10n} = -k_1^n\dot{y}_{1e0}$$

and the time elapsed in the n^{th} interval is

$$T_n = \frac{2k_1^n}{A}\dot{y}_{1e0} \tag{j}$$

At the heel, from (g), the heel velocity preceding the n^{th} impact is

$$\dot{y}_{2e(n-1)} = \dot{y}_{20(n-1)} + Bt' \tag{k}$$

The velocity change during the $(n - 1)$ interval is then equal to BT_{n-1}. From Equations (6), (7) and (12), the change in velocity during the n^{th} impact is $-P_{12}(1 + k_1)k_1^{n-1}\dot{y}_{1e0}$.

The total change of $\dot{y}_2$ between impacts is then

$$\dot{y}_{20n} - \dot{y}_{20(n-1)} = BT_{n-1} - P_{12}(1 + k_1)k_1^{n-1}\dot{y}_{1e0}$$

Similarly in preceding intervals:

$$\begin{aligned}
\dot{y}_{20(n-1)} - \dot{y}_{20(n-2)} &= BT_{n-2} - P_{12}(1 + k_1)k_1^{n-2}\dot{y}_{1e0} \\
&\vdots \\
\dot{y}_{202} - \dot{y}_{201} &= BT_1 - P_{12}(1 + k_1)k_1\dot{y}_{e0} \\
\dot{y}_{201} - \dot{y}_{2e0} &= \quad - P_{12}(1 + k_1)\dot{y}_{1e0}
\end{aligned}$$

By addition of the above

$$\begin{aligned}
\dot{y}_{20n} - \dot{y}_{2e0} &= B\sum_{m=1}^{n-1} T_m - P_{12}(1 + k_1)\sum_{m=0}^{n-1} k_1^m\dot{y}_{1e0} \\
&= \frac{2B}{A}\dot{y}_{1e0}\sum_{m=1}^{n-1} k_1^m - P_{12}(1 + k_1)\dot{y}_{1e0}\sum_{m=0}^{n-1} k^m
\end{aligned}$$

The summations may be evaluated, yielding

$$\dot{y}_{20n} - \dot{y}_{2e0} = \left[\frac{2B}{A}\,\frac{k_1 - k_1^n}{1 - k_1} - P_{12}(1 + k_1)\,\frac{1 - k_1^n}{1 - k_1}\right]\dot{y}_{1e0} \tag{l}$$

To evaluate the displacements at the heel, Equation (g) yields

$$y_{20n} - y_{20(n-1)} = \dot{y}_{20(n-1)}T_{n-1} + \tfrac{1}{2}BT_{n-1}^2$$

Adding these expressions for intervals 0 to n; the total change in y_2 is

$$\begin{aligned}
y_{20n} - y_{201} &= \sum_{m=1}^{n-1} \dot{y}_{20m}T_m + \tfrac{1}{2}B\sum_{m=1}^{n-1} T_m^2 \\
&= \frac{2k_1(1 - k_1^{n-1})}{A(1 - k_1)}\dot{y}_{1e0}\dot{y}_{2e0} \\
&\quad + \left[\frac{2B(k_1^2 - 2k_1^{n+1} + k_1^{2n})}{A^2(1 - k_1)^2}\right. \\
&\quad \left. - \frac{2P_{12}k_1(1 - k_1^n - k_1^{n-1} + k_1^{2n-1})}{A(1 - k_1)^2}\right]\dot{y}_{1e0}^2
\end{aligned} \tag{m}$$

Expressions for an initial series may be obtained by setting $\dot{y}_{1e0} = 1$ $\dot{y}_{2e0} = y_{2e0} = 0$, and, finally, for an initial complete series $m \to \infty$ and hence $k^m \to 0$, and Equations (l) and (m) become

$$\left.\begin{aligned} y_{2e\infty} &= \frac{2k_1}{A(1+k_1)^2}\left[\frac{Bk_1}{A} - P_{12}\right] \\ \dot{y}_{2e\infty} &= \frac{1}{1-k_1}\left[\frac{2Bk_1}{A} - P_{12}(1+k_1)\right] \end{aligned}\right\} \quad (17)$$

B. Complete Heel Series

For heel series, Equations (l) and (m) may be used by interchanging the initial velocities, accelerations, and impact transfer coefficients for those relating to heel motion:

$$\dot{y}_{10n} - \dot{y}_{1e0} = \left[\frac{2A}{B}\,\frac{k_2 - k_2^n}{1-k_2} - \frac{M_{12}(1+k_2)}{P_{12}}\,\frac{1-k_2^n}{1-k_2}\right]\dot{y}_{2e0} \quad (n)$$

$$\begin{aligned} y_{10n} - y_{101} &= \frac{2k_2(1-k_2^{n-1})}{B(1-k_2)}\,\dot{y}_{1e0}\dot{y}_{2e0} \\ &+ \left[\frac{2A(k_2^2 - 2k_2^{n+1} + k_2^{2n})}{B^2(1-k_2^2)} - \frac{2M_{12}k_2(1 - k_2^n - k_2^{n-1} + k_2^{2n-1})}{BP_{12}(1-k_2^2)}\right]\dot{y}_{2e0}^2 \end{aligned} \quad (o)$$

An initial heel series occurs when the heel strikes first after the "zero" impact. The first heel impact then occurs $T_1 = 2P_{12}/B(1 + k_1)$ after the zero impact and the initial conditions are

$$\dot{y}_{2e1} = P_{12}(1 + k_1)$$

$$\dot{y}_{1e1} = -k_1 + AT_1 = \frac{2AP_{12}}{B}(1 + k_1) - k_1$$

$$y_{1e1} = -k_1T_1 + \tfrac{1}{2}AT_1^2 = \frac{2P_{12}}{B}(1 + k_1)\left[\frac{AP_{12}}{B}(1 + k_1) - k_1\right]$$

Substitution of the above into (n) yields

$$\dot{y}_{10n} = -k_1 + \frac{1+k_1}{1-k_2}\left[\frac{2AP_{12}}{B}(1-k_2)^n - M_{12}(1+k_2)(1-k_2^n)\right] \quad (p)$$

The corresponding expression for y_{10n} is quite involved. For the special case of $k = k_1 = k_2$

$$y_{10n} = \frac{AP_{12}(1+k)^2}{B}\left[\frac{AP_{12}}{B}\left(\frac{1-k^n}{1-k}\right)^2 = \frac{k(1-k^n)}{1-k^2} - \frac{M_{12}(1-k^n)(k-k^n)}{(1-k)^2}\right] \tag{q}$$

If the initial series is a complete series, $n \to \infty$ and

$$\left.\begin{aligned} y_{10\infty} &= \frac{AP_{12}(1+k)^2}{B(1-k)^2}\left[\frac{AP_{12}}{B} - \frac{k(1-k)}{1+k} - M_{12}k\right] \\ \dot{y}_{1e\infty} &= \frac{1+k}{1-k}\left[\frac{2AP_{12}}{B} - \frac{k(1-k)}{(1+k)} - M_{12}(1+k)\right] \end{aligned}\right\} \tag{22}$$

C. Partial Front Series

The worst rebound occurs when heel and front impacts occur nearly simultaneously, with the front hitting first. From Equation (m) for an initial front series, this requires that

$$\frac{B}{AP_{12}} = Q = \frac{1-k^n}{k-k^n} \tag{30}$$

After n front impacts conditions are given by Equations (14) and (19) with $y_1 = y_2 = 0$, and

$$\frac{T+V}{T_0} = 1 - (1-M_{12})(1-k^{2n}) = k^{2n} + \frac{M_{12}\dot{y}_2^2}{P_{12}^2} - \frac{2M_{12}k^n\dot{y}_2}{P_{12}}$$

This may be solved for $\dot{y}_2 = P_{12}(1-k^n)$. The maximum front excursion now possible is that for a complete series of heel impacts. The above value of $\dot{y}_2$ in Equation (24) yields

$$2CY_1 = M_{12} + (1-M_{12})[k^{2n} - M_{12}(1-k^n)^2] \tag{31}$$

D. Partial Heel Series

The worst rebound occurs again when heel and front impacts occur nearly simultaneously, with the front hitting first. From Equation (9) for an initial heel series, this requires that

$$\frac{B}{AP_{12}} = Q = \frac{1-k^{n+1}}{\dfrac{k(1-k)}{1+k} + k(1-k^n)M_{12}} \tag{32}$$

After n heel impacts $\dot{y}_2 = P_{12}(1 + k)k^n$ and from Equations (19) and (23)

$$\frac{T + V}{T_0} = 1 - (1 - M_{12})(1 - k^2) - M_{12}(1 - M_{12})(1 + k)^2(1 - k^{2n})$$
$$= \dot{y}_1^2 - 2M_{12}(1 + k)k^n\dot{y}_1 + M_{12}(1 + k)^2k^{2n}$$

This may be solved for $\dot{y}_1 = k - M_{12}(1 + k)(1 - k^n)$ and after the front impact immediately following:

$$\dot{y}_2 = P_{12}(1 + k)k^n - P_{12}(1 + k)[k - M_{12}(1 + k)(1 - k^n)]$$

The maximum front excursion now possible is that for a complete series of heel impacts. The above value for $\dot{y}_2$ in Equation (24) yields

$$\begin{aligned} 2CY_1 = 1 &- (1 - M_{12})(1 - k^2)\{1 + [k - M_{12}(1 + k)(1 - k^n)]^2\} \\ &- M_{12}(1 - M_{12})(1 + k)^2\,\{1 - k^{2n} \\ &+ [k - k^n - M_{12}(1 + k)(1 - k^n)]^2\} \end{aligned} \tag{33}$$

Appendix IV

SUMMARY OF NOTATION

$A = C_{11} + C_{12} + C_{12}f$

$B = C_{12} = C_{22} + C_{22}f$

$C = C_{11} - \dfrac{C_{12}^2}{C_{22}}$

$C_{11} = 1 + \ell_1^2 \qquad C_{12} = C_{21} = [1 - \ell_1\ell_2]$

$C_{22} = 1 + \ell_2^2 \qquad C_{13} = C_{31} = \ell_1\ell_3$

$C_{33} = 1 + \ell_3^2 \qquad C_{23} = C_{32} - \ell_2\ell_2$

$f = \dfrac{F_2}{F_1}$

F_1 = front tensioning force

F_2 = heel tensioning force

k_1 = coefficient of restitution at vertical front stop

k_2 = coefficient of restitution at vertical heel stop

$K_{ji} = -\dfrac{C_{ji}}{C_{ii}}(1 + k_i)$

$\ell_1 R$ = vertical front stop location relative to c.g.

$\ell_2 R$ = vertical heel stop location relative to c.g.

$\ell_3 R$ = horizontal heel stop location relative to c.g.

m = mass of armature

$M_{ij} = \dfrac{C_{ij}^2}{C_{ii}C_{ij}}$

$P_{ij} = \dfrac{C_{ij}}{C_{ii}}$

$Q = \dfrac{1 + \dfrac{P_{12}}{M_{12}f}}{1 + P_{12}f} = \dfrac{t_1}{t_2}\dfrac{1 + k_1}{k_1}$

R = radius of gyration of armature about center of gravity

$\tau = \dfrac{\dot{x}_a m}{F_1}$

t = time

$t' = \dfrac{t}{\tau}$

t_1 = basic period of front after "zero" impact

t_2 = basic period of heel after "zero" impact

T_n = duration of n^{th} free interval

x_1 = vertical front displacement

x_2 = vertical heel displacement

x_3 = horizontal heel displacement

$\dot{x}_a$ = front velocity just prior to "zero" impact

$y = \dfrac{x}{\dot{x}_a \tau}$

Y_1 = greatest excursion (rebound) of front