Wave Propagation Along a Magnetically-Focused Cylindrical Electron Beam

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(Manuscript received August 24, 1953)

This paper analyses the nature of wave propagation along a cylindrical electron beam, focused in Brillouin flow by means of a finite axial magnetic field. Two different types of conducting boundaries external to the beam are treated: (1) the concentric cylindrical tube, forming a drift region; and (2) the sheath helix, forming a model of the helix traveling-wave tube. The field solution of the helix problem is used to evaluate the normal-mode parameters of an equivalent circuit seen by a thin beam, thereby permitting computation of the gain constant of growing waves. The gain constant of the cylindrical beam with Brillouin flow is found to exceed that of a similar beam with rectilinear flow, presumably because of the transverse component of electron motion in the former.

INTRODUCTION

The theory of the helix traveling-wave has been treated in previous papers,[1–4] for cases in which the electrons move along straight lines parallel to the axis of the helix, as though immersed in an infinitely strong magnetic field. In practice, however, the electron beam is focused by a magnetic field of finite intensity,[5, 6] such that the electrons follow spiral paths about the common axis. The purpose of this paper is to extend traveling-wave tube theory to the case of such focused beams, and to compare the gain constants for the two types of electron motion. The motion of the beam in an infinite field is usually described as rectilinear flow; that in a finite focusing field, as Brillouin flow.

The gain constant of the dominant mode in a traveling-wave tube may be computed from the field solution for the electron beam in the presence of its circuit structure. This procedure, however, requires the solution of cumbersome transcendental equations for each particular set of dimensions and operating conditions. A more flexible method of analysis has been provided by Pierce,[1] based on an expansion in terms of normal modes of propagation. For any particular *type* of beam and circuit, three circuit parameters must be evaluated from the field solution. The performance of the traveling-wave tube is then described quite accurately by a cubic equation containing these parameters, over a wide range of dimensions and operating conditions. The usefulness of this normal-mode method has been further enhanced by publication of a nomograph[7] for the calculation of the gain constant.

In its initial form, the normal-modes solution for a helix traveling-wave tube was greatly simplified by the assumption that the electron beam is so thin that the electric field acting on it is constant. Employing the field solution for a beam of finite thickness in a helix, Fletcher[4] was able to compute the circuit parameters for the solid and hollow cylindrical electron beams, respectively, confined to rectilinear flow.

This procedure will now be extended to cylindrical beams in Brillouin flow, in which transverse electron motion occurs. First, it will be necessary to solve the field equations for this type of beam in a helix. As a by-product of this computation, the solution of the field equations for the beam in a concentric drift tube will briefly be given. Finally, with some restrictions, the helix parameters will be evaluated, and the gain of helix amplifiers with such beams compared with that obtained with otherwise identical rectilinear beams.

FIELD EQUATIONS IN THE ELECTRON BEAM

When a small ac field is impressed upon a short length of electron beam, the electrons respond by executing small ac excursions about their steady-state trajectories. These ac motions of charged particles constitute a transverse distribution of ac currents, which in turn excites an ac field distribution. The propagation of an ac signal along a beam depends upon the reciprocal action of these currents and fields.

To find the propagation constants for a particular configuration of electron stream and enclosure, we must therefore solve Maxwell's equations in the presence of the ac driving currents in the beam, subject to the external boundary conditions. When the fields and currents possess circular symmetry, these equations may be formally separated into TE and TM groups.[2] In addition, as we are concerned only with "slow" waves, the equations may be simplified by neglecting all terms of relative magnitude k^2/γ^2, where k is the wave number in free space, and γ the propagation wave number.

TM WAVE

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) - \gamma^2 E_z = \frac{\gamma^2}{j\omega\epsilon} J_z + \frac{\gamma}{\omega\epsilon}\frac{1}{r}\frac{\partial}{\partial r}(rJ_r) \qquad (1)$$

$$E_r = \frac{j}{\gamma}\frac{\partial E_z}{\partial r} - \frac{j\omega\mu}{\gamma^2} J_r \tag{2}$$

$$H_\theta = \frac{\omega\epsilon}{\gamma} E_r - \frac{j}{\gamma} J_r \tag{3}$$

TE WAVE

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right) - \gamma^2 H_z = -\frac{1}{r}\frac{\partial}{\partial r}(rJ_\theta) \tag{4}$$

$$H_r = \frac{j}{\gamma}\left(\frac{\partial H_z}{\partial r} + J_\theta\right) \tag{5}$$

$$E_\theta = -\frac{\omega\mu}{\gamma} H_r = -\frac{j\omega\mu}{\gamma^2}\left(\frac{\partial H_z}{\partial r} + J_\theta\right) \tag{6}$$

Here (r, θ, z) are the polar cylindrical coordinates, ω the angular driving frequency, ϵ the dielectric constant and μ the permeability, of free space, in MKS units. The ac amplitudes of the electric and magnetic fields, and the convection-current density, respectively, are represented by the components of $\underline{E}$, $\underline{H}$, and $\underline{J}$. All ac quantities have been assumed to vary as $\exp j(\omega t\text{-}\gamma z)$.

When the assumption is made that the convection current density in the beam is of the same order of magnitude as the displacement current density, equations (2) and (6) reduce to the following:

$$E_r = \frac{j}{\gamma}\frac{\partial E_z}{\partial r} \tag{7}$$

$$E_\theta = -\frac{j\omega\mu}{\gamma^2}\frac{\partial H_z}{\partial r} \tag{8}$$

In order to evaluate the components of $\underline{J}$ in the beam, it is necessary to determine the velocity and charge distributions, first in the unmodulated, and then in the ac modulated beam.

The focusing of long cylindrical electron beams by axial magnetic fields of moderate strength has been fully described by Brillouin[5] and Samuel[6]. This type of electron motion, called "Brillouin flow", can be established when a parallel electron beam abruptly enters a suitable magnetic field. The electrons thereupon acquire an angular velocity component which leads to a balance of radial forces in the beam.

The equations of motion of electrons in an axial magnetic field B_0 are as follows:

$$\ddot{r} - r\dot{\theta}^2 = \eta(\partial V_0/\partial r - r\dot{\theta}B_0) \tag{9}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \eta\dot{r}B_0 \tag{10}$$

$$\ddot{z} = \eta\cdot\partial V_0/\partial z \tag{11}$$

In these equations (r, θ, z) is the position of an electron at time t; dots indicate differentiation with respect to t, following the electrons; $\eta = e/m$, where $-e$ is the electronic charge and m its mass; and V_0 is the potential describing the steady, axially symmetric electric field. Relativistic effects and the magnetic field resulting from electron motion have been neglected, as our interest is confined to beam velocities which are small compared to that of light.

It is readily verified that a solution of the above equation is:

$$\dot{r} = 0, \qquad \dot{\theta} = \dot{\theta}_0 = \eta B_0/2, \qquad \dot{z} = u_0 \tag{12}$$

$$\eta\, \partial V_0/\partial r = r\dot{\theta}_0^{\,2}, \qquad \partial V_0/\partial z = 0 \tag{13}$$

Thus all the particles in the beam have the same angular velocity, equal to the Larmor angular frequency, and the same axial velocity u_0. From Poisson's equation, we find the charge density:

$$\rho_0 = -\,2\epsilon\dot{\theta}_0^{\,2}/\eta \tag{14}$$

It is convenient to introduce the angular plasma frequency ω_p, defined by:

$$\omega_p^{\,2} = -\eta\rho_0/\epsilon = 2\dot{\theta}_0^{\,2} \tag{15}$$

In steady-state flow, an electron with initial position (r_0, θ_0, z_0) has the position $(r_0, \theta_0 + \dot{\theta}_0 t, z_0 + u_0 t)$ at time t. When the beam is modulated by a small ac signal, the electrons suffer small ac displacements from their steady-state trajectories. If we assume that the signal propagates along the axis of the beam as $\exp j(\omega t - \gamma z)$, we can write the perturbed electron coordinates in terms of the Lagrangian coordinates (r_0, θ_0, z_0) as follows:

$$r = r_0 + \tilde{r}(r_0)\cdot \exp j[\omega t - \gamma(z_0 + u_0 t)] \tag{16}$$

$$\theta = \theta_0 + \dot{\theta}_0 t + \tilde{\theta}(r_0)\cdot \exp j[\omega t - \gamma(z_0 + u_0 t)] \tag{17}$$

$$z = z_0 + u_0 t + \tilde{z}(r_0)\cdot \exp j[\omega t - \gamma(z_0 + u_0 t)] \tag{18}$$

where the tildes indicate ac amplitudes, and the dots indicate, as before, time differentiation at fixed r_0, θ_0, z_0. Thus the dots are equivalent to multiplication by $j(\omega - \gamma u_0)$, when applied to ac quantities.

The equations of motion for the ac modulated beam differ from the steady-state equations (9) — (11), in that the particle coordinates are now given by (16) — (18), and there are ac fields present in addition to the dc fields $-\partial V_0/\partial r$ and B_0. As is usual in small-signal theory, only first-order ac quantities are retained in any equation. To this approximation, the ac fields can be evaluated at the unperturbed particle position.

Not all of the ac fields need to appear in the force equations, however. Reference to the field equations shows that the contributions of the ac magnetic fields to the force components are smaller than those due to the electric fields by a factor of the order of $(u_0/c)^2$ or smaller (where c is the velocity of light), and hence may be neglected. In addition, the force exerted by E_θ is of the same order as that due to H_r, and may be neglected too.

Omitting the factor $\exp j[\omega t - \gamma(z_0 + u_0 t)]$ for brevity from all ac terms, we can write the equations of motion as follows:

$$\ddot{\tilde{r}} - (r_0 + \tilde{r})(\dot{\theta}_0 + \dot{\tilde{\theta}})^2 = -\eta[-\partial V_0/\partial r + E_r + (r_0 + \tilde{r})(\dot{\theta}_0 + \dot{\tilde{\theta}})B_0] \quad (19)$$

$$(r_0 + \tilde{r})\ddot{\tilde{\theta}} + 2\dot{\tilde{r}}(\dot{\theta}_0 + \dot{\tilde{\theta}}) = \eta\dot{\tilde{r}}B_0 \quad (20)$$

$$\ddot{\tilde{z}} = -\eta E_z \quad (21)$$

These equations may be simplified with the aid of (12):

$$\eta\, \partial V_0/\partial r = (r_0 + \tilde{r})\dot{\theta}_0^2 \quad (22)$$

and by recalling that the dots may be replaced by multiplication by $j(\omega - \gamma u_0)$. We obtain, finally:

$$\tilde{r} = \eta E_r/(\omega - \gamma u_0)^2 \quad (23)$$

$$\tilde{\theta} = 0 \quad (24)$$

$$\tilde{z} = \eta E_z/(\omega - \gamma u_0)^2 \quad (25)$$

Although the foregoing equations deal with the dynamics of individual electrons, the assumption that the beam behaves like a smoothed-out "fluid" of charge, with a single velocity at each point, enables us to assign values of velocity and all other ac quantities, to *fixed positions* in space, (r, θ, z). In these coordinates, the dc velocity is given by:

$$\underline{v}_0 = (0,\ r\dot{\theta}_0,\ u_0) \quad (26)$$

and the ac velocity by:

$$\begin{aligned} \underline{v} &= (\dot{\tilde{r}},\ r\dot{\tilde{\theta}},\ \dot{\tilde{z}}) \\ &= j(\omega - \gamma u_0)[(\tilde{r},\ r\tilde{\theta},\ \tilde{z})] \end{aligned} \quad (27)$$

Although the ac quantities are defined at r_0, they may be taken to be the same at r, to a linear approximation.

The same result, (27), might have been obtained by stating the equations of motion in terms of Eulerian coordinates, in which the perturbed variables are the components of fluid velocity at any fixed point. In this procedure, the "material" or total time derivative would be used in the expressions for acceleration.

The ac space-charge density ρ is found with the aid of the continuity equation:

$$\frac{\partial}{\partial t}(\rho_0 + \rho) = -\text{div}\,[(\rho_0 + \rho)(\underline{v}_0 + \underline{v})] \tag{28}$$

$$\rho = \frac{j\rho_0}{\omega - \gamma u_0}\,\text{div}\,\underline{v} \tag{29}$$

From (23)–(25) and (27), the ac velocity may be written:

$$\underline{v} = \frac{j\eta}{\omega - \gamma u_0}(E_r\,,\,0,\,E_z) \tag{30}$$

Combining these with Poisson's equation, we find:

$$\rho = -\frac{\eta\rho_0}{(\omega - \gamma u_0)^2}\,\text{div}\,\underline{E} = \frac{\omega_p^2}{(\omega - \gamma u_0)^2}\cdot\rho \tag{31}$$

There are two possible solutions to (31):

$$(\omega - \gamma u_0)^2 = \omega_p^2 \tag{32}$$

$$\rho = 0 \tag{33}$$

Solution (32) represents two longitudinal space-charge waves of arbitrary amplitude distribution, with plasma-frequency oscillations about the average beam velocity:

$$\gamma = \frac{\omega}{u_0} \pm \frac{\omega_p}{u_0} \tag{34}$$

The second solution, (33), however, permits us to evaluate the components of the ac convection current density $\underline{J}$, and thereby solve the field equations (1) — (8):

$$\underline{J} = \rho_0\underline{v} + \rho\underline{v}_0 \tag{35}$$

$$J_r = \rho_0 v_r = \frac{\omega_p^2\epsilon}{\gamma(\omega - \gamma u_0)}\frac{\partial E_z}{\partial r} \tag{36}$$

$$J_\theta = 0 \tag{37}$$

$$J_z = \rho_0 v_z = -\frac{j\omega_p^2\epsilon}{\omega - \gamma u_0}E_z \tag{38}$$

The wave equations (1) and (4) for E_z and H_z now reduce to the following.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) - \gamma^2 E_z = 0 \tag{39}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right) - \gamma^2 H_z = 0 \tag{40}$$

These equations have solutions for E_z and H_z, which are finite at $r = 0$, of the form $A \cdot I_0(\gamma r)$, where A is an arbitrary constant and I_0 the modified Bessel function of zero-th order.

It is not without interest to remark that the same pair of solutions, given by (32) and (39)— (40), has been found by L. R. Walker for a beam of arbitrary cross-section, with the same longitudinal velocity and space-charge density at every point, in the absence of any impressed dc magnetic field.

Due to the radial component of electron motion, the beam surface is rippled. For a steady-state radius b, this rippling can be expressed, in a linear approximation, by the perturbed radius:

$$r(b) = b + \tilde{r}(b) \exp j(\omega t - \gamma z) \tag{41}$$

The rippled beam is equivalent to a uniform cylindrical beam with an ac surface charge density $\rho_0\tilde{r}$, or a surface current density whose components are:

$$G_z = \rho_0 \tilde{r} u_0 \tag{42}$$

$$G_\theta = \rho_0 \tilde{r} \dot{\theta}_0 b \tag{43}$$

The total ac convection current may be written in a form which applies equally well to the cylindrical beam with purely rectilinear flow:

$$\begin{aligned} I_c &= \int_0^b J_z 2\pi r \, dr + 2\pi b \rho_0 u_0 \tilde{r}(b) \\ &= -j\omega\epsilon \cdot R \cdot 2\pi b \cdot A \cdot I_1(\gamma b)/\gamma \\ &= -j\omega\epsilon R \int_0^b E_z \, 2\pi r \, dr \end{aligned} \tag{44}$$

where R is a beam propagation function which will prove convenient:

$$R = \frac{\omega_p^2}{(\omega - \gamma u_0)^2} = \frac{(\beta_p b)^2}{(\gamma b - \beta_e b)^2} \tag{45}$$

and

$$\beta_e = \omega/u_0 \,, \quad \beta_p = \omega_p/u_0 \tag{46}$$

Thus we note that wave propagation along a cylindrical beam with Brillouin flow is accompanied by swelling and contracting of its boundary, with constant space-charge density, rather than by space-charge bunching. The second interesting result is that the dynamics and field equations for the focused beam are identical with those for a beam with zero dc magnetic field, except for the angular component of surface current density G_θ.

SPACE-CHARGE WAVES

We now consider the given beam, of radius b, in a concentric conducting tube of radius $a > b$. The boundary problem consists of matching the TM wave admittances inside and outside of the beam, at its boundary. (The TE fields are of no interest in the drift-tube problem, as they are not excited at the ends of the tube, and are not coupled to the TM fields.) Let I refer to the beam region $0 \leq r \leq b$, and II to the space between beam and conductor $b \leq r \leq a$. Then, at $r = b$,

$$\frac{H_\theta^I + G_z}{E_z^I} = \frac{H_\theta^{II}}{E_z^{II}} \tag{47}$$

The beam admittance on the left is evaluated with the aid of (3), (7), (36), and (42):

$$Y_e = \frac{j\omega\epsilon}{\gamma}(1 - R)\frac{I_1(\gamma b)}{I_0(\gamma b)} \tag{48}$$

In region II,

$$E_z = B \cdot I_0(\gamma r) + C \cdot K_0(\gamma r)$$

$$H_\theta = \frac{j\omega\epsilon}{\gamma}[B \cdot I_1(\gamma r) - C \cdot K_1(\gamma r)]$$

where K_0 and K_1 are modified Bessel functions of the second kind. The wave admittance at $r = b$ in II is therefore:

$$Y_c = \frac{j\omega\epsilon}{\gamma}\left[\frac{I_1(\gamma b) - (C/B) \cdot K_1(\gamma b)}{I_0(\gamma b) + (C/B) \cdot K_0(\gamma b)}\right] \tag{49}$$

At $r = a$, $E_z^{II} = 0$ or:

$$C/B = -I_0(\gamma a)/K_0(\gamma a) \tag{50}$$

Equating beam and circuit admittances (48) and (49), we obtain:

$$R = \frac{-\dfrac{I_0(\gamma a)}{K_0(\gamma a)}}{\gamma b \cdot I_1(\gamma b) \cdot \left[I_0(\gamma b) - \dfrac{I_0(\gamma a)}{K_0(\gamma a)} \cdot K_0(\gamma b)\right]} \tag{51}$$

This equation must be solved simultaneously with each of the following:

$$\begin{aligned} \gamma_1 b &= \beta_e b + \beta_p b/\sqrt{R_1} \\ \gamma_2 b &= \beta_e b - \beta_p b/\sqrt{R_2} \end{aligned} \tag{52}$$

Thus, for a given beam and frequency, the solution consists of two unattenuated waves, one faster and the other slower than the beam velocity. The wavelength of the interference pattern is given by:

$$\lambda_s = \frac{4\pi}{\gamma_1 - \gamma_2} \tag{53}$$

For a cylindrical beam,

$$\beta_p b = 174\sqrt{P} \tag{54}$$

where $P = I/V^{3/2}$ amps/(volts)$^{3/2}$, the perveance. In practice, P and hence $\beta_p b$ are usually so small that we can gain a fair estimate of λ_s by assuming $R_1 = R_2$:

$$\lambda_s \simeq \frac{2\pi\sqrt{R}}{\beta_p} \tag{55}$$

Fig. 1 shows the variation of $R^{1/2}$ with γb for several values of b/a. (The "intrinsic" solution (32) is included as a line at $R^{1/2} = 1$.) The ordinates of these curves are approximately proportional to the space-charge wavelength, and the abcissae to the frequency, as $\gamma \simeq \beta_e = \omega/u_0$ for small perveance.

Space-charge waves propagating along a cylindrical beam with rectilinear flow have been treated by Hahn[8] and Ramo[9]. In Fig. 2, their computations have been reformulated in the same way as in Fig. 1, and compared with the results for Brillouin flow, for two values of b/a. The space-charge wavelength is always greater in Brillouin flow, for the principal pair of waves and the same b/a and γb.

HELIX PROBLEM

In place of the drift tube at radius a, we now have a helically conducting sheet of zero thickness and pitch angle ψ. In addition to I ($0 \leq r \leq b$) and II ($b \leq r \leq a$), we shall use III to identify fields in the region ($a \leq r < \infty$). The boundary conditions at $r = b$ are:

$$\begin{aligned} H_\theta^I + G_z - H_\theta^{II} &= 0 \\ E_z^I - E_z^{II} &= 0 \\ H_z^I - G_\theta - H_z^{II} &= 0 \\ E_\theta^I - E_\theta^{II} &= 0 \end{aligned} \tag{56}$$

At $r = a$, the boundary conditions are:

$$\begin{aligned} E_z^{II} + E_\theta^{II} \cot \psi &= 0 \\ E_z^{III} + E_\theta^{III} \cot \psi &= 0 \\ E_z^{II} - E_z^{III} &= 0 \\ H_z^{II} + H_\theta^{II} \cot \psi - H_z^{III} - H_\theta^{III} \cot \psi &= 0 \end{aligned} \tag{57}$$

Inasmuch as $\cot \psi \sim \gamma/k$, the contribution of E_θ to the field at the helix can conceivably be comparable to that of E_z. The TE fields are coupled to the TM group, in addition, through the angular surface current G_θ, which depends on E_z. All 8 equations must therefore be solved simultaneously.

The procedure follows that of Chu and Jackson[2] for the field solution

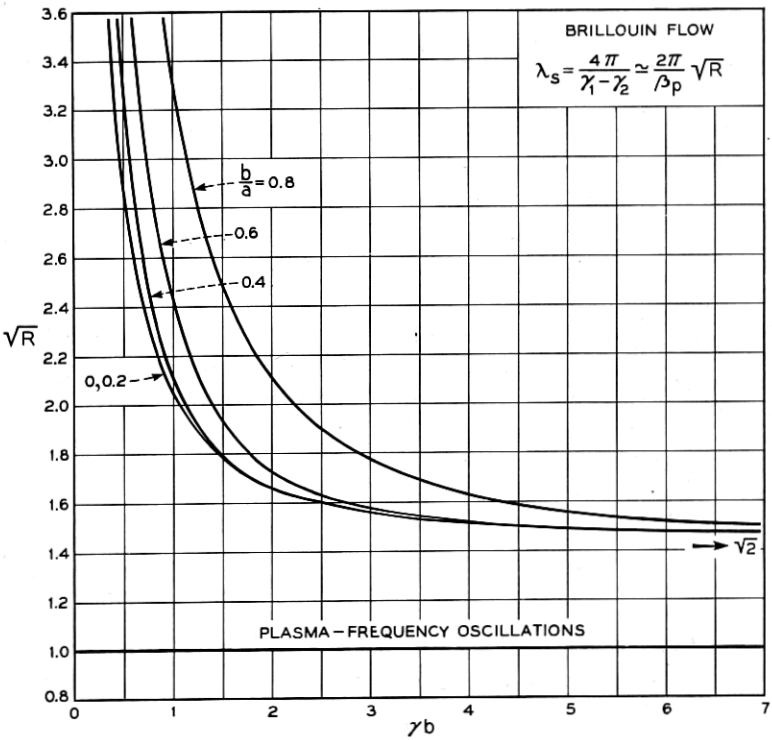

Fig. 1 — Space-charge wavelength λ_s for cylindrical beam with Brillouin flow, in a concentric drift tube. Here b and a are the beam and tube radii, respectively; $R^{1/2}$ is a dimensionless parameter; and the waves propagate as $\exp j(\omega t - \gamma z)$. To compute $\lambda_s \simeq 2\pi R^{1/2}/\beta_p$, use $\beta_p b = 174P^{1/2}$, where P is the beam perveance. The abscissae are approximately given by $\gamma \sim \beta e = \omega/u_0$. (Equations 52–55.)

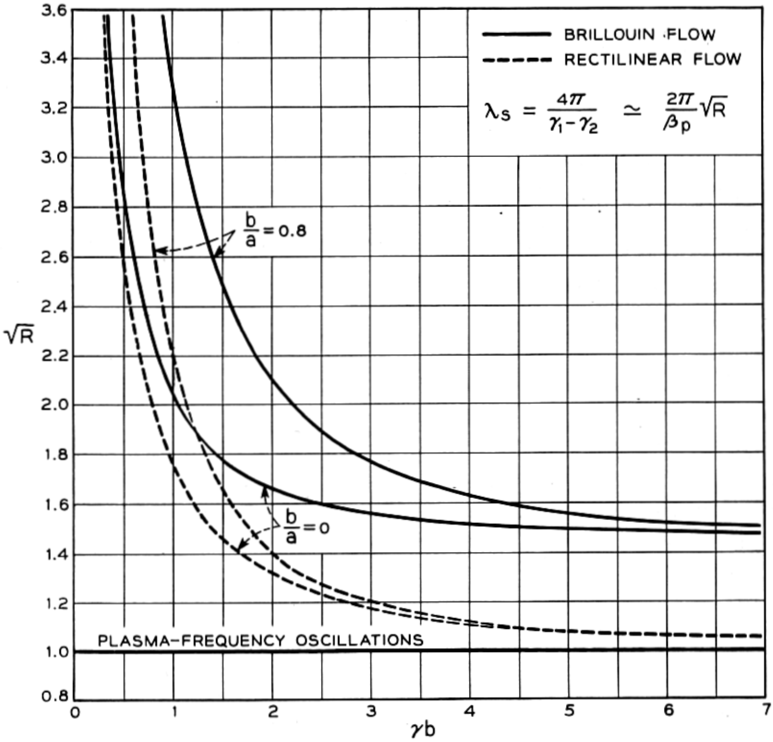

Fig. 2 — Comparison between space-charge wavelengths for cylindrical beams with Brillouin flow and those with rectilinear (confined) flow, respectively.

of the rectilinear beam. The 12 independent variables of (56–57) are reduced to 6 by expressing H_θ and E_θ in terms of E_z and H_z, respectively. The latter, however, require 2 arbitrary constants for a complete description in region II, making a total of 8 constants to be determined.

The eliminant of the 8 boundary-value equations can be written as a TM wave-admittance equation at the beam surface:

$$\frac{j\omega\epsilon}{\gamma}(1 - R)\frac{I_1(\gamma b)}{I_0(\gamma b)} = \frac{j\omega\epsilon}{\gamma}\frac{I_1(\gamma b) - \delta\cdot K_1(\gamma b)}{I_0(\gamma b) + \delta\cdot K_0(\gamma b)} \tag{58}$$

where

$$\delta = \frac{\delta_0 + RF}{1 - \dfrac{K_0(\gamma b)}{I_0(\gamma b)}RF} \tag{59}$$

$$\delta_0 = \frac{1}{K_0^2(\gamma a)}\left[\left(\frac{ka\cot\psi}{\gamma a}\right)^2 K_1(\gamma a)I_1(\gamma a) - K_0(\gamma a)I_0(\gamma a)\right] \tag{60}$$

$$F = (kb \cot \psi) \left(\frac{\dot{\theta}_0 b}{c} \right) \frac{I_1^2(\gamma b) K_1(\gamma a)}{K_0(\gamma a)} \tag{61}$$

In (61), c is the velocity of light.

The right side of (58), which is the admittance H_θ / E_z looking away from the beam surface, contains a term δ which depends on the helix geometry and the amplitude of the TE fields excited by the surface current G_θ. Thus, although the TE fields do not affect the electron paths, they are excited by the beam, and coupled to the TM fields at the helix. If G_θ were zero, δ would reduce to δ_0, and the circuit admittance in (58) would then be the same as for a cylindrical beam with rectilinear flow. In (59), δ is expressed in terms of δ_0 and the product, RF, where R is the beam propagation function, and F a factor dependent on the magnetic field and the geometry.

This is the complete field solution of the problem. Equation (58) has four roots: two complex and two real propagation wave numbers, one of the latter representing a backward wave. In addition, there are two unattenuated space-charge waves, given by (34); or a total of 6 waves in all.

EQUIVALENT THIN-BEAM SOLUTION

Pierce[1] has expressed the admittance equation for an ideally thin beam, interacting with an arbitrary distributed circuit, as follows:

$$\frac{q}{E} = \frac{j\beta_e}{(j\beta_e - \Gamma)^2} \frac{I_0}{2V_0} = -\left(\Gamma^2 K \left[\frac{\Gamma_0}{\Gamma^2 - \Gamma_0^2} + \frac{2jQ}{\beta_e} \right] \right)^{-1} \tag{62}$$

where q = total convection current

E = longitudinal electric field

Γ = propagation constant = $\sqrt{-\gamma^2 - k^2}$

I_0 = dc beam current

V_0 = dc beam potential

Γ_0, K, Q = normal-mode circuit parameters.

For slow waves, $\Gamma \simeq j\gamma$. For moderate values of perveance, the accelerating voltage may be replaced by the beam potential at the axis:

$$u_0 \simeq \sqrt{2\eta V_0}$$

$$\frac{j\beta_e}{(j\beta_e - \Gamma)^2} \frac{I_0}{2V_0} \simeq j\omega\epsilon\pi b^2 R \tag{63}$$

Then, dividing both sides of (62) by $2\pi b$, we may rewrite it as a wave-admittance equation:

$$-\frac{j\omega\epsilon}{\gamma}\cdot\frac{\gamma b}{2}\cdot R = -\left(2\pi b\Gamma^2 K\left[\frac{\Gamma_0}{\Gamma^2 - \Gamma_0^2} + \frac{2jQ}{\beta_e}\right]\right)^{-1} \tag{64}$$

With the aid of (59), we can solve the admittance equation (58) for R, and re-write it as follows:

$$Y_e = Y_B \tag{65}$$

with

$$Y_e = -\frac{j\omega\epsilon}{\gamma}\cdot\frac{\gamma b}{2}\cdot R \tag{66}$$

$$Y_B = \frac{j\omega\epsilon}{\gamma}\cdot\frac{\gamma b}{2}\cdot\frac{I_0(\gamma b)}{I_1(\gamma b)}\left(\frac{\delta_0}{\gamma b\cdot I_0(\gamma b)[I_0(\gamma b) + \delta_0\cdot K_0(\gamma b)] - \dfrac{FI_0(\gamma b)}{I_1(\gamma b)}}\right) \tag{67}$$

The solid-cylindrical Brillouin beam in a helix is thus equivalent to a thin beam whose circuit admittance is Y_B. By equating Y_B to the right side of (64), we can evaluate the normal-mode parameters for this admittance, and thereby use all the results of previous thin-beam calculations.[1,7] The equivalence of the two circuit expressions, however, requires that we replace the transcendental expression (67) by an algebraic one, with no more than three arbitrary constants. This can be done very effectively, in the region of interest, by means of the approximation:[2]

$$Y_B \simeq -(\gamma_p - \gamma_0)\left(\frac{\partial Y_B}{\partial \gamma}\right)_{\gamma=\gamma_0}\cdot\frac{\gamma - \gamma_0}{\gamma - \gamma_p} \tag{68}$$

in which γ_0 and γ_p are the zero and pole, respectively, of Y_B:

$$\delta_0(\gamma_0) = 0$$

$$\delta_0(\gamma_p) = \left(-\frac{I_0(\gamma b)}{K_0(\gamma b)} + \frac{F}{\gamma b\cdot I_1(\gamma b)\cdot K_0(\gamma b)}\right)_{\gamma=\gamma_p} \tag{70}$$

If we were to neglect the term containing F in (70), the error in the magnitude of $\delta_0(\gamma_p)$ would be measured by:

$$\frac{F}{\gamma b\cdot I_1(\gamma b)\cdot I_0(\gamma b)} \equiv (kb \cot\psi)\left(\frac{\dot{\theta}_0 b}{c}\right)\frac{I_1(\gamma b)\cdot K_1(\gamma a)}{I_0(\gamma b)\cdot K_0(\gamma a)} \tag{71}$$

In most low-power traveling-wave tubes, the first factor in parentheses is usually less than 3; the second factor less than 0.01; and the last factor always less than unity. The error in evaluating γ_p, moreover, is less than this product, for the slope of the curve δ_0 versus γb increases with γb. Putting $F = O$, therefore, leads at most to a very slight error in γ_p and

$$\left(\frac{\partial Y_B}{\partial \gamma}\right)_{\gamma=\gamma_0}$$

Outside of the region (γ_0, γ_p), δ_0 grows large rapidly, and the expression for Y_B is hardly affected at all by this assumption.

Physically, the negligible role played by F in the admittance equation means that $\delta \simeq \delta_0$, i.e., the TM helix admittance is not appreciably perturbed by the TE fields excited by G_θ.

With $F = 0$, (67) may be re-written:

$$Y_B = \frac{\gamma b}{2} \cdot \frac{I_0(\gamma b)}{I_1(\gamma b)} \cdot Y_H \tag{72}$$

$$Y_H = \frac{j\omega\epsilon}{\gamma}\left[\frac{I_1(\gamma b) - \delta_0 \cdot K_1(\gamma b)}{I_0(\gamma b) + \delta_0 \cdot K_0(\gamma b)} - \frac{I_1(\gamma b)}{I_0(\gamma b)}\right] \tag{73}$$

Here Y_H is the helix admittance seen by a thin cylindrical hollow beam, with rectilinear electron flow. As in the case of Y_B, it may be replaced in the vicinity of (γ_0, γ_p), by the approximation:

$$Y_H \simeq -(\gamma_p - \gamma_0)\left(\frac{\partial Y_H}{\partial \gamma}\right)_{\gamma=\gamma_0} \frac{\gamma - \gamma_0}{\gamma - \gamma_p} \tag{74}$$

Fletcher[4] has evaluated the normal-mode parameters for Y_H as follows:

$$\Gamma_0^2 = -\gamma_0^2 - k^2 \tag{75}$$

$$Q_H \frac{\gamma_0}{\beta_e}\left[1 + \frac{k^2}{\gamma_0^2}\right]^{-1/2} = \frac{1}{2} \frac{\gamma_0^2}{\gamma_p^2 - \gamma_0^2} \tag{76}$$

$$\frac{1}{K_H} = -j\pi b \gamma_0^2 \left[1 + \frac{k^2}{\gamma_0^2}\right]^{3/2} \left(\frac{\partial Y_H}{\partial \gamma}\right)_{\gamma=\gamma_0} \tag{77}$$

We have used the subscript H to refer the parameters to the hollow beam, and will use the subscript B to refer to the solid-cylindrical Brillouin beam.

As Y_B and Y_H have the same zero and pole, they have the same natural propagation constant Γ_0, and the same space-charge parameter Q:

$$Q_B = Q_H \tag{78}$$

This quantity can be found plotted in Fig. 1 of Reference 4, or in Fig. A6.1 of Reference 1.

From (68), (72), and (74), we find:

$$\left(\frac{Y_B}{Y_H}\right)_{\gamma=\gamma_0} = \left(\frac{\partial Y_B/\partial\gamma}{\partial Y_H/\partial\gamma}\right)_{\gamma=\gamma_0} = \left[\frac{\gamma b}{2}\frac{I_0(\gamma b)}{I_1(\gamma b)}\right]_{\gamma=\gamma_0} \tag{79}$$

The impedance parameters for the two beams are therefore related to each other by:

$$K_B = K_H\left[\frac{2}{\gamma b}\frac{I_1(\gamma b)}{I_0(\gamma b)}\right]_{\gamma=\gamma_0} \tag{80}$$

Both Pierce[10] and Fletcher[4] have found the impedance parameter of the hollow beam to be related to that of a thin beam along the axis of a helix, K_T, as follows:

$$K_H = K_T[I_0^2(\gamma b)]_{\gamma=\gamma_0} \tag{81}$$

The gain parameter C is defined by: $C^3 = (2K)(I_0/8V_0)$ (82)

Thus, for given I_0 and V_0, the factor by which the gain parameter of a thin beam should be multiplied to give that of a hollow beam, is:

$$(K_H/K_T)^{1/3} = [I_0^{2/3}(\gamma b)]_{\gamma=\gamma_0} \tag{83}$$

This "impedance reduction factor" can similarly be evaluated for the finite cylindrical beam with Brillouin flow:

$$(K_B/K_T)^{1/3} = \left[\frac{2}{\gamma b}\,I_1(\gamma b)\cdot I_0(\gamma b)\right]^{1/3}_{\gamma=\gamma_0} \tag{84}$$

Cutler,[7] who calls this quantity F_2, has described how it and the parameter Q can be used to compute the gain of traveling-wave tubes. The procedure depends upon the evaluation of C and QC. The expression for C_B, in Cutler's notation, is:

$$C_B \simeq (K_2/K)^{1/3}\ F_1F_2(I_0/8V_0)^{1/3} \tag{85}$$

Here K_2/K is a factor, of the order of 0.5, which corrects the impedance of the ideal sheath helix for the physical dimensions and support elements of the actual helix. It is best found by measurement. The factor F_1 is plotted in Fig. 3.4 of Reference 1, and obeys the empirical relation:

$$F_1(\gamma a) = 7.154\ \exp\ (-0.6664\ \gamma a) \tag{86}$$

Finally, the factor F_2 is the impedance reduction factor (84), which is plotted in Fig. 3 of this paper for various ratios of the radii, b/a.

It is of interest to compare the relative gain of beams with rectilinear and with Brillouin flow, respectively. Pierce[10] has computed a first approximation to the impedance reduction factor for the solid-cylindrical beam with rectilinear flow, by averaging E_z^2 over the beam area (with E_z for the empty helix):

$$(K_s/K_T)^{1/3} \sim [I_0^2(\gamma b) - I_1^2(\gamma b)]_{\gamma=\gamma_0}^{1/3} \tag{87}$$

Fletcher[4] has improved upon this calculation by replacing the solid beam with a thin hollow beam of different radius and dc current. This has the same electronic admittance Y_e and derivative $dY_e/d\gamma$ when $R = 1$.

The impedance reduction factors for the three types of beams have been plotted in Fig. 4, using a typical value of b/a. For the same I_0, V_0, and b/a, the gain parameters C are found to be greatest for the hollow beam, and least for the solid rectilinear beam.

The high gain of the hollow beam is due to its concentration in the region of greatest field strength. The greater gain of the beam with Brillouin flow, relative to that of a similar beam with confined flow, is probably due to transverse electron motion, in two ways:[11]

(1) causing electrons to interact with the transverse as well as longitudinal fields; and

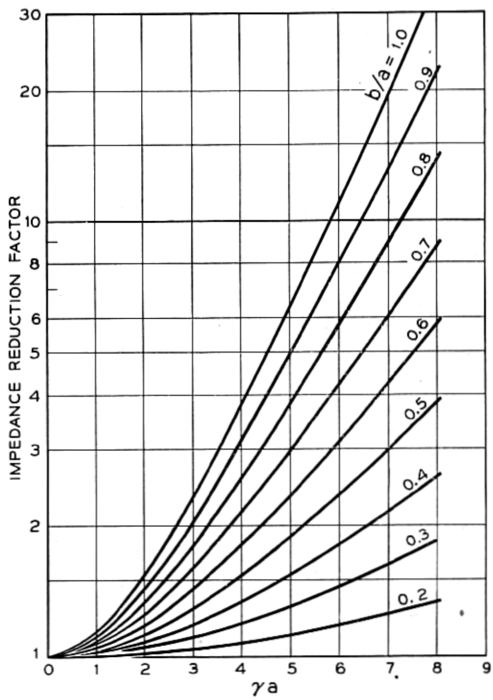

Fig. 3 — The factor $(K_B/K_T)^{1/3}$, or F_2, by which the gain parameter C_T for a thin beam should be multiplied to give C_B, the gain parameter for a cylindrical beam with Brillouin flow, of the same current and voltage. Computation of C_B using this factor is described in text following equation (85).

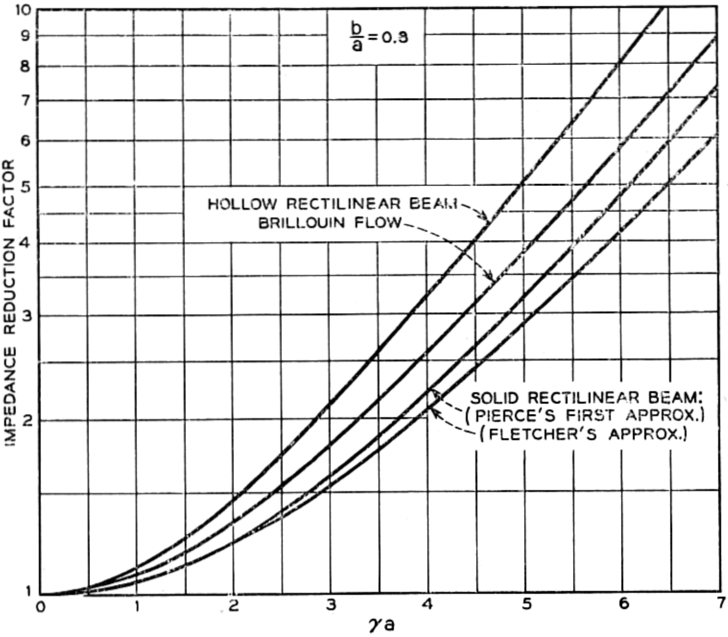

Fig. 4 — Comparative values of impedance reduction factor for several kinds of beams, of the same relative radii b/a.

(2) causing electrons to move preferentially into regions of retarding longitudinal fields, a process analogous to bunching.

CONCLUSIONS

Field solutions have been presented for the magnetically-focused cylindrical beam, when modulated by a small ac signal. Two types of beam enclosure have been treated: the concentric drift tube and the ideally thin (sheath) helix.

There are two pairs of unattenuated space-charge waves in the drift-tube: one with arbitrary amplitude distribution, and another pair which is coupled to the external field (Fig. 1). The space-charge wavelength of the latter pair is greater than that of space-charge waves in a similar beam with rectilinear flow (Fig. 2).

The solution of the helix problem consists of the aforementioned two space-charge waves with arbitrary amplitude, as well as the usual four waves of traveling-wave tube theory, or six waves in all. In order to compute the gain constant of the growing wave, the field solution has been re-written as the admittance equation of a thin beam in an artificial circuit. By means of two approximations, the normal-mode parameters of this circuit have been evaluated.

The first approximation amounts to neglecting the TE fields coupled to the TM wave, and is valid for most low-power traveling-wave tubes. The second approximation consists of replacing the circuit admittance function by an algebraic expression with the same zero and pole, and the same slope at the zero. Although thin-beam theory predicts small deviations of complex roots (of the admittance equation) from the natural propagation wave number, it is difficult to judge whether any such roots might occur outside of the region in which this approximation holds, for the finite beam.

The space-charge parameter Q_B is found to be the same as for a thin hollow beam with rectilinear flow (Fig. 1 of Reference 4, or Fig. A6.1 of Reference 1). The gain parameter C_B can be computed from Equation (85), Fig. 3.4 of Reference 1, and Fig. 3 of this paper. The gain of the cylindrical beam with Brillouin flow is found to be greater than that of a similar cylindrical beam with rectilinear flow, presumably because of transverse electron motion in the former. Its gain, however, is less than that of a thin hollow cylindrical beam with rectilinear flow, for the same radius, current, and voltage (Fig. 4).

ACKNOWLEDGEMENTS

The writers are indebted to J. R. Pierce for suggesting the general approach used in this paper, to H. Suhl and C. F. Quate for their helpful criticisms, and to Mrs. C. Lambert for computing the graph material.

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