Topics in Guided Wave Propagation Through Gyromagnetic Media

Part II — Transverse Magnetization and the Non-Reciprocal Helix

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Propagation through a gyromagnetic medium in a direction normal to a uniform magnetizing field is considered. Geometrical arrangements which make this propagation non-reciprocal are described. A few illustrative examples are discussed briefly. The non-reciprocal helix, of importance in traveling wave tube work, is treated at length.

1. INTRODUCTION

1.1. *General Remarks about Non-Reciprocal Propagation*

Part I of this paper began with a brief discussion of some of the microwave properties of two gyromagnetic media; the gas discharge plasma and the ferrite. The remainder of Part I was devoted to the analysis of the mode spectrum in a cylindrical waveguide filled with one of the media and placed in an axial magnetic field. It was demonstrated that the natural modes in such a guide are right- and left-circularly polarized waves which travel with different phase velocities. Accordingly a plane polarized mode, which to some approximation can be regarded as the sum of right and left circular modes, will, in traversing a section of the guide, undergo Faraday rotation, just like a plane wave in the unbounded medium. It is true that the presence of the guide wall has a drastic effect on the course of the rotation with magnetizing field, changing it, sometimes beyond recognition, from that prevailing in the unbounded medium. Nevertheless the principle remains the same; confinement of the wave to a guide merely modifies quantitatively the Faraday effect for plane waves. In optics, where practically plane waves are almost always employed in this connection, the non-reciprocal nature of this effect is so familiar that it hardly requires restatement here.

By contrast, Part II of this paper deals with devices whose non-reciprocal operation depends *in principle* as well as in numerical detail on the disposition of the boundary, or, more generally, on geometrical configuration. These devices employ magnetizing fields transverse to the propagation direction. Some electromagnetic field configurations are unaffected by such a dc field, but, whenever the rf magnetic field in the case of a ferrite (or the rf electric field for a plasma) has a component normal to the dc magnetic field, this is no longer the case. For, now, the magnetization (or the charges) will be caused to precess about the dc field, giving extra terms in Maxwell's equations and a resultant change in the propagation. This change may be simply an alteration in phase velocity, the propagation remaining reciprocal. This is the case, for example, for the propagation of plane waves in an infinitely extended medium [Cotton-Mouton effect]. Here, since every direction of propagation normal to the dc field is physically equivalent to any other and, in particular, to the opposite direction, no non-reciprocity can arise.

For reciprocity to be preserved in the presence of the dc magnetic field is, however, exceptional and requires a certain amount of geometrical symmetry in the system. That non-reciprocity may be expected in asymmetrical systems may be foreseen if we consider a system, typical of those to be treated in this paper, in which all the rf fields are independent of the coordinate along which the dc magnetic field is pointing. The relevant conducting boundaries and any interfaces between ferrite (plasma) and air are all surfaces parallel to the direction of propagation and lying in the dc magnetic field direction. Suppose the system to be divided into two parts by another surface of a similar kind and examine the surface impedance of one of the parts (which should contain some gyromagnetic material). If the propagation direction be reversed it is necessary to reverse the magnetic field to retrieve a situation in the part considered *geometrically* equivalent to the original. But, since the precession of the magnetization (or charges) about the dc magnetic field has a definite sense, the magnetic or electric current associated with this precession will be reversed when the dc field is reversed. Thus, the properties of the medium are altered and the surface impedance will be different for the two directions of propagation. In general, the surface impedance of the other part of the system will not compensate for this distinction between the two directions and we shall find different propagation constants for opposing directions. An exception will occur if the system contains a surface about which it has geometrical symmetry, for, then, compensation clearly takes place about this surface and the system is reciprocal.

An example of a simple non-reciprocal system is indicated in Fig. 1(a). Here a slab of ferrite is inserted into a rectangular waveguide parallel to the narrow walls and closer to one of them. Several workers have demonstrated that this arrangement and a similar arrangement in a circular waveguide are non-reciprocal for what is essentially the dominant mode.[1, 3, *] When the slab is centered in the guide we have a plane of symmetry and the non-reciprocity vanishes.

Another configuration of the transverse field type is represented by the system shown in Fig. 1(b). Here a hollow ferrite cylinder is magnetized circumferentially and propagates a TE_{0n}-mode. It is clear that any arrangement of this sort, which might, in principle, include conducting sheaths, internally or externally, or might have the ferrite extending to an indefinitely large or small radius, cannot have any symmetry

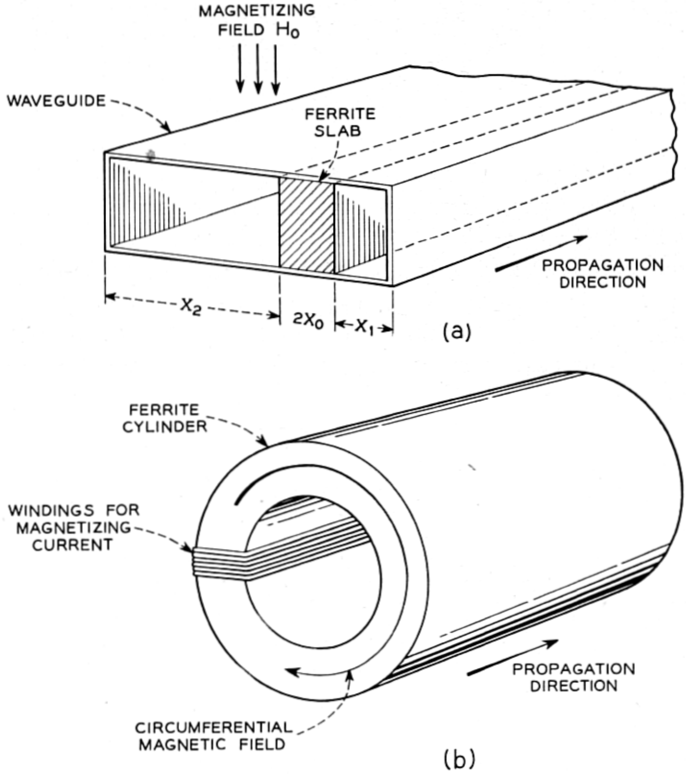

Fig. 1 — (a) Rectangular waveguide and ferrite slab. (b) Circumferentially magnetized ferrite cylinder.

* It is expected that an article on this subject by S. E. Miller, A. G. Fox and M. T. Weiss will appear in a forthcoming issue of the JOURNAL.

about a cylinder coaxial with the ferrite. Thus, the internal and external impedances of such a system at any coaxial cylinder can only compensate accidentally (perhaps at a single frequency) and non-reciprocity is the rule.

It is frequently asserted without qualification that for non-reciprocity a further condition upon the relevant rf field is that its projection upon a plane normal to the magnetizing field be elliptically or circularly polarized in the limit of vanishing magnetization. The argument is based on the consideration, in itself correct, that the effective material constants are different for right- and left-circularly polarized field vectors. Suppose that the magnetization direction is y. Then the tensor relating B to H is (see Part I, Section 2.1):

$$\begin{Vmatrix} \mu & 0 & -j\kappa \\ 0 & \mu_0 & 0 \\ j\kappa & 0 & \mu \end{Vmatrix}.$$

For right- and left-circular fields with $H_z = \pm jH_x$, therefore, the medium is isotropic in the plane transverse to the dc field with permeabilities $\mu + \kappa$, $\mu - \kappa$ respectively. Since opposite circular polarizations accompany opposite propagation directions, (see for example, Fig. 2) the permeabilities, and hence the propagation constants, are different for opposite propagation directions. It is then argued that the field must already be circularly, or at least elliptically polarized to start with, if non-reciprocal effects are to result from application of the magnetization. However, the argument is true only for effects of first order in the magnetization. For general values of magnetization, the rf field, even if linearly polarized to begin with, will become elliptically polarized, and

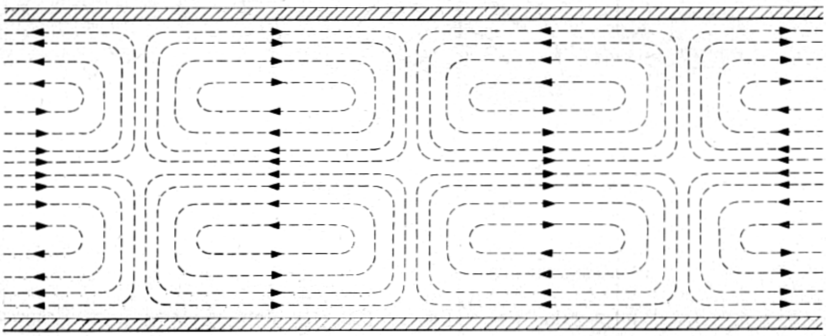

Fig. 2 — Magnetic lines of force parallel to the broad side of a rectangular waveguide.

non-reciprocity will occur. It is understandable that this double function of the magnetization (conversion to elliptic polarization plus creation of differences in permeability) leads to higher order effects. However, inasmuch as for the ferrite and plasma the magnetization can produce large changes, the requirement of elliptic polarization in zero magnetic field cannot be regarded as essential in practice. These considerations are demonstrated by a simple example in Section 2.2.

The devices considered in this paper actually are such that the electric or the magnetic field vector in the plane normal to the field is elliptically polarized even in the absence of the gyromagnetic medium. For example, the magnetic lines of force of the TE_{10} mode in a rectangular waveguide form two sets of closed loops in a plane parallel to the wide sides (see Fig. 2) and repeating every wavelength. This pattern moves bodily down the guide with the phase velocity of the mode, so that an observer stationary at any point not at the center or at the narrow walls of the guide sees a magnetic field rotating at the signal frequency and tracing out a generally elliptic path. The sense of the rotation is opposite on opposite sides of the center plane, and depends on the propagation direction. The conditions outlined in the previous paragraph are therefore satisfied; introduction of a ferrite slab magnetized as in Fig. 1(a) will yield first order non-reciprocal effects.

The problems considered here are such that the electromagnetic fields do not vary in the direction of magnetization. Under these conditions the field can be split into a TE and TM field satisfying different wave equations. In general, the two fields are coupled through the boundary conditions. Most of the paper is devoted to the analysis of the non-reciprocal helix, a problem that has recently gained importance in connection with high power traveling-wave-amplifiers.[4] The conventional amplifier suffers from a limitation on its maximum useful gain; waves reflected from the output end will make the tube "sing" above a certain critical gain. Ferrites offer the possibility of preferentially suppressing these backward waves and so of increasing the permissible gain by a large amount.* In section 2.3 the "flat" helix (one of infinite radius) is considered. For the slow waves employed in practice a rather complete treatment is possible in this case of planar geometry. In Section 3 the cylindrical helix is treated. Inasmuch as the solutions involve functions for which no extensive tables exist, the treatment has to be more sketchy.

* More specifically, in high-power traveling-wave tubes the large beam current employed may be above the critical value required for backward wave oscillations due to spatial harmonics of the helix structure. In such cases the larger attenuation of backward waves will permit a higher beam current and therefore stable amplification to higher power levels.

Thus the loss is neglected and only the non-reciprocal phase characteristic is considered. The losses have then to be determined approximately by differentiation of the phase characteristics.

A few further problems were considered as illustrations of the general principles. One case, that of the plasma filling the space above an impedance sheet can actually be solved analytically and provides a particularly clear demonstration of the non-reciprocity. The case of the rectangular waveguide with a ferrite slab has already been considered extensively elsewhere, and only the results for a thin slab are given here. A problem with cylindrical symmetry is taken up in Section 3.3: a cylindrical waveguide fitted with a circumferentially magnetized cylinder

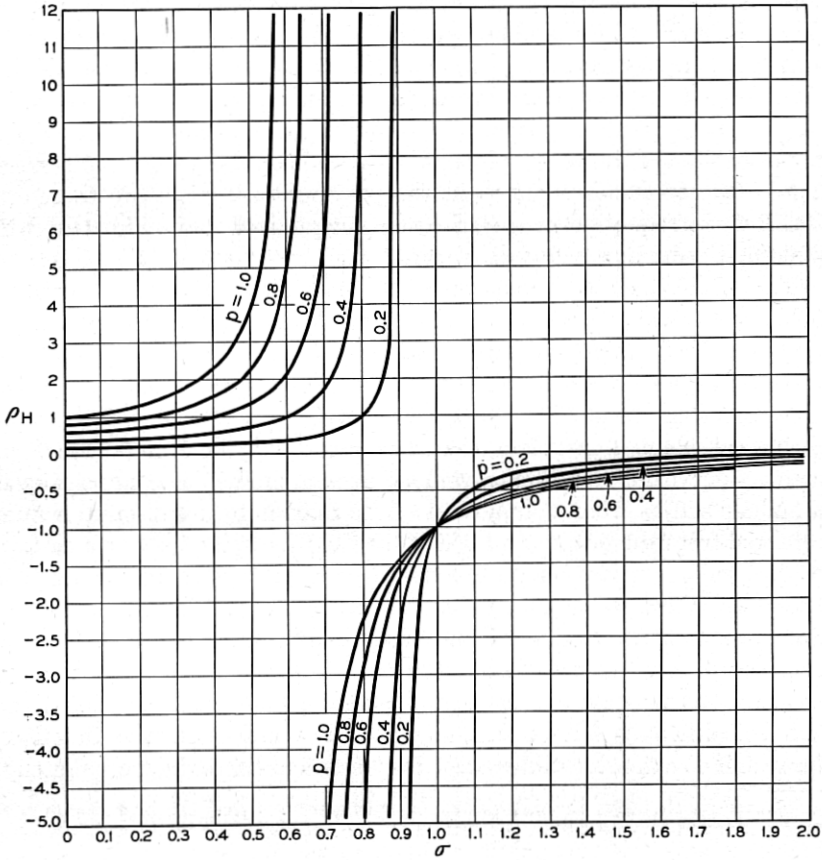

Fig. 3 — ρ_H versus σ for various p.

of ferrite. Again the discussion had to be sketchy in view of the scarcity of information on the functions that solve the problem.

2. PLANAR GEOMETRY

2.1. *Fields and Impedances*

In this section we consider planar transverse field problems which are characterized by the following conditions. The dc magnetic field is of uniform strength H_0 within the gyromagnetic medium and points along the y-axis. All rf field components are independent of the y coordinate. We discuss the ferrite case first, then indicate how the results are to be translated for the plasma.

For the orientation of the dc magnetic field which has been chosen the permeability matrix is of the form:

$$\begin{Vmatrix} \mu & 0 & -j\kappa \\ 0 & \mu_0 & 0 \\ j\kappa & 0 & \mu \end{Vmatrix},$$

μ and κ are, in general, even and odd functions of H_0; the permeability of unmagnetized ferrite is taken to be μ_0 as in free space. Following the procedure of Part I we shall assume specifically for μ and κ the formulae given by Polder's treatment of the dynamics of the medium. Thus, we have the expressions (for the case of no loss):

$$\frac{\mu}{\mu_0} = \frac{1 - p\sigma - \sigma^2}{1 - \sigma^2},$$

$$\frac{\kappa}{\mu_0} = \frac{p}{1 - \sigma^2}, \text{ and} \tag{1}$$

$$\rho_H = \frac{\kappa}{\mu} = \frac{p}{1 - p\sigma - \sigma^2},$$

where σ is the ratio of the precession frequency, $\frac{|\gamma|}{2\pi} H_0$, to the signal frequency and p is the ratio of a frequency, $\frac{|\gamma|}{2\pi} M_0/\mu_0$, associated with the saturation magnetization, M_0, to the signal frequency. It should be noted that p and σ have always the same sign. The behavior of μ and κ as functions of σ was shown in Fig. 1(a) and (b) of Part I. ρ_H is shown as a function of σ in Fig. 3. The dielectric constant of the ferrite is taken to be ϵ. For reasons given in Part I, $|p|$ is assumed less than unity.

When the condition, $\partial/(\partial y) \equiv 0$, is put into Maxwell's equations the latter are found to be separable into two sets:

$$-\frac{\partial H_y}{\partial z} = j\omega\epsilon E_x, \tag{2a}$$

$$\frac{\partial H_y}{\partial x} = j\omega\epsilon E_z, \tag{2b}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu_0 H_y, \tag{2c}$$

and

$$-\frac{\partial E_y}{\partial z} = -j\omega[\mu H_x - j\kappa H_z], \tag{3a}$$

$$\frac{\partial E_y}{\partial x} = -j\omega[j\kappa H_x + \mu H_z], \tag{3b}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y. \tag{3c}$$

It is to be stressed that such a separability is possible only when the rf fields do not vary along the dc magnetic field. The sets of equations (2) and (3) correspond to the separate equations for H_z and E_z which arise from (13) of Part I when β is there set equal to zero. The first set describes a TM field of the familiar type, whose propagation through the medium is unaffected by the magnetic field. The second set describes a TE field whose components, because of the presence of κ, are connected by different relations from those which exist in an unmagnetized medium. The separability of the two fields is equivalent to saying that they are not coupled by the medium itself, but they may, of course, be coupled at the boundaries.

We may write (3a) and (3b) in the form

$$-j\omega(\mu^2 - \kappa^2)H_x = j\kappa\frac{\partial E_y}{\partial x} - \mu\frac{\partial E_y}{\partial z}, \tag{4a}$$

$$-j\omega(\mu^2 - \kappa^2)H_z = \mu\frac{\partial E_y}{\partial x} + j\kappa\frac{\partial E_y}{\partial z}, \tag{4b}$$

and upon eliminating H_x and H_z, the wave equation for E_y is found to be:

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + \omega^2\epsilon\,\frac{\mu^2 - \kappa^2}{\mu}\,E_y = 0, \tag{5}$$

where E_y (and also the H's) are evidently propagated in the ferrite as

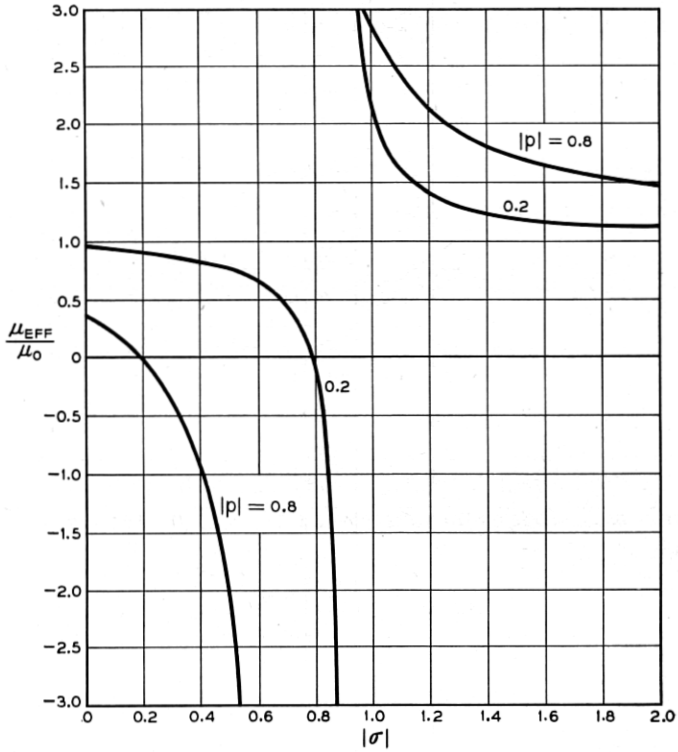

Fig. 4 — $\frac{\mu_{\text{eff}}}{\mu_0}$ versus $|\sigma|$.

though the latter had an effective permeability, $(\mu^2 - \kappa^2)/\mu = \mu_{\text{eff}}$. This permeability may assume any value from $+\infty$ to $-\infty$ as may be seen by writing it in terms of σ and p. From the Polder formulae, in fact,

$$\mu_{\text{eff}} = (\mu^2 - \kappa^2)/\mu = \mu_0 \frac{1 - (p + \sigma)^2}{1 - p\sigma - \sigma^2}. \tag{6}$$

As it should be, this is an even function of magnetic field. μ_{eff}/μ_0 decreases from $1 - p^2$ to 0 as $|\sigma|$ rises from 0 to $1 - |p|$; it decreases from 0 to $-\infty$ as $|\sigma|$ runs from $1 - |p|$ to $\sqrt{1 + p^2/4} - |p|/2$ and finally decreases from ∞ to 1 as $|\sigma|$ increases indefinitely above $\sqrt{1 + p^2/4} - |p|/2$. Its behavior is indicated in Fig. 4. It should be recalled from Part I that "$\sigma = 0$" is an abbreviation for the very small magnetic field necessary to saturate the ferrite.

A brief examination of the propagation of plane waves shows the more significant features of transmission in this medium. Since no direction in the x–z plane can be a preferred one, the plane wave may be assumed to travel in the z-direction with variation, $e^{-j\beta z}$. Then, from equations (4) and (5)

$$\beta^2 = \omega^2 \epsilon \mu_{\text{eff}} ,$$

and

$$H_x = \frac{-\beta\mu}{\omega(\mu^2 - \kappa^2)} E_y ,$$

$$H_y = \frac{j\beta\kappa}{\omega(\mu^2 - \kappa^2)} E_y ,$$

Since μ_{eff} is negative between $|\sigma| = 1 - |p|$ and $|\sigma| = \sqrt{1 + p^2/4} - |p|/2$, the medium is cut off for plane waves in this range of magnetic field (at a fixed signal frequency). H is elliptically polarized when the medium is magnetized and $|H_z|/|H_x| = |\kappa|/|\mu| = |\rho_H|$. We may also put

$$H_x + jH_z = \frac{-\beta E_y}{\omega(\mu - \kappa)} ,$$

$$H_x - jH_z = \frac{-\beta E_y}{\omega(\mu + \kappa)} ,$$

But $|H_x + jH_z|$ and $|H_x - jH_z|$ are proportional to the amplitudes of the left-handed and right-handed circularly polarized components of the magnetic field. The medium may thus be considered to exhibit the permeability,

$$\mu - \kappa = \mu_0\left(1 - \frac{p}{1 - \sigma}\right),$$

for left-handed components and the permeability,

$$\mu + \kappa = \mu_0\left(1 + \frac{p}{1 + \sigma}\right)$$

for right-handed components. The effective permeability is essentially a parallel combination of these two permeabilities and the medium may propagate ($\mu_{\text{eff}} > 0$) even when $\mu - \kappa$ is negative, as will happen for $\sqrt{1 + p^2/4} - |p|/2 < \sigma < 1$.

Since the medium itself has no non-reciprocal properties it is clear that if the latter are to arise they must do so as a result of interaction between the medium and its surroundings. The boundaries of the ferrite at which matching will be necessary are surfaces parallel to the y-axis and here we need an expression for H_{tang}/E_y where H_{tang} is a tangential magnetic field at the surface. For the moment we will consider the admittance looking into the ferrite and take tangential components in the counterclockwise sense. From equation (4), then,

$$\frac{H_{\text{tang}}}{E_y} = \frac{j\mu \dfrac{1}{E_y}\dfrac{\partial E_y}{\partial \nu}}{\omega(\mu^2 - \kappa^2)} - \frac{\kappa \dfrac{1}{E_y}\dfrac{\partial E_y}{\partial \sigma}}{\omega(\mu^2 - \kappa^2)}, \tag{7}$$

where $\partial/(\partial\nu)$ is a normal derivative (outward) and $\partial/(\partial\sigma)$, a tangential derivative. It is possible, although by no means essential, to interpret the terms of (7) in the following fashion: the first term is just the admittance of a normal TE mode propagating in the interior of the ferrite (which is to have the permeability, μ_{eff}); the second term is to be ascribed to an independent surface current,

$$\frac{-\kappa \dfrac{\partial E_y}{\partial \sigma}}{\omega(\mu^2 - \kappa^2)}.$$

Using this picture one may see how non-reciprocity arises in a simple case. If the ferrite be bounded by the planes $x = x_1$ and $x = x_2$, and the z-variation is of the form $e^{-j\beta z}$, the admittances due to the surface currents are $+j\kappa\beta/\omega(\mu^2 - \kappa^2)$. If β reverses its sign the surface currents are interchanged. If now the external admittances on the two sides of the slab are unequal (and, of course, themselves reciprocal) for given values of ω and $|\beta|$, there is no reason why β and $-\beta$ should simultaneously solve the matching problem.

Almost all the above considerations may be taken over to the case of the plasma. Here the TE fields will be undisturbed by the magnetic field (but the dielectric constant is altered by the presence of the charge from its free space value). Equations (4) and (5) are now replaced by

$$j\omega(\epsilon^2 - \eta^2)E_x = -\epsilon \frac{\partial H_y}{\partial z} + j\eta \frac{\partial H_y}{\partial x}, \tag{8a}$$

$$j\omega(\epsilon^2 - \eta^2)E_z = j\eta \frac{\partial H_y}{\partial z} + \epsilon \frac{\partial H_y}{\partial x}, \tag{8b}$$

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} + \omega^2\mu_0 \frac{\epsilon^2 - \eta^2}{\epsilon} H_y = 0 \tag{9}$$

for the TM fields. Here ϵ and η are the diagonal and off-diagonal terms of the dielectric matrix which is of the same form

$$\left\| \begin{matrix} \epsilon & 0 & -j\eta \\ 0 & \epsilon_1 & 0 \\ j\eta & 0 & \epsilon \end{matrix} \right\|$$

as the permeability matrix of the ferrite. The equations of motion for the plasma* lead to the expressions

$$\begin{aligned} \epsilon_1 &= \epsilon_0 (1 - q^2), \\ \epsilon &= \epsilon_0 \left(1 + \frac{q^2}{\sigma^2 - 1}\right), \\ \eta &= \epsilon_0 \frac{q^2 \sigma}{\sigma^2 - 1}, \\ \epsilon_{\text{eff}} &= \frac{\epsilon^2 - \eta^2}{\epsilon} = \epsilon_0 \frac{(1 - q^2)^2 - \sigma^2}{(1 - q^2) - \sigma^2}, \end{aligned} \tag{10}$$

where σ is now the ratio of the cyclotron resonance frequency;

$$\frac{1}{2\pi} \frac{|e| \mu_0}{m} H_0,$$

in a field H_0, to the applied frequency and q is the ratio of the plasma frequency to applied frequency; ϵ_{eff} behaves with magnetic field in much the same way as μ_{eff}. It is a constantly decreasing function of σ and is negative between $|\sigma| = 1 - q^2$ and $|\sigma| = \sqrt{1 - q^2}$, going to infinity at the latter value. The left and right handed dielectric constants, $\epsilon - \eta$ and $\epsilon + \eta$ are given by $\epsilon_0[1 - q^2/(1 - \sigma)]$ and $\epsilon_0[1 - q^2/(1 + \sigma)]$.

2.2. *Examples of Non-reciprocal Systems*

We now discuss briefly three examples of non-reciprocal systems as illustrations of particular points. As an example of a system which can be analyzed very easily and completely we consider a plasma occupying the region, $x > 0$ and bounded at $x = 0$ by a sheet of constant impedance. This impedance is to depend upon frequency but not on the propagation constant. It will be written as $j\sqrt{\mu_0/\epsilon_0}Z(\omega)$ where μ_0 and ϵ_0 are free space values. A practical realization of such a sheet might consist of a very large number of similar fins of negligible thickness and separation, parallel to the y-axis and attached normally to a conducting plane $x =$ constant. The fields between separate fins are uncoupled and E_z is uniform between fins. For such an arrangement, $Z(\omega) = \tan \omega\sqrt{\epsilon_0 \mu'_0} x_0$,

* See Section 2.2 of Part I.

where x_0 is the depth of the fins. If the z-variation of the fields is $e^{-j\beta z}$ and the waves are guided, the x-dependence in the plasma must be as $\exp - \sqrt{\beta^2 - \omega^2\mu_0\epsilon_{\text{eff}}}\, x$. From equation (8b)

$$j\omega(\epsilon^2 - \eta^2)E_z = [\eta\beta - \epsilon\sqrt{\beta^2 - \omega^2\mu_0\epsilon_{\text{eff}}}\,]H_y\,.$$

Matching at $x = 0$ gives

$$\frac{\eta\beta - \epsilon\sqrt{\beta^2 - \omega^2\mu_0\epsilon_{\text{eff}}}}{j\omega(\epsilon^2 - \eta^2)} = j\sqrt{\frac{\mu_0}{\epsilon_0}}\, Z(\omega).$$

This yields

$$\left[\beta - \eta\omega\sqrt{\frac{\mu_0}{\epsilon_0}}\, Z(\omega)\right]^2 = \omega^2\mu_0\epsilon + \omega^2\frac{\mu_0}{\epsilon_0}\,\epsilon^2 Z^2(\omega)$$

or

$$\frac{\beta}{\omega\sqrt{\mu_0\epsilon_0}} = \frac{\eta}{\epsilon_0}\, Z(\omega) \pm \sqrt{\frac{\epsilon}{\epsilon_0} + \frac{\epsilon^2}{\epsilon_0^2}\, Z^2(\omega)}\,. \tag{11}$$

The non-reciprocity is clearly exhibited, since the two values of β are not equal and opposite. The solution (11) is valid only if $\beta^2 > \omega^2\mu_0\epsilon_{\text{eff}}$, corresponding to guided waves.

In the second example we assume that the region between conducting planes at $x = 0$ and $x = x_0$ is filled by a plasma. When no magnetic field is present E_z is supposed to vanish and E_x is uniform across the gap. The unperturbed field is then plane polarized (TEM). The magnetic field is now applied parallel to the y axis to that part of the gap between $x = 0$ and $x = x_1$. E_z in the magnetized plasma is now given by

$$E_z = E_0 \sin \sqrt{\omega^2\mu_0\epsilon_{\text{eff}} - \beta^2}\, x$$

since it must vanish at $x = 0$. The z variation is again $\exp - j\beta z$. H_y may be found from equation (8b) and is

$$H_y = \frac{j\omega E_0}{\omega^2\mu_0\epsilon - \beta^2}\Big[\eta\beta \sin \sqrt{\omega^2\mu_0\epsilon_{\text{eff}} - \beta^2}\, x - \epsilon\sqrt{\omega^2\mu_0\epsilon_{\text{eff}} - \beta^2}\cos\sqrt{\omega^2\mu_0\epsilon_{\text{eff}} - \beta^2}\, x\Big].$$

The admittance of the magnetized section at $x = x_1$, is thus,

$$\frac{H_y}{E_z} = \frac{j\omega}{\omega^2\mu_0\epsilon - \beta^2}\Big[\eta\beta - \epsilon\sqrt{\omega^2\mu_0\epsilon_{\text{eff}} - \beta^2}\cot\sqrt{\omega^2\mu_0\epsilon_{\text{eff}} - \beta^2}\, x_1\Big],$$

and, analogously, that of the unmagnetized part is

$$\frac{H_y}{E_z} = \frac{j\omega}{\omega^2\mu_0\epsilon_1 - \beta^2}\Big[-\epsilon_1\sqrt{\omega^2\mu_0\epsilon_1 - \beta^2}\cot\sqrt{\omega^2\mu_0\epsilon_1 - \beta^2}\,(x_0 - x_1)\Big],$$

where $\epsilon_1 = \epsilon_0(1 - q^2)$. Suppose now that the applied field is weak, so that $\omega^2\mu_0\epsilon \sim \omega^2\mu_0\epsilon_1 \sim \beta^2$. Then, the cotangents may be expanded and by equating admittances one obtains

$$\frac{j\omega}{\omega^2\mu_0\epsilon - \beta^2}\left[\eta\beta - \frac{\epsilon}{x_1}\right] = \frac{j\omega\epsilon_1}{(\omega^2\mu_0\epsilon_1 - \beta^2)(x_0 - x_1)},$$

or

$$\frac{\beta^2}{\omega^2\mu_0\epsilon_1} = 1 + \frac{\epsilon - \epsilon_1}{-\epsilon_1 + \left(\eta\beta - \dfrac{\epsilon}{x_1}\right)(x_0 - x_1)}. \tag{12}$$

Since $\epsilon - \epsilon_1$ is of the second order in σ this equation may be solved by substituting the unperturbed value of β, $\pm\, \omega\sqrt{\mu_0\epsilon_1}$, in the right hand side. It is clear that although the system is non-reciprocal, it will be so only to third order in σ. This system, therefore, illustrates the fact pointed out in Section 1.1 that even when the fields are plane polarized in the absence of a dc magnetic field, non-reciprocity may arise, although it may be very small in weak fields.

The third example to be considered is one which has been referred to in Section 1.1, namely that in which a strip of ferrite is placed across the short dimension of rectangular wave guide, see Fig. 1(a). In view of the fact that this problem has been discussed with great thoroughness by Lax, Button and Roth, we shall, after deriving the characteristic equation, consider only the case of a very thin strip. Let the thickness of the strip be $2x_0$ and the distance from its two faces to the nearest guide wall be x_1 and x_2 respectively. The admittances at the two faces are then

$$\frac{j}{\omega\mu_0} \cot \sqrt{\beta_0^2 - \beta^2}\, x_1$$

and

$$\frac{-j}{\omega\mu_0} \cot \sqrt{\beta_0^2 - \beta^2}\, x_2$$

respectively, where $\beta_0^2 = \omega^2\mu_0\epsilon_0$. Inside the ferrite

$$\frac{\partial^2}{\partial x^2} \equiv -[\omega^2\epsilon\mu_{\text{eff}} - \beta^2] = -(\beta_f^2 - \beta^2),$$

and

$$-j\omega(\mu^2 - \kappa^2)H_z = \mu\,\frac{\partial E_y}{\partial x} + \kappa\beta E_y.$$

Thus, immediately within the ferrite,

$$\frac{1}{E_y}\frac{\partial E_y}{\partial x} = -j\omega\mu_{\text{eff}}\left(\frac{H_z}{E_y}\right)_{\text{external}} - \rho_H\beta.$$

If the two faces of the ferrite are $x = -x_0$ and $x = x_0$ we then have

$$\left(\frac{1}{E_y}\frac{\partial E_y}{\partial x}\right)_{x=-x_0} = \frac{\mu_{\text{eff}}}{\mu_0}\cot\sqrt{\beta_0^2 - \beta^2}\,x_1 - \rho_H\beta = A,$$

and

$$\left(\frac{1}{E_y}\frac{\partial E_y}{\partial x}\right)_{x=x_0} = -\frac{\mu_{\text{eff}}}{\mu_0}\cot\sqrt{\beta_0^2 - \beta^2}\,x_2 - \rho_H\beta = B.$$

If we write

$$\frac{1}{E_y}\frac{\partial E_y}{\partial x} = \sqrt{\beta_f^2 - \beta^2}\tan\left(\sqrt{\beta_f^2 - \beta^2}\,x\right),$$

and make use of the boundary conditions we obtain

$$\tan 2\sqrt{\beta_f^2 - \beta^2}\,x_0 = \frac{\sqrt{\beta_f^2 - \beta^2}\,(A - B)}{\beta_f^2 - \beta^2 - AB}\,. \tag{13}$$

The non-reciprocity is clearly contained in the odd power of β in A and B.

For small thickness we replace the tangent by its argument, and, substituting for A and B on one side of the equation, obtain

$$\frac{\mu_{\text{eff}}}{\mu_0}\left[\cot\sqrt{\beta_0^2 - \beta^2}\,x_1 + \cot\sqrt{\beta_0^2 - \beta^2}\,x_2\right] = 2x_0[\beta_f^2 - \beta^2 - AB],$$

or

$$\sin\sqrt{\beta_0^2 - \beta^2}\,(x_1 + x_2) = 2x_0\frac{\mu_0}{\mu_{\text{eff}}}\sin\sqrt{\beta_0^2 - \beta^2}\,x_1\sin\sqrt{\beta_0^2 - \beta^2}\,x_2[\beta_f^2 - \beta^2 - AB]. \tag{14}$$

Since the guide is almost empty, we may write

$$\sqrt{\beta_0^2 - \beta^2}(x_1 + x_2) = \pi - \delta,$$

where δ is small. Or,

$$\beta = \beta_1 + \frac{2\pi\delta}{\beta_1(x_1 + x_2)^2},$$

where $$\beta_1^2 = \beta_0^2 - \pi^2/(x_1 + x_2)^2.$$

Writing $\beta = \beta_1$ in the right hand side of equation (14) and noting that if

$$\sqrt{\beta_0^2 - \beta_1^2}\, x_1 = \frac{\pi x_1}{x_1 + x_2} = \theta$$

then $\sqrt{\beta_0^2 - \beta_1^2}\, x_2 = \pi - \theta$, we have

$$\delta = 2x_0 \frac{\mu_0}{\mu_{\text{eff}}} \sin^2\theta \left[\beta_f^2 - \beta_1^2 - \left(\frac{\mu_{\text{eff}}}{\mu_0}\cot\theta - \rho_H\beta_1\right)\left(\frac{\mu_{\text{eff}}}{\mu_0}\cot\theta - \rho_H\beta_1\right)\right],$$

$$= 2x_0 \frac{\mu_0}{\mu_{\text{eff}}} \left[\left(\beta_f^2 - (1 + \rho_H^2)\beta_1^2\right)\sin^2\theta - \left(\frac{\mu_{\text{eff}}}{\mu_0}\right)^2 \cos^2\theta + 2\frac{\mu_{\text{eff}}}{\mu_0}\rho_H\beta_1 \cos\theta \sin\theta \right]. \tag{15}$$

The non-reciprocal part of β is thus $4\pi x_0(x_1 + x_2)^{-2}\rho_H \sin 2\theta$. This has a maximum value for $\theta = \pi/4$ or $3\pi/4$ and, hence, $x_1 = (x_1 + x_2)/4$ or $3(x_1 + x_2)/4$. This result may be understood qualitatively by considering the fields in the guide before the ferrite is inserted. We then have $E_y = E_0 \sin(\pi x/a)$ where $a = x_1 + x_2$ and consequently,

$$H_x = -\frac{\beta}{\omega\mu}\sin\frac{\pi x}{a} \qquad \text{and} \qquad H_z = \frac{j\frac{\pi}{a}}{\omega\mu}\cos\frac{\pi x}{a}.$$

The amplitudes of the left- and right-handed components of circular polarization at a point are then proportional to

$$-\beta\sin\frac{\pi x}{a} - \frac{\pi}{a}\cos\frac{\pi x}{a} \qquad \text{and} \qquad -\beta\sin\frac{\pi x}{a} + \frac{\pi}{a}\cos\frac{\pi x}{a}.$$

The difference in the squares of these amplitudes is $2\beta(\pi/a)\sin(\pi x/a)\cos(\pi x/a)$ and this is proportional to the difference between the energy stored at x in the left-handed wave and in the right-handed wave. It is plausible that this should be a measure of the non-reciprocal effect produced by a thin piece of ferrite at x.

2.3. *The Plane Helix*

In dealing with transverse field problems with cylindrical geometry we shall consider non-reciprocal propagation along a helix which is surrounded by circumferentially magnetized ferrite. The analysis of this problem is rather cumbersome and it is advantageous to study first an analogous plane problem. The simplicity thus gained allows us to examine somewhat more complicated problems. The "plane helix," to be treated here, is a sheet of negligible thickness, lying in the plane $x = 0$, which conducts only in a single direction making an angle ψ with the y-axis. In this direction it will be supposed lossless. In addition we assume that the regions, $x < 0$ and $0 < x < x_0$ are empty, while the space $x > x_0$ is filled with ferrite. As usual the magnetic field is along the y-axis and the fields are independent of y. The problem is clearly the limit for very large radius of that of an empty, helically-conducting cylinder, surrounded by an infinitely thick shell of ferrite, circumferentially magnetized, the whole system carrying fields with no angular variation.

We first consider the boundary conditions for the plane helix, after noting that it is evident that both TE and TM fields will be required. The tangential electric field on either side of the sheet must necessarily be at right angles to the direction of conduction since the conductivity is infinite. Further, the tangential electric field must be continuous through the sheet. Hence, if the field normal to the direction of conduction is E_0 (omitting here and elsewhere the factor $e^{-j\beta z}$), we have

$$E_z^+ = E_z^- = E_0 \cos\psi,$$
$$E_y^+ = E_y^- = -E_0 \sin\psi,$$

where the symbols $+$ and $-$ refer to $x > 0$ and $x < 0$ respectively. Again, since current cannot flow normal to the direction of conduction, the tangential magnetic field along the latter must be continuous through the sheet or

$$(H_z^+ - H_z^-)\sin\psi + (H_y^+ - H_y^-)\cos\psi = 0.$$

The boundary conditions may be combined into a single equation, by introducing admittances, in the form

$$\left(\frac{H_z^+}{E_y^+} - \frac{H_z^-}{E_y^-}\right)\sin^2\psi = \left(\frac{H_y^+}{E_z^+} - \frac{H_y^-}{E_z^-}\right)\cos^2\psi. \tag{16}$$

The left-hand side refers to the TE fields and the right to TM fields. In the empty regions surrounding the sheet

$$\left[\frac{\partial^2}{\partial x^2} - (\beta^2 - \omega^2\epsilon_0\mu_0)\right]H_y = \left[\frac{\partial^2}{\partial x^2} - (\beta^2 - \beta_0^2)\right]H_y = 0,$$

$$E_z = \frac{1}{j\omega\epsilon_0}\frac{\partial H_y}{\partial x},$$

and

$$\left[\frac{\partial^2}{\partial x^2} - (\beta^2 - \beta_0^2)\right] E_y = 0, \qquad H_z = \frac{-1}{j\omega\mu_0}\frac{\partial E_y}{\partial x},$$

with $\beta_0^2 = \omega^2\epsilon_0\mu_0$. If the waves are to be guided, $\beta^2 > \beta_0^2$, and, then, for $x < 0$, we have $\partial/\partial x \equiv \sqrt{\beta^2 - \beta_0^2}$. Thus

$$\frac{H_y^-}{E_z^-} = \frac{j\omega\epsilon_0}{\sqrt{\beta^2 - \beta_0^2}},$$

and

$$\frac{H_z^-}{E_y^-} = -\frac{\sqrt{\beta^2 - \beta_0^2}}{j\omega\mu_0}.$$

If the admittances at the surface of the ferrite are H_y^f/E_z^f and H_z^f/E_y^f, then H_y^+/E_z^+ and H_z^+/E_y^+ are given by the impedance transformation:

$$\frac{H_y^+}{E_z^+} = \frac{j\omega\epsilon_0}{\sqrt{\beta^2 - \beta_0^2}} \cdot \frac{\dfrac{\sqrt{\beta^2 - \beta_0^2}}{j\omega\epsilon_0}\dfrac{H_y^f}{E_z^f} - \tanh\sqrt{\beta^2 - \beta_0^2}\,x_0}{1 - \dfrac{\sqrt{\beta^2 - \beta_0^2}}{j\omega\epsilon_0}\dfrac{H_y^f}{E_z^f}\cdot\tanh\sqrt{\beta^2 - \beta_0^2}\,x_0},$$

and

$$\frac{H_z^+}{E_y^+} = -\frac{\sqrt{\beta^2 - \beta_0^2}}{j\omega\epsilon_0} \cdot \frac{-\dfrac{j\omega\mu_0}{\sqrt{\beta^2 - \beta_0^2}}\dfrac{H_z^f}{E_y^f} - \tanh\sqrt{\beta^2 - \beta_0^2}\,x_0}{1 - \left[\dfrac{-j\omega\mu_0}{\sqrt{\beta^2 - \beta_0^2}}\dfrac{H_z^f}{E_y^f}\right]\cdot\tanh\sqrt{\beta^2 - \beta_0^2}\,x_0}.$$

Within the ferrite the TM-fields fall off as $\exp - \sqrt{\beta^2 - \omega^2\epsilon\mu_0}x$ and the TE-fields as $\exp - \sqrt{\beta^2 - \omega^2\epsilon\mu_{\text{eff}}}x$. We then have

$$\frac{H_y^f}{E_z^f} = \frac{-j\omega\epsilon}{\sqrt{\beta^2 - \omega^2\epsilon\mu_0}},$$

and

$$\frac{H_z^f}{E_y^f} = \frac{j\omega\mu_{\text{eff}}}{\sqrt{\beta^2 - \omega^2\epsilon\mu_{\text{eff}}} - \beta\kappa/\mu}.$$

These results may now be collected and substituted in equations (16). The equation of condition so obtained is

$$(\beta^2 - \beta_0^2)\frac{A + 1}{A + \tanh\sqrt{\beta^2 - \beta_0^2}x_0} = \beta_0^2\cot^2\psi\,\frac{1 - B}{1 - B\tanh\sqrt{\beta^2 - \beta_0^2}x_0}, \tag{17}$$

where

$$A = \frac{\mu_{\text{eff}}}{\mu_0} \frac{\sqrt{\beta^2 - \beta_0^2}}{\sqrt{\beta^2 - \omega^2 \epsilon \mu_{\text{eff}}} - \beta \rho_H},$$

and

$$B = -\frac{\epsilon}{\epsilon_0} \frac{\sqrt{\beta^2 - \beta_0^2}}{\sqrt{\beta^2 - \omega^2 \epsilon \mu_0}}.$$

We shall assume that we are dealing with slow waves (β large). This is the case of greatest practical interest and is usually ensured by making $\cot \psi$ large. Assuming that the waves are slow we simplify the equation (12) and find certain values for β. We may then ascertain in what ranges of σ and p the simplifying assumptions and the results are consistent.

In equation (17), then, we replace all square roots by $|\beta|$ and $\beta^2 - \beta_0^2$ by β^2, obtaining

$$\beta^2 = \beta_0^2 \cot^2 \psi \, \frac{A + \tanh |\beta| x_0}{A + 1} \cdot \frac{1 - B}{1 - B \tanh |\beta| x_0},$$

with

$$A = \frac{\mu_{\text{eff}}/\mu_0}{1 - \rho_H \operatorname{sgn} \beta} = \frac{\mu + \kappa \operatorname{sgn} \beta}{\mu_0},$$

and

$$B = -\epsilon/\epsilon_0$$

where $\operatorname{sgn} \beta = 1$ for $\beta > 0$ and $\operatorname{sgn} \beta = -1$ for $\beta < 0$. Taking first the simplest case of no separation ($x_0 = 0$), this becomes

$$\beta^2 = \frac{\beta_0^2(1 + \epsilon/\epsilon_0) \cot^2 \psi}{1 + \dfrac{1 - \rho_H \operatorname{sgn} \beta}{\mu_{\text{eff}}/\mu_0}} = \frac{\beta_0^2(1 + \epsilon/\epsilon_0) \cot^2 \psi}{1 + \dfrac{\mu_0}{\mu + \kappa \operatorname{sgn} \beta}}.$$

Substituting the expressions (6) and (1) for μ_{eff}/μ_0 and ρ_H we are led to*

$$\beta_+ = \beta_0 \cot \psi \cdot \sqrt{\frac{1}{2}(1 + \epsilon/\epsilon_0)} \sqrt{\frac{\sigma + 1 + p}{\sigma + 1 + p/2}}, \tag{18a}$$

$$\beta_- = -\beta_0 \cot \psi \cdot \sqrt{\frac{1}{2}(1 + \epsilon/\epsilon_0)} \sqrt{\frac{\sigma - 1 + p}{\sigma - 1 + p/2}} \tag{18b}$$

* Since reversal of the magnetization is equivalent to interchange of the propagation directions, we are at liberty to consider σ and p always positive, and to deal with the two cases $\beta > 0$ and $\beta < 0$ separately.

The β_+ mode propagates for all σ, $\dfrac{\sigma + 1 + p}{\sigma + 1 + p/2}$ declining steadily from $\dfrac{1 + p}{1 + p/2}$ to unity. The β_- mode on the other hand is cut off, with $\beta^2 = 0$ at $\sigma = 1 - p$ and then reappears with $\beta^2 = \infty$ at $\sigma = 1 - p/2$. The behavior of the two modes is shown in Fig. 5. Self-consistency requires that $\beta^2 \gg \omega^2 \epsilon \mu_0$ or $\omega^2 \epsilon \mid \mu_{\text{eff}} \mid$, whichever is the greater. The condition $\beta^2 \gg \omega^2 \epsilon \mid \mu_{\text{eff}} \mid$ fails to be met near

$$\sigma = \sigma_0 = \sqrt{\frac{p^2}{4} + 1} - \frac{p}{2},$$

when

$$\frac{\mid \sigma - \sigma_0 \mid}{\dfrac{p}{2} \sqrt{\dfrac{p^2}{4} + 1} \pm \left(\dfrac{p}{2} - 1\right)} < \frac{2}{(1 + \epsilon/\epsilon_0) \cot \psi}$$

The condition $\beta^2 \gg \omega^2 \epsilon \mu_0$ will fail for β_- near $\sigma = 1 - p$. The range for this to occur is given by

$$1 - p - \sigma < \frac{p/2}{\dfrac{1}{2}(1 + \epsilon/\epsilon_0) \cot \psi - 1}.$$

The extension of the Polder formulae to the case of a lossy ferrite was given in Part I, (Section 2.1). From the results given there one may write

$$\mu + \kappa \operatorname{sgn} \beta = 1 + p \frac{\sigma(1 + \alpha^2) - \operatorname{sgn} \beta + j\alpha}{\sigma^2(1 + \alpha^2) - 1 + 2j\alpha\sigma}$$

where α is a damping parameter. Substitution of these expressions in the slow wave formula for β^2 will give the effect of loss on the propagation constant. In Fig. 6 the complex value of β_- is shown for $\alpha = 0.1$ and several values of p. The imaginary part of β_+ is small and varies only slightly with σ. Fig. 7 shows the initial loss ($\sigma = 0$) for three values of α and a range of p values. From a knowledge of β as a function of $\sigma = \omega_H/\omega$ and $p = \omega_M/\omega$ it is possible to calculate the loss, Im $\beta_-/(\beta_0 \sqrt{1 + \epsilon/\epsilon_0} \cot \psi)$, as a function of frequency when the magnetic field and saturation magnetization of the ferrite are held constant. Fig. 8 shows the results of such calculations. It should be noted that the horizontal scale is linear in σ or $1/\omega$ and that the vertical scale implicitly contains the frequency in the form of $1/\beta_0$. Both of these distortions of scale tend to

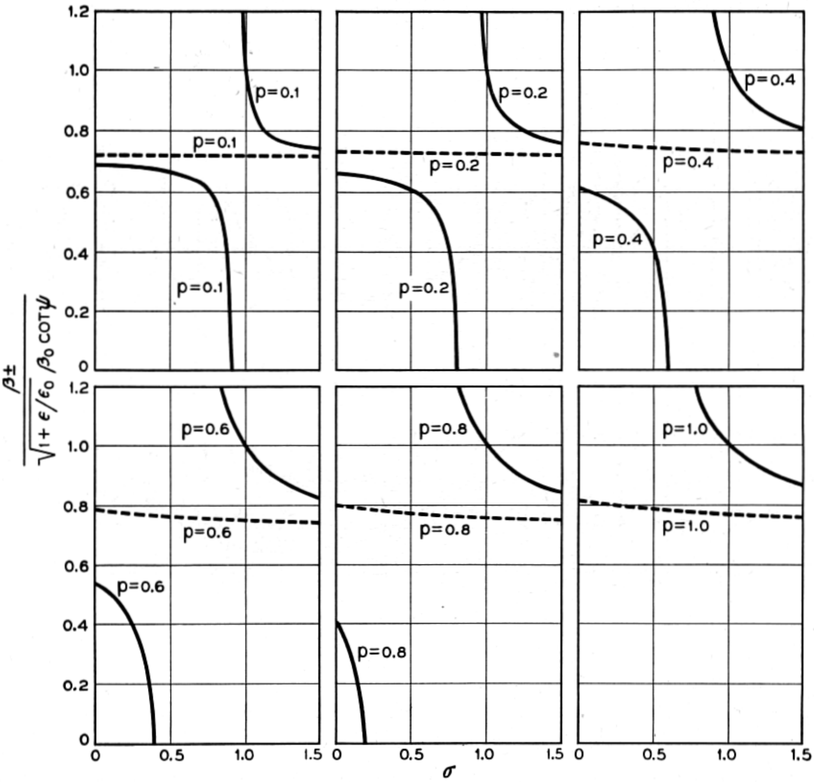

Fig. 5 — The non-reciprocal propagation constants of the flat helix. The dotted curves represent β_+, the solid curves β_-. (Loss free case).

produce an appearance of sharpness in the variation of the loss at higher frequencies.

If the slow wave assumption be made again it is possible to obtain solutions for the case in which the ferrite does not touch the helix and the latter is lossless. With the slow wave approximation, (17) becomes

$$\frac{\beta_\pm^2}{\beta_0^2 \cot^2 \Psi} = \frac{1 + \dfrac{\epsilon}{\epsilon_0}}{1 + \dfrac{\epsilon}{\epsilon_0} \tanh \mid \beta \mid x_0} \cdot \frac{\dfrac{\mu + \kappa \operatorname{sgn} \beta}{\mu_0} + \tanh \mid \beta \mid x_0}{\dfrac{\mu + \kappa \operatorname{sgn} \beta}{\mu_0} + 1}$$

If we write $\mid \beta \mid x_0 = u$, then the above equation expresses $\beta_\pm$ in terms of u. At the same time $x_0 = u / \mid \beta_\pm(u) \mid$ and we evidently have a parametric representation of $\beta_\pm$ and x_0. The results of such computations

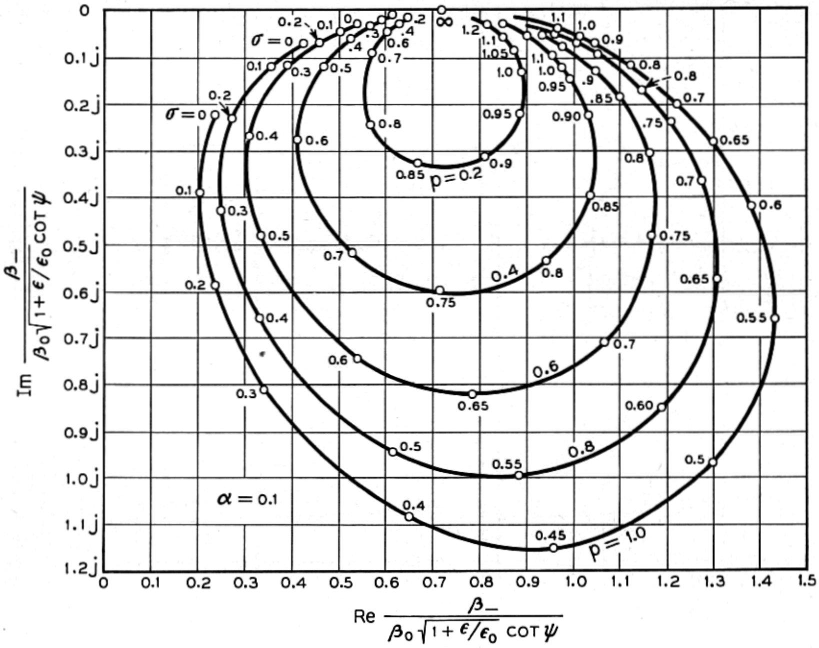

Fig. 6 — Real and imaginary parts of the reverse propagation constant β_- of a flat helix versus σ for various p and for $\alpha = 0.1$ (the parameter σ is marked along the curves). The forward propagation constant has a very small imaginary part which hardly varies with σ.

are shown in Figs. 9(a), (b) and (c), where β is plotted against x_0 for various fixed σ for two values of p. It is to be noted that the introduction of any characteristic length or scale into the problem, such as is provided here by the distance x_0 immediately produces a great complication in the mode spectrum. The plane helix with the ferrite in contact may be thought of as a highly degenerate problem.

To carry out loss calculations using the appropriate expressions for $\mu + \kappa \operatorname{sgn} \beta$ would be very tedious in the separated case. However, it was pointed out in Part 1 that to order α the expressions for μ and κ are given correctly if we put $\sigma + j\alpha$ in place of σ in the lossless formulae and that, in consequence, for small α, the imaginary part of the propagation constant is approximately given by

$$\alpha \frac{\partial \beta}{\partial \sigma}$$

In Figs. 10(a) and 10(b) the loss calculated in this way for the cases considered in Figs. 9 is shown.

3. CYLINDRICAL GEOMETRY

3.1. *Impedances*

In this section we consider systems with cylindrical symmetry about the propagation direction. All boundaries, those of the circuit as those of the medium, are coaxial circular cylinders, and the medium is assumed to be magnetized circumferentially. The practical means for bringing about such a magnetization — for example, thin wires threaded through a cylinder of ferrite and carrying a dc current, Fig. 1(b) — are assumed to effect the electromagnetic field to a negligible extent. As in the case of planar geometry, we restrict ourselves to fields which have no variation along the magnetizing field; that is, in the azimuthal direction. Only the ferrite is considered here; the results for a plasma are obvious corollaries. The magnetizing field and the dc magnetization

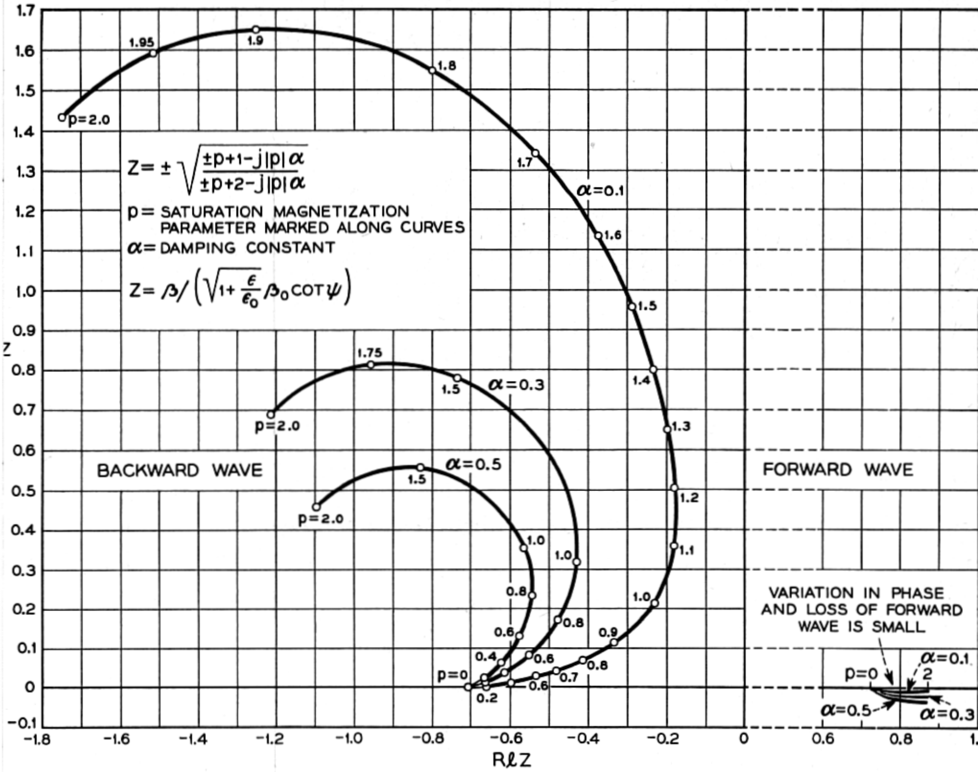

Fig. 7 — Real and imaginary parts of β_+ and β_- for a flat helix at the very small magnetizing field required to saturate the ferrite.

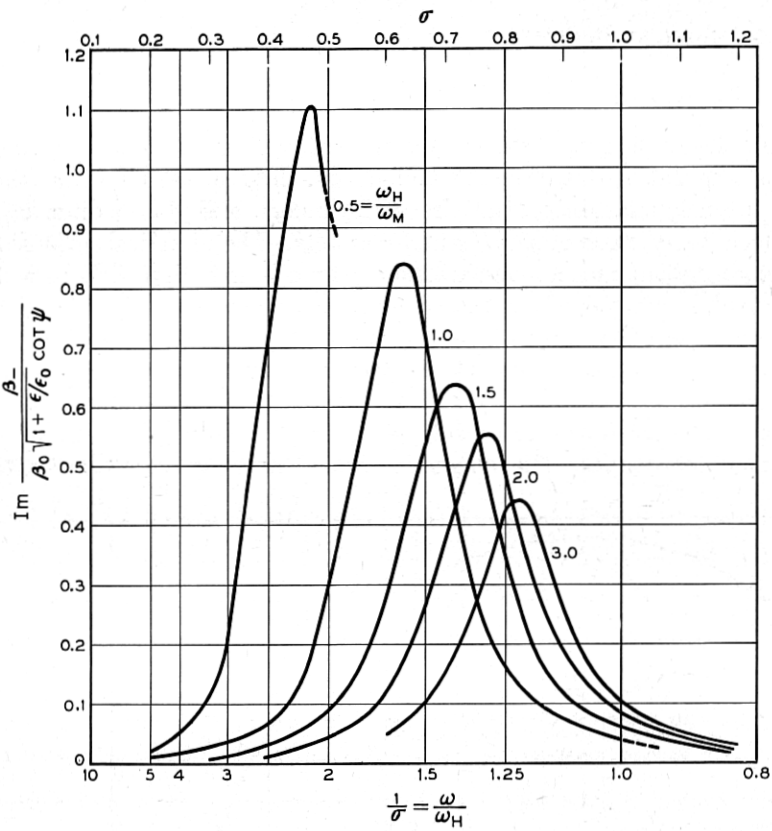

Fig. 8 — Attenuation of reverse wave versus frequency at a fixed magnetic field and saturation magnetization for various values of $\omega_H/\omega_M = (\mu_0 H)/M$. The curves for 0.5 and 1 are discontinued when p reaches unity.

are supposed to be independent of radial distance from the cylinder axis. For the geometries employed in practice, this will be a reasonably good assumption. Thus it is possible to relate the components of B and H in cylindrical coordinates (r, φ, z) through the tensor

$$\begin{Vmatrix} \mu & 0 & -j\kappa \\ 0 & \mu_0 & 0 \\ j\kappa & 0 & \mu \end{Vmatrix},$$

where μ and κ are given by the Polder relations (1) and are independent of position.

Written in cylindrical coordinates, Maxwell's equations in the ferrite are therefore

$$j\beta H_\varphi = j\omega\epsilon_1 E_r \,, \tag{20a}$$

$$-\,j\beta H_r - \frac{\partial H_z}{\partial r} = j\omega\epsilon_1 E_\varphi \,, \tag{20b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\, rH_\varphi = j\omega\epsilon_1 E_z \,, \tag{20c}$$

$$j\beta E_\varphi = -\,j\omega\mu(H_r - j\rho_H H_z), \tag{20d}$$

$$-\,j\beta E_r - \frac{\partial E_z}{\partial r} = -\,j\omega\mu_0 H_\varphi \,, \tag{20e}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\, rE_\varphi = -\,j\omega\mu(j\rho_H H_r + H_z). \tag{20f}$$

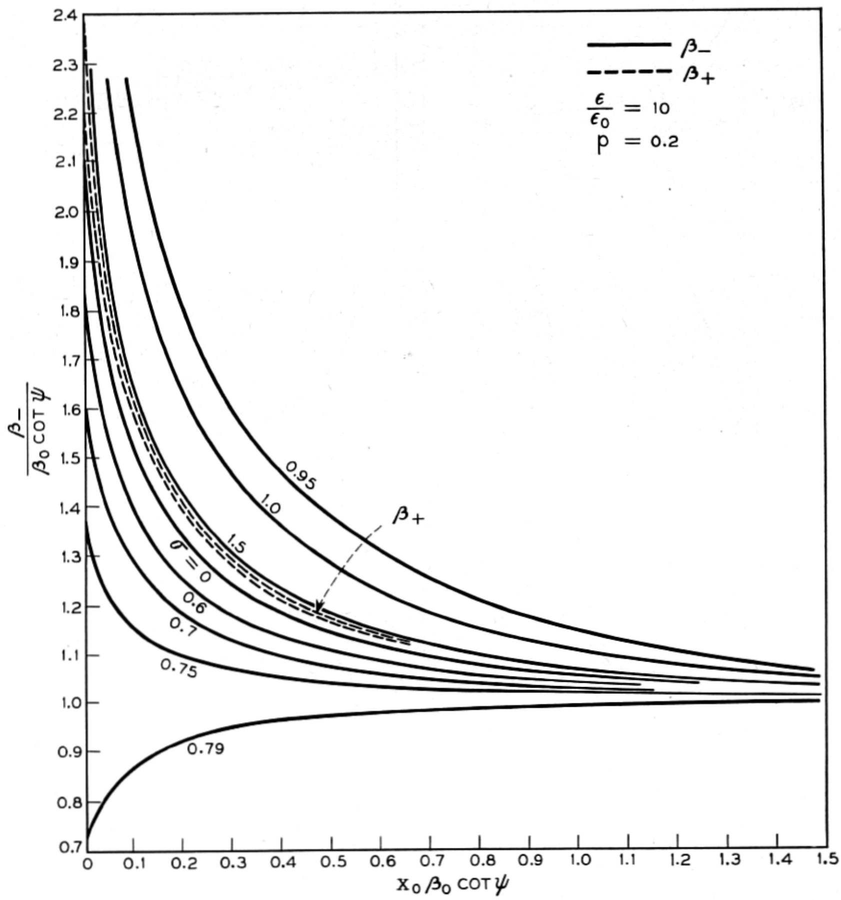

Fig. 9 — Propagation constants for a flat helix separated from the ferrite by a distance x_0 for various values of σ. (Loss-free case) (a) β_- and β_+ for $p = 0.2$, (b) β_- for $p = 0.8$, (c) β_+ for $p = 0.8$. In (a), above, the dotted lines bound all σ values.

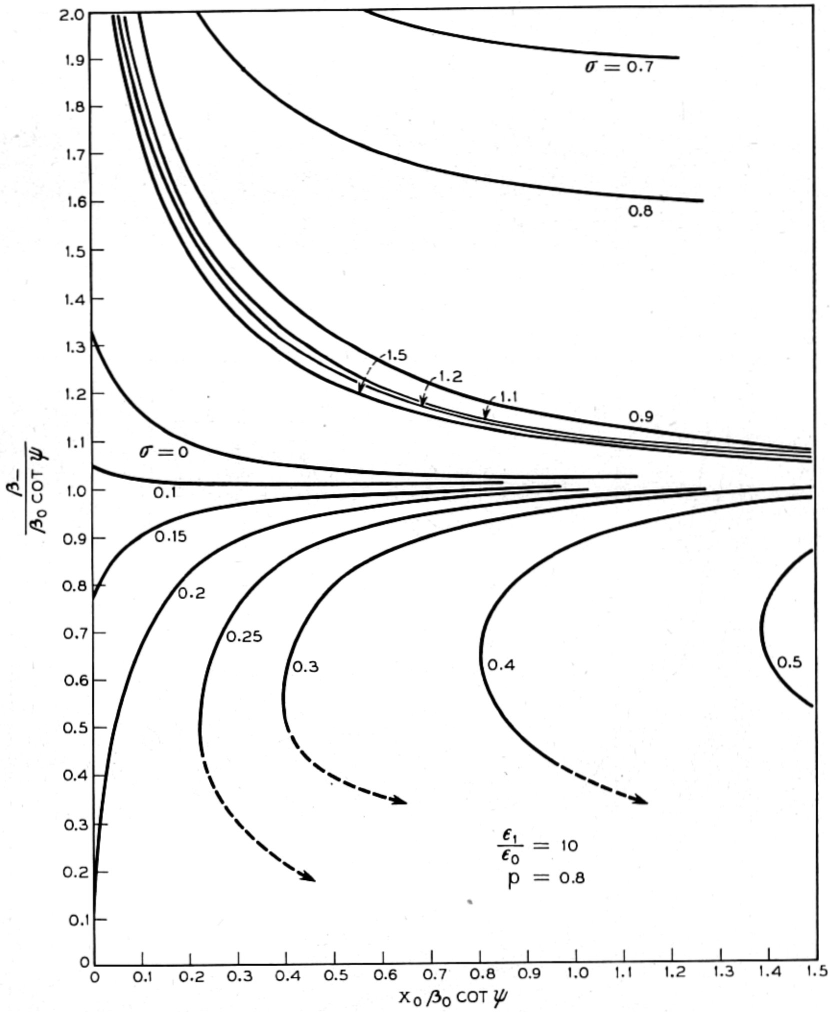

Fig. 9(b) — See Fig. 9.

As in the case of planar geometry, the field-components can be grouped into TE and TM parts; only the TE part will depend explicitly on the gyromagnetic properties of the medium. Equations (20a, 20c) and (20e) determine the TM field. E_r and H_φ can be eliminated from them, yielding the familiar wave equation

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial E_z}{\partial r} + (\beta_1^2 - \beta^2)E_z = 0; \qquad \beta_1^2 = \omega^2\mu_0\epsilon_1\,, \tag{21}$$

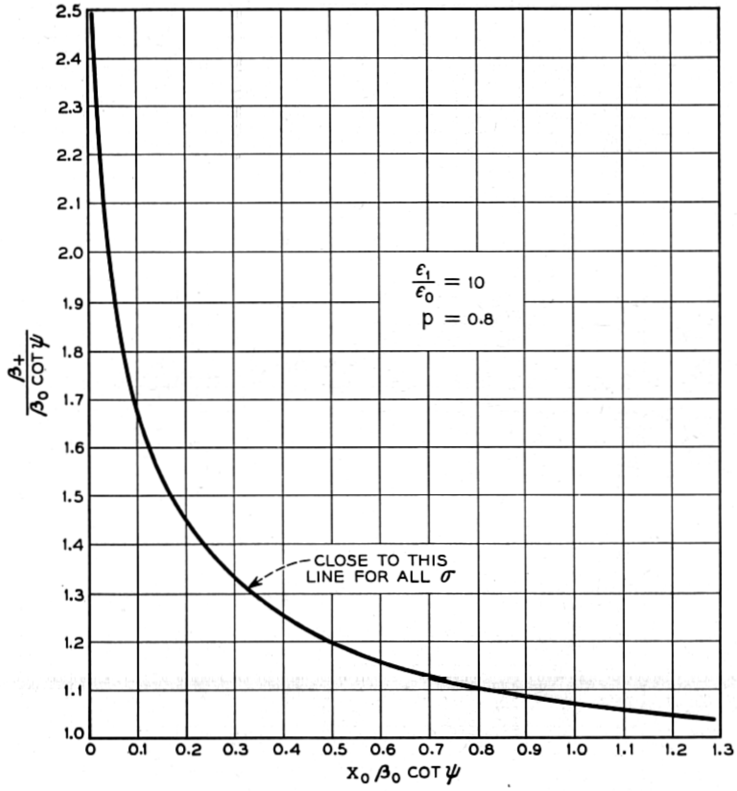

Fig. 9(c) — See Fig. 9.

whose solutions are zero order Bessel functions (or linear combinations thereof) of a kind depending on the region under consideration. Thus if $\beta > \beta_1$, and the region occupied by the medium includes the cylinder axis, the modified Bessel function $I_0\,(r\sqrt{\beta^2 - \beta_1^2})$ has to be chosen if the field is to be finite at $r = 0$; if the medium extends from a finite r to infinity, the function $K_0\,(r\sqrt{\beta^2 - \beta_1^2})$, regular at infinity, is selected. Correspondingly, if $\beta < \beta_1$, the Bessel and Hankel function $J_0\,(r\sqrt{\beta_1^2 - \beta^2})$ and $H_0^{1,2}\,(r\sqrt{\beta_1^2 - \beta^2})$ replace I_0 and K_0 respectively.

In terms of the appropriate solution of (21), the remaining field components are

$$E_r = -j\,\frac{\beta}{\beta_1^2 - \beta^2}\,\frac{\partial E_z}{\partial r}\,; \qquad H_\varphi = -\frac{j\omega\epsilon_1}{\beta_1^2 - \beta^2}\,\frac{\partial E_z}{\partial r}\,.$$

The tangential admittance for the TM field is thus

$$Y_{\text{TM}} = \frac{H_\varphi}{E_z} = -\frac{j\omega\epsilon_1}{\beta_1^2 - \beta^2}\,\frac{\partial}{\partial r}\log E_z\,. \tag{22}$$

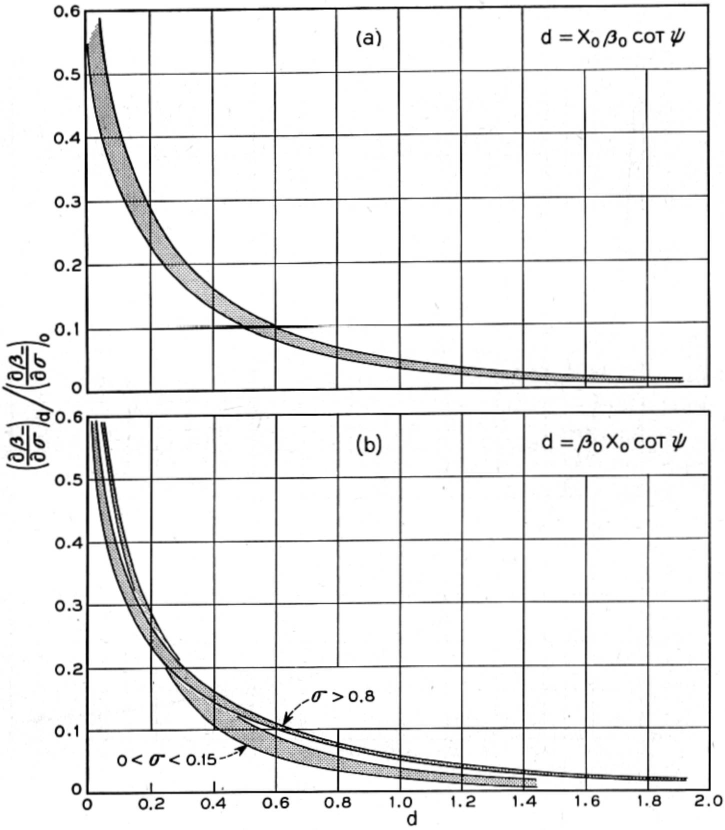

Fig. 10 — Ratios of reverse attenuation for a flat helix separated from the ferrite by a distance x_0 to the reverse attenuation at $x_0 = 0$. Computations made from the approximate slope-formula for the loss, where applicable. (a) $p = 0.2$. (b) $p = 0.8$. In (a) all applicable σ values lie in the shaded region.

For example, if $\beta > \beta_1$, and the medium occupies all space from a finite r to infinity

$$Y_{\text{TM}} = \frac{j\omega\epsilon_1}{\alpha_1} \frac{K'_0(\alpha_1 r)}{K_0(\alpha_1 r)} = \frac{-j\omega\epsilon_1 K_1(\alpha_1 r)}{\alpha_1 K_0(\alpha_1 r)}, \tag{23}$$

where $\alpha_1 = \sqrt{\beta^2 - \beta_1^2}$.

The field components of the TE field are determined by equations (20b), (20d) and (20f). Elimination of E_φ and H_r from these gives

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial H_z}{\partial r} + \left(\beta_f^2 - \beta^2 - \frac{\rho_H \beta}{r}\right) H_z = 0, \tag{24}$$

where $\beta_f^2 = \omega^2 \mu\epsilon_1 (1 - \rho_H^2)$. The term $\beta_f^2 - \beta^2$ in the bracket is already familiar from the planar case; it depends on the magnetic field and on the propagation direction in a purely reciprocal way. The term $\rho_H\beta/r$, however, reverses sign when either magnetizing field or propagation constant changes sign. The solutions of equation (24) are therefore different for opposite propagation directions (or opposite magnetizations). Thus, in contrast to the planar case, where non-reciprocity arose only through the boundary conditions, the cylindrical case is inherently non-reciprocal.

In the absence of the last term in the bracket, equation (24) would be solved by zero order Bessel functions, just like equation (21). In the presence of this term, the solutions become confluent hypergeometric functions. Different changes of variable bring these functions into forms known by different names and notations. One such change leads to Laguerre functions, another to Whittaker functions. We shall choose the latter representation, since it is closely related to Bessel functions, and numerical tables seem about equally scarce for all representations. In equation (24), let $\beta^2 - \beta_f^2 = \alpha_2^2$, and let $\alpha_2 r = y$. Then it becomes

$$\frac{1}{y}\frac{d}{dy}y\frac{dH_z}{dy} - \left(1 - \frac{2\chi}{y}\right)H_z = 0, \tag{25}$$

where $\chi = -\beta\rho_H/2\alpha_2$. Further, let $y = x/2$, and $H_z(y) = h(x)/\sqrt{x}$; then equation (25) becomes

$$\frac{d^2h}{dx^2} + \left(\frac{1}{4x^2} + \frac{\chi}{x} - \frac{1}{4}\right)h = 0, \tag{26}$$

which is the standard form of the equation for zero order Whittaker functions. It is a special case of the equation for μ^{th} order Whittaker functions:

$$\frac{d^2h}{dx^2} + \left(\frac{\frac{1}{4} - \mu^2}{x^2} + \frac{\chi}{x} - \frac{1}{4}\right)h = 0, \tag{27}$$

The solution of equation (27) which is regular near zero is denoted in the literature by $M_{\chi,\mu}(x)$; it is proportional, in the limit $\chi = 0$, to the Bessel function $I_\mu(x/2)$. The solution of equation (27) regular at infinity is denoted by $W_{\chi,\mu}(x)$, and in the limit $\chi = 0$, is proportional to $K_\mu(x/2)$. The factors of proportionality are found in Appendix I.

In this notation, the solutions for H_z are thus

$$\frac{M_{\chi,0}(2\alpha_2 r)}{\sqrt{2\alpha_2 r}}, \qquad \frac{W_{\chi,0}(2\alpha_2 r)}{\sqrt{2\alpha_2 r}}\,{}^*.$$

* If $\beta < \beta_2$, both argument and suffix χ are imaginary. These functions are then related to J_0 and H_0 respectively.

Once the appropriate H_z is determined, H_r and E_φ are given by

$$\begin{aligned} H_r &= \frac{j}{\beta^2 - \omega^2\mu\epsilon_1}\left(\beta\frac{\partial H_z}{\partial r} - \omega^2\mu\epsilon_1\rho_H H_z\right), \\ E_\varphi &= -\frac{j\omega\mu}{\beta^2 - \omega^2\mu\epsilon_1}\left(\frac{\partial H_z}{\partial r} - \rho_H\beta H_z\right). \end{aligned} \tag{28}$$

The tangential impedance for TE-fields is thus

$$Z_{\text{TE}} = \frac{E_\varphi}{H_z} = \frac{\omega\mu}{j(\beta^2 - \omega^2\mu\epsilon_1)}\left(\frac{\partial}{\partial r}\log H_z - \rho_H\beta\right). \tag{29}$$

The reader will recall that for isotropic media ($\rho_H = 0$) the numerator of the right hand side of equation (29) can always be expressed as the ratio of first order to zero order Bessel functions by virtue of relations like $K_0'(\chi) = -K_1(\chi)$, $I_0'(\chi) = I_1(\chi)$ and so on. Analogous results hold true in the present case. Suppose, for example, that we are dealing with a geometry such that the correct H_z is

$$H_z = \frac{W_{\chi,0}(2\alpha_2 r)}{\sqrt{2\alpha_2 r}} = R_{\chi 0}(2\alpha_2 r), \qquad \text{say}$$

Then

$$\frac{1}{H_z}\frac{\partial H_z}{\partial r} = 2\alpha_2 R_{\chi 0}'/R_{\chi 0},$$

and

$$Z_{\text{TE}} = \frac{2\alpha_2\omega\mu}{i(\beta^2 - \omega^2\mu\epsilon_1)}\frac{R_{\chi 0}' + \chi R_{\chi 0}}{R_{\chi 0}}.$$

It is shown in Appendix I that

$$R_{\chi 0}' + \chi R_{\chi 0} = (\chi - \tfrac{1}{2})\,R_{\chi 1}.$$

Therefore

$$\begin{aligned} Z_{\text{TE}} &= \frac{\omega\mu\alpha_2(2\chi - 1)}{j(\beta^2 - \omega^2\mu\epsilon_1)}\frac{R_{\chi 1}(2\alpha_2 r)}{R_{\chi 0}(2\alpha_2 r)}, \\ &= \frac{\omega\mu\alpha_2(2\chi - 1)}{j(\beta^2 - \omega^2\mu\epsilon_1)}\frac{W_{\chi,1}(2\alpha_2 r)}{W_{\chi,0}(2\alpha_2 r)}. \end{aligned} \tag{30}$$

A similar difference relation shows that if the region is such that

$$H_z = M_{\chi,0}(2\alpha_2 r)/\sqrt{2\alpha_2 r},$$

then

$$Z_{\mathrm{TE}} = \frac{\omega\mu\alpha_2(\tfrac{1}{4} - \chi^2)M_{\chi,1}(2\alpha_2 r)}{j(\beta^2 - \omega^2\mu\epsilon_1)M_{\chi,0}(2\alpha_2 r)}. \tag{31*}$$

In the unmagnetized case equations (30) and (31) reduce to

$$-\frac{\omega\mu_0\alpha_1}{j(\beta^2 - \omega^2\mu_0\epsilon_1)}\frac{K_1(\alpha_1 r)}{K_0(\alpha_1 r)} = -\frac{\omega\mu_0 K_1(\alpha_1 r)}{j\alpha_1 K_0(\alpha_1 r)}, \tag{32}$$

and to

$$\frac{\omega\mu_0}{j\alpha_1}\frac{I_1(\alpha_1 r)}{I_0(\alpha_1 r)}. \tag{33}$$

One might be led to believe that the search for solutions of Maxwell's equations with angular dependence $e^{jn\varphi}$ will lead to n^{th} order Whittaker functions (just as in the isotropic case this dependence leads to n^{th} order Bessel functions). Such is not the case, however. Unless $n = 0$, one is led to two simultaneous second order equations for E_z and H_z, and the character of the problem is changed completely.

3.2. *The Cylindrical Helix*

We are now in a position to derive the characteristic equation for a close wound cylindrical helix and approximated by a helically conducting sheet surrounded by ferrite. We confine the discussion to the case in which the ferrite is in actual contact with the helix, Fig. 11; the case of finite separation discussed for the planar helix (Section 2.3) would be too cumbersome here. Losses are neglected. If they are not excessive, they can be deduced from the curves for the propagation constant in the loss free sample by differentiation, as outlined in Sections 2.1 and 4.15, Part I.

The boundary conditions are just the same as in the planar case. In Section 2.3 they were stated in terms of admittances, and it is only necessary to substitute for these the admittances just derived for cylindrical geometry. Thus for H_y/E_z we substitute Y_{TM}, and for H_z/E_φ we write $Y_{\mathrm{TE}} = 1/Z_{\mathrm{TE}}$.

If superfixes i and e refer to the inside and outside of the helix (in Section 2.3 on the plane helix i and e were denoted by "$-$", and "$+$"), the characteristic equation is

$$\{Y_{\mathrm{TM}}^{(i)} - Y_{\mathrm{TM}}^{(e)}\}_{r=r_0}\cot^2\psi = (Y_{\mathrm{TE}}^{(i)} - Y_{\mathrm{TE}}^{(e)})_{r=r_0}, \tag{34}$$

where r_0 is the radius of the helix.

* The appearance of different factors $(2\chi - 1)$ and $(\tfrac{1}{4} - \chi^2)$ is simply due to the way the functions W, M are normalized in the literature.

For waves bound to the helix, $Y_{\text{TE}}^{(e)}$ is to be derived from that solution of (24) which tends to zero ar $r \to \infty$. This solution is $W_{\chi,0}(2\alpha_2 r)/\sqrt{2\alpha_2 r}$, and so $Y_{\text{TE}}^{(e)}$ is given by equation (30)

$$Y_{\text{TE}}^{(e)} = \frac{1}{Z_{\text{TE}}} = \frac{j}{\omega\mu} \frac{\beta^2 - \omega^2\mu\epsilon_1}{\alpha_2(2\chi - 1)} \frac{W_{\chi,0}(2\alpha_2 r)}{W_{\chi,1}(2\alpha_2 r)}.$$

Similarly, from equation (23), we have, for bound waves, with $E_z \sim K_0$,

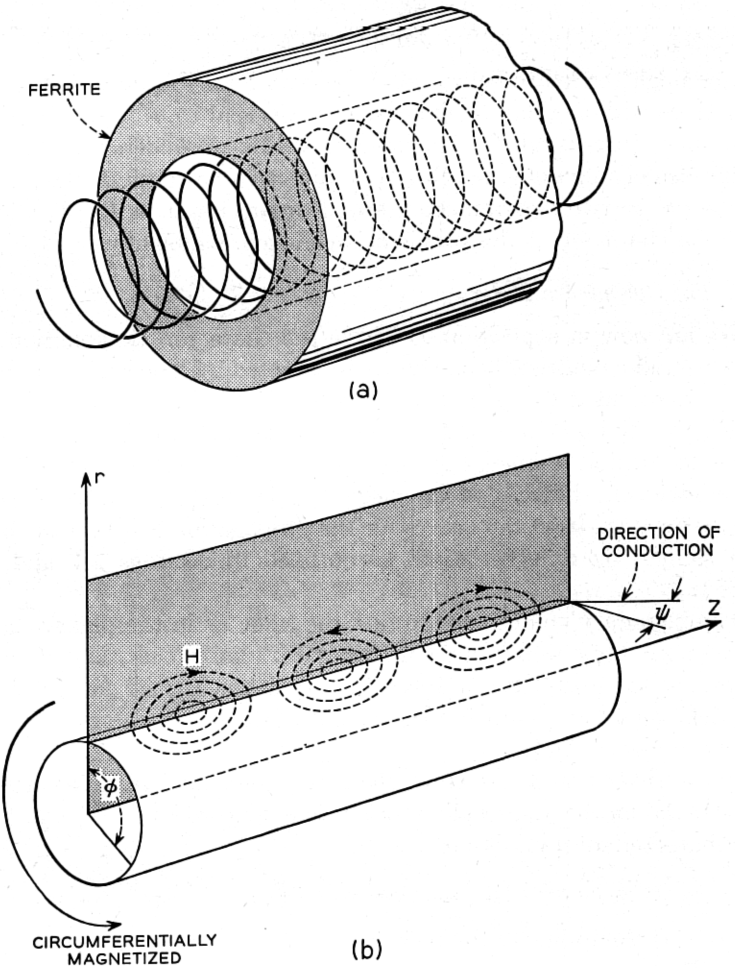

Fig. 11 — (a) Cylindrical helix surrounded by ferrite, (b) Magnetic field lines projected onto a plane containing the axis of the helix.

$$Y_{TM}^{(e)} = -\frac{j\omega\epsilon_1}{\alpha_1}\frac{K_1(\alpha_1 r)}{K_0(\alpha_1 r)}.$$

Inside the helix, we require the solution for an isotropic region which remains regular as $r \to 0$. Accordingly

$$Y_{TE}^{(i)} = \frac{j\alpha_0}{\omega\mu_0}\frac{I_0(\alpha_0 r)}{I_1(\alpha_0 r)}; \qquad \alpha_0 = \sqrt{\beta^2 - \beta_0^2},$$
$$\beta_0^2 = \omega^2\mu_0\epsilon_0$$

and

$$Y_{TM}^{(i)} = j\frac{\omega\epsilon_0}{\alpha_0}\frac{I_1(\alpha_0 r_0)}{I_0(\alpha_0 r_0)},$$

where ϵ_0, μ_0 are the dielectric constant, and permeability of vacuum. Combining these expressions in (34), we obtain after slight rearrangement

$$\frac{I_1(\alpha_0 r_0)}{I_0(\alpha_0 r_0)} + \frac{\alpha_0}{\alpha_1}\frac{\epsilon_1}{\epsilon_0}\frac{K_1(\alpha_1 r_0)}{K_0(\alpha_1 r_0)} = \left[\alpha_0^2\frac{I_0(\alpha_0 r_0)}{I_1(\alpha_0 r_0)} - \frac{\mu_0\alpha_0}{\mu\alpha_2}\frac{\beta^2 - \omega^2\mu\epsilon_1}{(2\chi - 1)}\frac{W_{\chi,0}(2\alpha_2 r)}{W_{\chi,1}(2\alpha_2 r)}\right]\tan^2\psi \tag{35}$$

which determines β.

A complete solution of equation (35) is out of the question. However, as in the planar case, for the slow waves used in traveling wave tube work, the equation may be simplified so that solutions may be computed rather easily. For electron velocities usually employed the resultant β must be about $10\beta_0$. Therefore in equation (35) it will be permissible to neglect all the quantities β_0^2, β_1^2, β_2^2, $\omega^2\mu\epsilon_1$, in comparison with β^2, except in the narrow ranges of magnetic field such that μ or $\mu(1 - \rho_H^2)$ becomes very large. This will occur near $\pm\sigma_0$ where $\sigma_0 = -p/2 + \sqrt{p^2/4 + 1}$, and near $\sigma = 1$. A solution obtained by assuming a large β must be self-consistent; that is, it can be credited only in regions where it does, in fact, predict large β. However, in Section 2.3 it was shown for the plane helix that in any practical case the ranges of magnetic fields so excluded are very narrow, even in the loss-free case, and one may suppose that this is true also in the cylindrical case.

For slow waves, each of the α's reduces to $|\beta|$; the absolute value sign derives from the fact that the positive square root is implied in the definition of the α's. Therefore

$$\chi \to \frac{-\rho_H\beta}{2|\beta|} = -\frac{\rho_H}{2}\operatorname{sgn}\beta. \tag{36}$$

Now the suffix χ of the Whittaker functions no longer depends on the magnitude of β, and it is chiefly for this reason that further progress is possible. For large β, equation (35) can be written

$$\beta^2 = \beta_0^2 \cot^2 \psi \frac{\dfrac{I_1(|\beta| r_0)}{I_0(|\beta| r_0)} + \dfrac{\epsilon_1}{\epsilon_0} \dfrac{K_1(|\beta| r_0)}{K_0(|\beta| r_0)}}{\dfrac{I_0(|\beta| r_0)}{I_1(|\beta| r_0)} - \dfrac{1}{\dfrac{\mu}{\mu_0}(2\chi - 1)} \dfrac{W_{\chi,0}(2|\beta| r_0)}{W_{\chi,0}'(2|\beta| r_0)}}. \tag{37}$$

where χ is now given by equation (36). Equation (37) is now solved by

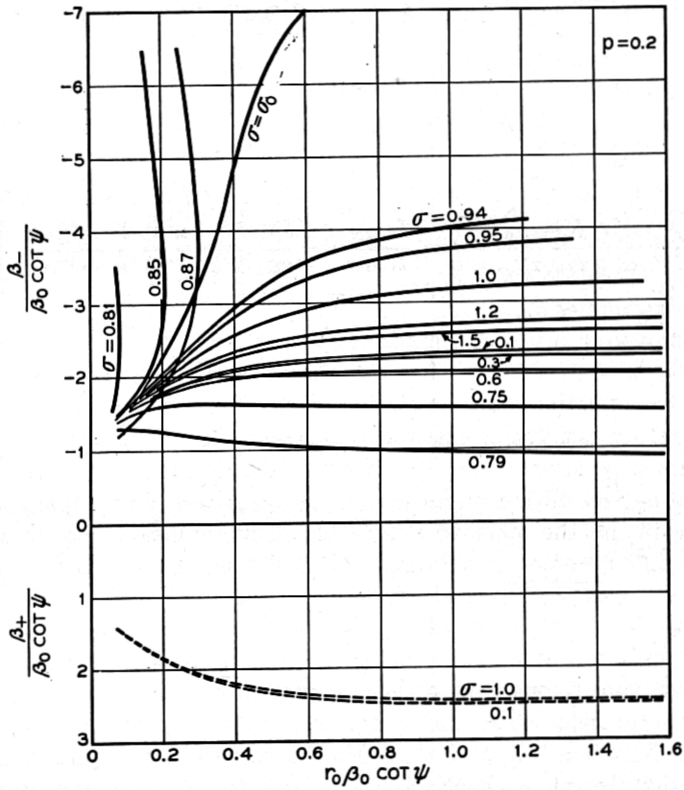

Fig. 12 — Reduced forward and reverse propagation constants versus reduced radius of a cylindrical helix (loss-free case) for various σ and p. The range $\sigma_1 < \sigma < \sigma_0$ where

$$\sigma_1 = \sqrt{1 + \frac{p^2}{4} - \frac{p}{3}} - \frac{p}{2} \quad \text{and} \quad \sigma_0 = \sqrt{1 + \frac{p^2}{4}} - \frac{p}{2}$$

contains an infinity of shape resonances and is not shown here. (a), above, $p = 0.2$. (b) $p = 0.6$. (c) $p = 1.0$.

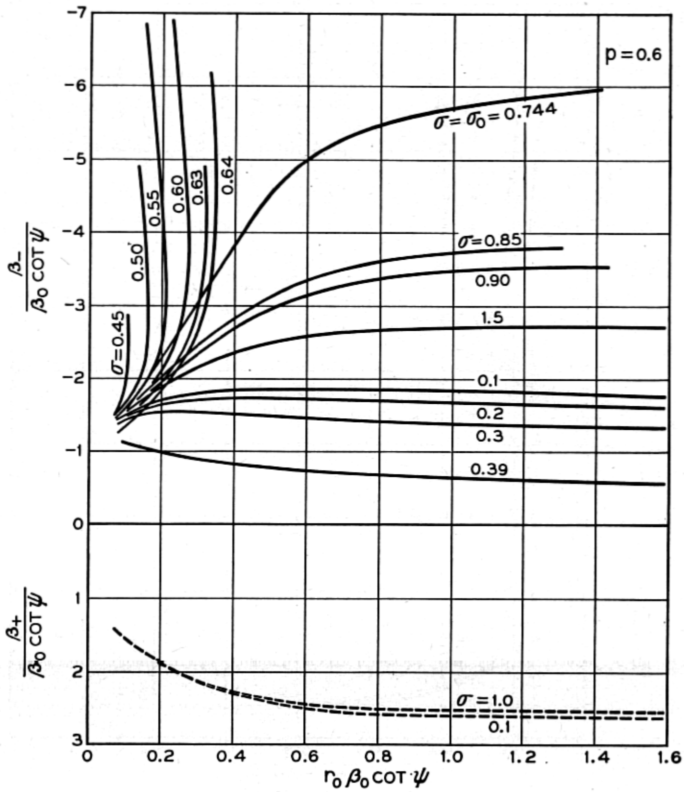

Fig. 12(b) — See Fig. 12.

the following procedure: Introduce the parameter

$$\frac{u}{2} = |\beta| r_0 . \tag{38}$$

For a given ρ_H and u (or σ and p), and a given sign of β, each value of u determines β through equation (37), and then r_0 through equation (38). Thus β can be plotted versus r_0. The procedure is repeated for the opposite sign of β (and therefore the opposite sign of χ). A different curve of β versus r_0 is then obtained. Thus for a given value of r_0, the "forward" and "backward" propagation constants are different in magnitude. The results (computed for a typical ratio $\epsilon_1/\epsilon_0 = 10$) are conveniently stated in terms of $\bar{\beta} = \beta/(\beta_0 \cot \psi)$ and $\bar{r}_0 = r_0\beta_0 \cot \psi$ and are shown in Fig. 12(a) to (c), and again, for fixed $\bar{r}_0$, in Fig. 13(a) to (e). We note that for $\bar{r}_0$ in excess of about 1.5, the results are almost the same as those

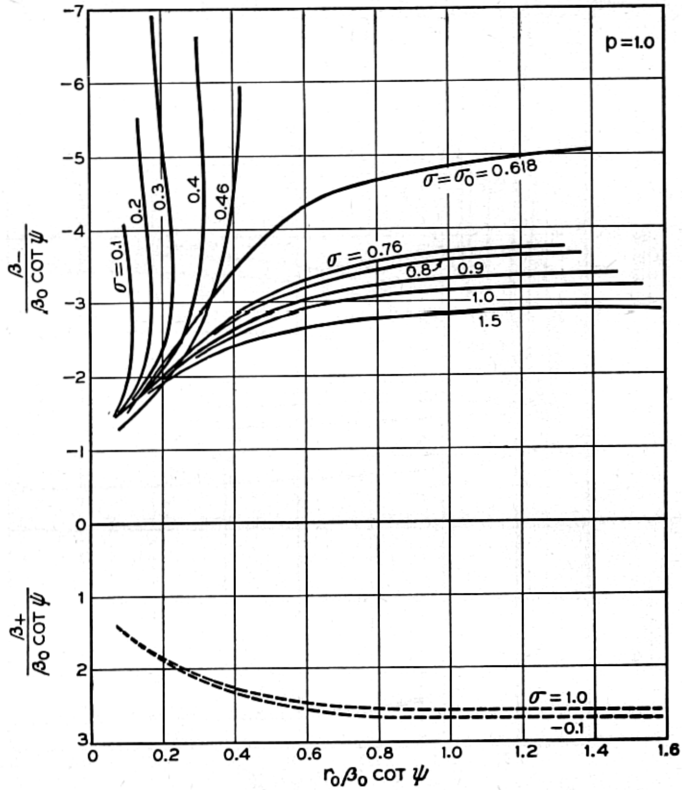

Fig. 12(c) — See Fig. 12.

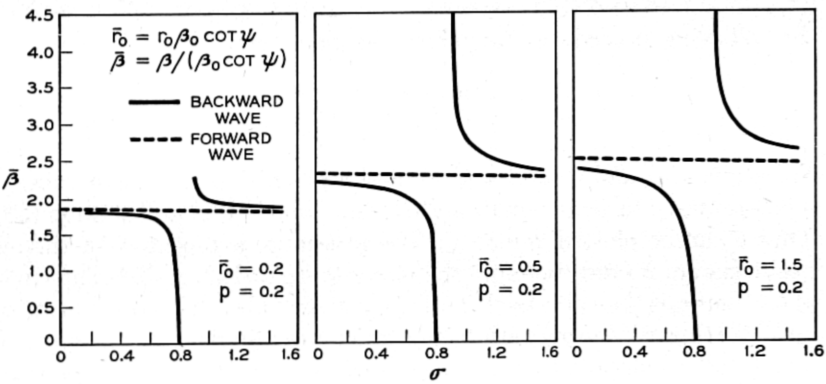

Fig. 13 — Reduced forward and reverse propagation constants versus σ for various reduced radii of a cylindrical helix. (Loss free case). The region $1 - p < \sigma < \sigma_0$ is omitted. (a), above, $p = 0.2$. (b) $p = 0.4$. (c) $p = 0.6$. (d) $p = 0.8$. (e) $p = 1.0$.

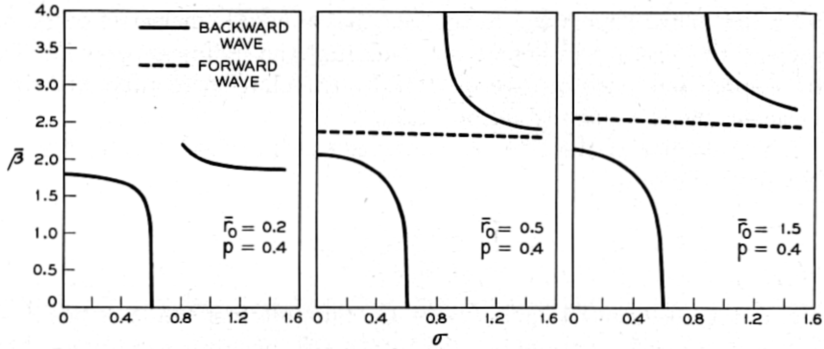

Fig. 13(b) — See Fig. 13.

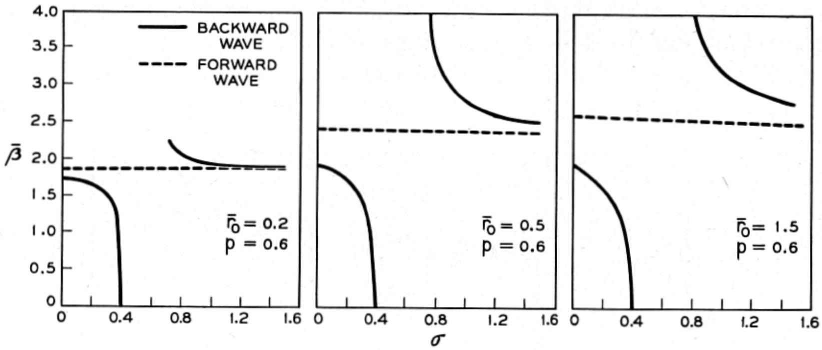

Fig. 13(c) — See Fig. 13.

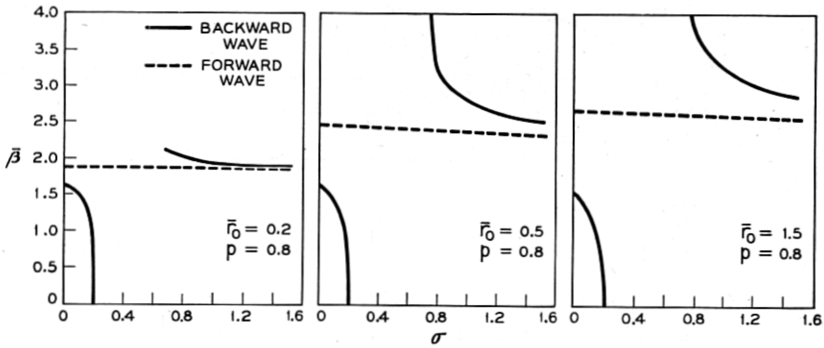

Fig. 13(d) — See Fig. 13.

for a flat helix. This is due to the fact that the large dielectric constant reduces the circuit wavelength so much that the radius appears infinite by comparison once it exceeds 1.5. In traveling-wave tube practice, however, $\bar{r}_0$ is generally below 1.5.

The behavior of the $\bar{\beta} - \bar{r}_0$ curves can be understood from the behavior of the ratio

$$\frac{W_{\chi,0}(u)}{W_{\chi,1}(u)} = Z_\chi(u)$$

and of the coefficient $1/[\mu/\mu_0\ (2\chi - 1)]$. Suppose first that χ is positive. When χ exceeds zero only slightly, $Z_\chi(u)$ behaves essentially like $K_0(u/2)/K_1(u/2)$. This function is always positive; it begins at 0 when $u = 0$ with a vertical tangent and steadily increases to unity as $u \rightarrow \infty$. $Z_\chi(u)$ varies in the same way, see Figs. 14 and 15, in the range $0 < \chi < \frac{1}{2}$. For $\frac{3}{2} > \chi \geqq \frac{1}{2}$, $W_{\chi,0}$, and therefore Z_χ, has a zero which increases from $u = 0$ at $\chi = \frac{1}{2}$ to $u = 1$ at $\chi = \frac{3}{2}$. Accordingly $Z_\chi(u)$ in $\frac{3}{2} >$

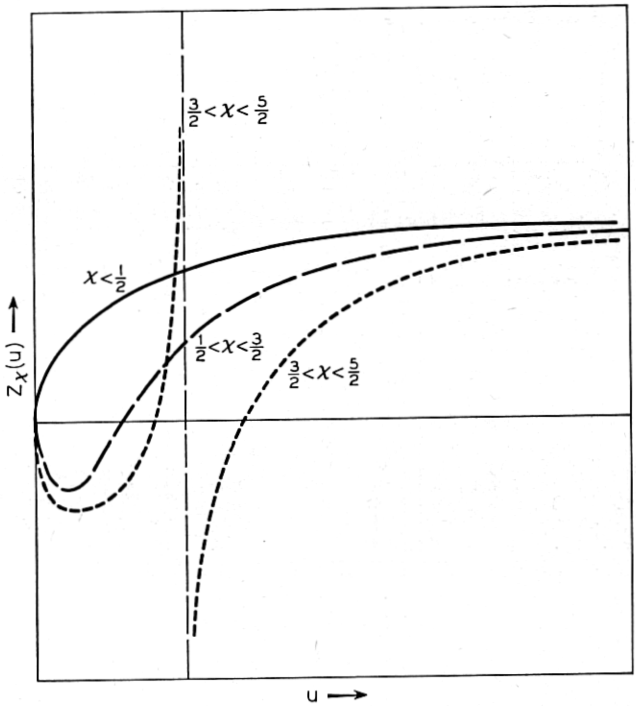

Fig. 14 — Schematic behavior of the function $Z_\chi(u) = W_{\chi,0}(u)/W_{\chi,1}(u)$ in various ranges of χ.

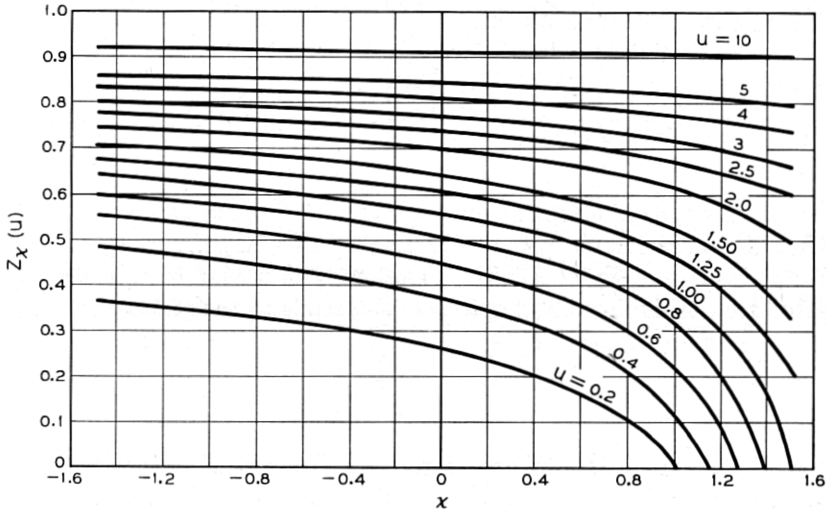

Fig. 15 — The function $Z_\chi(u)$ versus χ for various u, in the range $-\frac{3}{2} < \chi < \frac{3}{2}$.

$\chi > \frac{1}{2}$ starts from 0 at $u = 0$ with a downward-directed vertical tangent, achieves a negative minimum, then increases, through its zero, to the asymptotic value unity as $u \to \infty$. The minimum becomes deeper as χ approaches $\frac{3}{2}$. At $\chi = \frac{3}{2}$, $W_{\chi,1}(u)$ developes a zero at $u = 0$, which steadily moves to larger u as χ increases further towards $\frac{5}{2}$. At the same time the zero of $W_{\chi,0}$ already discussed moves from 1 to $2 + \sqrt{2}$, and a new zero arises at $u = 0$, $\chi = \frac{3}{2}$, which increases to $2 - \sqrt{2}$ as χ approaches $\frac{5}{2}$, but which lags behind the zero of $W_{\chi,1}(u)$. The function $Z_\chi(u)$ now has a pole and two zeros, and behaves as shown in Fig. 14. This process continues; each time χ passes $(2n + 1)/2$, a new zero and a new pole appear. (For a detailed list of poles and zeros the reader is referred to Appendix I). To apply these results, we first resort to the Polder relations.

In terms of σ, p, we have, for β negative

$$\chi = \frac{1}{2} \frac{p}{1 - p\sigma - \sigma^2}$$

and

$$\frac{1}{\frac{\mu}{\mu_0}(2\chi - 1)} = \frac{1 - \sigma}{p - 1 + \sigma} = A, \qquad \text{say.}$$

The characteristic equation is

$$\bar{\beta}^2 = \frac{\dfrac{I_1(u/2)}{I_0(u/2)} + \dfrac{\epsilon_1}{\epsilon_0}\dfrac{K_1(u/2)}{K_0(u/2)}}{\dfrac{I_0(u/2)}{I_1(u/2)} - AZ_\chi(u)}; \qquad |\bar{\beta}|\bar{r}_0 = u/2 \tag{37}$$

and can now be discussed in terms of σ at a fixed p. Suppose that $p < 1$. Then A is negative for $\sigma < 1 - p$. In the same range, $\chi < \frac{1}{2}$, so that Z behaves essentially as K_0/K_1. Therefore both numerator and denominator are positive for all u, and the ratio tends to the planar result

$$\bar{\beta}^2 = \frac{1 + \dfrac{\epsilon_1}{\epsilon_0}}{1 - A}$$

as $u \to \infty$. As $u \to 0$, $\bar{\beta}^2$ tends to zero along the vertical, as can be shown by an examination of the various functions near $u = 0$. For $\sigma < 1 - p$, the course of the $\bar{\beta}^2$ versus u-curves is as shown schematically in Fig. 16(a), and it is easily seen that the $\bar{\beta}$ versus $\bar{r}_0$ curves run in essentially the same way, Fig. 12(a) to (c). However, as σ approaches $1 - p$, the $\bar{\beta}^2$ versus u curves steadily fall, until at $\sigma = 1 - p$, $\bar{\beta}^2 = 0$ for all finite u, since $A = -\infty$.

As σ passes $1 - p$, A changes sign and at the same time χ passes $\frac{1}{2}$ so that Z_χ acquires a zero. As σ varies from $1 - p$ to $1 - p/2$, A decreases from $+\infty$ to unity. Therefore, while $u < u_1$, the zero of Z_χ, $1 - AZ_\chi$ is positive; however as u increases beyond u_1, $\{I_0(u/2)/I_1(u/2)\} - AZ_\chi(u)$ eventually passes zero, since $Z_\chi(u)$ and $I_0(u/2)/I_1(u/2)$ both approach unity. On the other hand the numerator of equation (37) is

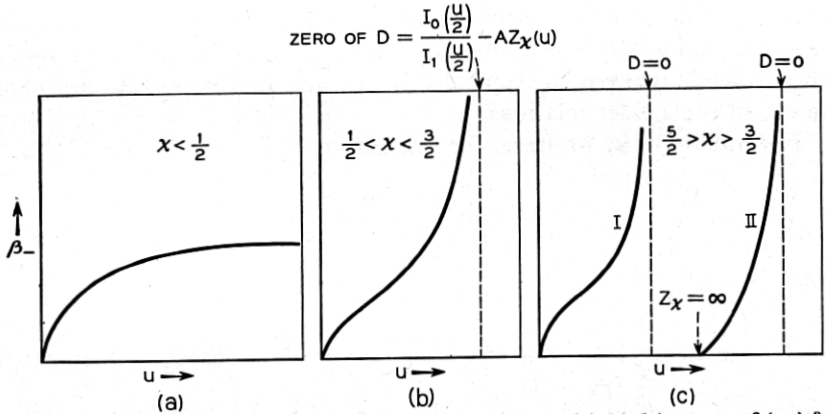

Fig. 16 — Schematic variation of β_- with u. a) $\chi < \frac{1}{2}$; b) $\frac{1}{2} < \chi < \frac{3}{2}$; c) $\frac{3}{2} < \chi < \frac{5}{2}$.

always positive; therefore $\bar{\beta}^2$ approaches infinity at $I_0(u/2)/I_1(u/2) - AZ_\chi(u) = 0$, and no real values of $\bar{\beta}$ exist thereafter (see Fig. 16b). Since this "cut-off" occurs at a finite value of u, the corresponding value of r_0 is zero. This explains the bulging of the corresponding $\bar{\beta} - \bar{r}_0$ curves in Fig. 12(a) to 12(c).

The next major change in the curves occurs when χ exceeds $3/2$, (that is, σ exceeds

$$\sigma_1 = -\frac{p}{2} + \sqrt{1 + \frac{p^2}{4} - \frac{p}{3}}.)$$

For $p < 2$, σ_1 is still less than $1 - (p/2)$, so that, initially at any rate, A is still greater than unity. In addition to the infinity of β^2 just discussed, a further infinity arises between $u = 0$, and the pole of $Z_\chi(u)$, as is seen from Fig. 16(c). β^2 increases from zero at $u = 0$ to this infinity, thereafter it is negative, until the pole of $Z_\chi(u)$ is reached. Thereupon it resumes at $\beta^2 = 0$ and approaches infinity at the zero of the denominator $[I_0(u/2)/I_1(u/2)] - AZ_\chi(u)$ already discussed. Thus there are now two branches of the $\bar{\beta}^2 - u$ curve; their corresponding traces in the $\bar{\beta} - \bar{r}_0$ plane are shown schematically in Fig. 17. (The computations on which Fig. 12(a) to 12(c) were based were broken off at $\sigma = \sigma_1$.)

A further branch is added each time χ increases beyond a number of the form $(2n + 1)/2$ (σ increases beyond

$$\sigma_n = -\frac{p}{2} + \sqrt{1 + \frac{p^2}{4} - \frac{p}{2n+1}}).$$

These all resemble the two branches just discussed, until $n >$

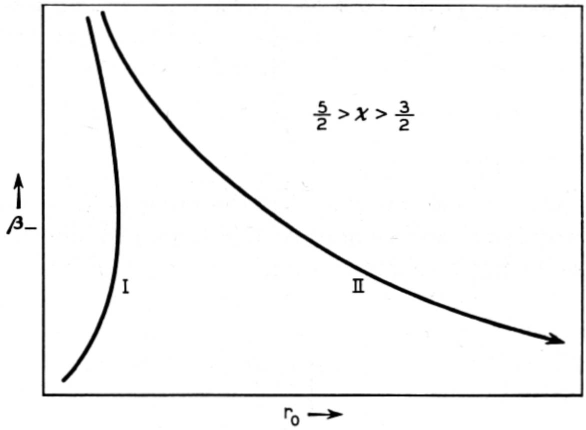

Fig. 17 — Schematic variation of β_- with r_0 for $3/2 < \chi < 5/2$.

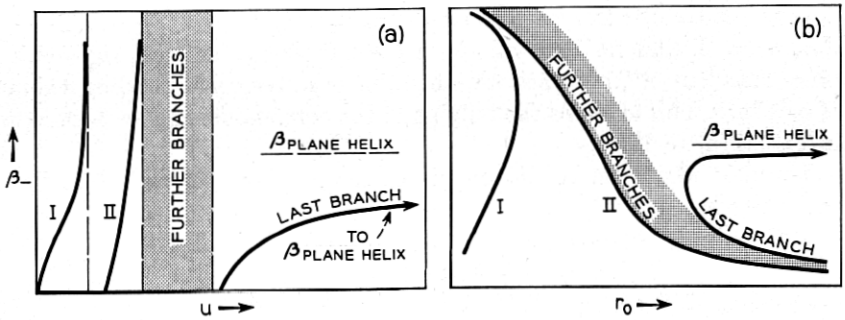

Fig. 18 — (a) β_- versus u when $1 - (p/2) < \sigma < \sigma_0$. (b) β_- versus r_0 under the same circumstances.

$\frac{1}{2}\left(\frac{4}{p} - 1\right)$. When this occurs, $\sigma_n > 1 - (p/2)$, so that there will be a value σ between σ_{n-1} and σ_n beyond which the denominator no longer decreases through zero as $u \to \infty$, but approaches a finite positive value, Fig. 18(a). Accordingly β^2 approaches a finite positive value, and cut-off of the extreme right hand branch, Fig. 16(c), no longer occurs. The corresponding $\bar{\beta}$ versus $\bar{r}_0$ branch is as in Fig. 18(b).

As $n \to \infty$ ($\sigma_n \to \sigma_0 = \sqrt{1 + (p^2/4)} - p/2$) the number of branches increases to infinity. This situation resembles that in the completely filled waveguide (Part I), where we found an infinity of modes ("Shape-resonances") in the range $\sigma_0 < \sigma < 1$. In the present case, however, they are to be found in the range $1 - p < \sigma < \sigma_0$.

When $\sigma = \sigma_0 + 0$, χ is infinite and negative. The function $Z_\chi(u)$ is then constant and equal to unity. A is less than unity, and the denominator of equation (27) has no zeros. The $\bar{\beta}$ versus $\bar{r}_0$ curve is now "normal" again, see Fig. 12(a). As σ increases further, the curve falls (since A decreases steadily to -1 as $\sigma \to \infty$), and no more qualitative changes occur.

3.3 *Cylindrical Waveguides*

As pointed out before, the fact that the propagation problem in the cylindrical case can always be integrated in terms of Whittaker functions when the fields show no angular variation is an accident, and in view of the lack of numerical tables, not a particularly fortunate one. Only in special cases (like that of the slow-wave helix) is the text-book information on these functions of any great utility. In general, it will be more convenient to solve the differential equations numerically. However, for completeness, we shall state some of the formal results for a cylindrical waveguide containing a cylinder of circumferentially magnetized ferrite, and propagating a TE_0 mode.

First we consider a waveguide, radius r_0, into which is fitted a hollow cylinder of ferrite, outer radius r_0, inner radius r_1. In that cylinder, the magnetic field H_z may be taken to be a superposition $AS_{\chi 0}(2\alpha_2 r) + BR_{\chi 0}(2\alpha_2 r)$ where, as before,

$$\alpha_2^2 = \beta^2 - \omega^2\epsilon_1\mu(1 - \rho_H^2); \qquad S(x) = \frac{M(x)}{\sqrt{x}}; \qquad R(x) = \frac{W(x)}{\sqrt{x}}.$$

$$\chi = -\frac{\beta\rho_H}{2\alpha_2}.$$

α_2 may be either real or positive imaginary. [In choosing this combination we depart from the usual practice of taking a superposition of J_0 and N_0 in the isotropic case. Were we to follow this practice, it would be necessary to define a new function $R_{\chi\mu}(2jx)e^{j(\mu\pi/2)} + R_{\chi\mu}(-2jx)e^{-j\mu\pi}$ to correspond to $N_\mu(x)$. Our choice corresponds to taking a combination of $J_0(x)$ and one of the Hankel functions $H_0(x)$ in the isotropic case. Since the functions H, J, N are linearly dependent, this will not affect the results.]

In view of the difference relations, equation (39) in Appendix I, and of equation (29) we obtain for the impedance in the ferrite

$$\frac{E_\varphi}{H_z} = -\frac{j\omega\mu\alpha_2}{(\beta^2 - \omega^2\mu\epsilon_1)} \frac{[A(\tfrac{1}{4} - \chi^2)S_{\chi 1}(2\alpha_2 r) + B(2\chi - 1)R_{\chi 1}(2\alpha_2 r)]}{[AS_{\chi 0}(2a_2 r) + BR_{\chi 0}(2a_2 r)]}.$$

A and B must be adjusted so that this quantity vanishes at r_0, the guide wall. This gives

$$\frac{E_\varphi}{H_z} = -(2\chi - 1)(\tfrac{1}{4} - \chi^2)\frac{j\omega\mu\alpha_2}{\beta^2 - \omega^2\mu\epsilon_1}$$

$$\frac{W_{\chi,1}(2\alpha_2 r_0)M_{\chi,1}(2\alpha_2 r) - M_{\chi,1}(2\alpha_2 r_0)W_{\chi,1}(2\alpha_2 r)}{(2_{\chi-1})W_{\chi,1}(2\alpha_2 r_0)M_{\chi,0}(2\alpha_2 r) - (\tfrac{1}{4} - \chi^2)M_{\chi,1}(2\alpha_\chi r_0)W_{\chi,0}(2\alpha_2 r)}.$$

In the vacuum, between $r = 0$ and $r = r_1$, H_z is $I_0(\alpha_0 r)$ and the impedance is $E_\varphi/H_z = -j(\omega\mu_0/\alpha_0)I_1(\alpha_0 r)/I_0(\alpha_0 r)$ where $\alpha_0^2 = \beta^2 - \omega^2\mu\epsilon_0$. At $r = r_1$, the ferrite-vacuum interface, the two impedances must be equal. Thus we obtain the characteristic equation

$$\frac{\mu_0}{\alpha_0}\frac{I_1(\alpha_0 r_1)}{I_0(\alpha_0 r_1)} = (2\chi - 1)(\tfrac{1}{4} - \chi^2)\frac{\alpha_2\mu}{\beta^2 - \omega^2\mu\epsilon_1}$$

$$\frac{W_{\chi,1}(2\alpha_2 r_0)M_{\chi,1}(2\alpha_2 r_1) - M_{\chi,1}(2\alpha_2 r_0)W_{\chi,1}(2\alpha_2 r_1)}{(2\chi - 1)W_{\chi,1}(2\alpha_2 r_0)M_{\chi,0}(2\alpha_2 r_1) - (\tfrac{1}{4} - \chi^2)M_{\chi,1}(2\alpha_2 r_0)W_{\chi,0}(2\alpha_2 r_1)}.$$

(It is understood that for "normal" waveguide propagation α_0 will be imaginary, and the I will be replaced by J). As a simple illustration we consider the case in which the ferrite cylinder fills the waveguide completely. E_φ is then proportional to $M_{\chi 1}(2\alpha_2 r)$, and so β is determined from the condition

$$M_{\chi 1}(2\alpha_2 r_0) = 0.$$

When "normal" propagation prevails (β less than the natural propagation constant of the medium $\omega\sqrt{\epsilon_1\mu(1 - \rho_H^2)}$) both α_2 and χ are imaginary, equal to $j\alpha_2'$ and $j\chi'$, say. Under these circumstances little is known about the zeros of M. However, it is possible to say something about the solution for large radial mode numbers. It follows from Erdélyi et al.[2] **1**, p. 278, formula (2), that for large argument

$$M_{j\chi',1}(2j\alpha_2' r_0) = \text{const} \cdot \sin[\alpha_2' r_0 + \chi' \log \alpha_2' r_0 + \chi' \log 2 + \Phi\,(\chi') - \pi/4].$$

where $\Phi(\chi') = \arg \Gamma(3/2 + j\chi')$. The zeros of this expression are at

$$\alpha_2' r_0 + \chi' \log \alpha_2' r_0 = \frac{5n\pi}{4} - \chi' \log 2 - \Phi(\chi'). \qquad n = \text{a large integer}$$

This equation may be solved graphically by setting $\alpha_2' r_0 = u$, assigning values to β, ρ_H, u (and hence to χ', α_2'). From a solution u one then finds

$$r_0 = u/\alpha_2'.$$

M also has zeros for real α_2, χ, if χ is large enough. Thus the waveguide will support waves with a β^2 greater than $\omega^2\mu\epsilon_1(1 - \rho_H^2)$. It is shown in Reference 2 (**1**, p. 289) that when χ is between $3/2$ and $5/2$, M has one zero, when χ is between $5/2$ and $7/2$, M has two zeros and so on. Suppose that ρ_H is negative, $= -\,|\,\rho_H\,|$. Then

$$\chi = \frac{\beta\,|\,\rho_H\,|}{\sqrt{\beta^2 - \beta_2^2}}.$$

For real positive β, this equation has a solution for β if $|\,\rho_H\,| < \chi$:

$$\beta = \frac{\chi\beta_2}{\sqrt{\chi^2 - \rho_H^2}}.$$

If $3/2 < \chi < 5/2$, M will have a zero $u(\chi)$ depending on the value of χ. Thus the equations

$$\beta = \frac{\chi\beta_2}{\sqrt{\chi^2 - \rho_H^2}}; \qquad r_0 = \frac{u(\chi)}{\sqrt{\beta^2 - \beta_2^2}} = \frac{\sqrt{\chi^2 - \rho_H^2}}{|\,\rho_H\,|\,\beta_2}\,u(\chi)$$

solve the propagation problem parametrically. Similarly when χ is between $5/2$ and $7/2$, there are two zeros of M given by two functions

$u_1(\chi)$ and $u_2(\chi)$. There are now two possible modes, with the same restrictions on ρ_H. An additional mode arises each time χ is allowed to pass a number of the form $(2n + 1)/2$. It is to be noted that these modes are not confined to the resonance range. For β positive, they can exist in the range $\infty > \sigma > \sigma_0$ and in the range $-\sigma_0 < \sigma < 0$.

Appendix I. Some Properties of Whittaker Functions Used in This Paper

I. RELATION TO BESSEL FUNCTIONS

$$\underset{x \to 0}{Lt} \frac{M_{\chi,\mu}(2jx)}{\sqrt{2jx}} = 2^{2\mu}\Gamma(\mu + 1)j^{\mu}J_\mu(x),$$

$$\underset{x \to 0}{Lt} \frac{M_{\chi,\mu}(2x)}{\sqrt{2x}} = 2^{2\mu}\Gamma(\mu + 1)I_\mu(x),$$

$$\underset{x \to 0}{Lt} \frac{W_{\chi,\mu}(2x)}{\sqrt{2x}} = \frac{K_\mu(x)}{\sqrt{\pi}},$$

$$\underset{x \to 0}{Lt} \frac{W_{\chi,\mu}(-2jx)}{\sqrt{-2jx}} = \frac{\sqrt{\pi}}{2} j^{\mu+1} H_\mu^{(1)}(x), \qquad \text{and}$$

$$\underset{x \to 0}{Lt} \left[\frac{W_{\chi,\mu}(2jx)}{\sqrt{2jx}} e^{j(\mu\pi/2)} + \frac{W_{\chi,\mu}(-2jx)}{\sqrt{-2jx}} e^{-j(\mu\pi/2)}\right] = -\sqrt{\pi}N_\mu(x)$$

II. DIFFERENCE RELATIONS

The following results can be obtained either by reference to Erdélyi,[2], 1 pp. 258, 254, by differentiation and subsequent integration by parts of integrals such as

$$W_{\chi,\mu}(x) = \frac{x^{\mu+1/2}\, e^{-x/2}}{\Gamma(\mu + ½ - \chi)} \int_0^\infty e^{-x\tau}\tau^{\mu-\chi-1/2}(H(1 + \tau)^{\mu+\chi-1/2}\, d\tau$$

or by observing that combinations of the form

$$\sqrt{x}\,\frac{d}{dx}\,\frac{W_{\chi,0}(x)}{\sqrt{x}} + \chi W_{\chi,0}(x)$$

satisfy Whittaker's equation with $\mu = 1$. In the last mentioned method, the required constant multiplying the first order Whittaker function can be obtained by reference to the limiting behavior for small x. If $R_{\chi\mu} = W_{\chi,\mu}(x)/\sqrt{x}$ and $S_{\chi\mu} = M_{\chi,\mu}(x)/\sqrt{x}$ the results are

$$\begin{aligned} R_{\chi 0}' + \chi R_{\chi 0} &= (\chi - ½)R_{\chi 1}, \\ S_{\chi 0}' + \chi S_{\chi 0} &= ½\,(¼ - \chi^2)\, S_{\chi 1}. \end{aligned} \tag{39}$$

III. ZEROS

When $\chi = (2n + 1)/2$, $n = 1, 2 \ldots$, $W_{\chi,\mu}/x^{\mu-1/2}$ reduces to a polynomial times a function of χ. This may be inferred from the asymptotic expansion,

$$W_{\chi,\mu} = e^{-(x/2)}x^{\chi}$$

$$\left(1 + \sum_{n=1}^{\infty} \frac{[\mu^2 - (\chi - \tfrac{1}{2})^2][\mu^2 - (\chi - \tfrac{3}{2})^2] \cdots [\mu^2 - (\chi - n + \tfrac{1}{2})^2]}{n!x^n}\right),$$

which terminates if $\mu = 1$ and $\chi = n + \tfrac{1}{2}$ $(n = 1, 2 \ldots)$, or from the fact that for these values of the suffixes, W reduces to the generalized Laguerre polynomial

$$L^{(2)}_{\chi-(3/2)}(x).$$

Similarly when $\chi = (2n + 1)/2$, $n = 0, 1, 2 \ldots$, $W_{\chi,0}$ reduces to the Laguerre polynomial

$$L_{\chi-(1/2)}(x).$$

The zeros at the critical values of χ are given in the following table

χ	Zeros of $W_{\chi,0}$	Zeros of $W_{\chi,1}$
$\tfrac{1}{2}$	0	None
$\tfrac{3}{2}$	0; 1	0
$\tfrac{5}{2}$	0; 0.586; 3.414	0; 3
$\tfrac{7}{2}$	0; 0.416; 2.294; 6.290	0; 6; 2
$\tfrac{9}{2}$	0; 0.323; 1.746; 4.537; 9.395	0; 1.517; 4.312; 9.171
$1\tfrac{1}{2}$	0; 0.26356; 1.413; 3.596; 7.086; 12.641	0; 1.227; 3.413; 6.903; 12.458

Between $n + \tfrac{1}{2}$ and $n + \tfrac{3}{2}$, $W_{\chi,0}$ has $n + 1$ zeros $(n = 0, 1, 2 \ldots)$ and $W_{\chi,1}$ has n zeros $(n = 1, 2 \ldots)$.

The zeros of $M_{\chi,1}$ coincide with those of $W_{\chi,1}$ when $\chi = n + \tfrac{1}{2}$, and at those values of χ only. The functions $M_{\chi,1}$ and $W_{\chi,1}$ then are proportional to each other.*

IV. THE RICCATI-EQUATION FOR THE IMPEDANCE FUNCTION $W_{\chi,0}/W_{\chi,1}$.

The computations concerning the cylindrical helix required a study of the function

$$Z(u) = \frac{W_{\chi,0}(u)}{W_{\chi,1}(u)}.$$

* At these critical values of χ, the solution to the problem of the hollow cylinder of ferrite in the waveguide breaks down, since M and W are then not independent. A further independent solution must then be constructed.

We will show that $Z_\chi(u)$ satisfies a non-linear first order differential equation of Riccati-type. From the difference relations, equation (36), we have

$$(\chi - \tfrac{1}{2}) \frac{W_{\chi,1}(u)}{W_{\chi,0}(u)} = \frac{W_{\chi,0}'(u)}{W_{\chi,0}(u)} - \frac{1}{2u} + \chi$$
$$= \frac{1}{g}, \qquad \text{say}$$

and from Whittaker's equation

$$\frac{d}{du}\left(\frac{W_{\chi,0}'}{W_{\chi,0}}\right) + \left(\frac{W_{\chi,0}'}{W_{\chi,0}}\right)^2 + \left(\frac{1}{4u^2} + \frac{\chi}{u} - \frac{1}{4}\right) = 0.$$

Therefore

$$\frac{d}{du}\frac{1}{g} + \frac{1}{g^2} + \frac{1}{g}\left(\frac{1}{u} - 2\chi\right) + (\chi^2 - \tfrac{1}{4}) = 0,$$

or

$$\frac{dg}{du} = 1 + \left(\frac{1}{u} - 2\chi\right) g + (\chi^2 - \tfrac{1}{4})\, g^2.$$

Finally, let

$$Z = (\chi - \tfrac{1}{2})\, g(u) = \frac{W_{\chi,0}(u)}{W_{\chi,1}(u)}.$$

Then

$$\frac{dZ}{du} = (\chi - \tfrac{1}{2}) + \left(\frac{1}{u} - 2\chi\right)Z + (\chi + \tfrac{1}{2})\, Z^2.$$

Since this equation is satisfied by $M_{\chi,0}(u)/M_{\chi,1}(u))$ as well as by $W_{\chi,0}(u)/W_{\chi,1}(u)$, a selection has to be made from all the possible solutions of this equation. We require the one which for large u approaches unity. But for large u the equation is

$$\frac{dZ}{du} = (1 - Z)\,[\chi - \tfrac{1}{2} - (\chi + \tfrac{1}{2})\, Z],$$

whose integral is

$$Z = \frac{(\chi - \tfrac{1}{2})Ae^u - 1}{(\chi + \tfrac{1}{2})Ae^u - 1}$$

For large u, therefore, the solution is either unity, when $A = 0$, or else $(\chi - \tfrac{1}{2})/(\chi + \tfrac{1}{2})$, $A \neq 0$. The solution with $A \neq 0$ corresponds to the M functions; that with $A = 0$ to the W-functions.

The case $A = 0$ was integrated on an analogue-computer, and the results are shown in Fig. 14(b). The computation was restricted to the range $|\chi| < 3/2$. Beyond these values, the helix-problem was discussed only qualitatively.

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