# Theory of Curved Circular Waveguide Containing an Inhomogeneous Dielectric

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*Generalized telegraphist's equations are derived, following Schelkunoff, for all modes in a curved circular waveguide containing an inhomogeneous dielectric. Particular attention is paid to the coupling between the* $TE_{01}$ *mode and other modes in the curved guide. The results are applied to the problem of preventing the mode conversion from* $TE_{01}$ *to* $TM_{11}$ *which normally occurs in a curved round waveguide, by partially filling the cross section of the curved guide with a suitably shaped dielectric, such as polystyrene foam. Design equations are given for various compensators, and criteria are set up for keeping the power levels of all spurious modes low in a compensated bend. Dielectric losses, which may be important at millimeter wavelengths, are briefly treated. The potentialities of different compensator designs are illustrated by numerical examples.*

## INTRODUCTION

It has been recognized for several years that a major problem in the transmission of circular electric waves through multimode round waveguides is the question of negotiating bends. Theoretical studies[1, 2, 3] have shown that a gentle bend couples the $TE_{01}$ mode to the $TE_{11}$, $TE_{12}$, $TE_{13}$, $\cdots$ modes and to the $TM_{11}$ mode. The $TM_{11}$ mode presents the most serious problem, since it has the same phase velocity as $TE_{01}$ in a perfectly conducting straight guide. It follows that power introduced in the $TE_{01}$ mode at the beginning of a gradual bend will be essentially completely transferred to the $TM_{11}$ mode at odd multiples of a certain critical bending angle $\vartheta_c$. The angle $\vartheta_c$ is proportional to the ratio of wavelength to guide radius but independent of bending radius; in other words, power transfer cannot be avoided merely by using a sufficiently gentle bend.

S. E. Miller[4] has discussed a number of methods for transmitting the circular electric wave around bends with small net power loss to $TM_{11}$. These methods are of two general types.

In the first type the $TE_{01}$ mode, which is not itself a normal mode of the curved guide, is deliberately converted to a combination of $TE_{01}$ and $TM_{11}$ which is a normal mode, or to a particular polarization of $TM_{11}$ which is another normal mode of the curved guide. After traversing the bend the energy is reconverted to $TE_{01}$. A disadvantage of the normal-mode approach is that the mode conversions necessary at the ends of the bend are frequency sensitive, so that bandwidths appear to be limited to the order of 10 per cent.

A second approach to the bend problem is to break up, by some modification of the guide, the degeneracy which exists between the propagation constants of the $TE_{01}$ and $TM_{11}$ modes in a perfectly conducting straight pipe. The two modes are still coupled by the curvature of the guide, but as Miller has shown, the maximum power transfer will be small if there is sufficient difference between the phase constants or between the attenuation constants of the coupled modes. A difference in phase constants may be provided, for example, by circular corrugations in the waveguide wall, or perhaps most easily by applying a thin layer of dielectric to the inner surface of the guide.[5] Differential attenuation may be introduced into the $TM_{11}$ mode by a number of methods, in particular by making the guide out of spaced copper rings or a closely-wound wire helix surrounded by a lossy sheath.[6] Unfortunately, the larger the guide diameter in wavelengths the more difficult it is to get the separation of propagation constants necessary to negotiate a bend of given radius satisfactorily.

Still another solution of the bend problem is to decouple the $TE_{01}$ and $TM_{11}$ modes in a curved guide by partially filling the cross section of the curved guide with dielectric material. The dielectric must be arranged to produce coupling between the $TE_{01}$ and $TM_{11}$ modes which is equal and opposite to the coupling produced by the curvature of the guide. This condition may be satisfied in a great variety of ways; but it is not the only requirement for a good bend compensator. Practical restrictions are that the power levels of all other modes which are coupled to $TE_{01}$ by the dielectric-compensated bend must be kept low, and of course dielectric losses in the compensator must not be excessive.

Part I of this paper treats the general problem of a curved circular waveguide containing an inhomogeneous dielectric. A convenient formulation of the problem is provided by S. A. Schelkunoff's generalized telegraphist's equations for waveguides.[7] The field at any cross section of the dielectric-compensated curved circular guide is represented as a superposition of the fields of the normal modes of an air-filled straight circular guide. A current amplitude and a voltage amplitude are associated with each normal mode, and the currents and voltages satisfy an infinite set of generalized telegraphist's equations. The coupling terms in these equations depend upon the curvature of the guide axis and upon the variation of dielectric permittivity over the cross section. In the present application, the distribution of dielectric is taken to be independent of distance along the bend.*

The generalized telegraphist's equations for all modes in a curved circular waveguide containing an inhomogeneous dielectric are set up in Section 1.1. As a special case one has the equations for an air-filled curved guide, or for a straight guide with an inhomogeneous dielectric. In Section 1.2 we transform from current and voltage amplitudes to the amplitudes of forward and backward traveling waves on a system of coupled transmission lines and in Section 1.3 we work out in some detail the coupling coefficients which involve the $TE_{01}$ mode. An approximate formula for dielectric loss in a compensated bend, which is valid at least in the important practical case when the relative permittivity of the dielectric differs but little from unity, is given in Section 1.4.

Part II applies the foregoing theory to the design of bend compensators for the $TE_{01}$ mode. In a well-designed compensator the amplitudes of the backward (reflected) waves are very low, so we shall neglect reflections. The amplitudes of the spurious forward waves should also be low compared to $TE_{01}$ , so that we may consider them one at a time. We assume that the $TE_{01}$ mode crosstalks independently into each spurious mode, and represent the interaction between modes by a pair of linear, first-order, differential equations in the wave amplitudes. Miller's treatment[8] of these equations is reviewed in Section 2.1, and applied in Section 2.2 to $TE_{01}$–$TM_{11}$ coupling in plain and compensated bends. Some results of the Jouguet-Rice theory[1, 2] for plain bends are confirmed by coupled-line theory. The condition for decoupling $TE_{01}$ and $TM_{11}$ in a compensated bend is written down, and the consequences of imperfect decoupling are discussed.

Three different compensator designs are described in Section 2.3, and evaluated with regard to mode conversions and approximate dielectric losses. In the first case, which may be called the "geometrical optics" solution, the permittivity is supposed to vary continuously in such a way that a bundle of parallel rays entering the bend would be bent into coaxial circular arcs all of the same optical length. This is not a perfect solution of the problem if the wavelength is finite, but it is of some

* We shall not consider the effects of random inhomogeneities, such as bubbles in polystyrene foam, although these might conceivably add to the mode conversion if their dimensions were comparable to the operating wavelength.

theoretical interest nevertheless. In the second case, a dielectric sector of constant permittivity is attached to the inner surface of the bend nearest the center of curvature; the angle of the sector is determined to satisfy the decoupling condition. Such a sector may be an effective compensator if the guide is small enough to propagate only 40 to 50 modes at the operating frequency, as, for example, a $\frac{7}{8}$-inch guide at a wavelength of 5.4 mm. Finally, we consider a compensator made of three dielectric sectors, whose angles and spacings are chosen to decouple the modes ($TE_{21}$ and $TE_{31}$) with phase velocities closest to $TE_{01}$. The three-sector compensator may be necessary if the guide is large enough to propagate 200 to 300 modes, say a 2-inch guide at 5.4 mm.

In Section 2.4 we investigate the effect of increasing the attenuation of the spurious modes generated by the compensated bend. The conclusion is that it is not feasible to add enough loss to the worst spurious modes to reduce appreciably the power which they abstract from $TE_{01}$.

As a sample of numerical results, it appears possible to negotiate a 90° bend of radius 20 inches in the $\frac{7}{8}$-inch guide at 5.4 mm with an insertion loss of about 0.3 db. This assumes a single-sector polyfoam compensator with a relative permittivity of 1.036 and a loss tangent of $5 \times 10^{-5}$ (polyfoam with approximately these constants is currently available). About 0.2 db of the quoted loss is due to mode conversions and 0.1 db to dielectric dissipation. For a 2-inch guide with a three-sector polyfoam compensator, a bending radius of about 12 feet appears feasible. The total loss in a 90° bend should be about 0.35 db, with approximately 0.1 db going into mode conversion and about 0.25 db into dielectric dissipation. The dielectric loss is proportional, of course, to the total bend angle, and for a 180° bend would be double the above figures.

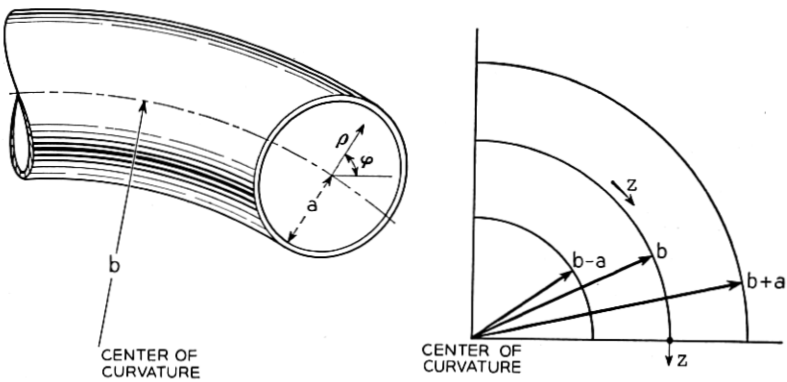


Fig. 1 — Coordinates used in circular bend in circular waveguide.

The appendix contains brief descriptions of three dielectric compensators which can be inserted in a straight section of guide adjacent to a bend. The first two are transducers which convert $TE_{01}$ to a normal mode of the curved guide; they are subject to bandwidth limitations as mentioned by Miller.[9] The third type merely takes the output mixture of $TE_{01}$ and $TM_{11}$ from a plain bend with a pure $TE_{01}$ input, and reconverts it all to $TE_{01}$ ; it is essentially a broadband device. The spurious modes generated by a bend plus compensator have not been calculated, but it is very unlikely that a smaller bending radius will be permitted when the compensator is outside the bend than when it is inside.

## I. THEORY

### 1.1 *Generalized Telegraphist's Equations*

To describe electromagnetic fields in a curved circular waveguide one is naturally led to use "bent cylindrical coordinates" $(\rho, \varphi, z)$,[1, 2, 3] in which the longitudinal coordinate $z$ is distance measured along the curved axis of the guide, while $\rho$ and $\varphi$ are polar coordinates in a plane normal to the axis of the guide, with origin at the guide axis. The lines $\varphi = 0$ and $\varphi = \pi$ lie in the plane of the bend. The radius of the guide is denoted by $a$ and the radius of the bend (i.e., the radius of curvature of the guide axis) is denoted by $b$. The coordinate system is shown in Fig. 1.

For the moment regarding $(\rho, \varphi, z)$ as general orthogonal curvilinear coordinates $(u, v, w)$, we let

$$u = \rho, \qquad v = \varphi, \qquad w = z. \tag{1}$$

The element of length in this system is

$$ds^2 = e_1^2du^2 + e_2^2dv^2 + e_3^2dw^2, \tag{2}$$

where

$$e_1 = 1, \qquad e_2 = \rho, \qquad e_3 = 1 + \xi, \tag{3}$$

and

$$\xi = (\rho/b) \cos \varphi. \tag{4}$$

Maxwell's equations for a field with time dependence $e^{i\omega t}$ may be written in the form:[10]

$$\frac{1}{e_2e_3}\left[\frac{\partial}{\partial v}(e_3E_w) - \frac{\partial}{\partial w}(e_2E_v)\right] = -i\omega\mu H_u,$$

$$\frac{1}{e_3e_1}\left[\frac{\partial}{\partial w}(e_1E_u) - \frac{\partial}{\partial u}(e_3E_w)\right] = -i\omega\mu H_v,$$

$$\frac{1}{e_1e_2}\left[\frac{\partial}{\partial u}(e_2E_v) - \frac{\partial}{\partial v}(e_1E_u)\right] = -i\omega\mu H_w,$$

$$\frac{1}{e_2e_3}\left[\frac{\partial}{\partial v}(e_3H_w) - \frac{\partial}{\partial w}(e_2H_v)\right] = i\omega\epsilon E_u, \tag{5}$$

$$\frac{1}{e_3e_1}\left[\frac{\partial}{\partial w}(e_1H_u) - \frac{\partial}{\partial u}(e_3H_w)\right] = i\omega\epsilon E_v,$$

$$\frac{1}{e_1e_2}\left[\frac{\partial}{\partial u}(e_2H_v) - \frac{\partial}{\partial v}(e_1H_u)\right] = i\omega\epsilon E_w.$$

In these equations the permeability $\mu$ and the permittivity $\epsilon$ may be functions of position. If there is dissipation in the medium, either $\epsilon$ or $\mu$ or both may be complex.

To convert Maxwell's equations into generalized telegraphist's equations, we introduce the field distributions characteristic of the normal modes of a straight, cylindrical guide filled with a homogeneous dielectric. The derivation follows very closely that given by Schelkunoff[11] for an inhomogeneously-filled straight guide. Each mode is described by a transverse field distribution pattern $T(u, v)$, where $T(u, v)$ satisfies

$$\nabla^2 T = \frac{1}{e_1e_2}\left[\frac{\partial}{\partial u}\left(\frac{e_2}{e_1}\frac{\partial T}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{e_1}{e_2}\frac{\partial T}{\partial v}\right)\right] = -\chi^2 T, \tag{6}$$

and $\chi$ is a separation constant which takes on discrete values for the various TE and TM modes. We shall denote the function corresponding to the $n$th TE mode by $T_{[n]}(u, v)$, and the separation constant by $\chi_{[n]}$, with the subscript in *brackets*. The normal derivative of $T_{[n]}$ vanishes on the perfectly conducting waveguide boundary. Similarly the function corresponding to the $n$th TM mode is denoted by $T_{(n)}(u, v)$, and the separation constant by $\chi_{(n)}$, with the subscript in *parentheses*. The function $T_{(n)}$ vanishes on the boundary of the waveguide. For the present the modes may be assumed to be numbered in order of increasing cutoff frequency; later when we have occasion to refer to specific modes we shall replace the subscript $n$ by the customary double subscript notation for TE and TM modes in a circular guide.

The $T$-functions are assumed to be so normalized that

$$\begin{aligned} \int_S (\text{grad } T)\cdot(\text{grad } T)\, dS &\equiv \int_S (\text{flux } T)\cdot(\text{flux } T)\, dS \\ &\equiv \chi^2 \int_S T^2\, dS = 1, \end{aligned} \tag{7}$$

where $S$ is the cross section of the guide. The gradient and flux of a scalar point-function $W$ are transverse vectors with the following components:

$$\begin{aligned} \text{grad}_u\, W &= \frac{\partial W}{e_1 \partial u}, \qquad & \text{grad}_v\, W &= \frac{\partial W}{e_2 \partial v}, \\ \text{flux}_u\, W &= \frac{\partial W}{e_2 \partial v}, \qquad & \text{flux}_v\, W &= -\frac{\partial W}{e_1 \partial u}. \end{aligned} \tag{8}$$

Various orthogonality relationships exist among the $T$-functions corresponding to different modes of the guide, and among their gradients and fluxes. These relationships have been listed by Schelkunoff.[12]

The transverse components of the fields in the curved guide may be derived from potential and stream functions, $U$ and $\Psi$ for TE waves and $V$ and $\Pi$ for TM waves. Thus

$$\begin{aligned} E_t &= -\text{grad } V - \text{flux } \Psi, \\ H_t &= \text{flux } \Pi - \text{grad } U. \end{aligned} \tag{9}$$

We now assume series expansions for the potential and stream functions in terms of the functions $T(u, v)$, with coefficients depending on $w$. Let

$$\begin{aligned} V &= -\sum_n V_{(n)}(w)\, T_{(n)}(u, v), \qquad & \Pi &= -\sum_n I_{(n)}(w)\, T_{(n)}(u, v), \\ \Psi &= -\sum_n V_{[n]}(w)\, T_{[n]}(u, v), \qquad & U &= -\sum_n I_{[n]}(w)\, T_{[n]}(u, v). \end{aligned} \tag{10}$$

The $I$'s and $V$'s have the dimensions and properties of transmission-line currents and voltages. If we substitute (10) into (9) and expand in components by (8), we obtain:

$$\begin{aligned} E_u &= \sum_n \left[ V_{(n)} \frac{\partial T_{(n)}}{e_1 \partial u} + V_{[n]} \frac{\partial T_{[n]}}{e_2 \partial v} \right], \\ E_v &= \sum_n \left[ V_{(n)} \frac{\partial T_{(n)}}{e_2 \partial v} - V_{[n]} \frac{\partial T_{[n]}}{e_1 \partial u} \right], \\ H_u &= \sum_n \left[ -I_{(n)} \frac{\partial T_{(n)}}{e_2 \partial v} + I_{[n]} \frac{\partial T_{[n]}}{e_1 \partial u} \right], \\ H_v &= \sum_n \left[ I_{(n)} \frac{\partial T_{(n)}}{e_1 \partial u} + I_{[n]} \frac{\partial T_{[n]}}{e_2 \partial v} \right]. \end{aligned} \tag{11}$$

For the longitudinal field components it is convenient to expand the combinations $e_3 E_w$ and $e_3 H_w$ in the following series:

$$\begin{aligned} e_3 E_w &= \sum_n \chi_{(n)} V_{w,(n)}(w) T_{(n)}(u, v), \\ e_3 H_w &= \sum_n \chi_{[n]} I_{w,[n]}(w) T_{[n]}(u, v). \end{aligned} \tag{12}$$

It should be noted that the boundary conditions in the curved circular guide are

$$E_v = E_w = 0 \tag{13}$$

at the boundary of the guide, and that these conditions are satisfied by the individual terms of the series for $E_v$ and $E_w$, on account of the boundary conditions already imposed on $T_{(n)}$ and $T_{[n]}$. Hence we do not have the problem of nonuniform convergence which sometimes arises in treating waveguides of varying cross section by the present method.

The procedure for transforming Maxwell's equations into generalized telegraphist's equations is now straightforward, if a trifle tedious. One substitutes the series (11) and (12) into equations (5), and integrates certain combinations of the latter equations over the cross section of the guide, taking account of the orthogonality properties of the $T$-functions. For example, subtracting $\partial T_{(m)}/e_2 \partial v$ times the first of equations (5) from $\partial T_{(m)}/e_1 \partial u$ times the second equation, and integrating over the cross section, yields

$$\begin{aligned} \frac{dV_{(m)}}{dw} - \chi_{(m)} V_{w,(m)} &= -i\omega \sum_n \left[ I_{(n)} \int_S \mu e_3 \, (\text{grad } T_{(n)}) \cdot (\text{grad } T_{(m)}) \, dS \right. \\ &\qquad \left. + I_{[n]} \int_S \mu e, \, (\text{grad } T_{(m)}) \cdot (\text{flux } T_{[n]}) \, dS \right]. \end{aligned} \tag{14}$$

Adding $\partial T_{[m]}/e_1 \partial u$ times the first equation to $\partial T_{[m]}/e_2 \partial v$ times the second and integrating gives

$$\frac{dV_{[m]}}{dw} = -i\omega \sum_n \left[ I_{(n)} \int_S \mu e_3 \,(\text{grad } T_{(n)}) \cdot (\text{flux } T_{[m]}) \, dS + I_{[n]} \int_S \mu e_3 \,(\text{grad } T_{[m]}) \cdot (\text{grad } T_{[n]}) \, dS \right]. \tag{15}$$

Similarly, from the fourth and fifth equations we get

$$\frac{dI_{(m)}}{dw} = -i\omega \sum_n \left[ V_{(n)} \int_S \epsilon e_3 \,(\text{grad } T_{(m)}) \cdot (\text{grad } T_{(n)}) \, dS + V_{[n]} \int_S \epsilon e_3 \,(\text{grad } T_{(m)}) \cdot (\text{flux } T_{[n]}) \, dS \right], \tag{16}$$

$$\frac{dI_{[m]}}{dw} - \chi_{[m]} I_{w,[m]} = -i\omega \sum_n \left[ V_{(n)} \int_S \epsilon e_3 \,(\text{grad } T_{(n)}) \cdot (\text{flux } T_{[m]}) \, dS - V_{[n]} \int_S \epsilon e_3 \,(\text{grad } T_{[m]}) \cdot (\text{grad } T_{[n]}) \, dS \right]. \tag{17}$$

From the third of equations (5) using the fact that the $T$-functions satisfy (6), then multiplying through by $T_{[m]}$ and integrating over the cross section, we get,

$$V_{[m]} = -i\omega \sum_n I_{w,[n]} \chi_{[n]} \int_S \frac{\mu T_{[n]} T_{[m]}}{e_3} \, dS. \tag{18}$$

Similarly, from the sixth equation,

$$I_{(m)} = -i\omega \sum_n V_{w,(n)} \chi_{(n)} \int_S \frac{\epsilon T_{(n)} T_{(m)}}{e_3} \, dS. \tag{19}$$

These equations may be written in the form

$$V_{[m]} = -\sum_n \frac{Z_{w,[m][n]} I_{w,[n]}}{\chi_{[m]}}, \qquad I_{(m)} = -\sum_n \frac{Y_{w,(m)(n)} V_{w,(n)}}{\chi_{(m)}}, \tag{20}$$

where we define

$$Z_{w,[m][n]} = i\omega \chi_{[m]} \chi_{[n]} \int_S \frac{\mu T_{[m]} T_{[n]}}{e_3} \, dS, \qquad Y_{w,(m)(n)} = i\omega \chi_{(m)} \chi_{(n)} \int_S \frac{\epsilon T_{(m)} T_{(n)}}{e_3} \, dS. \tag{21}$$

Solving (20) for $I_{w,[m]}$ and $V_{w,(m)}$ in terms of $V_{[n]}$ and $I_{(n)}$ respectively, we may write

$$
\begin{aligned}
I_{w,[m]} &= -\sum_n \frac{Y_{w,[m][n]}}{\chi_{[m]}} V_{[n]}, \\
V_{w,(m)} &= -\sum_n \frac{Z_{w,(m)(n)}}{\chi_{(m)}} I_{(n)},
\end{aligned}
\tag{22}
$$

where $Y_{w,[m][n]}$ and $Z_{w,(m)(n)}$ may be defined as the ratios of certain determinants involving $Z_{w,[m][n]}$ and $Y_{w,(m)(n)}$ respectively.*

We are now able to eliminate $V_{w,(m)}$ and $I_{w,[m]}$ from (14) and (17), and to write down the generalized telegraphist's equations for the curved circular waveguide filled with an inhomogeneous dielectric in the following form:

$$
\begin{aligned}
\frac{dV_{(m)}}{dw} &= -\sum_n [Z_{(m)(n)}I_{(n)} + Z_{(m)[n]}I_{[n]}], \\
\frac{dV_{[m]}}{dw} &= -\sum_n [Z_{[m](n)}I_{(n)} + Z_{[m][n]}I_{[n]}], \\
\frac{dI_{(m)}}{dw} &= -\sum_n [Y_{(m)(n)}V_{(n)} + Y_{(m)[n]}V_{[n]}], \\
\frac{dI_{[m]}}{dw} &= -\sum_n [Y_{[m](n)}V_{(n)} + Y_{[m][n]}V_{[n]}].
\end{aligned}
\tag{23}
$$

The impedance and admittance coefficients are defined by:

$$
\begin{aligned}
Z_{(m)(n)} &= i\omega \int_S \mu e_3 (\text{grad } T_{(m)}) \cdot (\text{grad } T_{(n)})\, dS + Z_{w,(m)(n)}, \\
Z_{(m)[n]} &= i\omega \int_S \mu e_3 (\text{grad } T_{(m)}) \cdot (\text{flux } T_{[n]})\, dS, \\
Z_{[m](n)} &= i\omega \int_S \mu e_3 (\text{flux } T_{[m]}) \cdot (\text{grad } T_{(n)})\, dS, \\
Z_{[m][n]} &= i\omega \int_S \mu e_3 (\text{grad } T_{[m]}) \cdot (\text{grad } T_{[n]})\, dS, \\
Y_{(m)(n)} &= i\omega \int_S \epsilon e_3 (\text{grad } T_{(m)}) \cdot (\text{grad } T_{(n)})\, dS,
\end{aligned}
\tag{24}
$$

* If (20) are actually regarded as infinite sets of equations, one has to pay some attention to defining the meaning of the statement that their solutions are given by (22). In practice we shall deal with a finite number of modes, and shall bypass all questions about infinite processes.

$$Y_{(m)[n]} = i\omega \int_S \epsilon e_3 (\text{grad } T_{(m)}) \cdot (\text{flux } T_{[n]}) \, dS,$$

$$Y_{[m](n)} = i\omega \int_S \epsilon e_3 (\text{flux } T_{[m]}) \cdot (\text{grad } T_{(n)}) \, dS,$$

$$Y_{[m][n]} = i\omega \int_S \epsilon e_3 (\text{grad } T_{[m]}) \cdot (\text{grad } T_{[n]}) \, dS + Y_{w,[m][n]}.$$

It may be noted that if the guide were straight, so that $e_3 = 1$, and if $\mu$ and $\epsilon$ were constant over the cross section, the $Y$'s and $Z$'s would all be zero except for those having equal subscripts. If the curvature of the guide is gentle, and if $\mu$ and $\epsilon$ do not vary much over the cross section (or if they vary extensively only in a small part of the cross section), then the $Y$'s and $Z$'s with unequal subscripts will be small, and we can obtain approximate solutions of (23) based upon the smallness of the coupling.

We shall now compute first-order approximations to the impedance and admittance coefficients under the following assumptions:

$$\begin{aligned} \mu &= \mu_0 , \\ \epsilon &= \epsilon_0(1 + \delta), \end{aligned} \tag{25}$$

where $\mu_0$ and $\epsilon_0$ are constants (usually but not necessarily the permeability and permittivity of free space), and $\delta$ is a dimensionless function of position. No mathematical difficulties would follow from assuming $\mu$ as well as $\epsilon$ to be variable, but since the case of varying permeability is not of such immediate practical interest we shall omit this slight additional complication. In order that the coupling per unit length due to curvature and to inhomogeneity of the dielectric be small, we further assume that

$$\begin{aligned} &| \xi | \leqq a/b \ll 1, \\ &\frac{1}{S} \int_S | \delta | \, dS \ll 1, \end{aligned} \tag{26}$$

where, as usual, $S$ is the cross-sectional area of the guide.

From (21), first-order approximations to $Z_{w,[m][n]}$ and $Y_{w,(m)(n)}$ are

$$\begin{aligned} Z_{w,[m][n]} &= i\omega\mu_0[\delta_{mn} - \xi_{[m][n]}], \\ Y_{w,(m)(n)} &= i\omega\epsilon_0[\delta_{mn} + \delta_{(m)(n)} - \xi_{(m)(n)}], \end{aligned} \tag{27}$$

where

$$\begin{aligned}
\xi_{[m][n]} &= \chi_{[m]}\chi_{[n]} \int_S \xi T_{[m]} T_{[n]}\, dS, \\
\xi_{(m)(n)} &= \chi_{(m)}\chi_{(n)} \int_S \xi T_{(m)} T_{(n)}\, dS, \\
\delta_{(m)(n)} &= \chi_{(m)}\chi_{(n)} \int_S \delta T_{(m)} T_{(n)}\, dS.
\end{aligned} \tag{28}$$

The quantity $\delta_{mn}$ is the Kronecker delta, and is not to be confused with $\delta_{(m)(n)}$, which is defined by the last of equations (28). Note that $\xi_{[m][n]}$ and $\xi_{(m)(n)}$ are zero unless the angular indices of the two modes involved differ by exactly unity.

It is not difficult to obtain approximate solutions of (20) in the forms (22), since the off-diagonal elements of the coefficient matrices of (20) are small compared to the diagonal elements. Using the expression derived by Rice[13] for the inverse of an almost-diagonal matrix (we shall not attempt to prove this result for infinite matrices), we find the first-order approximations

$$\begin{aligned}
Y_{w,[m][n]} &= \frac{\chi_{[m]}\chi_{[n]}}{i\omega\mu_0} [\delta_{mn} + \xi_{[m][n]}], \\
Z_{w,(m)(n)} &= \frac{\chi_{(m)}\chi_{(n)}}{i\omega\epsilon_0} [\delta_{mn} + \xi_{(m)(n)} - \delta_{(m)(n)}].
\end{aligned} \tag{29}$$

Approximate expressions for the impedance and admittance coefficients appearing in the generalized telegraphist's equations (23) are:

$$\begin{aligned}
Z_{(m)(n)} &= i\omega\mu_0[\delta_{mn} + \Xi_{(m)(n)}] + Z_{w,(m)(n)}, \\
Z_{(m)[n]} &= i\omega\mu_0\Xi_{(m)[n]}, \\
Z_{[m](n)} &= i\omega\mu_0\Xi_{[m](n)}, \\
Z_{[m][n]} &= i\omega\mu_0[\delta_{mn} + \Xi_{[m][n]}], \\
Y_{(m)(n)} &= i\omega\epsilon_0[\delta_{mn} + \Xi_{(m)(n)} + \Delta_{(m)(n)}], \\
Y_{(m)[n]} &= i\omega\epsilon_0[\Xi_{(m)[n]} + \Delta_{(m)[n]}], \\
Y_{[m](n)} &= i\omega\epsilon_0[\Xi_{[m](n)} + \Delta_{[m](n)}], \\
Y_{[m][n]} &= i\omega\epsilon_0[\delta_{mn} + \Xi_{[m][n]} + \Delta_{[m][n]}] + Y_{w,[m][n]},
\end{aligned} \tag{30}$$

where

$$
\begin{aligned}
\Xi_{(m)(n)} &= \int_S \xi\,(\text{grad}\ T_{(m)})\cdot(\text{grad}\ T_{(n)})\ dS,\\
\Xi_{(m)[n]} &= \int_S \xi\,(\text{grad}\ T_{(m)})\cdot(\text{flux}\ T_{[n]})\ dS,\\
\Xi_{[m](n)} &= \int_S \xi\,(\text{flux}\ T_{[m]})\cdot(\text{grad}\ T_{(n)})\ dS,\\
\Xi_{[m][n]} &= \int_S \xi\,(\text{grad}\ T_{[m]})\cdot(\text{grad}\ T_{[n]})\ dS,
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
\Delta_{(m)(n)} &= \int_S \delta\,(\text{grad}\ T_{(m)})\cdot(\text{grad}\ T_{(n)})\ dS,\\
\Delta_{(m)[n]} &= \int_S \delta\,(\text{grad}\ T_{(m)})\cdot(\text{flux}\ T_{[n]})\ dS,\\
\Delta_{[m](n)} &= \int_S \delta\,(\text{flux}\ T_{[m]})\cdot(\text{grad}\ T_{(n)})\ dS,\\
\Delta_{[m][n]} &= \int_S \delta\,(\text{grad}\ T_{[m]})\cdot(\text{grad}\ T_{[n]})\ dS.
\end{aligned}
\tag{32}
$$

The $\Xi$'s are zero unless the angular indices of the two modes involved differ by exactly unity.

### 1.2 *Representation in Terms of Coupled Traveling Waves*

From now on we shall assume that the distribution of dielectric over the cross section of the curved guide is independent of distance along the guide, so that the impedance and admittance coefficients are constants independent of $z$. (We shall henceforth designate the coordinates by $(\rho, \varphi, z)$, instead of the $(u, v, w)$ of the preceding section.) The generalized telegraphist's equations now represent an infinite set of coupled, uniform transmission lines, and their solution would be equivalent to the solution of an infinite system of linear algebraic equations and the corresponding characteristic equation.

For our purposes it is convenient to write the transmission-line equations not in terms of currents and voltages, but in terms of the amplitudes of forward and backward traveling waves, assumed to exist in a straight guide filled with a homogeneous medium. Thus let $a$ and $b$ be the amplitudes of the forward and backward waves of a typical mode at a certain point. In what follows the wave amplitudes $a$ and $b$ will always carry mode subscripts, so they need never be confused with guide radius and radius of curvature. The mode current and voltage are related to the wave amplitudes by

$$\begin{aligned} V &= K^{\frac{1}{2}}(a + b), \\ I &= K^{-\frac{1}{2}}(a - b), \end{aligned} \tag{33}$$

where $K$ is the wave impedance. We have

$$\begin{aligned} K_{(n)} &= h_{(n)}/\omega\epsilon_0 , \\ K_{[n]} &= \omega\mu_0/h_{[n]} , \end{aligned} \tag{34}$$

for TM and TE waves respectively, where $h_{(n)}$ and $h_{[n]}$ represent the unperturbed phase constants,

$$\begin{aligned} h_{(n)} &= (\beta^2 - \chi_{(n)}^2)^{\frac{1}{2}}, \\ h_{[n]} &= (\beta^2 - \chi_{[n]}^2)^{\frac{1}{2}}, \end{aligned} \tag{35}$$

and

$$\beta^2 = \omega^2\mu_0\epsilon_0 . \tag{36}$$

For a cutoff mode, $h$ is negative imaginary; but we shall deal only with propagating modes in the present analysis.

If we represent the currents and voltages in the generalized telegraphist's equations (23) in terms of the traveling-wave amplitudes, and then perform some obvious additions and subtractions, we obtain the following equations for coupled traveling waves:

$$\begin{aligned} \frac{da_{(m)}}{dz} &= -i \sum_n [\overset{+}{\kappa}_{(m)(n)}a_{(n)} + \overset{-}{\kappa}_{(m)(n)}b_{(n)} + \overset{+}{\kappa}_{(m)[n]}a_{[n]} + \overset{-}{\kappa}_{(m)[n]}b_{[n]}], \\ \frac{db_{(m)}}{dz} &= +i \sum_n [\overset{-}{\kappa}_{(m)(n)}a_{(n)} + \overset{+}{\kappa}_{(m)(n)}b_{(n)} + \overset{-}{\kappa}_{(m)[n]}a_{[n]} + \overset{+}{\kappa}_{(m)[n]}b_{[n]}], \\ \frac{da_{[m]}}{dz} &= -i \sum_n [\overset{+}{\kappa}_{[m](n)}a_{(n)} + \overset{-}{\kappa}_{[m](n)}b_{(n)} + \overset{+}{\kappa}_{[m][n]}a_{[n]} + \overset{-}{\kappa}_{[m][n]}b_{[n]}], \\ \frac{db_{[m]}}{dz} &= +i \sum_n [\overset{-}{\kappa}_{[m](n)}a_{(n)} + \overset{+}{\kappa}_{[m](n)}b_{(n)} + \overset{-}{\kappa}_{[m][n]}a_{[n]} + \overset{+}{\kappa}_{[m][n]}b_{[n]}]. \end{aligned} \tag{37}$$

The $\kappa$'s are coupling coefficients defined in terms of the impedance and admittance coefficients by

$$\begin{aligned} i\overset{\pm}{\kappa}_{(m)(n)} &= \tfrac{1}{2}[(K_{(m)}K_{(n)})^{\frac{1}{2}}\, Y_{(m)(n)} \pm (K_{(m)}K_{(n)})^{-\frac{1}{2}}\, Z_{(m)(n)}], \\ i\overset{\pm}{\kappa}_{(m)[n]} &= \tfrac{1}{2}[(K_{(m)}K_{[n]})^{\frac{1}{2}}\, Y_{(m)[n]} \pm (K_{(m)}K_{[n]})^{-\frac{1}{2}}\, Z_{(m)[n]}], \\ i\overset{\pm}{\kappa}_{[m](n)} &= \tfrac{1}{2}[(K_{[m]}K_{(n)})^{\frac{1}{2}}\, Y_{[m](n)} \pm (K_{[m]}K_{(n)})^{-\frac{1}{2}}\, Z_{[m](n)}], \\ i\overset{\pm}{\kappa}_{[m][n]} &= \tfrac{1}{2}[(K_{[m]}K_{[n]})^{\frac{1}{2}}\, Y_{[m][n]} \pm (K_{[m]}K_{[n]})^{-\frac{1}{2}}\, Z_{[m][n]}]. \end{aligned} \tag{38}$$

In these definitions the plus signs are taken together, likewise the minus signs. The factors of $i$ are introduced in order that the $\kappa$'s may be real for propagating modes in a lossless guide.

For the small-coupling case discussed at the end of the preceding section, it is convenient to separate a typical coupling coefficient into two parts; thus,

$$\kappa = c + d. \tag{39}$$

Here $c$ is the coupling coefficient due to curvature and is zero unless the angular indices of the two modes involved differ by unity. The coupling coefficient $d$ is due to the dielectric. All $d$'s vanish if the dielectric is homogeneous; otherwise particular symmetries may cause certain classes of $d$'s to be zero. The $c$'s and $d$'s may be expressed in terms of integrals written down in the preceding section if we substitute for the $Y$'s and $Z$'s in (38) their definitions according to (30).

The $\kappa^+$'s which have equal subscripts $(n)(n)$ or $[n][n]$ may be regarded as phase constants (of particular TM or TE modes) which have been modified by the presence of the dielectric. For the modified phase constants we introduce the symbols $\beta_{(n)}$ and $\beta_{[n]}$ ; thus,

$$\begin{aligned} \beta_{(n)} &\equiv \kappa^+_{(n)(n)} = h_{(n)} + \frac{\chi_{(n)}{}^2\delta_{(n)(n)} + h_{(n)}{}^2\Delta_{(n)(n)}}{2h_{(n)}}, \\ \beta_{[n]} &\equiv \kappa^+_{[n][n]} = h_{[n]} + \frac{\beta^2\Delta_{[n][n]}}{2h_{[n]}}. \end{aligned} \tag{40}$$

The general expressions for the coupling coefficients between any two different modes are as follows:

$$\begin{aligned} c^{\pm}_{(m)(n)} &= \frac{1}{2}\left[\sqrt{h_{(m)}h_{(n)}}\;\Xi_{(m)(n)} \pm \frac{\beta^2\Xi_{(m)(n)} - \chi_{(m)}\chi_{(n)}\xi_{(m)(n)}}{\sqrt{h_{(m)}h_{(n)}}}\right], \\ d^{\pm}_{(m)(n)} &= \frac{1}{2}\left[\sqrt{h_{(m)}h_{(n)}}\;\Delta_{(m)(n)} \pm \frac{\chi_{(m)}\chi_{(n)}\delta_{(m)(n)}}{\sqrt{h_{(m)}h_{(n)}}}\right], \\ c^{\pm}_{(m)[n]} &= \tfrac{1}{2}\,\beta\Xi_{(m)[n]}\,[\sqrt{h_{(m)}/h_{[n]}} \pm \sqrt{h_{[n]}/h_{(m)}}], \\ d^{\pm}_{(m)[n]} &= \tfrac{1}{2}\,\beta\Delta_{(m)[n]}\,\sqrt{h_{(m)}/h_{[n]}}, \\ c^{\pm}_{[m](n)} &= \tfrac{1}{2}\,\beta\Xi_{[m](n)}[\sqrt{h_{(n)}/h_{[m]}} \pm \sqrt{h_{[m]}/h_{(n)}}], \\ d^{\pm}_{[m](n)} &= \tfrac{1}{2}\,\beta\Delta_{[m](n)}\,\sqrt{h_{(n)}/h_{[m]}}, \\ c^{\pm}_{[m][n]} &= \frac{1}{2}\left[\frac{\beta^2\Xi_{[m][n]} - \chi_{[m]}\chi_{[n]}\xi_{[m][n]}}{\sqrt{h_{[m]}h_{[n]}}} \pm \Xi_{[m][n]}\sqrt{h_{[m]}h_{[n]}}\right], \\ d^{\pm}_{[m][n]} &= \frac{\beta^2\Delta_{[m][n]}}{2\sqrt{h_{[m]}h_{[n]}}}, \end{aligned} \tag{41}$$

where the symbols with double subscripts are defined by (28), (31), and (32).

### 1.3 *Coupling Coefficients Involving the* $TE_{01}$ *Mode*

In Part II we shall consider a well-compensated bend in which the total power in all spurious modes is everywhere low compared to the power level of $TE_{01}$. (This is somewhat more restrictive than merely assuming that the power in any one spurious mode is everywhere small compared to the power in $TE_{01}$.) To first order, therefore, we may compute the power abstracted from $TE_{01}$ by mode conversion by assuming that the $TE_{01}$ mode crosstalks into each spurious mode independently. For this calculation we need the values of the forward coupling coefficients between $TE_{01}$ and all other modes. The crosstalk into backward modes will be negligible in all practical cases.

We shall use the customary double-subscript notation for the various modes in a round guide, but shall continue to denote TM waves with parentheses and TE waves with brackets. *We assume that the distribution of dielectric is symmetric with respect to the plane of the bend,* so that $TE_{01}$ is coupled to a definite polarization of each spurious mode. The normalized $T$-functions are then:

$$\begin{aligned} T_{(nm)} &= \sqrt{\frac{\epsilon_n}{\pi}} \frac{J_n(\chi_{(nm)}\rho) \sin n\varphi}{k_{(nm)} J_{n-1}(k_{(nm)})}, \\ T_{[nm]} &= \sqrt{\frac{\epsilon_n}{\pi}} \frac{J_n(\chi_{[nm]}\rho) \cos n\varphi}{(k_{[nm]}^2 - n^2)^{\frac{1}{2}} J_n(k_{[nm]})}, \end{aligned} \tag{42}$$

where

$$\begin{aligned} k_{(nm)} &= \chi_{(nm)} a, \qquad J_n(k_{(nm)}) = 0, \\ k_{[nm]} &= \chi_{[nm]} a, \qquad J_n'(k_{[nm]}) = 0, \end{aligned} \tag{43}$$

and

$$\epsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n \neq 0. \end{cases} \tag{44}$$

#### 1.3.1 *Coupling Coefficients due to Curvature*

We know that to first order the curvature of the guide can couple the $TE_{01}$ mode only to modes of angular index unity. Let us calculate the coupling between $TE_{01}$ and $TM_{1m}$. Referring to (31), we have

$$\begin{aligned}\Xi_{(1m)[01]} &= \Xi_{[01](1m)} \\ &= \int_S \xi(\text{grad } T_{(1m)}) \cdot (\text{flux } T_{[01]})\, dS \\ &= \int_0^{2\pi} \int_0^a \frac{\sqrt{2} J_1(\chi_{[01]}\rho) J_1(\chi_{(1m)}\rho) \cos^2 \varphi}{\pi a b k_{(1m)} J_0(k_{[01]}) J_0(k_{(1m)})} \rho\, d\rho\, d\varphi \\ &= \begin{cases} a/\sqrt{2}\, k_{[01]} b & \text{if } m = 1, \\ 0 & \text{if } m \neq 1. \end{cases}\end{aligned} \tag{45}$$

Hence the only transverse magnetic mode coupled to $TE_{01}$ by the bend is $TM_{11}$, and from (41) the forward coupling coefficient is:

$$c^{+}_{(11)[01]} = c^{+}_{[01](11)} = \beta a/\sqrt{2}\, k_{[01]} b = 0.18454 \beta a/b. \tag{46}$$

To obtain the coupling between $TE_{01}$ and $TE_{1m}$, we must evaluate two integrals. From (28), the first is

$$\begin{aligned}\xi_{[01][1m]} &= \xi_{[1m][01]} \\ &= \chi_{[01]}\chi_{[1m]} \int_S \xi T_{[01]} T_{[1m]}\, dS \\ &= \int_0^{2\pi} \int_0^a \frac{\sqrt{2}\, \chi_{[1m]} \rho^2 J_0(\chi_{[01]}\rho) J_1(\chi_{[1m]}\rho) \cos^2 \varphi}{\pi a b (k_{[1m]}^2 - 1)^{\frac{1}{2}} J_0(k_{[01]}) J_1(k_{[1m]})}\, d\rho\, d\varphi \\ &= \frac{\sqrt{2} a}{b} \frac{k_{[1m]} (k_{[01]}^2 + k_{[1m]}^2)}{(k_{[1m]}^2 - 1)^{\frac{1}{2}} (k_{[01]}^2 - k_{[1m]}^2)^2},\end{aligned} \tag{47}$$

and from (31), the second is

$$\begin{aligned}\Xi_{[01][1m]} &= \Xi_{[1m][01]} \\ &= \int_S \xi(\text{grad } T_{[01]}) \cdot (\text{grad } T_{[1m]})\, dS \\ &= -\int_0^{2\pi} \int_0^a \frac{\sqrt{2}\, \chi_{[1m]} \rho^2 J_1(\chi_{[01]}\rho) J_1'(\chi_{[1m]}\rho) \cos^2 \varphi}{\pi a b (k_{[1m]}^2 - 1)^{\frac{1}{2}} J_0(k_{[01]}) J_1(k_{[1m]})}\, d\rho\, d\varphi \\ &= \frac{2\sqrt{2} a}{b} \frac{k_{[01]} k_{[1m]}^2}{(k_{[1m]}^2 - 1)^{\frac{1}{2}} (k_{[01]}^2 - k_{[1m]}^2)^2}.\end{aligned} \tag{48}$$

A short table of numerical values of the above two integrals follows:

| $m$ | $\xi_{[01][1m]}$ | $\Xi_{[01][1m]}$ |
|---|---|---|
| 1 | 0.23871 $a/b$ | 0.18638 $a/b$ |
| 2 | 0.32865 $a/b$ | 0.31150 $a/b$ |
| 3 | 0.03682 $a/b$ | 0.02751 $a/b$ |

Putting these values into the expression (41) for the forward coupling coefficient, we obtain:

$$
\begin{aligned}
c^{+}_{[01][11]} &= \left[\frac{0.09319(\beta a)^2 - 0.84204}{\sqrt{h_{[01]}ah_{[11]}a}} + 0.09319\sqrt{h_{[01]}ah_{[11]}a}\right]\frac{1}{b}, \\
c^{+}_{[01][12]} &= \left[\frac{0.15575(\beta a)^2 - 3.35688}{\sqrt{h_{[01]}ah_{[12]}a}} + 0.15575\sqrt{h_{[01]}ah_{[12]}a}\right]\frac{1}{b}, \qquad (49) \\
c^{+}_{[01][13]} &= \left[\frac{0.01376(\beta a)^2 - 0.60216}{\sqrt{h_{[01]}ah_{[13]}a}} + 0.01376\sqrt{h_{[01]}ah_{[13]}a}\right]\frac{1}{b}.
\end{aligned}
$$

### 1.3.2 *Coupling Coefficients due to Dielectric*

Depending upon the distribution of the dielectric material over the cross section of the guide, the $TE_{01}$ mode may be coupled to any mode except those of the $TM_{0m}$ family. The dielectric coupling coefficients, as given by equations (41), are

$$
\begin{aligned}
d^{+}_{[01](nm)} &= d^{+}_{(nm)[01]} = \tfrac{1}{2}\beta\Delta_{[01](nm)}\sqrt{h_{(nm)}/h_{[01]}}, \\
d^{+}_{[01][nm]} &= d^{+}_{[nm][01]} = \frac{\beta^2\Delta_{[01][nm]}}{2\sqrt{h_{[01]}h_{[nm]}}}.
\end{aligned} \qquad (50)
$$

The Δ's are obtained from equations (32); thus,

$$
\begin{aligned}
\Delta_{[01](nm)} &= \int_S \delta(\text{grad } \mathrm{T}_{(nm)})\cdot(\text{flux } \mathrm{T}_{[01]})\, dS \\
&= \int_0^{2\pi}\int_0^a \frac{\delta(\rho,\varphi) n\sqrt{\epsilon_n} J_n(\chi_{(nm)}\rho) J_1(\chi_{[01]}\rho) \cos n\varphi}{\pi a k_{(nm)} J_{n-1}(k_{(nm)}) J_0(k_{[01]})}\, d\rho\, d\varphi, \\
\Delta_{[01][nm]} &= \int_S \delta(\text{grad } \mathrm{T}_{[nm]})\cdot(\text{grad } \mathrm{T}_{[01]})\, dS \qquad (51) \\
&= -\int_0^{2\pi}\int_0^a \frac{\delta(\rho,\varphi)\sqrt{\epsilon_n}\chi_{[nm]} J_n'(\chi_{[nm]}\rho) J_1(\chi_{[01]}\rho) \cos n\varphi}{\pi a (k_{[nm]}^2 - n^2)^{\frac{1}{2}} J_n(k_{[nm]}) J_0(k_{[01]})}\, \rho\, d\rho\, d\varphi.
\end{aligned}
$$

## 1.4 *Dielectric Losses*

To take account of dielectric dissipation we may introduce a complex permittivity,

$$\epsilon = \epsilon' - i\epsilon'' = \epsilon'(1 - i \tan \varphi), \qquad (52)$$

where $\varphi$ is the loss angle of the dielectric and is not, of course, to be confused with the coordinate $\varphi$. If we let

$$\epsilon' = \epsilon_0(1 + \delta'), \qquad (53)$$

where $\delta'$ is real, then (52) may be written

$$\epsilon = \epsilon_0[1 + \delta' - i(1 + \delta') \tan \varphi] = \epsilon_0(1 + \delta), \tag{54}$$

where

$$\delta = \delta' - i(1 + \delta') \tan \varphi. \tag{55}$$

No changes in the formal analysis result from the fact that $\delta$ now has a complex value.

If the compensator is designed (assuming a lossless dielectric) so that the total power coupled from $TE_{01}$ into all spurious modes is small at all points, it is reasonable to assume that the principal effect of dielectric loss on the $TE_{01}$ mode will be seen in the modified phase constant $\beta_{[01]}$ of this mode in the presence of the compensator. If the compensator is made by filling a certain part $S_1$ of the guide cross section with a medium of constant (complex) permittivity, and the rest of the cross section with air, then from (32) and (40) the modified phase constant of the $TE_{01}$ mode is

$$\beta_{[01]} = h_{[01]} + \frac{\beta^2\delta}{2h_{[01]}} \int_{S_1} (\text{grad } T_{[01]})^2 \, dS, \tag{56}$$

where $\delta$ is given by (55). Since $\delta$ is complex, the attenuation constant is

$$\alpha_{[01]} = -\text{Im}\, \beta_{[01]} = \frac{\beta^2(1 + \delta') \tan \varphi}{2h_{[01]}} \int_{S_1} (\text{grad } T_{[01]})^2 \, dS, \tag{57}$$

where the integration may be carried out as soon as the area $S_1$ is specified.

The approximation (57) for the attenuation constant due to dielectric losses has a simple physical interpretation. It corresponds to the power which would be dissipated in a medium of conductivity $\omega\epsilon''$ if the electric field existing in the medium were the same as the field of the $TE_{01}$ mode in a straight, empty guide. This is probably a very good approximation if $\delta'$ is small, as it will be for the foam dielectrics from which compensators are most likely to be made.

It is doubtful that (57) furnishes a good approximation to the dielectric loss when the permittivity of the compensator is high ($\delta$ not small compared to unity). If the permittivity is high the cross section of the dielectric member will be small, but the field perturbation may be large in the immediate neighborhood of the dielectric. The series which represent the fields in terms of the normal modes of the empty guide may converge slowly; in other words, when using the telegraphist's equations one must consider the coupling between $TE_{01}$ and a large number of other modes, none of which appears by itself to be very strongly coupled to $TE_{01}$.

Of course if we had a single normal mode of the *compensated* guide, with a field pattern independent of distance along the guide, it might well be possible to calculate the field distribution and the dielectric losses approximately, without reference to the telegraphist's equations and regardless of the permittivity of the dielectric. However, we do not have a single normal mode of the compensated guide, but rather a mixture of modes. The field pattern varies along the guide as the modes phase in and out; and it is not easy to conclude from this picture what the actual dielectric losses will be.

Finally it should be remembered that we have said nothing about the possible effect of a dielectric compensator on eddy current losses in the waveguide walls. If one attempted to use a compensator of small physical size and correspondingly high permittivity, the resulting perturbation of the electric field might very well increase the eddy current losses in the wall adjacent to the compensator. On the other hand, the increase would probably be negligible for a compensator made out of a foam dielectric.

## II. APPLICATION

### 2.1 *Properties of Uniformly Coupled Transmission Lines*

We shall now apply the preceding theory to the calculation of $TE_{01}$ mode coupling in gentle bends. To describe propagation in a curved waveguide in terms of the modes of a straight guide requires, in general, the solution of the infinite set of equations (37); but we can give an adequate approximate treatment by considering just two modes at a time, one mode of each pair always being $TE_{01}$. Furthermore we need consider only the forward waves, since the relative power coupled from the forward waves into the backward waves is quite small.

The differential equations representing the forward waves on two uniformly coupled transmission lines are:

$$\begin{aligned} \frac{da_0}{dz} + \gamma_0 a_0 + i\kappa a_1 &= 0, \\ i\kappa a_0 + \frac{da_1}{dz} + \gamma_1 a_1 &= 0. \end{aligned} \tag{58}$$

In these equations $a_0(z)$ and $a_1(z)$ are the amplitudes of the forward traveling waves, normalized so that $| a_0 |^2$ and $| a_1 |^2$ represent power flow directly. We may think of the subscript 0 as always referring to the $TE_{01}$

mode. The complex constants $\gamma_0$ and $\gamma_1$ may be regarded as (modified) propagation constants; note that because of the coupling they are not necessarily equal to the propagation constants of the uncoupled modes. The coupling coefficient is denoted by $\kappa$; it will be real if the coupling mechanism is lossless, but is not required to be so in the general mathematical solution.

We are interested in the case in which line 0 contains unit power at $z = 0$, and line 1 contains no power. Subject to the initial conditions

$$a_0(0) = 1, \qquad a_1(0) = 0, \tag{59}$$

the solution of (58) is

$$\begin{aligned} a_0(z) &= \left[\frac{1}{2} + \frac{(\gamma_0 - \gamma_1)}{2r}\right] e^{m_2 z} + \left[\frac{1}{2} - \frac{(\gamma_0 - \gamma_1)}{2r}\right] e^{m_1 z}, \\ a_1(z) &= \frac{i\kappa}{r}[e^{m_2 z} - e^{m_1 z}], \end{aligned} \tag{60}$$

where

$$r = \sqrt{(\gamma_0 - \gamma_1)^2 - 4\kappa^2}, \tag{61}$$

and

$$\begin{aligned} m_1 &= \tfrac{1}{2}[-(\gamma_0 + \gamma_1) + r], \\ m_2 &= \tfrac{1}{2}[-(\gamma_0 + \gamma_1) - r]. \end{aligned} \tag{62}$$

For the case of two propagating modes without loss, $\kappa$ is real and we may write

$$\gamma_0 = i\beta_0\,, \qquad \gamma_1 = i\beta_1\,, \tag{63}$$

so that

$$r = i\sqrt{(\beta_0 - \beta_1)^2 + 4\kappa^2}. \tag{64}$$

A straightforward calculation now gives the power in each line at any point:

$$\begin{aligned} P_0 &= |\, a_0(z)\,|^2 = 1 - \frac{4\kappa^2}{(\beta_0 - \beta_1)^2 + 4\kappa^2} \sin^2 \tfrac{1}{2}[(\beta_0 - \beta_1)^2 + 4\kappa^2]^{\frac{1}{2}} z, \\ P_1 &= |\, a_1(z)\,|^2 = \frac{4\kappa^2}{(\beta_0 - \beta_1)^2 + 4\kappa^2} \sin^2 \tfrac{1}{2}[(\beta_0 - \beta_1)^2 + 4\kappa^2]^{\frac{1}{2}} z. \end{aligned} \tag{65}$$

Hence the maximum power transferred from line 0 to line 1 is

$$(P_1)_{\max} = \frac{4\kappa^2}{(\beta_0 - \beta_1)^2 + 4\kappa^2} = \frac{[2\kappa/(\beta_0 - \beta_1)]^2}{1 + [2\kappa/(\beta_0 - \beta_1)]^2}, \tag{66}$$

and the points of maximum power transfer are

$$z = \frac{(2n + 1)\pi}{\sqrt{(\beta_0 - \beta_1)^2 + 4\kappa^2}}, \tag{67}$$

where $n$ is any integer.

As is well known, complete power transfer from line 0 to line 1 is possible if and only if the (modified) phase constants $\beta_0$ and $\beta_1$ are equal. In general the maximum power transferred to line 1 depends on the ratio of the coupling coefficient to the difference in phase constants, and if this ratio is small, then

$$(P_1)_{\max} \approx 4\kappa^2/(\beta_0 - \beta_1)^2. \tag{68}$$

If the difference in phase constants is not large enough to prevent undesirable power loss from line 0, an alternative possibility, as Miller[8] has shown, is to increase the attenuation constant of line 1 while leaving the attenuation constant of line 0 as nearly unchanged as possible. We can get an idea of the required attenuation difference from the following approximate treatment.

Let

$$\begin{aligned} \gamma_0 &= \alpha_0 + i\beta_0 , \\ \gamma_1 &= \alpha_1 + i\beta_1 , \end{aligned} \tag{69}$$

and assume that

$$| 2\kappa/(\gamma_0 - \gamma_1) |^2 \ll 1. \tag{70}$$

Then

$$r \approx \gamma_0 - \gamma_1 - \frac{2\kappa^2}{(\gamma_0 - \gamma_1)}, \tag{71}$$

and

$$\begin{aligned} m_1 &\approx -\gamma_1 - \frac{\kappa^2}{\gamma_0 - \gamma_1}, \\ m_2 &\approx -\gamma_0 + \frac{\kappa^2}{\gamma_0 - \gamma_1}. \end{aligned} \tag{72}$$

We are interested in the power in line 0. From the first of equations (60), we have

$$a_0(z) \approx \left[1 + \frac{\kappa^2}{(\gamma_0 - \gamma_1)^2}\right] e^{m_2 z} - \frac{\kappa^2}{(\gamma_0 - \gamma_1)^2} e^{m_1 z}. \tag{73}$$

Let us consider the case in which line 1 has a much higher attenuation constant than line 0; that is,

$$\alpha_1 \gg \alpha_0 . \tag{74}$$

The second term on the right side of (73) is provided with a small coefficient, and also its exponential factor decays much faster than the exponential in the first term. The second term, therefore, rapidly becomes negligible as $z$ increases, and we may write,

$$a_0(z) \approx \left[1 + \frac{\kappa^2}{(\gamma_0 - \gamma_1)^2}\right] e^{\kappa^2 z/(\gamma_0-\gamma_1)} e^{-\gamma_0 z}. \tag{75}$$

If the attenuation constant of line 0 is not modified* by the presence of the coupled lossy line 1, then in the absence of line 1 the amplitude of $a_0(z)$ would be $e^{-\alpha_0 z}$, and the factor by which the amplitude is reduced owing to the presence of line 1 is

$$\left|1 + \frac{\kappa^2}{(\gamma_0 - \gamma_1)^2}\right| \left| e^{\kappa^2 z/(\gamma_0-\gamma_1)} \right|. \tag{76}$$

The first factor on the right is very nearly unity, but not less than unity if $\kappa$ is real (lossless coupling mechanism) and

$$(\alpha_1 - \alpha_0)^2 \geqq (\beta_1 - \beta_0)^2. \tag{77}$$

Hence the factor by which the amplitude is multiplied is not less than

$$\left| \exp \frac{\kappa^2 z}{\gamma_0 - \gamma_1} \right| = \exp \frac{(\alpha_0 - \alpha_1)\kappa^2 z}{(\alpha_0 - \alpha_1)^2 + (\beta_0 - \beta_1)^2}, \tag{78}$$

assuming that $\kappa$ is real. If the amplitude of the wave on line 0 is not to be down by more than $N$ nepers, after a distance $z$, from what it would have been in the absence of the coupled line, it suffices to have

$$\frac{(\alpha_1 - \alpha_0)\kappa^2 z}{(\alpha_1 - \alpha_0)^2 + (\beta_1 - \beta_0)^2} = N, \tag{79}$$

or

$$\alpha_1 - \alpha_0 = \tfrac{1}{2}\left[(\kappa^2 z/N) + \sqrt{(\kappa^2 z/N)^2 - 4(\beta_1 - \beta_0)^2}\right]. \tag{80}$$

### 2.2 $TE_{01}$–$TM_{11}$ *Coupling in Plain and Compensated Bends*

In Jouguet's[1] and Rice's[2] analysis of propagation in a curved waveguide, the curvature is treated as a perturbation and the field com-

* The value of $\alpha_0$ may very well be modified by the coupling; but if it is this can easily be taken into account when computing the over-all change in $|a_0(z)|$ due to the presence of line 1.

ponents are developed in powers of the small parameter $a/b$, but the field perturbations are not expressed in terms of the modes of the straight waveguide. We shall now consider propagation in plain and compensated bends from the coupled-mode viewpoint.

Denote the coefficient of coupling between the $TE_{01}$ mode and the $TM_{nm}$ mode or the $TE_{nm}$ mode by

$$\begin{aligned} \kappa_{(nm)} &= c_{(nm)} + d_{(nm)} , \\ \kappa_{[nm]} &= c_{[nm]} + d_{[nm]} , \end{aligned} \tag{81}$$

respectively, where as usual we indicate TM modes by enclosing the subscripts in *parentheses* and TE modes by enclosing the subscripts in *brackets*. In Part I the coupling coefficients are written with two pairs of subscripts, since in general they may refer to any two modes, but here one pair of subscripts would always be [01] and will be omitted. The coefficient $c$ represents that part of the coupling (if any) which is due to the curvature of the guide, and $d$ represents the coupling (if any) which is due to the inhomogeneity of the dielectric. We assume that the dielectric loading, if present, is symmetric with respect to the plane of the bend, so that coupling to only one polarization of each mode need be considered.

The phase constants of the $TM_{nm}$ and $TE_{nm}$ modes in a straight, empty guide are, respectively,

$$h_{(nm)} = \sqrt{\beta^2 - \chi_{(nm)}^2}, \qquad h_{[nm]} = \sqrt{\beta^2 - \chi_{[nm]}^2}, \tag{82}$$

where

$$\chi_{(nm)} = k_{(nm)}/a, \qquad \chi_{[nm]} = k_{[nm]}/a, \tag{83}$$

and

$$J_n\,(k_{(nm)}) = 0, \qquad J_n'(k_{[nm]}) = 0. \tag{84}$$

Also

$$\beta = 2\pi/\lambda, \tag{85}$$

where $\lambda$ is the free-space wavelength.

As noted in the preceding section, the presence of coupling may cause the modified phase constants $\beta_{(nm)}$ and $\beta_{[nm]}$ of the coupled modes to differ slightly from the unperturbed phase constants $h_{(nm)}$ and $h_{[nm]}$. In a plain bend (curvature coupling only), the $\beta$'s are equal to the $h$'s, and in most cases the effect of a small amount of dielectric coupling on the phase constants may be neglected. Exact values of $\beta_{[nm]}$ and $\beta_{(nm)}$ may be obtained if necessary from (40).

The coupling coefficient between the $TE_{01}$ and $TM_{11}$ modes in a plain bend is given by equation (46) as

$$c_{(11)} = \beta a/\sqrt{2}k_{[01]}b = 0.18454\ \beta a/b = 1.1595a/\lambda b. \tag{86}$$

The (smallest) critical distance for maximum power transfer is given by (67) with $\beta_0 = \beta_1$, namely

$$z_{c_0} = \frac{\pi}{2c_{(11)}} = \frac{k_{[01]}\, b\lambda}{2\sqrt{2}a} = \frac{1.3547b\lambda}{a}, \tag{87}$$

and the critical angle $\vartheta_{c_0}$ is

$$\vartheta_{c_0} = z_{c_0}/b = 1.3547\ \lambda/a \text{ radians} = 77.62\ \lambda/a \text{ degrees}. \tag{88}$$

This expression agrees, as it must, with that obtained by Jouguet and Rice. (We write $\vartheta_{c_0}$ for the uncompensated bend in order to reserve $\vartheta_c$ for a bend with dielectric loading.)

It should be pointed out that $c_{(11)}$ is not necessarily the largest of the coupling coefficients due to curvature. In a guide sufficiently far above cutoff, it appears from (49) that $c_{[11]}$ is approximately equal to $c_{(11)}$, and $c_{[12]}$ is one and two-thirds times as large as $c_{(11)}$. If two transmission lines are coupled over a distance $z$ which is small compared to the distance required for maximum power transfer, then by (65) the relative power transferred to line 1 is

$$P_1(z) \approx \kappa^2 z^2, \tag{89}$$

which is proportional to the square of the coupling coefficient. It follows that for a sufficiently small bending angle the largest amount of power will go into the mode which has the largest coupling coefficient to $TE_{01}$ (in the above example, $TE_{12}$). Each coupling coefficient, however, is proportional to $1/b$, and can be made as small as desired by increasing the radius of curvature of the bend. Since the phase constants are unaltered to first approximation by the curvature, the maximum power transferred tends to zero with $1/b^2$ for every mode whose unperturbed phase constant differs from $h_{[01]}$. The only mode with finite power transfer for a finite bending angle with an arbitrarily large bending radius is $TM_{11}$, since $h_{(11)} = h_{[01]}$. For the present we shall assume a bending radius so large that power transfer to modes other than $TM_{11}$ is negligible. Complete power transfer from $TE_{01}$ to $TM_{11}$ will then take place in a plain bend at odd multiples of the critical bending angle $\vartheta_{c_0}$.

We now consider a dielectric-loaded bend in which the permittivity $\epsilon$ is a function of the transverse coordinates $(\rho, \varphi)$, but does not vary from one cross section to another. The permeability $\mu$ is assumed to be constant. Thus we shall write, as in (25),

$$\begin{aligned} \mu &= \mu_0 , \\ \epsilon &= \epsilon_0[1 + \delta(\rho, \varphi)], \end{aligned} \tag{90}$$

and assume that

$$\frac{1}{S} \int_S |\, \delta \,|\, dS \ll 1, \tag{91}$$

where $S$ is the cross-sectional area of the guide. The inequality (91) implies either that $\epsilon$ does not vary much over the cross section, or that it varies extensively over only a small part of the cross section. The first alternative corresponds to a dielectric whose relative permittivity differs but little from unity, and is the most likely case in practice. If, on the other hand, $\delta$ is large in a small region (a thin sliver of high-permittivity material), the dielectric coupling coefficients will not diminish very rapidly with increasing mode number. The $TE_{01}$ mode will be appreciably coupled to a large number of modes, and it may not be safe to assume that the total power converted into spurious modes is small just because the conversion into any given mode is small. We shall not try to decide here what maximum value of $|\, \delta \,|$ is practicable.

If the distribution of dielectric is symmetric with respect to the plane of the bend, the dielectric couples the $TE_{01}$ mode to a definite polarization of each spurious mode, and in particular to the same polarization of the $TM_{11}$ mode that is coupled by curvature alone.* The dielectric coupling coefficient between $TE_{01}$ and $TM_{11}$ is, from (50) and (51),

$$d_{(11)} = \frac{\beta}{\sqrt{2}\pi a k_{[01]} J_0^{\,2}(k_{[01]})} \int_S \delta(\rho, \varphi) J_1^{\,2}(\chi_{[01]}\rho) \frac{\cos \varphi}{\rho}\, dS. \tag{92}$$

It is obvious that the *decoupling condition*, namely:

$$\kappa_{(11)} \equiv c_{(11)} + d_{(11)} = 0, \tag{93}$$

may be satisfied by an infinite number of different distributions of permittivity. Ingenuity is required, however, to find a configuration which is easy to fabricate and which does not couple the $TE_{01}$ mode too strongly to any other mode in the guide. The spurious mode problem is quite serious when the diameter of the guide (in wavelengths) is so large that

* A dielectric insert which is not symmetric with respect to the plane of the bend will couple $TE_{01}$ to the other polarization of $TM_{11}$, with potentially complete power transfer to this mode on account of the equality of phase velocities. A small accidental lack of symmetry should not lead to serious mode conversion in a bend of moderate angle.

many modes have phase velocities close to $TE_{01}$. In the next section we shall compute the coupling to various modes for a number of cases.

Mention may be made here of the effect of imperfect decoupling. If we could satisfy the decoupling condition (93) exactly, then to first order there would be no transfer of power from $TE_{01}$ to $TM_{11}$ in a bend of any angle. In practice we cannot expect to satisfy (93) exactly, on account of uncertainties in the permittivity and dimensions of the compensator. If the coupling coefficients are constant along the bend, the effect of reducing $\kappa_{(11)}$ by making $d_{(11)}$ nearly equal and opposite to $c_{(11)}$ is to increase the distance required for maximum power transfer to $TM_{11}$ to take place. In the simple case where $\beta_{(11)} = \beta_{[01]}$, as, for example, when the compensator is made of a bent half-cylinder of dielectric, the critical angle for a fixed bend radius is inversely proportional to $\kappa_{(11)}$. The critical angle $\vartheta_c$ for an imperfectly compensated bend, in terms of the critical angle $\vartheta_{c_0}$ for an uncompensated bend, is

$$\vartheta_c = \left| \frac{c_{(11)}}{c_{(11)} + d_{(11)}} \right| \vartheta_{c_0} . \tag{94}$$

If $d_{(11)}$ can be made equal and opposite to $c_{(11)}$ within 1 per cent, say, then $\vartheta_c = 100\, \vartheta_{c_0}$. The power transferred in a bend of angle $\vartheta$ is simply proportional to $\sin^2 (\vartheta/\vartheta_c)$.

### 2.3 *Various Compensator Designs*

From now on we shall consider a compensated bend in which the $TE_{01}$ and $TM_{11}$ modes are completely decoupled, and we shall investigate the coupling between $TE_{01}$ and spurious modes. (By "spurious mode" we mean any mode of the straight round guide except $TE_{01}$ or $TM_{11}$.) We shall assume that the power in all spurious modes is everywhere low compared to the power level of $TE_{01}$. To first order, therefore, we may compute the power abstracted from $TE_{01}$ by mode conversion by assuming that the $TE_{01}$ mode crosstalks into each spurious mode independently. In practice the crosstalk is greatest into modes whose phase velocities are nearest to the phase velocity of $TE_{01}$; and it seems more than adequate, at least for foam dielectric compensators, to consider about a dozen modes.

From (68), the maximum relative power (assumed small) which crosstalks from $TE_{01}$ into a given spurious mode is

$$(P_1)_{\max} \approx [2\kappa/(\beta_0 - \beta_1)]^2, \tag{95}$$

where $\kappa$ is the total coupling coefficient. For all spurious modes we assume that the difference in phase constants is not much changed by the dielectric loading; this assumption obviates the somewhat laborious calculation of $\beta_1$ for each spurious mode from (40). Thus,

$$\beta_0 - \beta_1 \approx h_0 - h_1 , \tag{96}$$

and a good estimate of the maximum power which crosstalks into any spurious mode is

$$(P_1)_{\max} \approx [2(c + d)/(h_0 - h_1)]^2. \tag{97}$$

If the form and dimensions of a bend compensator are fixed, and the $TE_{01}$—$TM_{11}$ decoupling condition is assumed to be met by adjusting the permittivity, it turns out that the maximum power which crosstalks into a given mode is proportional to $(a/b)^2$. It is thus easy to calculate the bending radius which makes $(P_1)_{\max}$ for any given mode equal to a (small) preassigned value. The total power abstracted from the $TE_{01}$ mode by mode conversion will be a complicated, fluctuating function of distance along the bend, or of frequency at a fixed distance, because of the different critical distances for maximum power transfer into the different spurious modes, but we can get an idea of the minimum tolerable bending radius by considering just the crosstalk into the one or two most troublesome modes.

It seems likely that with present-day dielectrics at millimeter wavelengths, dielectric losses in a compensated bend will be comparable to mode conversion losses. For this reason an estimate of dielectric losses is given in connection with each type of compensator design discussed below.

### 2.3.1 *The Geometrical Optics Solution*

An obvious way to design a bend compensator on paper is to load the bend with a medium of continuously varying permittivity for which*

$$\delta = -2(\rho/b) \cos \varphi. \tag{98}$$

This may be called the geometrical optics solution of the bend problem, since to first order it equalizes the optical length of all circular paths which are coaxial with the curved center line of the waveguide. Physically the required variation of permittivity is rather simple; the permittivity at each point depends only on the distance of the point from a line through the center of curvature perpendicular to the plane of the bend. An attempt to indicate this variation by shading has been made

* In order that the relative permittivity of the medium be nowhere less than unity, the constant $\epsilon_0$ in the expression $\epsilon = \epsilon_0(1 + \delta)$ must be greater than the permittivity of free space; but this does not affect the analysis in any way.

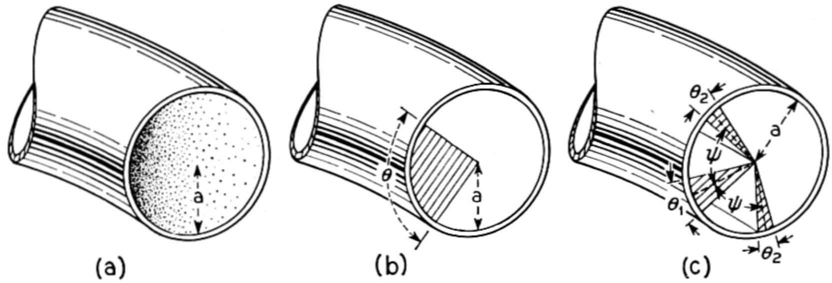


Fig. 2 — Various bend compensators.

in Fig. 2(a). One could approximate the continuous variation with a laminated structure consisting of a number of layers of different permittivities, the permittivity varying slightly from one layer to the next.

Although the geometrical optics approach provides a very good theoretical solution to the bend problem, it does not lead to a perfect compensator. It is shown at the end of this section that a perfect compensator, in the sense of one which does not couple $TE_{01}$ to any other mode at any frequency, does not exist. The geometrical optics compensator couples the same modes to $TE_{01}$ that are coupled by the curvature of the bend itself, namely $TM_{11}$ and the $TE_{1m}$ family; but the coupling coefficients of the various modes are not in exactly the same ratios. Thus if $\delta$ is given by (98), the net coupling between $TE_{01}$ and $TM_{11}$ in the compensated bend is zero, but there will be a small residual coupling between $TE_{01}$ and each of the $TE_{1m}$ modes.

The curvature coupling coefficients are given by (49). Table I contains numerical values which have been worked out for $\beta a = 12.930$ and $\beta a = 29.554$, corresponding respectively to guide diameters of $\frac{7}{8}$ inch and 2 inches at an operating wavelength of 5.4 mm.

Dielectric coupling coefficients can be worked out without much dif-

TABLE I — COUPLING COEFFICIENTS AND POWER TRANSFER IN GEOMETRICAL OPTICS COMPENSATOR

| Mode | $\beta a = 12.930$ | | | $\beta a = 29.554$ | | |
|---|---|---|---|---|---|---|
| | $c$ | $d$ | $2\frac{(c+d)}{(h_0-h_1)}$ | $c$ | $d$ | $2\frac{(c+d)}{(h_0-h_1)}$ |
| $TM_{11}$ | $2.386/b$ | $-2.386/b$ | — | $5.454/b$ | $-5.454/b$ | — |
| $TE_{11}$ | $2.344/b$ | $-2.479/b$ | $0.600a/b$ | $5.480/b$ | $-5.537/b$ | $0.597a/b$ |
| $TE_{12}$ | $3.759/b$ | $-4.318/b$ | $-1.962a/b$ | $9.092/b$ | $-9.322/b$ | $-1.954a/b$ |
| $TE_{13}$ | $0.306/b$ | $-0.420/b$ | $-0.087a/b$ | $0.793/b$ | $-0.834/b$ | $-0.083a/b$ |

ficulty from (50), (51), and (98). The first few coefficients are:

$$
\begin{aligned}
d_{(11)} &= -0.18454\beta a/b, \\
d_{[11]} &= -\frac{0.18638(\beta a)^2/b}{\sqrt{h_{[01]}ah_{[11]}a}}, \\
d_{[12]} &= -\frac{0.31150(\beta a)^2/b}{\sqrt{h_{[01]}ah_{[12]}a}}, \\
d_{[13]} &= -\frac{0.02751(\beta a)^2/b}{\sqrt{h_{[01]}ah_{[13]}a}}.
\end{aligned} \tag{99}
$$

Some numerical values are recorded in Table I.

The phase constant of any mode in the compensated bend is equal to the phase constant of the same mode in a guide filled with material of constant permittivity $\epsilon_0$. We may therefore set $\beta_{[01]} - \beta_{[1m]}$ equal to $h_{[01]} - h_{[1m]}$. Strictly speaking, $\lambda$ is then the wavelength of a free wave in a medium of permittivity $\epsilon_0$; but $\epsilon_0$ differs little from the permittivity of vacuum if the compensator is made from a foam dielectric. The ratio of total coupling coefficient to difference in phase constants, namely $2(c + d)/(h_0 - h_1)$, which determines the maximum power transfer by (97), is given in Table I for the $\frac{7}{8}$-inch and 2-inch guides at $\lambda = 5.4$ mm.

For large $\beta a$ the leading terms in $c_{[1m]}$ and $d_{[1m]}$, which are proportional to $\beta a$, cancel each other, and $c_{[1m]} + d_{[1m]}$ decreases like $1/\beta a$. The difference in phase constants, $h_{[01]} - h_{[1m]}$, is also proportional to $1/\beta a$ for large $\beta a$. Hence the ratio of coupling coefficient to difference in phase constants approaches a finite limit as $\beta a$ approaches infinity; to three decimal places the limiting values are the same as those given in Table I for $\beta a = 29.554$.

If we choose a sufficiently large value of $a/b$, the maximum power transferred to a given mode may be made to approach any preassigned small value as $\lambda/a$ approaches zero. *This is a special property of the geometrical optics compensator*. For other compensator designs $c + d$ will be of the order of $\beta a$ while $h_0 - h_1$ will be of the order of $1/\beta a$, so that $2(c + d)/(h_0 - h_1)$ will tend to infinity like $(\beta a)^2$. Hence in the short-wavelength limit the bend output will be a jumble of modes all at comparable power levels. Practically, one must expect the same end result with a geometrical optics compensator, because of the impossibility of meeting exact mathematical specifications; but a carefully-designed geometrical optics compensator should work satisfactorily up to a considerably higher frequency than any other type.

To get an idea of tolerable bending radii with a geometrical optics compensator, we shall calculate the radius at which the maximum mode conversion loss from $TE_{01}$ into $TE_{12}$ (the worst spurious mode) is 0.1 db. Setting $P_1 = 0.02276$, we find from (68) that the minimum bending radius for a $\frac{7}{8}$-inch guide is 5.69 inches, and for a 2-inch guide, 12.95 inches, both at a wavelength of 5.4 mm. It is worth noting that if $|d_{[12]}|$ is increased by 5 per cent of its theoretical value, the minimum bending radius becomes 7.89 inches for the $\frac{7}{8}$-inch guide and 39.2 inches for the 2-inch guide. (This assumes that the $TM_{11}$ mode is still properly decoupled and that $TE_{12}$ is still the worst spurious mode.)

Dielectric losses are likely to be a serious problem in a geometrical optics compensator, inasmuch as the whole volume of the bent guide has to be filled with dielectric. Relatively large values of $\delta$ are required to negotiate bends as sharp as those just discussed. For example, if $b = 13a$, in a practical case $\delta$ might range from 0.058 at the inner surface of the bend to 0.250 at the center of the guide to 0.442 at the outer surface (referred to $\epsilon_0$ as the permittivity of free space). The loss tangent of present-day dielectrics in this range is approximately $2 \times 10^{-4}$. A large (i.e., far above cutoff) waveguide filled with a dielectric of relative permittivity 1.25 and loss tangent $2 \times 10^{-4}$ will show a dielectric loss of about 1.13 db/meter or 0.34 db/ft at 5.4 mm. The dielectric loss in a 90° bend with a bending radius of 1 foot would be about 0.54 db, and for other bends the loss would be directly proportional to the length of the bend, and to the loss tangent if different from $2 \times 10^{-4}$. It is true that loss tangents as low as $5 \times 10^{-5}$ may be obtained with lower values of permittivity, say $\delta = 0.033$; but with such a small $\delta$ the bend radius must be proportionately larger, and the total dielectric losses in a bend of given angle would be larger than with a higher permittivity material.

We proceed now to demonstrate the assertion made earlier that a perfect bend compensator does not exist. More precisely, we shall show that it is impossible to compensate the bend with an isotropic medium whose permittivity and permeability are everywhere finite, but otherwise arbitrary, in such a way that there is no conversion from $TE_{01}$ to any other mode at any point at any frequency.

If there is no mode conversion at any point of the bend then the fields at all points must be those of the $TE_{01}$ mode, referred to the bent cylindrical coordinate system described at the beginning of Section 1.1. In other words, we have prescribed the electromagnetic field and are asking whether it is possible to choose the permeability and permittivity so that the given field will satisfy Maxwell's equations. Usually the answer to this question will be "No." The Maxwell equations,

$$\vec{\nabla} \times \vec{E} = -i\omega\mu\vec{H},$$
$$\vec{\nabla} \times \vec{H} = i\omega\epsilon\vec{E}, \tag{100}$$

are equivalent to six scalar equations, and if the components of $E$ and $\vec{H}$ are prescribed, one cannot in general satisfy all these equations by merely adjusting the two scalar functions $\mu$ and $\epsilon$.

It is particularly easy to see the difficulty for the $TE_{01}$ mode in a curved guide. Recall that the $TE_{01}$ mode fields are independent of the coordinate $\varphi$, and that the only non-vanishing field components are $E_\varphi(\rho, z)$, $H_\rho(\rho, z)$, and $H_z(\rho, z)$. The fourth of equations (5) is:

$$\frac{1}{\rho[1 + (\rho/b) \cos \varphi]} \left[ \frac{\partial}{\partial \varphi} ([1 + (\rho/b) \cos \varphi] H_z) - \frac{\partial}{\partial z} (\rho H_\varphi) \right] = i\omega\epsilon E_\rho . \tag{101}$$

Since $E_\rho = 0$, the right side of the equation is zero for any finite value of $\epsilon$ at any finite frequency, and the whole equation reduces to

$$-\frac{H_z \sin \varphi}{b + \rho \cos \varphi} = 0, \tag{102}$$

which can be true for all values of $\rho$ and $\varphi$ only if the radius of curvature of the bend is infinite. Hence a perfect compensator cannot be designed with *any* value of $\epsilon$.

The practical importance of this result does not appear to be great, since theoretically the geometrical optics solution would provide an extraordinarily good compensator. Until one has a dielectric whose permittivity is continuously variable and precisely controllable, and whose loss tangent is very low, even this solution is of only academic interest.

### 2.3.2 *The Single-Sector Compensator*

In practice the simplest way to compensate a bend is to fill part of the cross section of the guide with homogeneous dielectric material of relative permittivity $(1 + \delta)$, and leave the remainder empty. In many cases a suitable shape for the cross section of the dielectric is a sector of a circle, inserted on the side of the guide nearest the center of curvature of the bend, and symmetrically placed with respect to the plane of the bend. Such a sector, of total angle $\theta$, is shown in Fig. 2(b). We shall now discuss the properties of a single-sector compensator.

For future reference, the modified phase constants of the $TE_{01}$ and $TM_{11}$ modes may be calculated from (40) as:

$$\beta_{(11)} = h_{(11)} + \frac{\delta}{4\pi h_{(11)}} [(\theta - 0.29646 \sin \theta)\beta^2 - (0.70354 \sin \theta)\chi_{(11)}^2], \tag{103}$$

$$\beta_{[01]} = h_{[01]} + \frac{\beta^2 \delta\theta}{4\pi h_{[01]}}.$$

The dielectric coupling coefficients for the single-sector compensator are given by (50) and (51), provided that some of the Bessel functions are integrated by Simpson's rule. In particular, the dielectric coupling coefficient between $TE_{01}$ and $TM_{11}$ is

$$d_{(11)} = -0.12066\, \beta\delta \sin \theta/2. \tag{104}$$

Substituting (86) and (104) into (93), we obtain *the decoupling condition for a single-sector compensator*, namely

$$\delta = \frac{1.5295}{\sin \theta/2} \frac{a}{b}. \tag{105}$$

No circular magnetic modes ($TM_{0m}$) and no higher circular electric modes ($TE_{02}$, $TE_{03}$, etc.) are coupled to $TE_{01}$ by the single-sector compensator. We also observe that the coupling coefficients of the $TE_{1m}$ and $TM_{1m}$ modes do not depend upon the sector angle $\theta$ so long as $\delta$ and $\theta$ are related by (105). This is because these modes have the same angular dependence as the $TM_{11}$ mode, which we are trying to compensate.

The most troublesome spurious modes are those whose phase velocities are closest to $TE_{01}$, namely $TE_{21}$ and $TE_{31}$. The coupling coefficients and power transfer ratios for these two modes in $\frac{7}{8}$-inch and 2-inch guide are given in Table II. Either of these modes could be decoupled by proper choice of the sector angle, provided we had a uniform, low-dissipation dielectric with the required value of $\delta$, but it is impossible to decouple both modes at once with a single sector. As a compromise, we might adopt the sector angle which equalizes the maximum power transferred to $TE_{21}$ and $TE_{31}$ (the distances for maximum power transfer are of course not the same for the two modes). This angle is approximately 144°.

Suppose we wish to employ a 144° sector in a $\frac{7}{8}$-inch guide at 5.4 mm. If the criterion is that $TE_{01}$ is to lose a maximum of 0.1 db each to $TE_{21}$ and $TE_{31}$, so that the total mode conversion losses are of the order of 0.2

TABLE II — COUPLING COEFFICIENTS AND POWER TRANSFER TO $TE_{21}$ AND $TE_{31}$ MODES DUE TO SINGLE-SECTOR COMPENSATOR

| Mode | $\beta a = 12.930$ | | $\beta a = 29.554$ | |
|---|---|---|---|---|
| | $d$ | $2d/(h_0 - h_1)$ | $d$ | $2d/(h_0 - h_1)$ |
| $TE_{21}$ | $\frac{1.1152}{b} \frac{\sin\theta}{\sin\theta/2}$ | $-10.38 \frac{a}{b} \frac{\sin\theta}{\sin\theta/2}$ | $\frac{2.4727}{b} \frac{\sin\theta}{\sin\theta/2}$ | $-54.23 \frac{a}{b} \frac{\sin\theta}{\sin\theta/2}$ |
| $TE_{31}$ | $-\frac{0.6579}{b} \frac{\sin 3\theta/2}{\sin\theta/2}$ | $-10.89 \frac{a}{b} \frac{\sin 3\theta/2}{\sin\theta/2}$ | $-\frac{1.4426}{b} \frac{\sin 3\theta/2}{\sin\theta/2}$ | $-56.91 \frac{a}{b} \frac{\sin 3\theta/2}{\sin\theta/2}$ |

db, then the bending radius can be 19.5 inches. The corresponding value of $\delta$ is 0.036.

If we try to use a 144° sector in a 2-inch guide at 5.4 mm, with the same mode conversion criterion as before, the bending radius must be so large that no currently available dielectric has a small enough value of $\delta$ to satisfy the decoupling condition. We are therefore forced to use a smaller sector angle. With a sector of small angle, $TE_{31}$ is the worst spurious mode. It turns out that if $\delta = 0.033$ and if $TE_{01}$ is not to lose more than 0.1 db by conversion into $TE_{31}$, the minimum bending radius is 1131 inches or 94.3 feet, and the corresponding sector angle is 4.70°.

An approximate formula for the attenuation constant due to dielectric losses in a single-sector compensator is given by (57), provided that $\delta$ is small. The result is

$$\alpha_d = \frac{\beta^2(1 + \delta) \tan\varphi}{2h_{[01]}} \frac{\theta°}{360} \text{ nepers/meter,} \tag{106}$$

where $\tan\varphi$ is the loss tangent of the dielectric, and $\theta°$ is the sector angle in degrees. As numerical examples, we find that the dielectric loss at 5.4 mm in a $\frac{7}{8}$-inch guide compensated with a 144° sector having $\delta = 0.036$, $\tan\varphi = 5 \times 10^{-5}$, amounts to 0.085 db in a 90° bend of radius 19.5 inches. In a 2-inch guide compensated with a 4.70° sector ($\delta = 0.033$, $\tan\varphi = 5 \times 10^{-5}$), the dielectric loss is 0.155 db in a 90° bend of radius 94.3 feet.

### 2.3.3 *The Three-Sector Compensator*

Although the single-sector compensator should work well in a guide which will propagate only 40 to 50 modes, it does not look so promising for a 200- to 300-mode guide, chiefly because of the unavoidable crosstalk into $TE_{21}$ and/or $TE_{31}$. We are therefore led to consider the design of a three-sector compensator which will not couple either $TE_{21}$ or $TE_{31}$ to $TE_{01}$ .

A three-sector compensator is shown schematically in Fig. 2(c). The angle of the center sector is called $\theta_1$ and the angle of each outer sector $\theta_2$ . Each outer sector makes an angle $\psi$, measured between center planes, with the center sector.

The condition for decoupling $TE_{01}$ from $TM_{11}$ with the three-sector compensator is

$$\delta = \frac{1.5295}{(2 \cos \psi \sin \theta_2/2 + \sin \theta_1/2)} \frac{a}{b}. \tag{107}$$

If $\delta$ and $a/b$ are given, two additional conditions may be imposed upon $\theta_1$ , $\theta_2$ , and $\psi$. The conditions are taken to be:

$$\begin{aligned} 2 \cos 2\psi \sin \theta_2 + \sin \theta_1 &= 0, \\ 2 \cos 3\psi \sin 3\theta_2/2 + \sin 3\theta_1/2 &= 0. \end{aligned} \tag{108}$$

If equations (108) are satisfied, the compensator does not couple $TE_{01}$ to any modes of the $TE_{2m}$ , $TE_{3m}$ , $TM_{2m}$ , or $TM_{3m}$ families.

To design a three-sector compensator with given values of $a/b$ and $\delta$, one can solve (107) and (108) numerically for $\theta_1$ , $\theta_2$ , and $\psi$. However, if $\theta_1$ is small ($\leqq 20°$, for example), a simpler design procedure may be used. The following equations are approximately true:

$$\begin{aligned} \theta_1 &= \frac{126.8a}{b\delta} \text{ degees}, \\ \theta_2 &= 0.618\theta_1 , \\ \psi &= 72°. \end{aligned} \tag{109}$$

It should perhaps be pointed out that the precision of $\theta_1$ , $\theta_2$ , and $\psi$ individually is not critical, since there is no necessity for the coupling to $TE_{21}$ and $TE_{31}$ to be exactly zero, so long as it is reasonably small. One should, however, strive to make the $TE_{01}$–$TM_{11}$ coupling as nearly zero as possible, and it is therefore important to satisfy (107) with the greatest possible precision.

Expressions for the dielectric coupling coefficients due to the three-sector compensator may be obtained from those for the single-sector compensator if we merely replace $\sin n\theta/2$, wherever it occurs, by $\sin n\theta_1/2 + 2 \cos n\psi \sin n\theta_2/2$, where $n$ is the angular mode index, as usual. If $\theta_1$ , $\theta_2$ , and $\psi$ satisfy (108), then the coupling to $TE_{2m}$ , $TE_{3m}$ , $TM_{2m}$ , and $TM_{3m}$ is zero, and the mode having the largest value of $2(c + d)/(h_0 - h_1)$ is $TE_{12}$ . As noted earlier, since the fields of $TM_{11}$

TABLE III — COUPLING COEFFICIENT AND POWER TRANSFER TO $TE_{12}$ MODE DUE TO THREE-SECTOR COMPENSATOR

| Mode | $\beta a = 12.930$ | | | $\beta a = 29.554$ | | |
|---|---|---|---|---|---|---|
| | $c$ | $d$ | $2\frac{(c+d)}{(h_0 - h_1)}$ | $c$ | $d$ | $2\frac{(c+d)}{(h_0 - h_1)}$ |
| $TE_{12}$ | $3.7591/b$ | $-3.0334/b$ | $2.549a/b$ | $9.0919/b$ | $-6.5491/b$ | $21.59a/b$ |

and $TE_{12}$ have the same angular dependence, the coupling to $TE_{12}$ is independent of the number and arrangement of sectors used in the compensator so, long as the decoupling condition for $TM_{11}$ is satisfied. Numerical values are given in Table III.

The formula analogous to (106) for the attenuation constant due to dielectric loss in a three-sector compensator is

$$\alpha_d = \frac{\beta^2(1 + \delta) \tan \varphi}{2h_{[01]}} \frac{(\theta_1° + 2\theta_2°)}{360°} \text{ nepers/meter.} \tag{110}$$

As a numerical example, let us design a three-sector compensator for a $\frac{7}{8}$-inch guide at 5.4 mm. Under the requirement that the $TE_{01}$ loss due to conversion into $TE_{12}$ must not be greater than 0.1 db, the minimum bending radius is 7.39 inches. If the angles are *

$$\theta_1 = 60°,$$
$$\theta_2 = 30°,$$
$$\psi = 75°,$$

the value of $\delta$ should be 0.143, which is not difficult to obtain with foam dielectrics. Assuming a loss tangent of $2 \times 10^{-4}$ we find that dielectric losses in a 90° bend are about 0.12 db.

As a second example, for a 2-inch guide at 5.4 mm with the same mode conversion criterion, one needs a bending radius of 143.1 inches or 11.92 feet. With $\delta = 0.033$, the compensator angles are

$$\theta_1 = 27.6°,$$
$$\theta_2 = 16.4°,$$
$$\psi = 72.5°;$$

* It is not practicable to use larger angles, because if $\theta_1 > 60°$ portions of the outer sectors counteract the effect of the rest of the compensator on $TM_{11}$ , and dielectric losses make the design inefficient.

and if $\tan \varphi = 5 \times 10^{-5}$, the dielectric loss in a 90° bend is about 0.25 db.

### 2.4 *Can Dissipation Be Used to Discourage Spurious Modes?*

It was shown in Section 2.1 that the effect of markedly increasing the attenuation constant of one of two coupled transmission lines is to reduce the over-all attenuation of a wave introduced on the other line. One might wonder whether it would be practicable to decrease the permissible radius of a compensated bend by introducing loss into the spurious modes. The answer is "No", at least for guides large enough to propagate 200 to 300 modes at the operating wavelength. One simply cannot get the required magnitude of loss into the spurious modes without simultaneously introducing intolerable loss into $TE_{01}$. A numerical example will make this clear.

We found in Section 2.3.3 that with a three-sector compensator in 2-inch guide at 5.4 mm it would be possible to negotiate a bend of radius about 12 feet with a maximum loss of 0.1 db by mode conversion to $TE_{12}$ (the worst spurious mode). Let us now ask what the attenuation constant of $TE_{12}$ would have to be if we wished to transmit around a bend of radius 6 feet with a three-sector compensator, and have the mode conversion loss suffered by $TE_{01}$ not greater than 0.1 db in a 90° bend. Preparing to substitute into (80) of Section 2.1, we have the following values:

$$b = 72 \text{ inches},$$

$$\kappa_{[12]} = c_{[12]} + d_{[12]} = 2.54/b = 0.0353 \text{ in}^{-1},$$

$$z = \pi b/2 = 113.1 \text{ inches},$$

$$\beta_0 - \beta_1 \approx h_0 - h_1 = 0.236 \text{ radians/inch},$$

$$N = 0.1 \text{ db} = 0.0115 \text{ nepers}.$$

From (80) we get

$$\alpha_{[12]} - \alpha_{[01]} = 12.2 \text{ nepers/inch} \approx 4200 \text{ db/meter}.$$

Since the maximum $TE_{12}$ attenuation which can be achieved in a 2-inch guide by a mode filter which transmits $TE_{01}$ freely is of the order of 10 db/meter,* the value of $\alpha_{[12]}$ called for by the above calculation is obviously out of the question.

* This estimate is based on calculations described in Reference 6 for modes in a helix surrounded by a lossy sheath; but it is doubtful that much greater loss could be produced by other types of filter, such as resistance card "killers".

It should be noted that a moderate amount of loss in the spurious modes may be worse than none, so far as the effect on $TE_{01}$ is concerned. Miller[14] has shown that the total power dissipated in the system goes through a maximum when $(\alpha_1 - \alpha_0)/\kappa \approx 2$. It appears that $\alpha_1 - \alpha_0$ must exceed $\kappa$ by a couple of orders of magnitude before the loss in the driven line (i.e., $TE_{01}$) becomes really small, if we are counting on dissipation to counteract the coupling to spurious modes.

Since by use of the compensator we are attempting to make the $TE_{01} - TM_{11}$ coupling coefficient zero, we may expect that this coefficient, if not exactly zero, will at least be small compared to the coupling coefficients of spurious modes such as $TE_{12}$. Because $\kappa_{(11)}$ is very small, it may be that a practicable amount of loss in the $TM_{11}$ mode would improve the performance of the bend. But in view of the preceding paragraph we must be careful, when introducing loss into $TM_{11}$, not to introduce the wrong amount of loss into some spurious mode which has a larger coupling coefficient to $TE_{01}$.

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## APPENDIX

### *Compensation of a Gradual Bend by a Dielectric Insert in the Adjacent Straight Pipe*

We shall discuss briefly three different ways of transmitting the $TE_{01}$ mode around a plain (i.e., air-filled) bend with the aid of dielectric mode transducers inserted into the straight sections of guide on one or both sides of the bend. The first two methods involve converting the $TE_{01}$ mode to a normal mode of the bend and reconverting to $TE_{01}$ on the other side.[9] In the third method, the input to the bend is pure $TE_{01}$, and the output mixture of $TE_{01}$ and $TM_{11}$, whatever it may be, is reconverted to $TE_{01}$ by a dielectric transducer.

#### A.1 *The* $TM_{11}'$ *Normal Mode Solution*

One of the normal modes of the bend is a pure $TM_{11}$ mode ($TM_{11}'$) which is polarized at right angles to the $TM_{11}$ mode ($TM_{11}''$) that the bend couples to $TE_{01}$. Clearly if one has a transducer in a straight guide which converts $TE_{01}$ entirely to $TM_{11}$, it is a mere matter of rotating the transducer about the guide axis to insure that the polarization which enters the bend is $TM_{11}'$. We shall design such a transducer using a dielectric sector in a straight pipe.

From Section 2.1, for complete power transfer from $TE_{01}$ to $TM_{11}$ we must have

$$\beta_{[01]} - \beta_{(11)} = 0; \tag{111}$$

the transfer then takes place in a distance

$$l = \pi/2 \mid \kappa_{(11)} \mid . \tag{112}$$

The modified phase constants $\beta_{[01]}$ and $\beta_{(11)}$ for a single dielectric sector of angle $\theta$ are given by (103) of Section 2.3.2. Substituting these values into (111), we find that the only condition under which it is satisfied is

$$\begin{aligned} \sin \theta &= 0, \\ \theta &= 180°. \end{aligned} \tag{113}$$

The transducer must therefore be a half cylinder. From (104) we have

$$\kappa_{(11)} = d_{(11)} = -0.12066\, \beta\delta, \tag{114}$$

and so (112) gives for the length of the transducer,

$$l = 2.072\, \lambda/\delta. \tag{115}$$

The $TE_{01} - TM_{11}'$ transducer should be placed on either side of the diametral plane of the straight guide which lies in the plane of the bend. An exactly similar transducer on the other side of the bend will reconvert $TM_{11}'$ into $TE_{01}$. Since $TE_{01}$ and $TM_{11}$ have the same velocity in a straight guide, the transducer can be made of a number of sections with arbitrary spacing and of total length $l$; but in practice one will not wish to have too long a run of $TM_{11}$ in the empty guide because of the higher heat losses of this mode.

### A.2 *The* $TE_{01} \pm TM_{11}''$ *Normal Mode Solution*

If we write the coupled wave equations for $TE_{01}$ and $TM_{11}''$ in a plain bend in the form

$$\begin{aligned} \frac{da_0}{dz} + iha_0 + ica_1 &= 0, \\ ica_0 + \frac{da_1}{dz} + iha_1 &= 0, \end{aligned} \tag{116}$$

where

$$\begin{aligned} h &= h_{[01]} = h_{(11)}\,, \\ c &= c_{(11)}\,, \end{aligned} \tag{117}$$

it is evident that we can add and subtract to get the equivalent pair of equations:

$$\begin{aligned} \frac{d(a_0 + a_1)}{dz} + i(h + c)(a_0 + a_1) &= 0, \\ \frac{d(a_0 - a_1)}{dz} + i(h - c)(a_0 - a_1) &= 0. \end{aligned} \tag{118}$$

Hence the normal modes of the curved guide are the combinations $a_0 \pm a_1$, or $\mathrm{TE}_{01} \pm \mathrm{TM}''$.

In order to launch only the normal mode $\mathrm{TE}_{01} + \mathrm{TM}_{11}''$ at the input of the bend, the amplitude of the other mode must be zero, or $a_0 = a_1$. Similarly, to launch only the mode $\mathrm{TE}_{01} - \mathrm{TM}_{11}''$, the condition is $a_0 = -a_1$. Hence the output of the normal mode transducer of length $l$, say, must be

$$a_0(l) = \pm a_1(l). \tag{119}$$

We return to the solution of the coupled line equations in Section 2.1 and write

$$\begin{aligned} \kappa &= d_{(11)} = d, \\ \gamma_0 &= i\beta_{[01]} = i\beta_0\,, \\ \gamma_1 &= i\beta_{(11)} = i\beta_1\,, \\ r &= is = i\sqrt{(\beta_0 - \beta_1)^2 + 4d^2}. \end{aligned} \tag{120}$$

Then equations (60) of Section 2.1 become:

$$\begin{aligned} a_0(l) &= \left[\cos \tfrac{1}{2}sl - i\,\frac{(\beta_0 - \beta_1)}{s} \sin \tfrac{1}{2}sl\right] e^{-\frac{1}{2}i(\beta_0+\beta_1)l}, \\ a_1(l) &= -i\frac{2d}{s} \sin \tfrac{1}{2}sl\; e^{-\frac{1}{2}i(\beta_0+\beta_1)l}. \end{aligned} \tag{121}$$

Substituting into (119) and equating real and imaginary parts gives:

$$l = \pi/s, \tag{122}$$

$$|\,\beta_0 - \beta_1\,| = |\,2d\,|\,. \tag{123}$$

In view of (103) and (104) of Section 2.3.2, (123) is equivalent to

$$\cos \theta/2 = \frac{\pm\sqrt{1 - \nu^2}}{0.4640\nu^2 + 0.1955}, \tag{124}$$

and (122) becomes

$$l = \frac{1.465\lambda}{\delta \mid \sin \theta/2 \mid}, \tag{125}$$

where

$$\nu = \lambda/\lambda_c = 3.8317\ \lambda/2\pi a \tag{126}$$

is the cutoff ratio for $TE_{01}$ and $TM_{11}$ waves in a straight, empty guide.

A $TE_{01}$ to $TE_{01} \pm TM_{11}''$ mode transducer may thus consist of a dielectric sector, attached to the surface of the straight guide on the side nearest the center of curvature of the bend, if the angle of the sector satisfies (124) and the length satisfies (125). However, the condition (124) can be satisfied by a real angle only if

$$0.8483 \leqq \lambda/\lambda_c \leqq 1; \tag{127}$$

that is, only if the guide is very near cutoff; and the value of $\theta$ which satisfies (124) varies rapidly with $\lambda$ over the above range. This form of normal mode transducer is therefore too narrow band to be of much practical interest.

### A.3 *A Broadband Compensator*

We shall now show how to design a dielectric compensator, in a straight section of guide, which takes the mixture of $TE_{01}$ and $TM_{11}''$ put out by an adjacent bend and reconverts it to pure $TE_{01}$, independent of frequency.

First let us write the solution of (116) for a plain bend in terms of arbitrary initial values $a_0(0)$ and $a_1(0)$. We have

$$\begin{aligned} a_0(z) + a_1(z) &= [a_0(0) + a_1(0)]e^{-ihz-icz}, \\ a_0(z) - a_1(z) &= [a_0(0) - a_1(0)]e^{-ihz+icz}, \end{aligned} \tag{128}$$

and hence

$$\begin{aligned} a_0(z) &= [a_0(0) \cos cz - ia_1(0) \sin cz]e^{-ihz}, \\ a_1(z) &= [-ia_0(0) \sin cz + a_1(0) \cos cz]e^{-ihz}. \end{aligned} \tag{129}$$

The bend may be compensated with a dielectric-loaded straight guide, provided that* $\beta_0 = \beta_1$ in the straight guide (this may be arranged, for example, by using a half-cylinder of dielectric), and provided that the length of the compensator and the coupling coefficient $d$ are properly chosen. The amplitudes of the two modes in the straight guide, in terms of arbitrary initial values $a_0(0)$ and $a_1(0)$, are

$$\begin{aligned} a_0(z) &= [a_0(0) \cos dz - ia_1(0) \sin dz]e^{-i\beta_0 z}, \\ a_1(z) &= [-ia_0(0) \sin dz + a_1(0) \cos dz]e^{-i\beta_0 z}. \end{aligned} \tag{130}$$

Now suppose that the length of the bend is $l_1$ and the length of the compensator $l_2$, and take the origin of $z$ at the input to the bend. A pure $TE_{01}$ input is represented by

$$\begin{aligned} a_0(0) &= 1, \\ a_1(0) &= 0, \end{aligned} \tag{131}$$

and so, applying (129) and (130) in succession,

$$\begin{aligned} a_0(l_1) &= \cos cl_1 \, e^{-ihl_1}, \\ a_1(l_1) &= -i \sin cl_1 \, e^{-ihl_1}, \end{aligned} \tag{132}$$

$$\begin{aligned} a_0(l_1 + l_2) &= \cos (cl_1 + dl_2)e^{-i(hl_1+\beta_0 l_2)}, \\ a_1(l_1 + l_2) &= -i \sin (cl_1 + dl_2)e^{-i(hl_1+\beta_0 l_2)}. \end{aligned} \tag{133}$$

The condition that all the power be in $TE_{01}$ at the output of the compensator at every frequency is

$$cl_1 + dl_2 = 0, \tag{134}$$

or

$$dl_2 = -0.18454 \, \beta a l_1/b, \tag{135}$$

on referring to equation (86) of Section 2.2 for the value of $c$.

A convenient form of compensator would be a half cylinder of dielectric whose diametral plane is perpendicular to the plane of the bend. The coupling coefficient of the half cylinder is given by (104), and the condition for complete compensation becomes

$$l_2 \delta = 1.5295 \, l_1 a/b. \tag{136}$$

The most easily adjustable parameter is probably the length $l_2$ of the compensator.

* The necessity for $\beta_0 = \beta_1$ is apparent if we consider that under certain conditions the bend may put out a pure $TM_{11}''$ mode, and complete reconversion is possible only if the compensator has $\beta_0 = \beta_1$.

We have analyzed the compensator as if it were all on one side of the bend; but it may evidently be divided into sections of total length $l_2$ which are distributed arbitrarily on both sides of the bend. An obvious configuration would be to put a section of length $l_2/2$ on each side of the bend immediately adjacent to it.

Limitations on the usefulness of this kind of compensator will be dielectric losses and mode conversions. The former can presumably be reduced as the dissipation of available dielectrics is reduced. Mode conversions could be calculated by the general methods of Part I, but one would have to work out the values of the coupling coefficients between $TM_{11}$ and all other modes, which has not yet been done. In any case it is likely that the minimum tolerable bending radius would be no less than for the within-the-bend compensators discussed earlier in this paper.

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