# Normal Mode Bends for Circular Electric Waves

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*In dielectric-coated round waveguide the degeneracy or equality of phase constants of* $TE_{01}$ *and* $TM_{11}$ *waves is removed. In such a non-degenerate waveguide, mode conversion in bends can be reduced by changing the curvature gradually instead of abruptly. With curvature tapers, which are of the order of, or longer than, the largest beat wavelength between* $TE_{01}$ *and any of the coupled waves, propagation of only one normal mode is maintained throughout the bend. Linear curvature tapers can easily be made by bending the pipe within the limit of elastic deformation.*

*Changes in the direction of a waveguide line can thus be made by inserting a dielectric-coated guide section which is elastically bent over a fixed center point. A thirty degree change of direction of a 2-inch I.D. pipe with 30 ft of a dielectric-coated guide yields a total bend loss at 5.4-mm wavelength that is twice the* $TE_{01}$ *loss in 30 ft of straight pipe. An optimum bend geometry is found which minimizes the total bend loss. The normal mode bend is a broadband device.*

## I. INTRODUCTION AND SUMMARY

A major problem in circular electric wave transmission is the question of negotiating bends. In curved sections of a round waveguide the $TE_{01}$ mode couples to the $TE_{11}$ , $TE_{12}$ , $TE_{13}$ $\cdots$ modes and to the $TM_{11}$ mode. The coupling to the $TM_{11}$ mode presents the most serious problem, since the $TE_{01}$ and $TM_{11}$ modes are degenerate in that they have equal phase velocities in a perfectly conducting straight guide. $TE_{01}$ power introduced at the beginning of the bend will be almost completely transferred to the $TM_{11}$ mode at odd multiples of a certain critical bending angle. This power transfer can be reduced by removing the degeneracy of equal phase velocity of $TE_{01}$ and $TM_{11}$ modes. There are various methods to remove the degeneracy; a simple and a very effective one is a thin dielectric layer next to the walls of the waveguide. As a study of the dielectric-coated waveguide has shown,[1] this dielectric layer changes the phase constant of the $TM_{11}$ wave without appreciably affecting the propagation characteristics of the $TE_{01}$ mode.

With removal of the $TE_{01}$-$TM_{11}$ degeneracy the mode conversion which occurs when a $TE_{01}$ wave passes through a bend of constant curvature may be considerably reduced, but it will not be completely eliminated. To design a bend with still lower mode conversion losses, we consider the effect of tapering the curvature along the guide.

Guided wave propagation is most easily explained in terms of normal modes. Normal modes are solutions of the wave equation in a particular waveguide structure, and represent waves propagating without loss of power except for dissipation. In the straight waveguide the normal mode, in which we are mainly interested here is the $TE_{01}$ mode. The normal modes of the curved section are not as simple as the straight guide modes, but they can be expressed as the sum of the normal modes in the straight waveguide. Here the mode of our main concern is the one which, when represented as a sum of straight guide modes, has most of the power in the $TE_{01}$ part of the sum; in other words, is most similar to the $TE_{01}$ mode of the straight waveguide.

At a transition from a straight waveguide to a curved waveguide the normal ($TE_{01}$) mode of the straight waveguide will certainly excite this normal mode in the curved waveguide but it will also excite a series of other normal modes. All these modes propagate in the curved section, and at the other transition to the straight guide excite not only the $TE_{01}$ mode but a series of other normal modes of the straight waveguide. All the power in the other normal modes represents mode conversion loss of the bend.

A transition which transforms the normal ($TE_{01}$) mode of the straight waveguide into only one of the normal modes of the curved guide and vice versa will avoid all mode conversion losses. Such a transition can be realized approximately by tapering the curvature. Beginning with zero curvature at the straight guide end of the taper, the curvature is increased gradually along the taper to the finite value of the bend. The normal ($TE_{01}$) mode incident from the straight guide will be gradually transformed into the particular normal mode of the bent guide which is most similar in field configuration to the circular electric wave. At each point along the taper there is essentially only one local normal mode corresponding in its configuration to the value of curvature at that point.

This taper, unless it is infinitely long, is however only an approximation of the ideal normal mode transition. There will still be other modes excited with a very low power level. In the next section we shall analyze the propagation in the normal mode taper. We will find that the taper should have a certain minimum length to work properly. Usually it has to be long compared to the beat wavelength between the $TE_{01}$ normal mode and any of the other normal modes into which power may be converted.

In the plain waveguide the degeneracy between $TE_{01}$ and $TM_{11}$ causes an infinitely long beat wavelength. Hence, the normal mode taper would not work there. A nondegenerate waveguide is an essential condition for the normal mode bend.

We shall confine our attention to the linear taper. This is not the optimum taper form, but it is most easily built.

The residual mode conversion in the bend is to be accounted for as bend loss. This loss and the loss caused by the normal mode attenuation in the bend add up to a total bend loss. We shall evaluate the total bend loss for bend configurations which might be useful in circular electric wave transmission. For specified waveguide dimensions the total bend loss can be minimized by choosing the proper bend geometry.

The normal mode bend is an inherently broadband device. The total bend loss shows the same order of frequency dependence as the loss in the straight waveguide.

## II. ANALYSIS OF THE NORMAL MODE TAPER

In the curved waveguide, wave propagation can be described in terms of the normal modes of the straight waveguide. The relation between these modes is then given by an infinite system of simultaneous first order linear differential equations. It represents the mutual coupling of the straight guide modes in the curved waveguide. We are interested mainly in $TE_{01}$ propagation and shall restrict ourselves to a low power level in all other modes. Consequently, an adequate procedure is to consider only coupling between $TE_{01}$ and one of the coupled modes at a time. Thus, the infinite system of equations reduces to the well known coupled line equations:[2]

$$\begin{aligned} \frac{dE_1}{dz} &= -\gamma_1 E_1 + jcE_2, \\ \frac{dE_2}{dz} &= jcE_1 - \gamma_2 E_2, \end{aligned} \tag{1}$$

in which

$E_{1,2}(z)$ = wave amplitudes in mode 1 (here always $TE_{01}$) and mode 2 ($TM_{11}$ or one of the $TE_{1m}$), respectively;

$\gamma_{1,2} = j\beta_{1,2} + \alpha_{1,2}$ = propagation constants of modes 1 and 2, respectively, (the small perturbations of $\gamma_1$ and $\gamma_2$ caused by the coupling may be neglected here); and

$c(z)$ = coupling coefficient between modes 1 and 2.

In the curved waveguide the coupling coefficient is proportional to the curvature $k$:

$$c(z) = c'k(z) \equiv c' \frac{d\theta}{dz}, \tag{2}$$

in which $\theta$ is the direction of the guide axis. The coupled line equations (1) with varying coupling coefficient have been solved by W. H. Louisell and we shall borrow freely from his treatment.[3]

We define local normal modes $w_1(z)$ and $w_2(z)$:

$$\begin{aligned} E_1 &= [w_1 \cos \tfrac{1}{2}\,\xi - w_2 \sin \tfrac{1}{2}\,\xi]\, e^{-\gamma z}, \\ E_2 &= [w_1 \sin \tfrac{1}{2}\,\xi + w_2 \cos \tfrac{1}{2}\,\xi]\, e^{-\gamma z}, \end{aligned} \tag{3}$$

in which

$$\gamma = \frac{\gamma_1 + \gamma_2}{2},$$

$$\tan \xi = j2 \frac{c}{\gamma_2 - \gamma_1} \equiv j2 \frac{c}{\Delta\gamma}.$$

Substituting (3) into (1), we find that $w_1(z)$ and $w_2(z)$ must satisfy

$$\begin{aligned} \frac{dw_1}{dz} - \Gamma(z)w_1 &= \frac{1}{2}\frac{d\xi}{dz}\, w_2, \\ \frac{dw_2}{dz} + \Gamma(z)w_2 &= -\frac{1}{2}\frac{d\xi}{dz}\, w_1, \end{aligned} \tag{4}$$

where $\Gamma(z) = \frac{1}{2}\sqrt{\Delta\gamma^2 - 4c^2}$. In (4) the local normal modes are coupled only through the terms poportional to $d\xi/dz$. When $\xi$ is constant they are uncoupled and true normal modes. For small values of $d\xi/dz$ or more specifically when

$$\left| \frac{1}{2\Gamma}\frac{d\xi}{dz} \right| \ll 1 \tag{5}$$

approximate solutions of (4) can be written down, which proceed essentially in powers of $d\xi/dz$. These solutions are:

$$
\begin{aligned}
w_1(z) &= e^{\rho(z)}\left[w_1(0) + \frac{1}{2}\,w_2(0)\int_0^z \frac{d\xi}{dz'}\,e^{-2\rho(z')}\,dz'\right.\\
&\qquad \left. - \frac{1}{4}\,w_1(0)\int_0^z \frac{d\xi}{dz'}\,e^{-2\rho(z')}\int_0^{z'} \frac{d\xi}{dz''}\,e^{2\rho(z'')}\,dz''\,dz'\right],\\
w_2(z) &= e^{-\rho(z)}\left[w_2(0) - \frac{1}{2}\,w_1(0)\int_0^z \frac{d\xi}{dz'}\,e^{2\rho(z')}\,dz'\right.\\
&\qquad \left. - \frac{1}{4}\,w_2(0)\int_0^z \frac{d\xi}{dz'}\,e^{2\rho(z')}\int_0^{z'} \frac{d\xi}{dz''}\,e^{-2\rho(z'')}\,dz''\,dz'\right],
\end{aligned}
\tag{6}
$$

in which

$$
\rho(z) = \int_0^z \Gamma(z')\,dz'.
$$

The initial conditions in the normal mode taper are $E_1(0) = 1$ and $E_2(0) = 0$. The taper begins with zero curvature, $\xi(0) = 0$. Hence from (3) $w_1(0) = 1$ and $w_2(0) = 0$. The wanted local normal mode is $w_1$ while $w_2$ is an unwanted mode. At the end, $z_1$, of the taper the unwanted mode

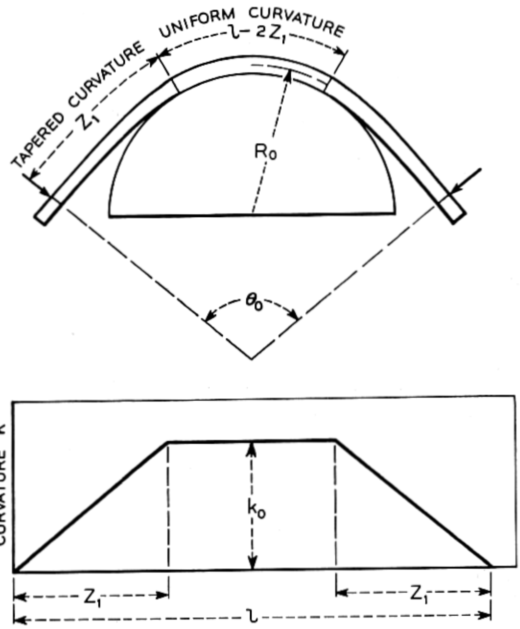


Fig. 1 — Normal mode bend with linear curvature taper.

amplitude is

$$| w_2(z_1) | = \frac{1}{2} \left| \int_0^{z_1} \frac{d\xi}{dz} e^{2\rho(z)} dz \right|. \tag{7}$$

This amplitude represents mode conversion loss and therefore has to be kept as small as possible.

In (7) the function $\xi(z)$, i.e., the taper function, is still undetermined. Obviously it can be chosen so as to optimize the taper performance. A taper of optimal design keeps the unwanted mode below a certain value with as short a taper length as possible. From (7) the relation between this optimizing problem and the problem of the transmission line taper of optimal design[4] is at once evident. The transmission line taper is a low reflection transition between lines of different characteristic impedances. To minimize the length of the transition for a specified maximal reflection, the characteristic impedance has to change along the transition according to a function which is essentially the Fourier transform of a Tschebyscheff polynomial of infinite degree. The same procedure can be applied here and it will result in a curvature taper of optimal design.

We are, however, at present not as much interested in a transition of optimal design as in a curvature taper which can easily be built. Suppose we bend the pipe to a bending radius $R_o$ which causes only elastic deformation. We do this on a form of radius $R_o$, Fig. 1. The forces acting on both ends of the pipe cause a torque and hence a curvature of the pipe which increases approximately linearly from the pipe end ($z = 0$) to the point of contact ($z = z_1$) between pipe and form:

$$k = k_0 \frac{z}{z_1}. \tag{8}$$

The corresponding curve which the pipe forms along the taper is Cornu's spiral.

We shall evaluate (7) for a curvature as given by (8). In considering the mode conversion we may neglect all heat losses, that is $\gamma = j\beta$, etc. With

$$c = c_0 \frac{z}{z_1}, \tag{9}$$

we get

$$\frac{d\xi}{dz} = \frac{2c_0}{\Delta\beta z_1} \frac{1}{1 + 4 \dfrac{c_0^2 z^2}{\Delta\beta^2 z_1^2}}, \tag{10}$$

and

$$\rho(z) = j \frac{\Delta\beta^2 z_1}{4c_0} \left[ \frac{c_0 z}{\Delta\beta z_1} \sqrt{1 + 4\left(\frac{c_0 z}{\Delta\beta z_1}\right)^2} + \frac{1}{2} \sinh^{-1} \frac{2c_0 z}{\Delta\beta z_1} \right]. \tag{11}$$

We introduce (10) and (11) into (7) and take advantage of

$$2 \left| \frac{c_0}{\Delta\beta} \right| \ll 1, \tag{12}$$

which is satisfied for a gentle enough bend. The unwanted mode level is then given by

$$| w_2(z_1) | = \left| \frac{c_0}{\Delta\beta^2 z_1} (1 - e^{j\Delta\beta z_1}) \right| + 0\left(8 \frac{c_0^3}{\Delta\beta^3}\right). \tag{13}$$

The general restriction (5) for the solution (6) in case of the linear taper is

$$\frac{2c_0}{\Delta\beta^2 z_1} \left(1 + 4 \frac{c_0^2 z^2}{\Delta\beta^2 z_1^2}\right)^{-3/2} \ll 1,$$

and in view of (12) only

$$| \Delta\beta z_1 | \geqq 1 \tag{14}$$

is required. The length of the transition has to be of the order or greater than the largest beat wavelength.

In addition to mode conversion loss the normal mode suffers heat loss along the taper. From (3) and (6) this heat loss is given by the real part of $[\gamma z - \rho(z)]$. If $\Delta\beta \gg \Delta\alpha$ we get

$$R[\gamma z - \rho] = \alpha_1 z + \Delta\alpha \int_0^z \frac{c^2}{\Delta\beta^2} dz'. \tag{15}$$

It follows from (15) that the attenuation of the coupled straight guide modes should not be too high for the normal mode bend to work properly.

## III. THE TOTAL BEND LOSS

We will consider bends of the dielectric-coated waveguide only. The normal modes of the dielectric-coated waveguide have phase constants which are slightly different from the phase constants of the modes in the plain guide[1]

$$\begin{aligned} \mathrm{TM}_{nm} \quad & \frac{\Delta\beta}{\beta_{nm}} = \frac{\epsilon' - 1}{\epsilon'} \delta, \\ \underset{\text{with } n\neq 0}{\mathrm{TE}_{nm}} \quad & \frac{\Delta\beta}{\beta_{nm}} = \frac{n^2}{p_{nm}^2 - n^2} \frac{\epsilon' - 1}{\epsilon'(1 - \nu^2)} \delta, \\ \mathrm{TE}_{0m} \quad & \frac{\Delta\beta}{\beta_{0m}} = \frac{p_{0m}^2}{3} \frac{\epsilon' - 1}{1 - \nu_{0m}^2} \delta^3. \end{aligned} \tag{16}$$

The losses in the dielectric coat increase the attenuation constants by:

$$\text{TM}_{nm} \quad \frac{\alpha_D}{\beta_{nm}} = \frac{\epsilon''}{\epsilon'^2}\,\delta,$$

$$\underset{\text{with } n\neq 0}{\text{TE}_{nm}} \quad \frac{\alpha_D}{\beta_{nm}} = \frac{n^2}{p_{nm}^2 - n^2}\,\frac{\epsilon''}{\epsilon'^2(1-\nu_{nm}^2)}\,\delta, \tag{17}$$

$$\text{TE}_{0m} \quad \frac{\alpha_D}{\beta_{0m}} = \frac{p_{0m}^2}{3}\,\frac{\epsilon''}{1-\nu_{0m}^2}\,\delta^3.$$

For the circular electric waves the dielectric coat increases the wall current attenuation by

$$TE_{0m}\,\frac{\Delta\alpha}{\alpha_{0m}} = (\epsilon' - 1)\,\frac{p_{0m}^2}{\nu_{0m}^2}\,\delta^2. \tag{18}$$

The various symbols in (16), (17) and (18) are:

$\beta_{nm}$ = plain guide phase constants of $\text{TM}_{nm}$ and $\text{TE}_{nm}$ respectively;
$p_{nm}$ = $m$th root of $J_n(x) = 0$ for $\text{TM}_{nm}$, and $m$th root of $J_n'(x) = 0$ for $\text{TE}_{nm}$ waves respectively;
$\nu_{nm} = \dfrac{\lambda}{\lambda_c} = \dfrac{p_{nm}\lambda}{2\pi a}$ cutoff factor in plain waveguide of radius $a$;
$\delta = \dfrac{d}{a}$ relative thickness of dielectric coat;
$\epsilon = \epsilon' - j\epsilon''$ relative dielectric permittivity of dielectric coat;
$\alpha_{om}$ = attenuation constant of $\text{TE}_{om}$ in the plain waveguide.

The coupling coefficient between the straight guide modes in a curved waveguide is $c = c'/R$ in which:[5]

$$\text{TE}_{01} \rightleftarrows \text{TM}_{11} \quad c' = 0.18454\,\beta a,$$

$$\text{TE}_{01} \rightleftarrows \text{TE}_{11} \quad c' = \frac{0.09319(\beta a)^2 - 0.84204}{\sqrt{\beta_{01}a\beta_{11}a}} + 0.09319\sqrt{\beta_{01}a\beta_{11}a},$$

$$\text{TE}_{01} \rightleftarrows \text{TE}_{12} \quad c' = \frac{0.15575(\beta a)^2 - 3.35688}{\sqrt{\beta_{01}a\beta_{12}a}} + 0.15575\sqrt{\beta_{01}a\beta_{12}a}, \tag{19}$$

$$\text{TE}_{01} \rightleftarrows \text{TE}_{13} \quad c' = \frac{0.01376(\beta a)^2 - 0.60216}{\sqrt{\beta_{01}a\beta_{13}a}} + 0.01376\sqrt{\beta_{01}a\beta_{13}a},$$

where $\beta$ = free-space phase constant.

We consider the bend configuration of Fig. 1, with the curvature being a trapezoidal function of length. The maximum power loss due to conversion to one of the unwanted modes in the first transition is, by (13),

$$|\,w_2(z_1)\,|^2_{\max} = \frac{4}{(\Delta\beta z_1)^2}\,\frac{c_0^2}{\Delta\beta^2}.$$

By the law of reciprocity the same conversion loss occurs in the second transition. Both conversion parts phase with each other. To get an average total conversion we may add them in quadrature and the total conversion loss to one of the unwanted modes is, expressed in nepers,

$$A_c = \frac{4}{(\Delta\beta z_1)^2} \frac{c_0^2}{\Delta\beta^2}. \tag{20}$$

Under most unfavorable phase conditions the conversion loss may be twice this value, but it is very unlikely that such phase conditions will be satisfied for all coupled modes simultaneously.

Besides mode conversion loss the local normal mode suffers attenuation in the bend. This attenuation is larger than the straight guide attenuation. Each straight guide mode contained in the local normal mode of the bend causes an increase in attenuation. From (15) the loss caused by one of these straight guide modes is:

$$A_b = (\alpha_{1m} - \alpha_{01}) \int_0^l \frac{c^2}{\Delta\beta^2} dz. \tag{21}$$

Where $\alpha_{1m}$ is the attenuation constant of the $TE_{1m}$ and $TM_{11}$ waves respectively in the dielectric coated waveguide.

Introducing the trapezoidal curvature function of Fig. 1 into (21) we get

$$A_b = \frac{c_0^2}{\Delta\beta^2} (\alpha_{1m} - \alpha_{01}) \left(l - \frac{4}{3} z_1\right). \tag{22}$$

The loss caused by the $TE_{01}$ attenuation in the straight dielectric coated guide is from (17) and (18)

$$A_s = \alpha_{01} l \left[1 + (\epsilon - 1) \frac{p_{0m}^2}{\nu_{0m}^2} \delta^2\right] + \frac{p_{01}^2}{3} \frac{\epsilon'' \beta_{01}}{1 - \nu_{01}^2} \delta^3 l. \tag{23}$$

The total bend loss is finally obtained by summing up all the terms of (20), (22), and (23),

$$A = A_s + \sum A_b + \sum A_c . \tag{24}$$

The summation signs indicate that all coupled modes ($TM_{11}$ and $TE_{1m}$) have to be taken into account.

The effectiveness of the normal mode bend is best demonstrated by a practical example. A copper pipe now in experimental use at Bell Telephone Laboratories for circular electric wave transmission near 5.4 mm wavelength has 2-inch I. D. and $2\frac{3}{8}$-inch O.D. Suppose we want to change the direction of a waveguide line with this copper pipe bn an angle $\theta_0$. We can do this most easily by inserting a dielectric-coated section, which is bent around a fixed support in the center by forces acting on both ends. In order not to exceed elastic deformation the bending radius must not be smaller than

$$R_{\min} = \frac{E}{f_{\max}} a_1 , \tag{25}$$

where $f_{\max}$ = flexural stress at elastic limit,
$E$ = modulus of elasticity,
$a_1$ = outside radius of pipe.

This minimum bending radius requires a minimum length to change the pipe direction by a specified angle $\theta_0$ given by

$$l_{\min} = 2\theta_0 R_{\min} . \tag{26}$$

The total bend loss (24) has been evaluated for a bend configuration as specified by (25) and (26). The result is shown in Fig. 2. The total additional bend loss is only of the order of the $TE_{01}$ loss in the plain straight waveguide. For small bending angles the curvature taper becomes shorter and consequently the mode conversion loss increases. The mode conversion loss, however, does not go to infinity for zero bending angle. In this case (14) is no longer satisfied, and the mode conversion loss is no longer described by (20).

The level of the various unwanted modes which can be calculated from (20) is plotted in Fig. 2.

For a practical waveguide one would decide on a standard length of dielectric-coated pipe, one or several of which would be inserted whenever a change in direction has to be made. Take, in our example, a standard length of 15 feet. With one such section a change of direction up to 15° could be made. For a change in direction up to 30° two such sections would have to be inserted and bent around a fixed support at the center joint. The total loss of Fig. 2 is then a maximum value, which would only occur when the pipe is bent to the highest allowable bending angle.

## IV. A NORMAL MODE BEND OF OPTIMUM DESIGN

The various terms of the total bend loss (24) depend on the bend geometry in quite different ways. It is therefore likely that for a given bending angle $\theta_0$ a bend geometry can be found, which minimizes the total bend loss. The total bend loss can generally be written as:

$$A = Sl + B \frac{1 - \frac{4}{3} u}{l(1 - u)^2} + C \frac{1}{l^4(u - u^2)^2} , \tag{27}$$

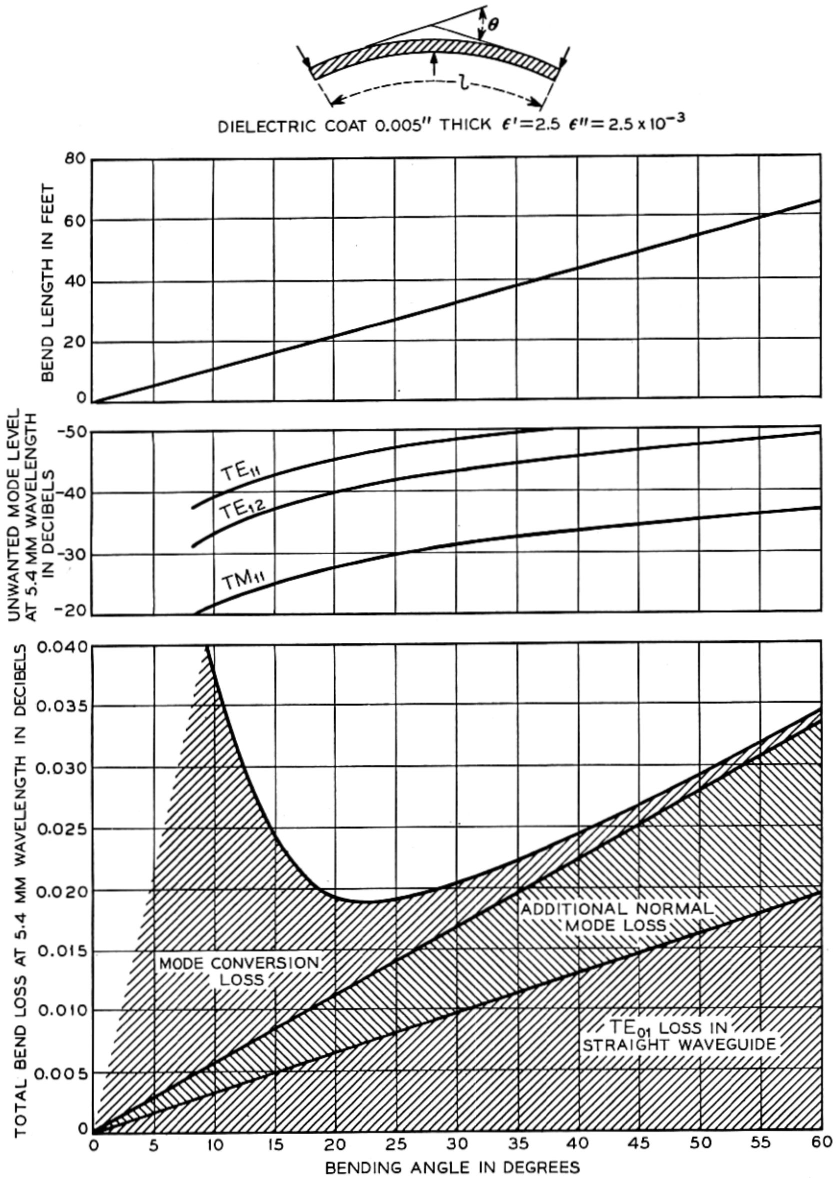


Fig. 2 — Normal mode bend. Dielectric-coated copper pipe with 2-inch I.D. and $2\frac{3}{8}$-inch O.D. deflected to limit of elastic deformation ($\lambda = 5.4$ mm).

in which

$$
\begin{aligned}
S &= \alpha_{01}\left[1 + (\epsilon' - 1)\frac{{p_{01}}^2}{{\nu_{01}}^2}\delta^2\right] + \frac{{p_{01}}^2}{3}\frac{\epsilon''\beta_{01}}{1 - {\nu_{01}}^2}\delta^3, \\
B &= \Sigma \frac{c'^2}{\Delta\beta^2}{\theta_0}^2(\alpha_{1m} - \alpha_{01}), \\
C &= \Sigma \frac{c'^2}{\Delta\beta^4}{\theta_0}^2, \\
u &= \frac{z_1}{l}.
\end{aligned}
\tag{28}
$$

Here again the summation signs indicate that all coupled modes have to be taken into account. The factors $S$, $B$, and $C$ do not depend on the bend geometry, but only on the total bending angle, the waveguide properties, and the frequency. Necessary conditions for $A(u, l)$ to be a minimum are:

$$\frac{\partial A}{\partial u} \equiv \frac{2(1 - 2u)}{l(1 - u^3)}\left[\frac{B}{3} - \frac{C}{l^3u^3}\right] = 0, \tag{29}$$

with the two roots

$$u = \frac{1}{2}, \tag{30}$$

$$ul = \left(3\,\frac{C}{B}\right)^{\frac{1}{3}}, \tag{31}$$

and

$$\frac{\partial A}{\partial l} \equiv S - B\,\frac{1 - \frac{4}{3}u}{(1 - u)^2}\frac{1}{l^2} - \frac{4C}{(u - u^2)^2}\frac{1}{l^5} = 0. \tag{32}$$

If $u = \frac{1}{2}$, the solutions of (32) are the roots of

$$S\,l^5 - \tfrac{4}{3}\,B\,l^3 - 64C = 0. \tag{33}$$

If $(lu)^3 = 3(C/B)$, the solutions of (32) are the roots of

$$l = \left(3\,\frac{C}{B}\right)^{\frac{1}{3}} \pm \left(\frac{B}{S}\right)^{\frac{1}{2}}. \tag{34}$$

Sufficient conditions for $A(u, l)$ to be a minimum are:

$$\frac{\partial^2 A}{\partial u^2}\frac{\partial^2 A}{\partial l^2} - \left(\frac{\partial^2 A}{\partial u \partial l}\right)^2 > 0, \tag{35}$$

$$\frac{\partial^2 A}{\partial u^2} > 0, \qquad \frac{\partial^2 A}{\partial l^2} > 0, \tag{36}$$

If $u = \frac{1}{2}$, we have:

$$\frac{\partial^2 A}{\partial u \partial l} \equiv 0,$$

$$\frac{\partial^2 A}{\partial l^2} = \frac{8}{l^3}\left(\frac{B}{3} + 40\frac{C}{l^3}\right), \tag{37}$$

$$\frac{\partial^2 A}{\partial u^2} = \frac{32}{l}\left(8\frac{C}{l^3} - \frac{B}{3}\right).$$

Substituting

$$x = \frac{1}{2}\left(\frac{S}{B}\right)^{\frac{1}{2}} l,$$

$$r = \left(3\frac{C}{B}\right)^{\frac{1}{3}}\left(\frac{S}{B}\right)^{\frac{1}{2}}, \tag{38}$$

we get instead of (33)

$$x^3(3x^2 - 1) - 2r^3 = 0, \tag{39}$$

and instead of (37)

$$\frac{\partial^2 A}{\partial u^2} = 16\frac{B}{l}(x^2 - 1).$$

The positive root of (39) is plotted in Fig. 3. It follows that if $r > 1$ we have $x > 1$, consequently $\partial^2 A/\partial u^2 > 0$; and if $r < 1$ we have $x < 1$, consequently $\partial^2 A/\partial u^2 < 0$. Consequently if, and only if, $r > 1$ the values $u = \frac{1}{2}$ and $x$ from Fig. 3 minimize the total bend loss A.

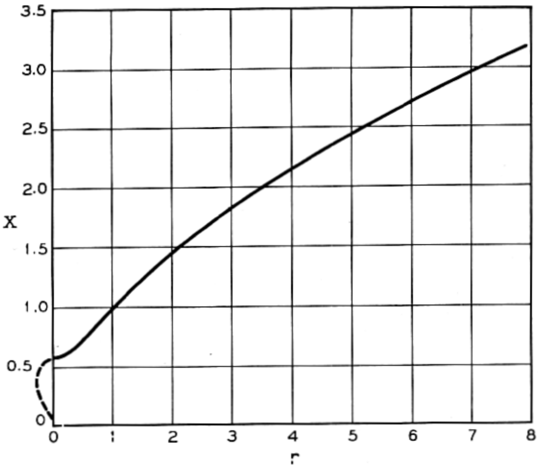


Fig. 3 — Positive root of $x^3(3x^2 - 1) - 2r^3 = 0$.

If $(ul)^3$ is equal to $3(C/B)$,

$$\frac{\partial^2 A}{\partial u^2}\frac{\partial^2 A}{\partial l^2} - \left(\frac{\partial^2 A}{\partial u \partial l}\right)^2 \equiv \frac{4B^2}{l^4 u}\frac{1 - 2u}{(1 - u)^6},$$

and

$$\frac{\partial^2 A}{\partial u^2} \equiv \frac{2B}{lu}\frac{1 - 2u}{(1 - u)^3}.$$

Hence, if $u < \frac{1}{2}$ or, because of (31) and (34) $r < 1$, a minimum of $A(u, l)$ is located at

$$(ul)^3 = 3\frac{C}{B}, \qquad l = \left(3\frac{C}{B}\right)^{\frac{1}{3}} + \left(\frac{B}{S}\right)^{\frac{1}{2}}.$$

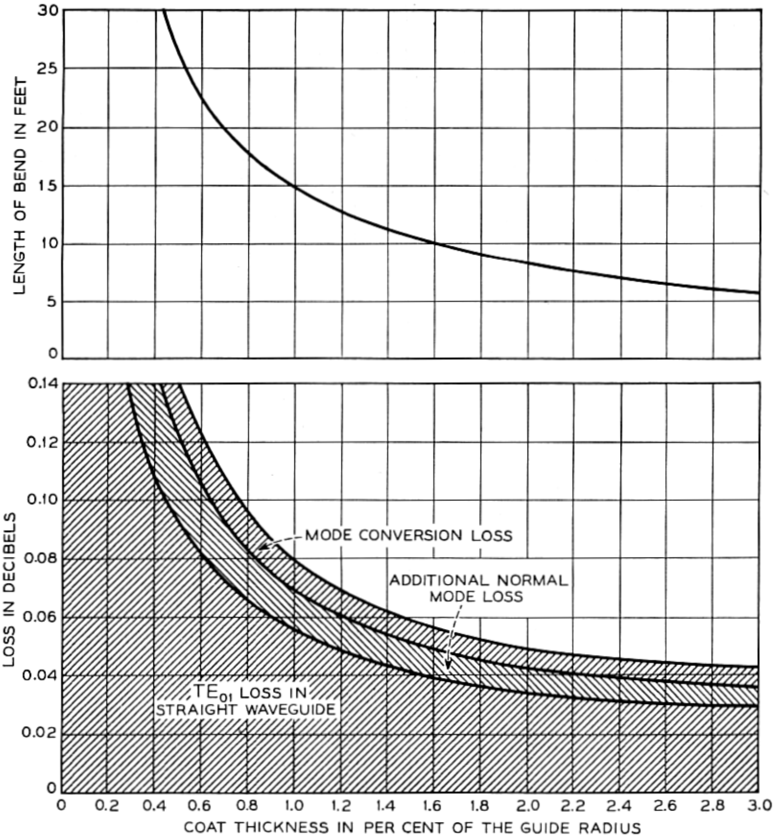


Fig. 4 — 90° normal mode bend of $\frac{7}{8}$-inch I.D. copper pipe with a dielectric coat of $\epsilon' = 2.5$, $\epsilon'' = 2.5 \times 10^{-3}$. Optimum design for 5.4-mm wavelength.

To find the optimum bend geometry for a given dielectric coated guide and a specified bending angle we calculate $r$ from (38). If $r > 1$ the optimum geometry is

$$l = 2\sqrt{\frac{B}{S}}\,x \quad \text{and} \quad z_1 = \sqrt{\frac{B}{S}}\,x$$

with $x$ from Fig. 3. If $r < 1$,

$$l = \left(3\,\frac{C}{B}\right)^{\frac{1}{3}} + \left(\frac{B}{S}\right)^{\frac{1}{3}}, \quad \text{and} \quad z_1 = \left(3\,\frac{C}{B}\right)^{\frac{1}{3}}.$$

A numerical example, the 90° bend of a $\frac{7}{8}$-inch I. D. copper pipe, is shown in Fig. 4. The total bend loss in the optimally designed bend decreases steadily with increasing thickness of the dielectric coat. This indicates that there is also an optimum coat thickness, which minimizes the total loss of the normal mode bend of optimum total length and taper length. Unfortunately, however, several approximations made in calculating phase constants and coupling coefficients in the dielectric-coated

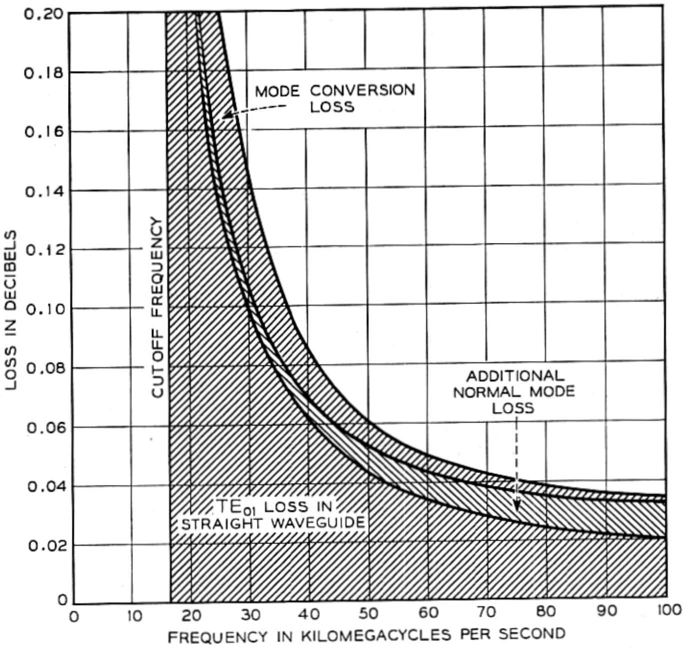


Fig. 5 — 90° normal mode bend of $\frac{7}{8}$-inch I.D. copper pipe with a dielectric coat 0.0075 inch thick ($\epsilon' = 2.5$, $\epsilon'' = 2.5 \times 10^{-3}$), designed for optimum performance at $\lambda = 5.4$ mm (55.5 kmc).

waveguide usually break down at smaller than optimum values of the coat thickness.

It should be mentioned finally that the normal mode bend is an inherently broad band device. Except for the oscillations of the mode conversion portion of the total loss as caused by spurious mode phasing, there is only a gradual change of the loss with frequency.

Some terms contributing to the total loss decrease with frequency, others increase. The over-all frequency dependence is of the same order as the frequency dependence of the loss in the straight waveguide. As an example, in Fig. 5 the bend loss has been plotted versus frequency for the normal mode bend of Fig. 4.

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