Statistics of Regenerative Digital Transmission

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Statistics of a synchronous binary message pulse train applied to a regenerative repeater are related to those of the original binary message, which is assumed ergodic. It is shown that the ensemble of possible message pulse trains is a nonstationary random process having a periodically varying mean and autocovariance. A spectral density is calculated which shows line spectral components at harmonics of the pulse rate and a continuous density function, both with intensity proportional to the square of the absolute value of the Fourier transform of the standard pulse at the frequency considered. The continuous component has many properties similar to thermal noise but differs in that, under certain conditions described, it can exhibit regularly spaced axis crossings, can be exactly predicted over finite intervals and is capable of producing discrete components when nonlinear operations are performed on it, even though no line spectral terms are originally present. The analytical results are applied to the problem of deriving a timing wave from the message pulse train by shock-exciting a tuned circuit with impulses occurring at the axis crossings.

I. INTRODUCTION

An ideal regenerative digital repeater is defined here as a nonlinear time-varying device with response expressed as the product of a staircase output vs. input function such as shown in Fig. 1(a) and a time-sampling function such as shown in Fig. 1(b). The first function (a) converts a continuous range of possible input values to a discrete or quantized set of output values. The second function (b) is a time-varying response function which ideally makes the repeater sensitive only during infinitesimally narrow time intervals with regular spacing $T = 1/f_r$. Taken together, the two functions quantize and sample the input wave at uniformly spaced instants of time. The input wave consists of a discrete-valued message component plus noise and distortion accumulated in transmission from the previous repeater. If the noise and distortion are sufficiently small to confine the departure from the proper discrete message value within one quantization interval at the sampling instants, the regenerated wave matches the message and errorless transmission is achieved for any number of repeater spans.

To economize bandwidth, the actual pulses transmitted are not infinitesimally sharp, nor even rectangular as indicated in Fig. 1(b), but are multiples of a standard finite width pulse $g(t)$ which may be considered to be generated by a linear network from the sharp samples. A typical outgoing wave from the repeater is therefore expressible by:

$$x(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT), \tag{1}$$

where the sequence a_n, n going from $-\infty$ to ∞, represents the message values. In much of our work we shall specialize our attention to the binary case, in which there are only two possible values of a_n, which we shall usually take as zero and unity. The corresponding output vs. input function of Fig. 1(a) then takes the form shown in Fig. 2. A typical outgoing wave train is shown in Fig. 3 for the message sequence

$$\cdots 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0 \cdots .$$

The ideal repeater requires an absolute clock to supply the timing control exemplified by Fig. 1(b). In practical transmission systems of considerable length, it may not be feasible to supply such absolute timing information. An alternate procedure, for example, derives the timing wave from the signal pulse train itself by applying a portion of the wave

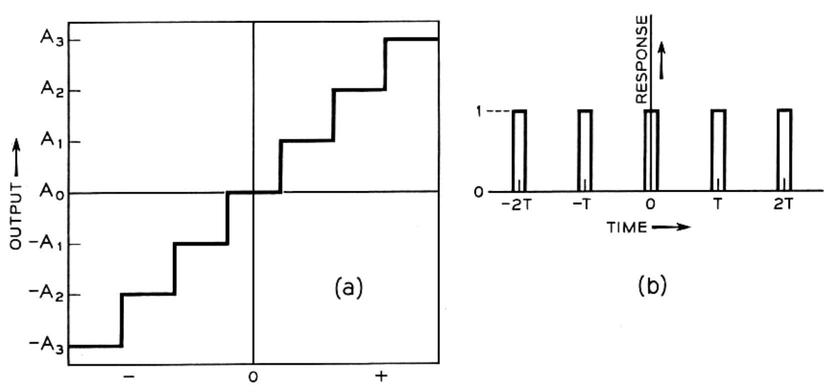

Fig. 1 — Specification of ideal regenerative repeater characteristic.

train to a circuit tuned to the pulse repetition frequency. If the circuit has a sufficiently high Q it continues to oscillate at the desired frequency during the gaps when no message pulses are present. Design of the tuned circuit requires a statistical analysis of the pulse train. This is one of many important statistical problems associated with practical, as distinguished from ideal, regenerative repeaters.

In this paper, we consider the following specific problems:

1. Properties of a digital message pulse train as a random noise source.
2. Derivation of the pulse repetition frequency from a pulse train by shock excitation of a tuned circuit. Under this topic, the following effects are studied:
   - (a) message statistics
   - (b) message pulse shape
   - (c) Q
   - (d) mistuning
   - (e) noise.
3. Effect of time jitter in received pulse train on recovered analog signal.

The important problem of analyzing accumulated effects in a chain of nonideal regenerative repeaters is discussed in a companion paper by H. E. Rowe.[1] Both this paper and that of Rowe are intended to pro-

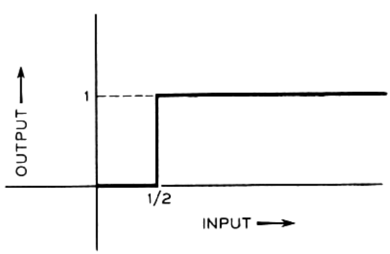

Fig. 2 — Binary output-vs.-input response.

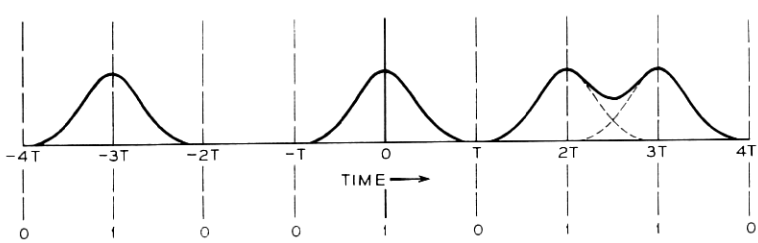

Fig. 3 — Typical binary wave train.

vide analytical background for the two preceding papers in this issue.[2, 3] Reference is also made to a paper by E. D. Sunde[4] which treats many of the same problems in a somewhat different way and gives results for specific repeater embodiments.

II. A DIGITAL MESSAGE AS A RANDOM NOISE SOURCE

The digital message may be regarded as a random time series of discrete numbers. In the notation of stochastic processes, consider the ensemble of possible messages $M(t)$, where a typical $M(t)$ consists of the infinite discrete time sequence: $\cdots a_{-2}, a_{-1}, a_0, a_1, a_2 \cdots$. The values of the a's belong to a specified set of discrete numbers and each possible sequence has a probability of occurrence associated with it. We assume that the message ensemble is ergodic; that is, averages over the ensemble at fixed time are identical with averages over time in any member of the ensemble, except for a set of probability zero. It follows that the ensemble is also stationary; that is, averages over the ensemble do not depend on position in the sequence. For our purposes we do not go beyond second-order statistics and shall base our calculations on two ensemble averages, the ordinary mean and the autocovariance.

The mean m_1 is defined by the ensemble average for any n:

$$m_1 = \text{av } a_n . \tag{2}$$

The value of m_1 is a constant for the message ensemble. For example, in the binary case in which zero and one occur with equal probability m_1 is equal to one half. The autocovariance $R(n)$ is defined by the ensemble average

$$R(n) = \text{av } (a_k a_{k+n}) \quad \text{for fixed } k. \tag{3}$$

The value of $R(n)$ depends on n but not on k. It is determined by the dependency of successive message values. If the message consists of independent numbers, the value of $R(0)$ is the average squared message value and $R(n)$ is equal to the square of the mean for n not equal to zero, since the average of the product of independent quantities is equal to the product of their averages. From the ergodic property, these quantities may also be derived from almost all members of the ensemble individually, thus:

$$m_1 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} a_n , \tag{4}$$

$$R(n) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} a_k a_{k+n} . \tag{5}$$

The actual wave sent over the line from one repeater to the next is the function $x(t)$ given by (1). We may regard $x(t)$ as a typical member of the pulse ensemble $\{x(t)\}$. Unlike the message ensemble $\{M(t)\}$, the pulse ensemble $\{x(t)\}$ consists of continuous functions of time rather than discrete time series. We shall now relate the statistics of the pulse ensemble to those of the discrete message ensemble.

We first evaluate the ensemble average of $\{x(t)\}$ by holding t fixed and averaging over all members. Then, since the average of a sum is equal to the sum of individual averages, and since the standard pulse $g(t)$ is the same for all members of the ensemble, we find

$$\begin{aligned} \text{av } \{x(t)\} &= \text{av} \sum_{n=-\infty}^{\infty} a_n g(t - nT) \\ &= \sum_{n=-\infty}^{\infty} \text{av } (a_n) g(t - nT) \\ &= m_1 \sum_{n=-\infty}^{\infty} g(t - nT). \end{aligned} \tag{6}$$

It thus appears that the ensemble average varies with time. This is not really unexpected since it seems reasonable that we should find values in the middle of a pulse interval to be distributed quite differently from those at the beginning or end. In fact, we should expect the ensemble average to vary periodically at the pulsing rate, and we readily demonstrate this by noting that

$$\begin{aligned} \text{av } \{x(t + T)\} &= m_1 \sum_{n=-\infty}^{\infty} g(t + T - nT) \\ &= m_1 \sum_{n=-\infty}^{\infty} g[t - (n - 1)T] = \text{av } x(t) \end{aligned} \tag{7}$$

by substitution of the summation index $n' = n - 1$.

The periodicity of the ensemble average suggests a formal Fourier series expansion, thus

$$\text{av } x(t) = \sum_{m=-\infty}^{\infty} c_m \exp (2m\pi j f_r t), \tag{8}$$

$$\begin{aligned} c_m &= f_r \int_0^T \text{av } x(t) \exp (-2m\pi j f_r t) \, dt \\ &= m_1 f_r \sum_{n=-\infty}^{\infty} \int_0^T g(t - nT) \exp (-2m\pi j f_r t) \, dt. \end{aligned} \tag{9}$$

We now substitute $t - nT = u$ in the integral. The limits then become $-nT$ to $-(n - 1)T$ while the exponential becomes $\exp (-2m\pi j f_r u)$, since $f_r T = 1$. The summation of finite integrals with adjoining limits may then be replaced by a single infinite integral, giving

$$c_m = m_1 f_r \int_{-\infty}^{\infty} g(u) \exp(-2m\pi j f_r u)\, du. \tag{10}$$

If we introduce the Fourier transform of the standard unit pulse by

$$G(f) = \int_{-\infty}^{\infty} g(t) \exp(-2\pi j f t)\, dt \tag{11}$$

we observe that

$$c_m = m_1 f_r G(m f_r), \tag{12}$$

and, hence,

$$\text{av}\,\{x(t)\} = m_1 f_r \sum_{m=-\infty}^{\infty} G(m f_r) \exp(2m\pi j f_r t). \tag{13}$$

The ensemble average is thus expressed as a Fourier series in time with the amplitude of the mth harmonic of the signaling frequency proportional to the amplitude of the spectral representation of the unit pulse at that same harmonic frequency. We note that an ensemble consisting of precisely this set of harmonics and no other terms would exhibit just this same result for its ensemble average. It is convenient, therefore, to resolve our pulse ensemble into a set of such discrete nonrandom harmonic components plus a remainder which has random properties and zero mean. We consider therefore the ensemble

$$\{y(t)\} = \{x(t)\} - \text{av}\,\{x(t)\} = \sum_{n=-\infty}^{\infty} (a_n - m_1) g(t - nT). \tag{14}$$

Then,

$$\text{av}\,\{y(t)\} = 0. \tag{15}$$

We explore farther by evaluating the autocovariance function $R_y(\tau, t)$ for the ensemble $\{y(t)\}$. By definition,

$$\begin{aligned} &R_y(\tau, t) \\ &= \text{av}\,\{y(t)y(t + \tau)\} \\ &= \text{av}\left\{\sum_{m=-\infty}^{\infty} (a_m - m_1) g(t - mT) \sum_{n=-\infty}^{\infty} (a_n - m_1) g(t - nT + \tau)\right\} \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \text{av}[(a_m - m_1)(a_n - m_1)] g(t - mT) g(t - nT + \tau) \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} [R(m - n) - m_1^2] g(t - mT) g(t - nT + \tau). \end{aligned} \tag{16}$$

By replacing t by $t + T$ and increasing both summation indices by unity, we see that $R_y(\tau, t)$ is periodic in t with period T. Hence, as in the case of the mean,

$$R_y(\tau, t) = \sum_{k=-\infty}^{\infty} d_k \exp(2k\pi j f_r t), \tag{17}$$

$$\begin{aligned} d_k &= f_r \int_0^T R_y(\tau, t) \exp(-2k\pi j f_r t)\, dt \\ &= f_r \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} [R(m - n) - m_1^2] \\ &\qquad \int_0^T g(t - mT) g(t - nT + \tau) \exp(-2k\pi j f_r t)\, dt. \end{aligned} \tag{18}$$

We substitute $m - n = m'$, $t - nT = t'$, and find after some rearrangement analogous to that used in obtaining (10):

$$d_k = f_r \sum_{m=-\infty}^{\infty} [R(m) - m_1^2] F(\tau + mT, kf_r), \tag{19}$$

where

$$F(\tau, f) = \int_{-\infty}^{\infty} g(t) g(t + \tau) \exp(-j2\pi f t)\, dt. \tag{20}$$

By the convolution theorem,

$$F(\tau, f) = \int_{-\infty}^{\infty} G(\lambda) G^*(\lambda + f) \exp[-j2\pi\tau(\lambda + f)]\, d\lambda. \tag{21}$$

In the special case of independent message values,

$$R(0) = \text{av}(a_n^2) = m_2; \qquad R(n) = \text{av}^2(a_n) = m_1^2,\ n \neq 0 \tag{22}$$

and

$$d_k = f_r \text{ var}(a_n)\, F(\tau, kf_r), \tag{23}$$

where var (a_n) is the variance of the a's defined by

$$\text{var}(a_n) = m_2 - m_1^2 = \text{av}(a_n^2) - \text{av}^2 a_n. \tag{24}$$

There is no reason, in general, for $F(\tau, kf_r)$ to vanish for all nonzero values of k, and hence the autocovariance of the y-ensemble does not reduce to an expression independent of time. It appears, therefore, that separating out the periodic components which account for the fluctuating mean does not make the remainder a stationary process. We therefore cannot invoke the Fourier transform relationship between autocovariance and power spectrum to find a power spectrum for our pulse ensemble, even after the periodic components have been removed.

A procedure which has often been used in similar situations is to make the process stationary by assuming a random phase relationship between the members of the ensemble. We do not wish to do this here because synchronous phase is an important property to be preserved in digital regeneration. However, there is a meaningful definition of a power spectrum which can be used without reference to the fluctuating autocovariance. We shall calculate the spectral density in this way, but, as might be expected, the resulting function is the same as would be obtained by randomizing the phases of the ensemble.

Consider first the ensemble $y_N(t)$ which includes only the pulses from $n = -N$ to $n = N$:

$$y_N(t) = \sum_{n=-N}^{N} (a_n - m_1)g(t - nT). \tag{25}$$

Then, for unit pulses which possess a Fourier transform, the Fourier transform $S_N(f)$ of $y_N(t)$ exists and is given by

$$S_N(f) = \sum_{n=-N}^{N} (a_n - m_1)g(f) \exp(-2n\pi jfT). \tag{26}$$

By Parseval's theorem,

$$\int_{-\infty}^{\infty} y_N^2(t)\, dt = \int_{-\infty}^{\infty} |\, S_N(f)\,|^2\, df. \tag{27}$$

Let the average of the ensemble $y_N^2(t)$ over the interval $-NT$ to NT be represented by $\text{av}_N y_N^2(t)$. Then,

$$\begin{aligned} \text{av}_N\, y_N^2(t) &= \frac{1}{(2N+1)T} \int_{-NT}^{NT} \text{av}\, y_N^2(t)\, dt \\ &= \frac{1}{(2N+1)T} \int_{-\infty}^{\infty} \text{av}\, |\, S_N(f)\,|^2\, df \\ &= \frac{1}{(2N+1)T} \left[\int_{NT}^{\infty} + \int_{-\infty}^{NT} \right] \text{av}\, y_N^2(t)\, dt. \end{aligned} \tag{28}$$

Also,

$$\begin{aligned} \text{av}_\infty\, y_\infty^2(t) &= \lim_{N\to\infty} \text{av}_N\, y_N^2(t) \\ &= \lim_{N\to\infty} \int_{-\infty}^{\infty} \frac{\text{av}\, |\, S_N(f)\,|^2}{(2N+1)T}\, df \\ &= \int_{-\infty}^{\infty} w(f)\, df, \end{aligned} \tag{29}$$

where

$$w(f) = \lim_{N\to\infty} \frac{\text{av}\, |\, S_N(f)\,|^2}{(2N+1)T}. \tag{30}$$

From (29), $w(f)$ as defined by (30) satisfies the requirement for a spectral density. From (26),

$$\begin{aligned}
&\text{av} \mid S_N(f) \mid^2 \\
&= \text{av}\,[S_N(f) S_N{}^*(f)] \\
&= \sum_{m=-N}^{N} \sum_{n=-N}^{N} \text{av}\,[(a_n - m_1)(a_m - m_1)] G(f) G^*(f) \exp\,[2\pi j f T(m-n)] \qquad (31) \\
&= \sum_{m=-N}^{N} \sum_{n=-N}^{N} [R(m-n) - m_1{}^2] \mid G(f) \mid^2 \exp\,[2\pi j f T(m-n)].
\end{aligned}$$

Let $m - n = k$, giving

$$\text{av} \mid S_N(f) \mid^2 = \sum_{m=-N}^{N} \sum_{k=m-N}^{m+N} [R(k) - m_1{}^2] \mid G(f) \mid^2 \exp\,(2\pi j k f T). \qquad (32)$$

The order of summation can be interchanged by the formula:

$$\sum_{m=-N}^{N} \sum_{k=m-N}^{m+N} = \sum_{k=-2N}^{0} \sum_{m=-N}^{k+N} + \sum_{k=1}^{2N} \sum_{m=k-N}^{N}. \qquad (33)$$

Since the summand does not depend on m, we perform the summation on m by counting the terms. The result is:

$$\begin{aligned}
\text{av} \mid S_N(f) \mid^2 = (2N+1) \mid G(f) \mid^2 \Bigg\{ R(0) - m_1{}^2 \\
+ 2 \sum_{k=1}^{2N} \left(1 - \frac{k}{2N+1}\right) [R(k) - m_1{}^2] \cos 2\pi k f T \Bigg\}.
\end{aligned} \qquad (34)$$

From (30), then,

$$w(f) \doteq f_r \mid G(f) \mid^2 \left\{ R(0) - m_1{}^2 + 2 \sum_{k=1}^{\infty} [R(k) - m_1{}^2] \cos 2\pi k f T \right\}. \qquad (35)$$

In the special case of independent message values, $R(0) = m_2 = m_1$ and $R(k) = m_1{}^2$ for $k \neq 0$, giving a result in agreement with the well-known solution[5] based on a random phase ensemble average, namely

$$w(f) = f_r\, m_1(1 - m_1) \mid G(f) \mid^2. \qquad (36)$$

This spectral density function is defined for both positive and negative frequencies. If the densities at negative frequencies are added to those of the corresponding positive frequencies, the above result is multiplied by two.

The spectrum of $\{y(t)\}$ is thus continuous and is proportional to the squared absolute value of the Fourier transform of the unit pulse. The spectrum of the actual pulse ensemble has added to the above continuous spectrum a set of line spectral components given by (13). The line spectral terms occur at harmonics of the signaling frequency and have amplitudes proportional to the unit pulse spectrum evaluated at the harmonic frequencies. If the pulse spectrum vanishes at any harmonic that harmonic does not appear in the $\{x(t)\}$ ensemble spectrum. Equation (13) can also be written in the form

$$\text{av}\ \{x(t)\} = m_1 f_r G(0) + 2m_1 f_r \sum_{m=1}^{\infty} |\, G(mf_r) \,| \cos\,[m\omega_r t + \text{ph}\ G(mf_r)]. \tag{37}$$

From this expression it is clear that the dc component has mean square $m_1^2 f_r^2 G^2(0)$, while the mean square value of the mth harmonic is

$$2m_1^2 f_r^2 \,|\, G(mf_r) \,|^2.$$

Fig. 4 gives curves of the continuous and line spectral density components for independent binary on-off signaling with various standard pulse shapes. For this case, if p_1 is the probability of the value unity, it follows that

$$m_2 = p_1\ ; \qquad m_2 - m_1^2 = p_1(1 - p_1). \tag{38}$$

The continuous part of the pulse train spectrum has much in common with thermal noise. An audible band selected between adjacent harmonics sounds to the ear like thermal noise. If the signaling rate is placed well above the noise band a random on-off pulse train serves adequately as a hiss source in a vocoder synthesizer. Oscillograms and spectrograms of narrow-band telegraph noise not including a harmonic would appear to the eye to be indistinguishable from thermal noise. The curves of Fig. 4 provide useful quantitative data relating the amount of noise produced in this way for specific cases.

It must not be concluded, however, that random telegraph noise is in all respects equivalent to thermal noise with the same spectral density. The telegraph noise remains a nonstationary process and has a phase structure among its components which thermal noise does not possess. The nonstationary properties of the telegraph wave ensemble are not the most general but are of a special kind which have sometimes been described as "periodically stationary." We have suggested the name "cyclostationary" for this type of process, i.e., one in which the ensemble statistics vary periodically with time. Analogously, the name "cycloergodic" will be used for cyclostationary processes in which statistics over the ensemble for fixed time are the same as statistics over the in-

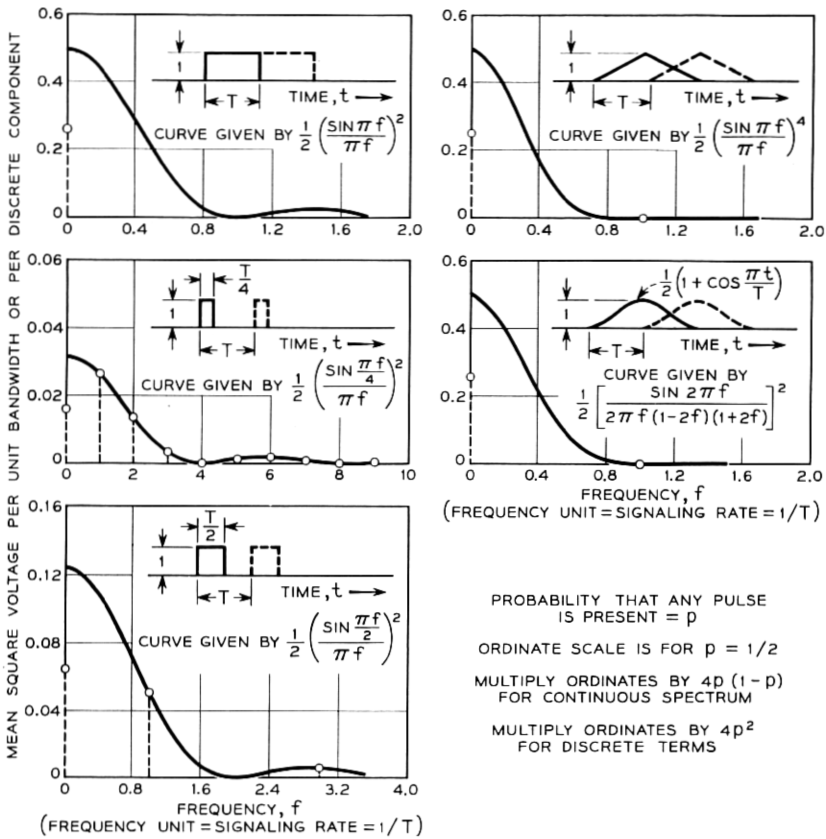

Fig. 4 — Continuous and line spectral density components for independent binary on-off signaling with specific pulse shapes.

stants of time differing from the fixed instant by multiples of the period in almost all individual members of the ensemble. A cycloergodic ensemble is cyclostationary but the converse is not necessarily true.

An important practical difference between telegraph noise and thermal noise is that the former can be generated deterministically if the message values are known or can be found. This follows because the wave form is completely determined by the sequence of message values and the standard unit pulse wave form. Use of telegraph noise for masking other waves is therefore subject to the qualification that, if the telegraph message can be read, the noise can be removed. Reading the message becomes more and more difficult as the ratio of bandwidth to signaling frequency is made smaller, and the overlapping of adjacent pulses in time becomes correspondingly greater. From the standpoint of digital transmission, reading the message is, of course, a paramount objective and the bandwidth of the transmission medium must be made large enough and the noise and distortion small enough to enable a correct reading to be made.

A further salient feature of the telegraph noise ensemble as contrasted with thermal noise is the possibility of regularly spaced axis crossings or zeros of the received wave. From (1) and (14) we see that both $x(t)$ and $y(t)$ will have zeros at points spaced T seconds apart if the standard pulse $g(t)$ has such zeros. That is, if for some t_0

$$g(t_0 + nT) = 0, \qquad n = 0, \pm 1, \pm 2, \cdots , \tag{39}$$

then

$$x(t_0 + nT) = y(t_0 + nT) = 0, \qquad n = 0, \pm 1, \pm 2, \cdots . \tag{40}$$

This does not mean that there cannot be other zeros as well, and the other zeros in general can have irregular spacings. It is possible, however, to exclude irregular zeros in specific cases. For example, if the standard pulse is the unit impulse response of a series-tuned circuit consisting of resistance R, inductance L, and capacitance C, the function $g(t)$ becomes $g_0(t)$ defined by

$$\begin{aligned} \omega_0 L g_0(t) &= (\omega_0^2 + \alpha^2)^{1/2} \exp(-\alpha t) \cos(\omega_0 t + \theta); \qquad t > 0 \\ \alpha &= R/2L; \qquad \omega_0^2 LC = 1 - R^2C/4L; \qquad \tan\theta = R/(2\omega_0 L). \end{aligned} \tag{41}$$

By setting $\omega_0 = 2\pi/T$, we obtain nulls at t_0 and t_1 in the interval

$$0 \leqq t \leqq T,$$

where

$$\begin{aligned} 2\pi t_0/T &= \pi/2 - \theta, \\ 2\pi t_1/T &= 3\pi/2 - \theta. \end{aligned} \tag{42}$$

These represent the only nulls in the interval from zero to T and they repeat at $t_0 + nT$ and $t_1 + nT$ for all integer values of n, as shown in the full-line waveform of Fig. 5. Furthermore, $g_0(t)$ changes in sign from positive to negative at $t = t_0 + nT$ and from negative to positive at $t = t_1 + nT$. No other sign changes occur. If we consider $g_0(t - mT)$ with m any positive or negative integer, we note that the values between $t_0 + nT$ and $t_1 + nT$ are the same as those of $g_0(t)$ between $t_0 + (n - m)T$ and $t_1 + (n - m)T$, and hence they are all negative. Likewise, the values of $g_0(t - mT)$ between $t_1 + nT$ and $t_0 + (n + 1)T$ are all positive. The

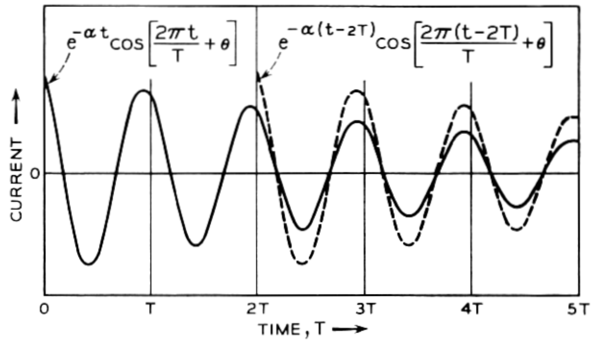

Fig. 5 — Response of tuned circuit to synchronous impulses.

dashed curve of Fig. 5 shows the relation for $m = 2$. The conclusion is that the summations from which $x(t)$ and $y(t)$ are formed consist of waves having common nulls and the same sign between nulls. It follows that both $x(t)$ and $y(t)$ for the case of $g(t) - g_0(t)$ have nulls only at $t_0 + nT$ and $t_1 + nT$, the axis crossings are from plus to minus at $t_0 + nT$, and the axis crossings are from minus to plus at $t_1 + nT$.

We have thus proved that, if $g_0(t)$ is the standard pulse, the continuous noise spectrum arising from a random choice of signaling pulses has no effect on the nulls of the composite wave. Telegraph noise therefore is profoundly different from thermal noise in that it does not perturb axis crossings in a situation in which thermal noise certainly would. Production of an invariant null spacing permits the recovery of the timing or clock wave from the pulse train and is accordingly of vital importance in the self-timed regenerative repeater.

In general, the standard pulse will not have the invariant null property exhibited by the special function $g_0(t)$. It may, however, be possible to convert the actual $g(t)$ into a new pulse which does have the required property. We note that the Fourier transform of $g_0(t)$ is given by

$$G_0(f) = \frac{j2\pi fC}{1 + j2\pi fRC - 4\pi^2 f^2 LC}. \tag{43}$$

An ensemble with standard pulse $g(t)$ and transform $G(f)$ can be converted into one with standard pulse $g_0(t - T_0)$ by inserting a linear network with transmittance function

$$Y(f) = \frac{G_0(f)}{G(f)} \exp(-j2\pi fT_0), \tag{44}$$

provided that $Y(f)$ is physically realizable. Since $G_0(f)$ is the response of a tuned circuit to a constant spectrum, the problem reduces to the realizability of the reciprocal of $G(f)$ with an arbitrary delay factor.

Suppose, however, that the received pulse shape is such that $Y(f)$ can not be realized. For example, (3) for the discrete or line spectrum components shows that, if the pulse transform $G(f)$ vanishes at the pulse repetition frequency f_r and all its harmonics, there are no discrete frequency components representing the signaling frequency in the pulse train. In (44) we would then have $G(f) = 0$ at $f = f_r$ and, since $F_0(f_r)$ is not zero, the value of $Y(f_r)$ would have to be infinite. It might seem that in this case the timing information is lost. However, we shall show that this difficulty only exists for linear time-invariant methods of deriving the timing information.

To see how we can materialize a discrete component from a continuous spectrum, consider a specific case of an uncurbed rectangular signaling pulse

$$g(t) = 1; \qquad 0 < t < T. \tag{45}$$

Then, from (11),

$$G(f) = (\pi f)^{-1} \exp(-j\pi f/f_r) \sin(\pi f/f_r). \tag{46}$$

It is clear that $G(nf_r)$ vanishes for all integer values of n. If a pulse by itself contains zero amplitude at all harmonics of f_r, no succession of pulses can have nonzero amplitude at these frequencies. However, we note that, in a binary on-off system using these uncurbed pulses, a wave such as that shown in Fig. 6 is obtained, in which every message digram

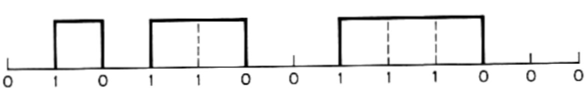

Fig. 6 — Train of uncurbed rectangular pulses.

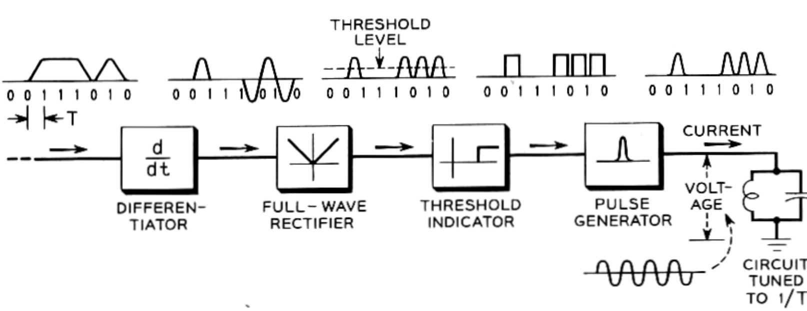

Fig. 7 — Recovery of timing wave from transitions of binary pulse train.

01 produces an upward transition, every digram 10 produces a downward transition, and the wave is constant for unbroken sequences of 1's or 0's. If this wave is applied to the circuit of Fig. 7, which includes a differentiator, full-wave rectifier, threshold triggered pulse generator and tuned circuit, an output wave containing pulses of form

$$g_0(t - nT + T_0)$$

is obtained where n represents the integer values in the message sequence at which 01 or 10 digrams occur. The effective message source is transformed from the original sequence of zero and unity values to a new sequence in which the unity values are associated with 01 or 10 transitions and the zeros with 11 or 00 digrams. A new continuous spectrum is thereby obtained and a discrete line spectrum absent from the original wave is created. Nonlinear operations have been used to attain this end. The system will still operate if either (but not both) the rectifier or the nonlinear triggered pulse generator is omitted. If we were dealing with thermal noise, even nonlinear operations would not suffice to construct a discrete spectrum.

The impulse response of a tuned circuit is only one example of a standard pulse shape which yields correct timing information from a synchronous telegraph pulse train. Another standard pulse having the same essential properties is the response of the same tuned circuit to a rectangular pulse of duration equal to half the signaling period. When the exciting pulse has unit height, the response becomes $g_1(t)$, defined by

$$\omega_0 L g_1(t) = \exp(-\alpha t) \begin{cases} \sin \omega_0 t; & 0 < t < T/2 \\ \sin \omega_0 t - \exp(\alpha T/2) \sin \omega_0 (t - T/2); & t > T/2. \end{cases} \tag{47}$$

When $\omega_0 T = 2\pi$, it may be verified that $g_1(t)$ has positive-going axis crossings at $t = 0, T, 2T, \cdots$; negative going axis crossings at $t = T/2, 3T/2, 5T/2, \cdots$; and no other axis crossings. Hence the summation of any such pulses beginning only at multiples of T has the same nulls and no others.

The problem of producing regularly spaced axis crossings from a received pulse train is studied in more detail in the next section, which treats the problems associated with derivation of timing information. We conclude our discussion here with some further remarks about secondary pulse trains triggered from transitions in a primary pulse train. Such secondary trains are of major importance in pulse frequency derivation techniques. The example given above of a primary uncurbed rectangular pulse train is not of much interest in digital transmission, since it would require excessive bandwidth to approximate the square waveforms. A more practical type of pulse is the raised cosine of Fig. 8. Such a waveform can be transmitted with fair accuracy over a bandwidth slightly less than the signaling frequency. Closest spacing without overlapping is obtained when one such pulse is started at the instant a preceding one reaches its peak. A flat-topped resultant wave then occurs between successive individual pulse peaks, as shown in Fig. 8. A train of raised cosine pulses can therefore give timing information only on the transitions 01 and 10, and the circuit of Fig. 7 is appropriate for signaling frequency derivation. It may be verified from Fig. 4 that the raised cosine pulse has null values in its Fourier transform at the signaling frequency and its harmonics.

The message associated with the transition pulse train is, of course, different from the original message and has a different set of statistics. It is of interest to calculate the relation between the two for the case of binary on-off message source with independent signal values. For such a

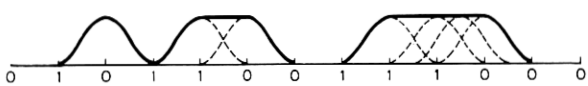

Fig. 8 — Train of raised cosine pulses.

source, the values of a_n are selected independently, with the probability p_1 that unity occurs and $1 - p_1$ that zero occurs in any position. Then the mean first and second powers are given by

$$m_1 = m_2 = p_1 \tag{48}$$

and the autocovariance by

$$R(0) = m_1, \qquad R(n) = m_1^2, \qquad n \neq 0. \tag{49}$$

The rule for constructing the transition message is given by the following table:

Original Message	Transition Message
1 preceded by 0	1
1 preceded by 1	0
0 preceded by 0	0
0 preceded by 1	1

The value 1 occurs in the transition message from two events each having probability $m_1(1 - m_1)$ and hence has probability $2m_1(1 - m_1)$. The mean first and second powers of the transition series are therefore given by

$$r_1 = r_2 = 2m_1(1 - m_1). \tag{50}$$

We shall represent the autocovariance of the transition message ensemble by $\rho(n)$. The value of $\rho(0)$ is simply the mean square, and hence

$$\rho(0) = 2m_1(1 - m_1) = r_1 . \tag{51}$$

To evaluate $\rho(1)$ we note that the product of adjacent values in the transition message is zero except when the original values are 10 preceded by 0, or 01 preceded by 1. The probability of obtaining the first is $m_1(1 - m_1)^2$ and of obtaining the second $m_1^2(1 - m_1)$. Hence the probability of a 11 sequence in the transition messages is given by the sum of these two probabilities, and since we obtain unity for the product in this case and zero otherwise,

$$\rho(1) = m_1(1 - m_1)^2 + m_1^2(1 - m_1) = m_1(1 - m_1) = r_1/2. \tag{52}$$

For values of n greater than one the constraint imposed by the transitions effectively disappears, since we may fill in all possible 0 and 1 values between the critical end points. Thus, in the case of $n = 2$ we obtain values of unity in the transition series two intervals apart for the following original message sequences:

$$\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0. \end{array}$$

These have total probability

$$\rho(2) = 4m_1^2(1 - m_1)^2 = r_1^2. \tag{53}$$

To evaluate $\rho(3)$, we merely fill in all possible 1's and 0's in a column between the second and third above and obtain the same total probability because the compound event represented by the first and last two columns is independent of that represented by the middle column. The probability of the first compound event is r_1^2, as calculated above, and that of a second is unity. Hence the complete autocovariance is expressed by

$$\begin{aligned} \rho(0) &= r_1 , \\ \rho(1) &= \rho(-1) = r_1/2, \qquad \text{for} \quad |n| > 1. \\ \rho(n) &= \rho(-n) = r_1^2 \end{aligned} \tag{54}$$

Thus, the transition series formed from the independent binary series becomes a digram series; that is, one in which adjacent values are dependent but nonadjacent values are independent. The discrete components in the spectrum are the same as in the independent case except that m_1 is replaced by r_1. The continuous spectrum is evaluated by substituting ρ for R and r_1 for m_1 in (35). The result after r_1 has been replaced by its value from (50) is

$$w(f) = 2m_1(1 - m_1) f_r \mid G(f) \mid^2 [1 - 2m_1 + 2m_1^2 + (1 - 2m_1)^2 \cos 2\pi fT]. \quad (55)$$

In the special case in which $m_1 = \frac{1}{2}$, this reduces to the same spectral density obtained for the independent message case, (36).

III. DERIVATION OF TIMING INFORMATION FROM A RANDOM TELEGRAPH MESSAGE

The existence of standard telegraph pulses which shock-excite a tuned circuit to produce a sustained oscillation at the signaling rate for almost all message sequences has been demonstrated by example in Section II.

In practical timing recovery circuits the ideal of a constant-amplitude correct-frequency output is not perfectly realizable. Factors which influence the result include the message pulse pattern, the Q of the tuned circuit, the presence of noise and the precision to which the natural oscillation frequency of the tuned circuit can be made to match the signaling frequency.

We shall analyze the problem of timing recovery under the conditions in which the ideal performance is almost, but not quite, obtained. The errors in this case are relatively small in magnitude and corresponding approximations can be made in the calculations which will show with sufficient accuracy what the quantitative requirements must be to hold the errors within specified small ranges.

Assume that the tuned circuit oscillates at a frequency very near to f_r, the signaling frequency, and that Q is sufficiently high to make the influence of any one pulse on the amplitude of oscillation slight. Consider the ensemble of possible responses in the interval from $t = 0$ to $t = T$, with the process having begun at $t = -\infty$. Almost all the waveforms are approximately sinusoidal and the negative-going axis crossings cluster about $t = t_c$, where t_c is defined by

$$\text{av } x(t_c) = 0; \qquad \text{av } x'(t_c) < 0. \quad (56)$$

The actual axis crossing for the typical member of the ensemble occurs at $t_c + \epsilon/2\pi f_r$, where ϵ is a small angle representing the phase error and has a distribution of values over the ensemble. We assume that the waveform through this axis crossing is that of a sine wave of frequency f_r and amplitude equal to the value of $x(t)$ one-quarter period after the axis crossing, i.e.:

$$x(t) \cong -x(t_c + \epsilon/\omega_r + T/4) \sin [\omega_r(t - t_c) - \epsilon], \tag{57}$$

with $|\epsilon| \ll 1$. Then, retaining only first powers of ϵ, we have

$$x(t_c) \doteq \epsilon\, x(t_c + T/4). \tag{58}$$

We further assume that $x(t_c + T/4)$ can be written in the form

$$x(t_c + T/4) = (1 + \zeta) \text{ av } x(t_c + T/4), \tag{59}$$

where $|\zeta| \ll 1$. By averaging both sides of this equation over the ensemble we deduce that av $\zeta = 0$.

It follows from the above assumptions that

$$\begin{aligned} \text{av } x(t_c) &\doteq \text{av } [\epsilon(1 + \zeta) \text{ av } x(t_c + T/4)] \\ &\doteq \text{av } \epsilon \text{ av } x(t_c + T/4). \end{aligned} \tag{60}$$

Hence, av $x(t_c) = 0$ implies av $\epsilon \doteq 0$. The principal quantity of interest is the rms phase error, which is the square root of av ϵ^2. By the assumed cycloergodicity of the process, an ensemble average of ϵ^2 is equal to an average of the value of ϵ^2 over all intervals of any member of the ensemble. From (58) and (59),

$$\begin{aligned} \text{av } x^2(t_c) &\doteq [\epsilon^2(1 + 2\zeta + \zeta^2) \text{ av}^2\, x(t_c + T/4)] \\ &\doteq \text{av } \epsilon^2 \text{ av}^2\, x(t_c + T/4). \end{aligned} \tag{61}$$

Hence, to the first order of small quantities,

$$\text{av } \epsilon^2 = \frac{\text{av } x^2(t_c)}{\text{av}^2\, x(t_c + T/4)}. \tag{62}$$

The mean square phase error is thus expressed in terms calculable from the general expressions previously derived for means and covariances of the pulse ensemble.

It is also of interest to evaluate the mean square value of ζ to validate the treatment of ζ as a small quantity and to estimate the amount of amplitude variation. From (59),

$$\begin{aligned} \text{av } x^2(t_c + T/4) &= \text{av } (1 + 2\zeta + \zeta^2) \text{ av}^2\, x(t_c + T/4) \\ &= (1 + \text{av } \zeta^2) \text{ av}^2\, x(t_c + T/4). \end{aligned} \tag{63}$$

Hence,

$$\text{av } \zeta^2 = \frac{\text{av } y^2(t_c + T/4)}{\text{av}^2 x(t_c + T/4)} = \frac{R_y(0, t_c + T/4)}{\text{av}^2 x(t_c + T/4)}. \tag{64}$$

It is to be noted that the above treatment evaluates only the mean square total phase error and does not give a frequency resolution. The spectral composition is important, particularly in the case of a chain of repeaters and is discussed in the companion paper by Rowe.[1]

We next calculate the average values appearing in the equations for mean square phase error and mean square amplitude ratio variation for specific pulse shapes. We assume that the circuit is not tuned to the pulse repetition frequency so that a basis for estimating the error from mistuning can be established. For the case in which $g(t) = g_0(t)$, the unit impulse response of a single tuned circuit, we write (41) in the form

$$g_0(t) = A \exp(-\alpha t) \cos(\omega_0 t + \theta); \qquad t > 0, \tag{65}$$

where

$$\begin{aligned} A &= \omega_0(4Q^2 + 1)^{1/2}/2RQ^2; \qquad \alpha = \omega_0/2Q; \\ Q &= \omega_0 L/R, \qquad \tan\theta = 1/2Q. \end{aligned} \tag{66}$$

From (6), the ensemble average of $x(t)$ in the interval $0 \leqq t \leqq T$ is

$$\begin{aligned} \text{av } \{x(t)\} &= m_1 \sum_{n=-\infty}^{\infty} g_0(t - nT) \\ &= m_1 A \sum_{n=-\infty}^{0} \text{Re} \exp[(j - \alpha)\omega_0(t - nT) + j\theta]. \end{aligned} \tag{67}$$

The series is geometric and may be summed to obtain

$$\begin{aligned} \text{av } x(t) = {} & \frac{m_1 A \exp[-\omega_0(t - T/2)/2Q]}{2RQ^2(\cosh^2 \omega_0 T/4Q - \cos^2 \omega_0 T/2)} \\ & \cdot \left\{ \sinh\frac{\omega_0 T}{4Q} \cos\frac{\omega_0 T}{2} \cos\left[\omega_0\left(t - \frac{T}{2}\right) + \theta\right] \right. \\ & \left. - \cosh\frac{\omega_0 T}{4Q} \sin\frac{\omega_0 T}{2} \sin[\omega_0(t - T/2) + \theta] \right\}. \end{aligned} \tag{68}$$

The value of t_c in the noise-free mistuned case can be obtained readily from (68) by equating the terms inclosed in the braces to zero, giving

$$t_c = T/2 - \theta/\omega_0 + \omega_0^{-1} \text{ arc tan}[\tanh(\omega_0 T/4Q) \cot(\omega_0 T/2)]. \tag{69}$$

We note that t_c approaches $T/4$ as Q approaches infinity. If we assume perfect tuning by setting $\omega_0 T = 2\pi$, (68) becomes

$$\text{av } x(t) = \frac{m_1\omega_r(4Q^2+1)^{1/2} \exp[-(\omega_r t - \pi)/2Q] \cos[\omega_r(t - T/2) + \theta]}{4RQ^2 \sinh \pi/2Q}. \tag{70}$$

That is, the average waveform of the tuned circuit response becomes a damped sine wave of the correct frequency. When Q is large, the damping is small. The limiting form as Q approaches infinity is an undamped sine wave:

$$\lim_{Q\to\infty} \text{av } x(t) = \frac{m_1\omega_r}{\pi R} \cos \omega_r(t - T/2). \tag{71}$$

This result is perfect for frequency recovery. It is to be noted that it does not matter whether the message values are independent or not. In the special case in which pulses are generated by transitions only, m_1 would be replaced by $r_1 = 2m_1(1 - m_1)$, as given by (50).

To complete the estimate of rms phase error and amplitude ratio variation, we must evaluate the average value of $x(t)$ one-quarter period away from t_c, and the average squares of $x(t)$ at $t = t_c$ and $t = t_c + T/4$. The value of av $x(t_c + T/4)$ is obtained by substituting $t = t_c + T/4$ in (58). The evaluation of average squares may be done in general from the covariance formula of either (16) or (17)–(21), recalling that

$$\text{av } x^2(t) = \text{av}^2\, x(t) + \text{av } y^2(t), \tag{72}$$

$$\text{av } y^2(t) = R_y(0, t). \tag{73}$$

Two binary cases of considerable importance which simplify considerably are those of completely independent message values and the message derived from transitions between independent values. These are:

Independent Binary Message

$$R(0) = m_1; \qquad R(n) = m_1^2; \qquad n \neq 0,$$

$$R_y(\tau, t) = m_1(1 - m_1) \sum_{m=-\infty}^{\infty} g(t - mT)g(t - mT + \tau), \tag{74}$$

$$\text{av } y^2(t) = R_y(0, t) = m_1(1 - m_1) \sum_{m=-\infty}^{\infty} g^2(t - mT).$$

Transition Message for Independent Binary Values

$$R(0) = r_1 = 2m_1(1 - m_1), R(1) = R(-1) = r_1/2, R(n) = r_1^2; |n| > 1, \tag{75}$$

$$R_y(\tau, t) = r_1 \sum_{m=-\infty}^{\infty} g(t - mT)(1 - r_1)g(t - mT + \tau) + (\tfrac{1}{2} - r_1)g[t - (m-1)T + \tau] + (\tfrac{1}{2} - r_1)g[t - (m+1)T + \tau], \tag{76}$$

$$\text{av } y^2(t) = R_y(0, t) = r_1 \sum_{m=-\infty}^{\infty} [(1 - r_1)g^2(t - mT) + (1 - 2r_1)g(t - mT)g[t - (m+1)T]. \tag{77}$$

For the independent binary message ensemble with $g(t) = g_0(t)$, $0 < t < T$,

$$\begin{aligned} \text{av } y^2(t) &= R_y(0, t) = m_1(1 - m_1) \sum_{m=-\infty}^{\infty} g_0^2(t - mT) \\ &= \frac{m_1(1 - m_1) \exp[-\omega_0(t - T/2)/Q]}{4 \sinh \omega_0 T/2Q (\cosh^2 \omega_0 T/2Q - \cos^2 \omega_0 T)} \\ &\quad \cdot \{\cosh^2 \omega_0 T/2Q - \cos^2 \omega_0 T + \sinh^2 (\omega_0 T/2Q) \cos \omega_0 T \\ &\quad \cdot \cos [\omega_0(2t - T) + 2\theta] - \sinh (\omega_0 T/Q) \cosh (\omega_0 T/2Q) \\ &\quad \cdot \sin \omega_0 T \sin [\omega_0(2t - T) + 2\theta]\}. \end{aligned} \tag{78}$$

If $\omega_0 T = 2\pi$, this reduces to:

$$\text{av } y^2(t) = \frac{m_1(1 - m_1)\omega_r^2(4Q^2 + 1) \exp[-\omega_r(t - T/2)/Q] \cos^2(\omega_r t + \theta)}{8R^2Q^4 \sinh \pi/Q}. \tag{79}$$

In the limit as Q approaches infinity, then,

$$\text{av } y^2(t) \to \frac{m_1(1 - m_1)\omega_r^2}{2\pi R^2 Q} \cos^2 \omega_r t. \tag{80}$$

IV. EVALUATION OF RMS PHASE ERROR WHEN INDEPENDENT ON-OR-OFF UNIT IMPULSES ARE APPLIED TO MISTUNED CIRCUIT

We illustrate the use of the equations derived above by carrying through the evaluation of (62) when the message pulse source consists of independent on-or-off unit impulses and the ratio of tuning error to the signaling rate is small compared with unity. We adopt the following notation:

$$\begin{aligned} k(\omega_0 - \omega_r)/\omega_r &= \text{fractional tuning error}, \\ \alpha &= \omega_0 T/4Q = (1 + k)\pi/2Q, \\ \beta &= (1 + k)\pi, \\ \omega_0[(t_c - T/2) + \theta] &= \Phi. \end{aligned} \tag{81}$$

From (69), then,

$$\begin{aligned} \sin \Phi &= \tanh \alpha \cot \beta \, (1 + \tanh^2 \alpha \cot^2 \beta)^{-1/2}, \\ \cos \Phi &= (1 + \tanh^2 \alpha \cot^2 \beta)^{-1/2}. \end{aligned} \tag{82}$$

Then, from (78), we may write, after noting that av $x(t_c) = 0$ implies av $x^2(t_c) =$ av $y^2(t_c)$:

$$\text{av } x^2(t_c) = \frac{m_1(1 - m_1)\omega_0^2(4Q^2 + 1) \exp[-\omega_0(t_c - T/2)/Q]}{16R^2Q^4 \sinh 2\alpha \, (\cosh^2 2\alpha - \cos^2 2\beta)} \{\sinh^2 2\alpha \tag{83}$$

$$+ \sin^2 2\beta + \sinh^2 2\alpha \cos 2\beta \cos 2\Phi - \sinh 2\alpha \cosh 2\alpha \sin 2\beta \sin 2\Phi\}.$$

From straightforward trigonometrical manipulations,

$$\begin{aligned} \sin 2\Phi &= \frac{2 \tanh \alpha \cot \beta}{1 + \tanh^2 \alpha \cot^2 \beta}, \\ \cos 2\Phi &= \frac{1 - \tanh^2 \alpha \cot^2 \beta}{1 + \tanh^2 \alpha \cot^2 \beta}. \end{aligned} \tag{84}$$

When the values of sin 2Φ and cos 2Φ are substituted in (83) we find, after some reduction:

$$\text{av } x^2(t_c) = \frac{m_1(1 - m_1)\omega_0^2(4Q^2 + 1) \sin 2\beta \; \exp\{-Q^{-1}[\theta + \text{arc tan}(\tanh \alpha \cot \beta)]\}}{16R^2Q^4 \sinh 2\alpha \, (\cosh^2 2\alpha - \cos^2 2\beta)}. \tag{85}$$

In the high-Q case, the value of av $x(t_c + T/4)$ will be very nearly equal to the maximum value of av $x(t)$ as given by (68). Making this assumption and simplifying, we obtain:

$$\text{av}^2 \, x(t_c + T/4) \doteq \frac{m_1^2\omega_0^2(4Q^2 + 1) \exp\{-Q^{-1}[\omega_0/4f_r + \theta + \text{arc tan}(\tanh \alpha \cot \beta)]\}}{16R^2Q^2(\cosh^2 \alpha - \cos^2 \beta)}. \tag{86}$$

Then, from (62),

$$\begin{aligned} \text{av } \epsilon^2 &\doteq \frac{(1 - m_1) \sin^2 2\beta \exp(-\omega_0/4 f_r Q)}{4m_1 \sinh 2\alpha (\cosh^2 \alpha - \sin^2 \beta)} \\ &= \frac{(1 - m_1) \sin^2 2k\pi \exp[-(1 + k)\pi/2Q]}{4m_1 \sinh (1 + k)\pi/Q [\cosh^2 (1 + k)\pi/2Q - \sin^2 k\pi]}. \end{aligned} \tag{87}$$

In the limit for $k \ll 1$ and $Q \gg 1$, which is the region of practical interest,

$$\text{av } \epsilon^2 \to (1 - m_1)\pi k^2 Q/m_1 . \tag{88}$$

The rms phase error is the square root of this quantity. For equal probability of pulses and spaces, $m_1 = \frac{1}{2}$, and

$$\epsilon_{\text{rms}} = (\text{av } \epsilon^2)^{1/2} \doteq k\sqrt{\pi Q}. \tag{89}$$

Fig. 9 shows a set of curves of rms phase error vs. Q for fixed values of tuning error. It is to be noted that this performance is attained when the reference phase of negative axis-crossings is t_c defined by av $x(t_c) = 0$. This value of time is displaced from the time t_0 defined by (42), which would be the correct reference if there were no mistuning. The difference is accounted for by the steady-state phase shift of the network at the

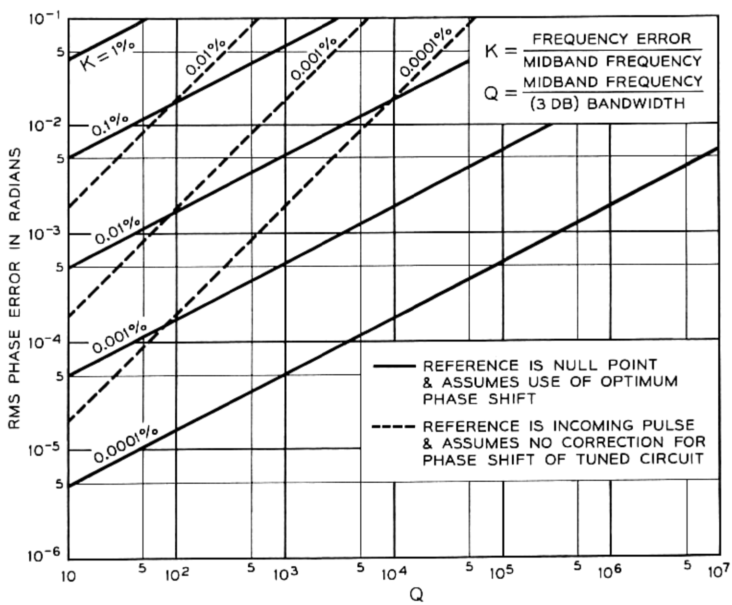

Fig. 9 — Phase error in timing wave from mistuned recovery circuit.

pulse frequency. If we were to take the axis-crossing reference time as t_0, the mean square phase error would be determined by av $x^2(t_0)$ instead of av $x^2(t_c)$ in (62). Since av $x(t_0)$ does not vanish, the expression for av $x^2(t_0)$ must be written as

$$\text{av } x^2(t_0) = \text{av}^2 x(t_0) + \text{av } y^2(t_0). \tag{90}$$

From (42),

$$\begin{aligned} t_0 &= T/4 - T\theta/2\pi, \\ t_0 - T/2 &= -T/4 - \theta/\omega_r = -(\theta + \pi/2)/\omega_r, \\ \cos[\omega_0(t_0 - T/2) + \theta] &= \cos[(1 + k)\pi/2 - k\theta] \\ &= -\sin[k(\pi/2 - \theta)], \\ \sin[\omega_0(t - T/2) + \theta] &= \cos[k(\pi/2 - \theta)], \\ \cos 2[\omega_0(t_0 - T/2) + \theta] &= -\cos k(\pi - 2\theta), \\ \sin 2[\omega_0(t - T/2) + \theta] &= -\sin k(\pi - 2\theta). \end{aligned} \tag{91}$$

Substituting these values and taking the limit for $k \ll 1$, and $Q \gg 1$, we find the rms phase error in the region of interest to be given approximately by

$$\epsilon'_{\text{rms}} \doteq \frac{k(1 + 1/4Q)}{(k^2 + 1/4Q^2)^{1/2}} \doteq 2Qk. \tag{92}$$

The dashed curves on Fig. 9 show the values of error thus obtained and indicate the importance of a phase adjustment to optimize for the tuned circuit in use.

It remains to evaluate the variation in peak height of the tuned circuit response. This requires evaluating av ζ^2 from (64), which in turn requires calculation of av $y^2(t_c + T/4)$ from (78). As in the case of av^2 $y(t_c + T/4)$ it is sufficient to take the maximum value of (78) over a period in t. The evaluation of av ζ^2 is facilitated by immediate comparison of the resulting expression for av^2 $y(t_c + T/4)$ with that for av^2 $x(t_c + T/4)$ given by (86). We find

$$\text{av } y^2(t_c + T/4) = \text{av}^2 x(t_c + T/4) \frac{B(1 - m_1)(\cosh^6 \alpha - \cosh^2 \beta)}{m_1 \sinh^2 \alpha (\cosh^6 2\alpha - \cos^2 2\beta)}, \tag{93}$$

where

$$\begin{aligned} B = {} & \cosh^2 2\alpha - \cos^2 2\beta \\ & + \sinh 2\alpha[\sinh^2 2\alpha \cos^2 2\beta + \cosh^2 2\alpha \sin^2 2\beta]^{1/2}. \end{aligned} \tag{94}$$

Then, by some further manipulation, we find:

$$\text{av } \zeta^2 = \frac{(1 - m_1)[\sinh^2 2\alpha + \sin^2 2\beta + \sinh 2\alpha (\sinh^2 2\alpha + \sin^2 2\beta)^{1/2}]}{4m_1 \sinh 2\alpha (\cosh^2 \alpha - \sin^2 \beta)}. \tag{95}$$

In the limit when $Q \gg 1$ and $k \ll 1$, we obtain

$$\text{av } \zeta^2 = \frac{(1 - m_1)\pi Q}{4m_1(1 - k^2\pi^2)} \left[\frac{1}{Q^2} + 4k^2 + \frac{1}{Q} \left(\frac{1}{Q^2} + 4k^2 \right)^{1/2} \right]. \tag{96}$$

With no tuning error, the ratio

$$\text{av } \zeta^2 \to \frac{(1 - m_1)\pi}{2m_1 Q} \tag{97}$$

can also be obtained by taking the ratio of (80) to the square of (71). When $k \ll \frac{1}{2}Q$, the limiting form is

$$\text{av } \zeta^2 \doteq \frac{(1 - m_1)\pi}{2m_1 Q} (1 + 3k^2Q^2). \tag{98}$$

If k is fixed and not equal to zero, the limit as Q is made very large is

$$\text{av } \zeta^2 = \frac{(1 - m_1)\pi Q k^2}{m_1}. \tag{99}$$

The analysis given here is valid only when av $\zeta^2 \ll 1$.

V. EFFECT OF NOISE ON TIMING

The question of timing errors caused by noise superimposed on the received pulse train can be resolved into two parts: (1) the shift in reference time at the input to the tuned circuit and (2) the resultant variation

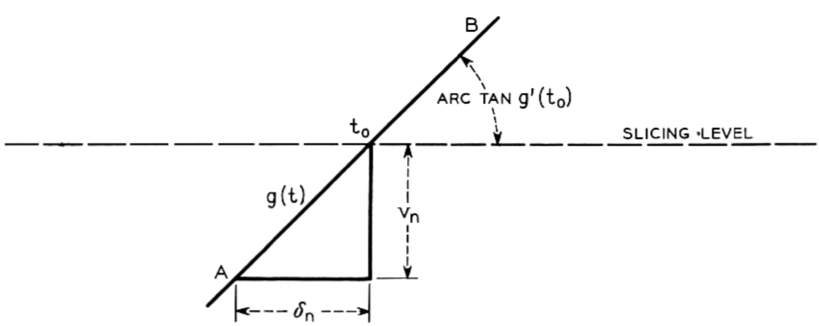

Fig. 10 — Geometrical construction showing relation between noise and timing error.

in the phase of the tuned circuit response. The first question can be conveniently answered by a geometrical argument based on Fig. 10. Here an enlarged section of the message pulse wave front is approximated by a straight line AB of slope equal to that of the pulse at the unperturbed critical time t_0. The instantaneous noise voltage v_n advances the critical time by δ_n, where $v_n/\delta_n =$ slope, and hence

$$\delta_n = v_n/g'(t_0). \tag{100}$$

Statistics of δ_n can be evaluated from the statistics of the noise and the specification of the pulse waveform. In particular, if the instantaneous noise values at corresponding instants of signaling intervals n intervals apart are not independent but have an autocorrelation or autocovariance function $R_v(n)$, the resulting time shifts have the corresponding function $R_\delta(n)$, where

$$R_\delta(n) = \text{av } (\delta_k\delta_{k+n}) = \text{av } (v_k v_{k+n})/g'(t_0). \tag{101}$$

The average is taken over all time for one member of the ensemble to give the autocorrelation or over the ensemble at fixed time to give the autocovariance. We assume here that the process is ergodic and hence that the two averages are equal. We have also assumed that the slope of the signal pulse is constant in the neighborhood of the slicing level.

In the second part of the problem, we assume perfect tuning of the recovery circuit and write for its response

$$z(t) = \sum_{n=-\infty}^{\infty} a_n g_0(t - nT + \delta_n). \tag{102}$$

Here $g_0(t)$ represents any response function which reproduces the axis crossings correctly and, in particular cases evaluated, it will be assumed as the impulse response of a tuned circuit. We assume that the individual values of δ_n are sufficiently small to enable an accurate representation by first-order terms in a power series, thus:

$$\begin{aligned} z(t) &\doteq \sum_{n=-\infty}^{\infty} a_n[g_0(t - nT) + \delta_n g_0'(t - nT)] \\ &= x(t) + \sum_{n=-\infty}^{\infty} a_n \delta_n g_0'(t - nT). \end{aligned} \tag{103}$$

Without loss of generality, we may take av $\delta_n = 0$, since a constant time shift may be compensated by a change in the reference time. Then, if the noise and message ensembles are independent,

$$\text{av } z(t) = \text{av } x(t). \tag{104}$$

We next calculate the autocovariance of the perturbed tuned circuit response as

$$\text{av}\,[z(t)z(t+\tau)] = \text{av}\,[x(t)x(t+\tau)] + \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \text{av}\,(a_m a_n)\,\text{av}\,(\delta_m \delta_n) g_0'(t-mT) g_0'(t-nT+\tau). \tag{105}$$

The first term on the right represents the unperturbed autocovariance and has already been evaluated. The effect of the noise is given by the second term and, by use of the same technique as in obtaining (16), we obtain its autocovariance as

$$R_{z-x}(\tau, t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} R(m-n) R_\delta(m-n) g_0'(t-mT) g_0'(t-nT+\tau). \tag{106}$$

In particular for the case of independent message values,

$$R_{z-x}(\tau, t) = m_1 \sum_{m=-\infty}^{\infty} R_\delta(m) g_0'(t-mT) g_0'(t-mT+\tau), \tag{107}$$

and

$$\text{av}\,[z^2(t) - x^2(t)] = R_{z-x}(0, t) = m_1 \sum_{m=-\infty}^{\infty} R_\delta(m)[g_0'(t-mT)]^2. \tag{108}$$

We may treat the phase error caused by noise in the same way as we did the mistuning error in deriving (62). Since, with no tuning error, $\text{av}\,x^2(t_c) = 0$ and $t_c = T/4 - \theta/2\pi f_r$,

$$\text{av}\,\epsilon_n^{\ 2} = \frac{\text{av}\,z^2(t_c)}{\text{av}^2\,x(t_c + T/4)}. \tag{109}$$

If the input noise values are assumed independent,

$$R_\delta(m) = \text{av}\,\delta_n^{\ 2} = \text{av}\,v_n^{\ 2}/[g'(t_0)]^2 \equiv \delta^2. \tag{110}$$

Then

$$\text{av}\,z^2(t_c) = m_1\delta^2 \sum_{m=-\infty}^{\infty} [g_0'(t_c - mT)]^2. \tag{111}$$

If

$$g_0(t) = Ae^{-\alpha t}\cos(\omega_r t + \theta), \tag{112}$$

it follows that

$$g_0'(t) = Be^{-\alpha t}\cos(\omega_r t + \theta - \Phi), \tag{113}$$

where

$$B = (\alpha^2 + \omega_r^2)^{1/2} A, \qquad \tan \Phi = \omega_r/\alpha. \tag{114}$$

Hence, the previous evaluation of (79) can be used for av $z^2(t_c)$, replacing θ by $\theta - \Phi$ and multiplying the result by $(\alpha^2 + \omega_r^2)\delta^2$. Then

$$\text{av } z^2(t_c) = \frac{m_1 \omega_r^4 \delta^2 (4Q^2 + 1) \exp (\theta + \pi)/2Q}{R^2 Q^4 \sinh \pi/Q}. \tag{115}$$

The limit as Q approaches infinity is

$$\text{av } z^2(t_c) = \frac{m_1 \omega_r^4 \delta^2}{2\pi R^2 Q}. \tag{116}$$

The expression for av ϵ_n^2 is found from division by the square of the amplitude factor of (70), giving

$$\begin{aligned} \text{av } \epsilon_n^2 &= m_1^{-1} \omega_r^2 \delta^2 \exp [(3\pi - 2\theta)/4Q] \tanh (\pi/2Q) \\ &\rightarrow \frac{\omega_r^2 \pi}{2m_1 Q} \delta^2 \quad \text{as} \quad Q \rightarrow \infty \end{aligned} \tag{117}$$

The effect of noise on the recovered timing is seen to vary inversely with Q and hence may be made arbitrarily small by using a sufficiently high-Q circuit. The effect is opposite to that of tuning error which becomes worse as Q is increased. The possibility of an optimum Q taking both effects into account thus exists.

VI. EFFECT OF TIMING VARIATIONS ON A DECODED ANALOG SIGNAL

The effects of timing errors are of two main kinds. First, they increase the difficulty of reading the message correctly because the decisions are made at a less favorable point on the pulse wave. This is called the alignment error and is particularly important in a long chain of regenerative repeaters. It has been discussed by E. D. Sunde[4] and is also analyzed in the companion paper by H. E. Rowe.[1] It will not be treated here. The second effect is to impair the usefulness of the received digital message even though the correct sequence of values is delivered. The seriousness of this effect varies widely with the type of message. At one extreme, the recovery of printed text from a telegraph message would hardly be affected. An analog signal transmitted by pulse code modulation, on the other hand, is in some danger because the decoded signal is not correctly reproduced by irregularly spaced samples. A single-channel system would undergo phase modulation from this cause. In the case of a time-division multiplex system, some interchannel crosstalk may result if the jitter varies with the signals in the channels. The most severe requirement occurs when PCM is applied to transmit a group of frequency-multiplexed channels, since any waveform distortion can mean interchannel crosstalk after the channels are separated.

We should point out at the beginning that we are not forced to accept time jitter in the received digital wave as finally used. A master clock could be inserted at any stage to force the pulse back into a proper time-reference framework. By means of resampling combined with pulse stretching and delay line distribution techniques, such corrections can be made even when the extent of the tuning variations exceeds the signaling interval. It would, of course, be desirable to avoid these operations and hence it is important to determine the permissible extent of time jitter in specific situations. We shall consider here the case of multiplex speech channels in frequency division transmitted by pulse code modulation, the FDM-PCM system.

Our procedure will be to represent the multiplex signal by an ensemble of band-limited functions $\{s(t)\}$ having a spectral density $w_s(f)$ which vanishes for $|f| \geqq f_0$. Any member of the ensemble is completely defined by its samples taken $1/2f_0$ apart; in fact, by the sampling theorem, we have the identity:

$$s(t) = \sum_{n=-\infty}^{\infty} s(n/2f_0) \frac{\sin 2\pi f_0(t + n/2f_0)}{2\pi f_0(t + n/2f_0)} . \tag{118}$$

The right-hand member is the response of an ideal low-pass filter to impulses of weight proportional to the samples. The effect of time jitter is to introduce an irregular spacing of the recovered samples which, when applied to an ideal filter, produce the distorted wave

$$\sigma(t) = \sum_{n=-\infty}^{\infty} s(n/2f_0) \frac{\sin 2\pi f_0(t + u_n + n/2f_0)}{2\pi f_0(t + u_n + n/2f_0)} . \tag{119}$$

The timing errors are represented by $u_n = u(n/2f_0)$, where $\{u(t)\}$ represents a jitter ensemble with specified statistical properties.

The evaluation of interchannel crosstalk is conveniently accomplished by assuming signals present in all but one of the FDM channels. The spectral density $w_s(f)$ is set equal to zero throughout the frequencies occupied by the idle channel. Values in this frequency range of $w_\sigma(f)$, the spectral density of the recovered signals, then give the interchannel interference caused by the time jitter. This method has the advantage of separating the interchannel interference sought from in-band distortion which is of no consequence. We refer to a previous paper[6] for a description of some of the necessary mathematical techniques.

We assume that $s(t)$ and $u(t)$ have autocovariance functions expressed in terms of the spectral densities by:

$$R_s(\tau) = \int_{-f_0}^{f_0} w_s(f) \exp(j2\pi\tau f)\, df, \tag{120}$$

$$R_u(\tau) = \int_{-f_0}^{f_0} w_u(f) \exp(j2\pi\tau f)\, df. \tag{121}$$

It is no loss of generality to assume that $u(t)$ is band-limited to the range $-f_0$ to f_0, since higher frequency components would produce effects indistinguishable from those in the band of half the sampling rate. The spectral densities are in turn expressible in terms of the autocovariances by

$$w_s(f) = \int_{-\infty}^{\infty} R_s(\tau) \exp(-j2\pi f\tau)\, d\tau, \tag{122}$$

$$w_u(f) = \int_{-\infty}^{\infty} R_u(\tau) \exp(-j2\pi f\tau)\, d\tau. \tag{123}$$

The s- and u- ensembles are assumed to be independent and ergodic. Note also that the signal density spectrum is defined in terms of mean squared voltage or current values, while the jitter spectrum is expressed in terms of mean square values of time.

Our procedure is to calculate first the autocorrelation function of the $\sigma(t)$- ensemble by the definition

$$\begin{aligned} R_\sigma(\tau) = \text{av}\ \sigma(t)\sigma(t+\tau) = \text{av} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s(m/2f_0)s(n/2f_0) \\ \cdot \frac{\sin 2\pi f_0(t + u_m + m/2f_0) \sin 2\pi f_0(t + \tau + u_n + n/2f_0)}{4\pi^2 f_0^2(t + u_m + m/2f_0)(t + \tau + u_n + n/2f_0)}. \end{aligned} \tag{124}$$

The spectral density of the distorted signal ensemble is then given by

$$w_\sigma(f) = \int_{-\infty}^{\infty} R_\sigma(\tau) \exp(-j2\pi f\tau)\, d\tau. \tag{125}$$

We assume ensemble and time averages to be interchangeable by the ergodic assumption. The averaging over the s- and u- ensembles may be done separately because of their independence. Furthermore, the double series may be averaged term by term since, in any case, the average of the sum is equal to the sum of the individual averages. We first note that, for the s- ensemble averaging,

$$\begin{aligned} \text{av}\ [s(m/2f_0)s(n/2f_0)] &= \text{av}\ [s(m/2f_u)s(m + n - m)/2f_0] \\ &= R_s[(n - m)/2f_0] = R_s[(m - n)/2f_0]. \end{aligned} \tag{126}$$

The averaging over the u- ensemble is more troublesome.

It has been found possible to obtain a reasonably simple answer only in the case of a gaussian jitter ensemble. By an involved calculation, which is explained in the Appendix, we obtain for this case, setting $T_0 = 1/2f_0$,

$$R_\sigma(\tau) = \frac{T_0}{\pi} \sum_{n=-\infty}^{\infty} R_s(nT_0) \int_0^{\pi/T_0} \exp(-z^2 U_n^2) \cos(\tau - nT_0)z \, dz, \quad (127)$$

where

$$U_n^2 = R_u(0) - R_u(nT_0). \quad (128)$$

In terms of the complex error function,

$$R_\sigma(\tau) = \frac{T_0}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \frac{R_s(nT_0)}{U_n} \operatorname{Re} \operatorname{erf}\left(\frac{\pi U_n}{T_0} - i\,\frac{\tau + nT_0}{2U_n}\right). \quad (129)$$

In the case of PCM transmission of an FDM group, the allowable jitter is small. This means that the integral of the spectral density $w_u(f)$ must be small and, hence, that $R_u(0)$, which is equal to this integral, must be small; $R_u(\tau)$ assumes its largest values at $\tau = 0$ and we conclude *a fortiori* that U_n^2 must be small. Hence, we approximate by replacing $\exp(-z^2U_n^2)$ by zero and first-power terms in its power series expansion and obtain an approximate value of the integral in (127):

$$\begin{aligned} R_\sigma(\tau) \doteq \sum_{n=-\infty}^{\infty} R_s(nT_0) \frac{\sin 2\pi f_0(\tau + nT_0)}{2\pi f_0(\tau + nT_0)} (t - 4\pi^2 f_0^2 U_n^2) \\ - 2U_n^2 \left[\frac{\cos 2\pi f_0(\tau + nT_0)}{(\tau + nT_0)^2} - \frac{T_0 \sin 2\pi f_0(\tau + nT_0)}{\pi(\tau + nT_0)}\right]. \end{aligned} \quad (130)$$

By a further manipulation also given in the Appendix, we find that this approximation yields for the spectral density

$$\begin{aligned} w_\sigma(f) &\doteq (1 - 4\pi^2 f^2 u_0^2) w_s(f) \\ &+ 4\pi^2 f^2 [w_{s*u}(f) + w_{s*u}(2f_0 + f) + w_{s*u}(2f_0 - f)]; \ |f| < f_0, \end{aligned} \quad (131)$$

where

$$u_0 = \sqrt{R_y(0)} \text{ is the rms time jitter and} \quad (132)$$

$$\begin{aligned} w_{s*y}(f) &= \int_{-\infty}^{\infty} w_s(\lambda + f) w_u(\lambda) \, d\lambda \\ &= \text{convolution of signal and jitter spectrum.} \end{aligned} \quad (133)$$

We note that the first term consists of the original power spectrum of the signal depressed by the "crowding" effect factor $(2\pi f u_0)^2$. The crowding effect does not introduce new frequencies not present in the original signal and therefore does not contribute to interchannel interference. The interference is given entirely by the second term, which consists of differentiated convolution spectra of signal and jitter. The differentiation is expressed by the multiplying factor $(2\pi f)^2$, which gives a triangular voltage spectrum like that of thermal noise in frequency modulation. The term $w_{s*y}(f)$ represents the direct convolution of signal and jitter spectra, while $w_{s*u}(2f_0 \pm f)$ represents sidebands of the convolution spectrum on the sampling frequency.

In testing of multichannel systems by noise-band loading, as previously mentioned, the signal spectrum applied is flat except for the narrow band assigned to the channel under test. The power spectrum is made equal to zero throughout the test-channel frequencies (see Figs. 11 and 12). The expression for $w_s(f)$ is then:

$$w_s(f) = \begin{cases} K; & |f| < f_t \quad \text{and} \quad f_t + f_c < |f| < f_0 \\ 0; & f_t < |f| < f_t + f_c , \end{cases} \tag{134}$$

where f_t is the test channel carrier frequency and f_c is the bandwidth of one channel. (We have assumed a single sideband system with upper sideband transmitted.) Then (133) gives

$$w_{s*u}(f) = K\left[\int_{-f_0}^{-f_t-f_c} + \int_{-f_t}^{f_t} + \int_{f_t+f_c}^{f_0}\right] w_u(f+\lambda)\, d\lambda. \tag{135}$$

It is sufficient to evaluate the interference spectrum in the test channel range f_t to $f_t + f_c$ and $-f_t - f_c$ to $-f_t$. Since power spectra are even

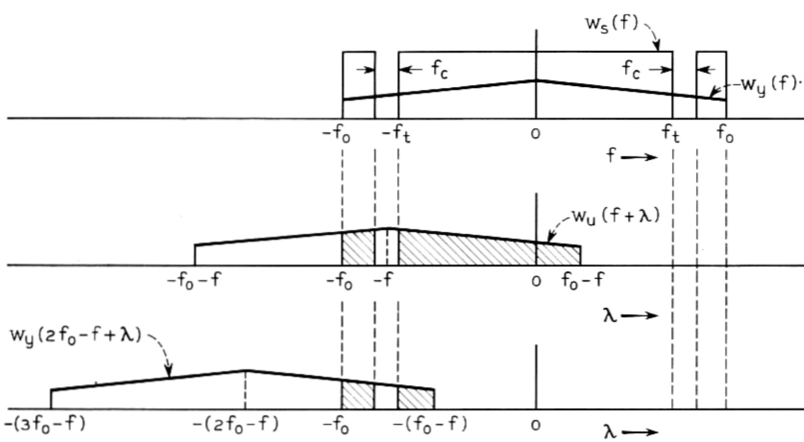

Fig. 11 — Spectral relations for noise band as test signal with $f_t > f_0/2$.

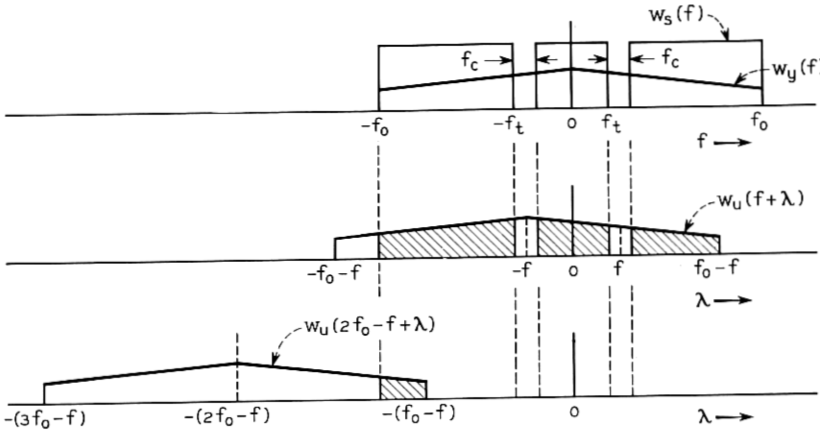

Fig. 12 — Spectral relations for noise band as test signal with $f_t < f_0/2$.

functions, we actually need to calculate for only one of these intervals. There are two contributions from (131) to the positive interval — one for $w_{s*u}(f)$ and the other from $w_{s*u}(2f_0 - f)$. The term $w_{s*u}(2f_0 + f)$ vanishes for $f > 0$ since $w_{s*u}(\nu)$ is zero for $\nu > 2f_0$.

The contribution from $w_{s*u}(2f_0 - f)$ to the positive interval f_t to $f_t + f_c$ is defined by the interval $2f_0 - f = f_t + f_c$ to $2f_0 - f = f_t$ or $f = 2f_0 - f_t - f_c$ to $f = 2f_0 - f_t$. We must therefore evaluate $w_{s*u}(f)$ in the ranges f_t to $f_t + f_c$ and $2f_0 - f_t - f_c$ to $2f_0 - f_t$.

Figs. 11 and 12 show the limits of integration which apply for the cases of f_t greater than and less than $f_0/2$, respectively. From these diagrams, we deduce that the distortion spectrum in the range $f_t < f < f_t + f_c$, is given by

$$w_{\sigma d}(f) = 4\pi^2 f^2 K \left\{ \int_{-f_0}^{-f_t - f_c} [w_u(f + \lambda) + w_u(2f_0 - f + \lambda)]\, d\lambda \right.$$
$$+ \int_{-f}^{f_0 - f_t} w_u(f + \lambda)\, d\lambda \tag{136}$$
$$\left. + \int_{-f_t}^{f - f_0} w_u(2f_0 - f + \lambda)\, d\lambda \right\} \qquad \text{for } f > \frac{f_0}{2},$$

and

$$w_{\sigma d}(f) = 4\pi^2 f^2 K \left\{ \left[\int_{-f_0}^{-f_t - f_c} + \int_{-f_t}^{f_t} + \int_{f_t + f_0}^{f_0 - f} \right] w_u(f + \lambda)\, d\lambda \right.$$
$$\left. + \int_{-f_0}^{f - f_0} w_u(2f_0 - f + \lambda)\, d\lambda \right\} \qquad \text{for } f < \frac{f_0}{2}. \tag{137}$$

A special case of interest is that of a flat jitter band over a low-frequency portion of the signal band. Rowe's previously cited work[1] on regenerative repeater chains indicates that the jitter spectrum peaks at the low-frequency edge of the signal band as the number of repeaters becomes large. Suppose the jitter spectrum is uniform from $-F$ to F, $f < f_0$ and consider the case of $f_t > f_0/2$ and $F < f_0 - f_t$. This corresponds to a test channel in the upper half of the signal band but below the top by at least the width of the jitter band. The distortion spectrum is given by

$$\begin{aligned} w_{\sigma d}(f) &= \frac{4\pi^2 f^2 K u_0^2}{2F} \left(\int_{-f-F}^{-f_t - f_c} + \int_{-f_t}^{-f+F} \right) d\lambda \\ &= \frac{4\pi^2 f^2 K u_0^2}{2F} (-f_t - f_c + f + F - f + F + f_t) \qquad (138) \\ &= \frac{4\pi^2 f^2 K u_0^2}{2F} (2F - f_c) \\ &\doteq 4\pi^2 f^2 K u_0^2 \qquad \text{if } f_c \ll 2F.^* \end{aligned}$$

The mean square total interference voltage in the idle channel is

$$u_c^2 = 2 \int_{f_t}^{f_t + f_c} w_{\sigma d}(f)\, df \doteq 8\pi^2 f_t^2 K u_0^2 f_c . \qquad (139)$$

If the mean square signal is s_0^2, we have

$$s_0^2 = 2Kf_0 , \qquad (140)$$

and

$$\frac{u_c^2}{s_0^2} = 4\pi^2 f_t^2 u_0^2 f_c / f_0 . \qquad (141)$$

The dependence on the square of test-channel frequency shows that we have a triangular voltage spectrum like that obtained in FM.

A quantity of interest defining signal quality in the typical channel of the frequency-division multiplex system is the ratio M of mean square value of the sine wave test tone which fully loads the channel to the mean square interference voltage when the channel is idle and the other channels are being operated in a normal manner. The mean square full load test tone voltage on one channel of an FDM telephone system is related

* The implications of this approximation should not be overlooked. If the bandwidth of the jitter spectrum is small compared to a channel width, interchannel interference would result only from high-order modulation neglected in the above approximation.

to the mean square full load test tone for the entire system by the Holbrook-Dixon[7] curves, which we shall express in the form

$$M_n = n\,P_1/P_n\,, \tag{142}$$

where

P_1 = full load test tone mean square value necessary to transmit one voice channel,
P_n = full load test tone mean square value necessary to transmit n superimposed voice channels.

When the multiplex system is operating normally the active channels are loaded with voice signals and not by sine waves. The value of s_0^2, the mean square total signal voltage with speech loading, may be expressed as

$$s_0^2 = P_n/H^2, \tag{143}$$

where H is the ratio of the peak factor of a sine wave to that of a composite multichannel speech wave. This is based on the assumption that the normal speech loading reaches system overload occasionally. The peak factor of a sine wave is $\sqrt{2}$, corresponding to 3 db, while the peak factor of superimposed speech channels approaches that of thermal noise as the number of channels is made large. The peak factor of thermal noise depends on the value of the probability of excess chosen. A value often used is 4, corresponding to 12 db, for which the probability of observing greater peaks is one in 10,000. This choice fixes H^2 at eight, corresponding to 9 db.

We write, in accordance with the above,

$$M = P_1/u_c^2, \tag{144}$$

and assume M specified. It follows that

$$M = \frac{M_n P_n f_0}{4\pi^2 f_t^2 u_0^2 n f_c s_0^2} = \frac{H^2 M_n f_0}{4\pi^2 f_t^2 u_0^2 n f_c}. \tag{145}$$

Solving for u_0^2, we obtain the minimum requirement for mean square time jitter:

$$u_0^2 = \frac{H^2 f_0 M_n}{4\pi^2 f_t^2 f_c n M}. \tag{146}$$

Because of the triangular distortion voltage spectrum, the requirement is most severe at the highest channel carrier frequency.

It is convenient to express this requirement in terms of rms phase jitter Φ allowed at the digital pulse frequency, which is $2Nf_0$ for an N-digit PCM system. Since $\Phi = 2\pi(2Nf_0)u_0$,

$$\Phi^2 = \frac{4N^2H^2f_0^3M_n}{nf_t^2f_cM}. \tag{147}$$

Consider, for example, an 8-digit binary PCM system transmitting a 2000-channel single-sideband FDM telephone signal with 4-kc carrier spacing. The top channel is at 8 mc and imposes the most severe requirement. We ask for a 60-db ratio of full load test tone power to idle channel interference. Assume the sampling frequency is 20 mc. We evaluate M_n by extrapolating from Table I of Ref. 8: $H^2 = 8$, $N = 8$, $f_0 = 10^7$, $M_n = 2000/80$, $n = 2000$, $f_t = 8 \times 10^6$, $f_c = 4 \times 10^3$, $M = 10^6$, and we calculate $\Phi = 0.3$ radian. This result is close to estimates made by O. E. DeLange and E. D. Sunde by different methods.

VII. ACKNOWLEDGMENT

The work described here had its inception in the work on PCM done at Bell Laboratories in the early 1940's and its beginnings were influenced by discussions with C. B. Feldman, R. L. Dietzold and L. A. MacColl. Later benefits were received by comparison with results obtained on various features of the problem by J. R. Pierce, H. E. Rowe and E. D. Sunde. The present form has profited from suggestions by D. W. Tufts.

APPENDIX

By the definition of a time average,

$$R_\sigma(\tau) = \lim_{N=\infty} \frac{1}{(2N+1)T_0} \sum_{m=-N}^{N} \sum_{n=-N}^{N} R_s[(m-n)T_0]$$
$$\int_{-NT_0}^{NT_0} \frac{\sin \frac{\pi}{T_0}(t + mT_0 + u_m) \sin \frac{\pi}{T_0}(t + \tau + nT_0 + u_n)\, dt}{\left(\frac{\pi}{T_0}\right)^2 (t + mT_0 + u_m)(t + \tau + nT_0 + u_n)}. \tag{148}$$

The integrand is an analytic function of t in the finite plane and hence the path of integration may be deformed without changing the value of the integral. We replace the part of the path passing through the zeros of the denominator by downward indentations. Sum and difference formulas may then be applied to the sines to resolve the integral into the sum of separate components without introducing singularities in the integrands. Designating the resulting path of integration by C, we then have:

$$R_x(\tau) = \lim_{N=\infty} \sum_{m=-\infty}^{N} \sum_{n=-\infty}^{N} \frac{T_0 R_s[(m - n)T_0]}{4N\pi^2} \int_C \frac{dt}{(t + mT_0 + u_m)(t + \tau + nT_0 + u_n)} \left\{ \cos \frac{\pi}{T_0} [\tau + (n - m)T_0 + u_n - u_m] - \cos \frac{\pi}{T_0} [2t + \tau + (m + n)T + u_m + u_n] \right\}. \tag{149}$$

To evaluate the integral, write:

$$\left.\begin{aligned} mT_0 + u_m &= a \\ \tau + nT_0 + u_n &= b \end{aligned}\right\}. \tag{150}$$

Note that a and b are real numbers. We must calculate:

$$\int_C \frac{\cos \frac{\pi}{T_0} (b - a)}{(t + a)(t + b)} dt - \int_C \frac{\cos \frac{\pi}{T_0} (2t + a + b)}{(t + a)(t + b)} dt.$$

The first integral vanishes as we let N approach infinity, as may be seen by closing the contour in an infinite semicircle below the real t-axis. The absolute value of the integral around the semicircle of radius r cannot exceed the product of length of path πr and the maximum absolute value of the integrand, which is less than $1/r^2$. The integral around the semicircle is therefore less in absolute value than π/r and approaches zero as r goes to infinity. The integral around the closed contour including C and the semicircle must vanish because there are no singularities inside. Hence the integral over C also vanishes and the first integral is zero.

The second integral may be written as the sum of two terms:

$$\frac{e^{j\pi(a+b)/T_0}}{2} \int_C \frac{e^{j2\pi t/T_0}\, dt}{(t + a)(t + d)} + \frac{e^{-j\pi(a+b)/T_0}}{2} \int_C \frac{e^{-j2\pi t/T_0}\, dt}{(t + a)(t + b)}. \tag{151}$$

In the first term we close the contour in an infinite semicircle above the real t-axis and obtain an integrand which vanishes exponentially as the radius is made large. In the limit the integral along C is then equal to the integral around the closed contour, which in turn is equal to $2\pi/j$ times the sum of the residues at the poles $t = -a$ and $t = -b$. In the second term we close the contour below the real t-axis and show that the integral vanishes. Therefore the complete result is given by:

$$-\frac{2\pi j}{2} e^{j\pi(a+b)/T_0}\left(\frac{e^{-j2\pi a/T_0}}{b-a}+\frac{e^{-j2\pi b/T_0}}{a-b}\right)$$

$$= -\frac{\pi j}{a-b}\left[e^{j\pi(a-b)T_0} - e^{-j\pi(a-b)/T_0}\right] \quad (152)$$

$$= \frac{2\pi \sin \pi(a-b)/T_0}{a-b}.$$

Hence,

$$R_\sigma(\tau) = \lim_{N=\infty} \sum_{n=-N}^{N} \sum_{b=-N}^{N} \frac{T_0 R_s[(m-n)T_0] \sin \frac{\pi}{T_0}[(m-n)T_0+u_m-u_n-\tau]}{2N\pi[(m-n)T_0+u_m-u_n-\tau]}. \quad (153)$$

For purposes of calculation it is convenient to introduce the dummy variable z and write the equivalent form:

$$R_\sigma(\tau) = \int_0^{\pi/T_0} dz \sum_{m=-N}^{N} \sum_{n=-N}^{N} \frac{T_0 R_s[(m-n)T_0] \cos[(m-n)T_0+u_m-u_n-\tau]z}{2N\pi}. \quad (154)$$

Next we change the order of summation by letting $m - n = m'$ and dropping the prime after eliminating m to obtain

$$R_\sigma(\tau) = \int_0^{\pi/T_0} dz \sum_{n=-N}^{N} \sum_{m=-N-n}^{n-N} \frac{T_0 R_s(mT_0) \cos(mT_0+u_{m+n}-u_n-\tau)z}{2N\pi}. \quad (155)$$

The summation can be rearranged into the equivalent form

$$\sum_{m=-2N}^{0} \sum_{n=-N-m}^{N} + \sum_{m=1}^{2N} \sum_{n=-N}^{N-m},$$

so that the summation may be performed with respect to n first. For each value of m there are $2N + 1$ values of n. When N is large, summing over the values of the function

$$G(nT_0) = \frac{T_0 R_s(mT_0)}{\pi} \cos\{mT_0 + u[(m+n)T_0] - u(nT_0) - \tau\}z \quad (156)$$

for $2N + 1$ values of n with m fixed and then dividing by $2N$ approaches the procedure for calculating the average of $G(nT)$ over the u-ensemble.

To evaluate the ensemble average of $G(nT_0)$, we invoke a theorem expressed by equation (1.14) of Ref. 6, which states that, if $u(t)$ is a gaussian ensemble,

$$\text{av}\,\{\cos[au(t) + bu(t+\tau) + \beta]\} = \exp\left[\frac{a^2+b^2}{2}R_u(0) - abR_u(\tau)\right]\cos\beta. \tag{157}$$

We shall assume from this point on that $u(t)$ is a gaussian ensemble. Then the above result fits our case if we set $t = nT_0$, $a = -z$, $b = z$, $\beta = (mT_0 - \tau)z$, $\tau = mT_0$. We then find

$$\text{av}\,G(nT_0) = \frac{T_0 R_s(mT_0)}{\pi}\, e^{-z^2[R_u(0) - R_u(mT_0)]}\cos(mT_0 - \tau)z, \tag{158}$$

and

$$R_\sigma(\tau) = \frac{T_0}{\pi}\sum_{m=-\infty}^{\infty} R_s(mT_0) \int_0^{\pi/T_0} e^{-z^2[R_u(0)-R_u(mT_0)]}\cos[(\tau - mT_0)z]\,dz. \tag{159}$$

The integral is now evaluated by writing the cosine in exponential form, completing the square in the exponent, and substituting variables to obtain the error function. The result is given as (129) of Section VI. We have replaced m by $-n$ and made use of the fact that the autocorrelation function is even.

We next calculate the power spectrum $w_\sigma(f)$ by taking the Fourier transform of the approximate autocorrelation (130). The complete integrand has no singularities in τ and, by indenting the path of integration below each point $\tau = -nT_0$, we can separate the integral into component parts without singularities on the path. Denoting the depressed path by C, we evaluate

$$\int_C \frac{e^{-j2\pi f\tau}\sin\frac{\pi}{T_0}(\tau + nT_0)}{\pi(\tau + nT_0)}\,d\tau$$

$$= \int_C [e^{-j[2\pi f-(\pi/T_0)]\tau + jn\pi} - e^{-j[2\pi f+(\pi/T_0)]\tau - jn\pi}]\frac{d\tau}{2\pi j(\tau + nT_0)} \tag{160}$$

$$= \begin{cases} 0; & |f| > \dfrac{1}{2T_0}, \\ e^{j2n\pi fT_0}; & |f| < \dfrac{1}{2T_0} \end{cases}$$

$$\int_C \frac{e^{-j2\pi f\tau} \cos \frac{\pi}{T_0}(\tau + nT_0)}{(\tau + nT_0)^2}\, d\tau$$

$$= \int_C [e^{-j[2\pi f-(\pi/T_0)]\tau+jn\pi} + e^{-j[2\pi f+(\pi/T_0)]\tau-jn\pi}] \frac{d\tau}{2(\tau + nT_0)^2} \quad (161)$$

$$= \begin{cases} 0; & |f| > \dfrac{1}{2T_0}, \\ \pi\left(2\pi f - \dfrac{\pi}{T_0}\right)e^{j2n\pi fT_0}; & |f| < \dfrac{1}{2T_0} \end{cases}$$

$$\int_C \frac{e^{-j2\pi f\tau} \sin \frac{\pi}{T_0}(\tau + nT_0)}{(\tau + nT_0)^3}\, d\tau$$

$$= \int_C [e^{-j[2\pi f-(\pi/T_0)]\tau+jn\pi} - e^{-j[2\pi f+(\pi/T_0)]\tau-jn\pi}] \frac{d\tau}{2j(\tau + nT_0)^3} \quad (162)$$

$$= \begin{cases} 0; & |f| > \dfrac{\pi}{2T_0}. \\ -\dfrac{\pi}{2}\left(2\pi f - \dfrac{\pi}{T_0}\right)^2 e^{j2n\pi fT_0}; & |f| < \dfrac{1}{2T_0} \end{cases}$$

Then, for $|f| < 1/2T_0$; that is, $|f| < f_0$:

$$w_\sigma(f) \doteq \sum_{n=-\infty}^{\infty} T_0 R_s(nT_0)\left[1 - \frac{\pi^2}{T_0^2} U_n^2 - 2U_n^2\pi^2\left(2f - \frac{1}{T_0}\right)\right.$$

$$\left. - 2\frac{T_0}{\pi}\frac{\pi}{2}\pi^2\left(2f - \frac{1}{T_0}\right)^2 U_n^2\right] e^{j2n\pi fT} \quad (163)$$

$$= T_0 \sum_{n=-\infty}^{\infty} R_s(nT_0)(1 - 4\pi^2 f^2 U_n^2)e^{j2n\pi fT}.$$

From (120), we note that $w_s(t)$ vanishes outside the range $-f_0$ to f_0,

$$R_s(nT_0) = \int_{-\infty}^{\infty} w_s(\nu)e^{j2\pi nT_0\nu}\, d\nu, \quad (164)$$

and from the analogous relation for $R_u(t)$,

$$R_u(nT_0) = \int_{-\infty}^{\infty} w_u(\nu)e^{j2\pi nT_0\nu}\, d\nu. \quad (165)$$

Application of the convolution theorem shows that

$$R_s(nT_0)R_u(nT_0) = \int_{-\infty}^{\infty} w_{s*y}(\nu)e^{j2\pi nT_0\nu}\, d\nu, \tag{166}$$

where

$$\begin{aligned} w_{s*u}(\nu) &= \int_{-\infty}^{\infty} w_s(\lambda)w_u(\lambda + \nu)\, d\lambda \\ &= \int_{-\infty}^{\infty} w_s(\lambda + \nu)w_u(\lambda)\, d\lambda. \end{aligned} \tag{167}$$

We substitute the above expressions for $R_s(nT_0)$ and $R_s(nT_0)R_u(nT_0)$ in (163) and then apply Poisson's summation formula,

$$\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(z)e^{inz}\, dz = 2\pi \sum_{n=-\infty}^{\infty} \varphi(2n\pi), \tag{168}$$

to obtain, for $|f| < f_0$,

$$\begin{aligned} W_\sigma(f) \doteq [1 - 4\pi^2 f^2 R_u(0)] \sum_{n=-\infty}^{\infty} w_s\left(\frac{n}{T_0} - f\right) \\ + 4\pi^2 f^2 \sum_{n=-\infty}^{\infty} w_{s*u}\left(\frac{n}{T_0} - f\right). \end{aligned} \tag{169}$$

Since both $w_s(f)$ and $w_u(f)$ have been assumed to vanish for $|f| > f_0 = 1/2T_0$, the only term of the first sum which falls in the signal band $-f_0$ to f_0 is that for $n = 0$, while in the second sum the signal-band contributors are $n = 0, -1$ and 1. The latter conclusion follows from the fact that the convolution of two functions limited to the same low-pass band is limited to a low-pass band twice as great. Also, noting that power spectra are even functions, we obtain (131).

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