On Trunks with Negative Exponential Holding Times Serving a Renewal Process

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A group of N trunks serves calls arriving in a renewal process, and lost calls are cleared. The number, $N(k)$, of trunks found busy by the kth arriving customer is studied as a Markov process imbedded in a (usually) non-Markov process $N(t)$, the number of trunks busy at t. Results of C. Palm and F. Pollaczek on the distribution of $N(k)$ are generalized, and a study is made of bounds for, and approximations to, the probability of loss. The probability of loss is studied as a functional of the interarrival distribution function, and certain extremal properties are proven. Formulas for the mean of $N(k)$ and for the covariance function are given, together with equilibrium curves for the probability of loss, for the mean and variance of $N(k)$, and for the first four values of the covariance function. Some applications to switch counting are discussed.

I. INTRODUCTION

We shall study a mathematical model for the random behavior of the occupancy of trunk groups. The principal results are complete descriptions (in principle) of (a) the variations of the traffic in time, (b) the equilibrium probabilities and (c), the covariance function of the traffic found by arriving customers. These mathematical results have practical application in engineering trunk groups to have a given probability of loss, and in estimating the sampling error incurred in certain ways of measuring traffic.

A "trunk group" is a set of transmission channels (trunks) between central offices. The trunks in a group are often equivalent in the sense that a call handled on one idle trunk could as well have been assigned another. A "holding time" of a trunk is a length of time during which it is continuously unavailable because it is being seized and used as a talking path. By "interarrival times" we mean the time intervals elapsing between successive epochs at which attempts are made to place a call on the trunk group. With these definitions in mind, the theoretical model we use to describe the trunk group involves four assumptions:

i. The holding times of trunks are independent random quantities having a negative exponential distribution, with mean value, γ^{-1} (γ is the hang-up rate). This means that if a trunk is in use at time x, the chance that it is still in use at $(x + dx)$ is $1 - \gamma dx - o(dx)$, $o(dx)$ denoting a quantity of order smaller than dx, irrespective of how long the trunk has been in use. The probability that a holding time is less than t is then $1 - \exp\{-\gamma t\}$ for $t \geqq 0$, and 0 otherwise.

ii. The interarrival times of calls are independent positive variates; each has the general distribution $A(u)$, where $A(u)$ is arbitrary except for the condition $A(0) = 0$. If t_k and t_{k+1} are successive arrival times, then

$$\Pr\{t_{k+1} - t_k \leqq u\} = A(u),$$

for all k, independently. This assumption covers Poisson (or completely random) arrivals as a special case. In accordance with the usage in the literature, we call a sequence of mutually independent, identically distributed, positive variates, a "renewal process." The interarrival times in our model then form a renewal process. It has been shown by Palm[1] and noted by Feller[2] that non-Poisson renewal processes arise in their own right in the study of overflow traffic from a trunk group, even when the original offered traffic is Poisson in character.

iii. There are $N < \infty$ trunks in the group.

iv. Calls which find all N trunks busy are lost, and are cleared from the system.

A model like the above, but without the strong simplifying assumption of exponential holding time, was studied by Pollaczek.[3] The model described in (i) through (iv) above has been considered by Palm,[1] and also by Takács,[4] who used a functional equation. Takács' paper was apparently written without knowledge of the prior work of Palm and Pollaczek; in a recent paper,[5] Takács thanks R. Syski for calling his attention to Refs. 1 and 3. The same model has also been treated by Cohen.[6] For convenience and unity of exposition, some of the results of these authors shall be rederived here, and attributed to the appropriate author as they arise.

II. SUMMARY OF RESULTS

It is natural to use the number $N(t)$ of calls in progress on the trunk group at time t as an indicator of traffic; $N(t)$ is a random step function, fluctuating in unit steps from 0 to N.

Unless the arrivals form a Poisson process; that is, unless

$$A(u) = 1 - \exp\{-u/\mu_1\}$$

for $u \geqq 0$ and $\mu_1 > 0$, $N(t)$ is not a Markov process. However, let t_k be the epoch of the kth arrival, and suppose that $N(t_k - 0)$ is known. Thus, we know how many busy trunks were found by the kth call. Until the next call arrives at t_{k+1}, the number of calls in progress forms essentially a simple death process, with death rate γ per head of population. The conditional distribution of $N(t_{k+1} - 0)$, given $N(t_k - 0)$, can then be calculated from the known transition probabilities of the death process (see Feller[7]). No additional knowledge of $N(t)$ for $t < t_k$ is of prognostic relevance to $N(t)$ for $t > t_k$, when $N(t_k - 0)$ is known. We define

$$N(k) = N(t_k - 0),$$

where $N(k)$ is the number of trunks found busy by the kth arriving call. The variates $N(k)$ form a Markov chain imbedded in the non-Markov process $N(t)$. This Markov chain is the basic random process considered in this paper.

Let the numbers a_n, $n = 1, \cdots, N$ be defined by

$$a_n = \int_0^\infty e^{-n\gamma u}\, dA(u),$$

so that a_n is the Laplace-Stieltjes transform of the interarrival distribution $A(u)$, evaluated at the point $n\gamma$, where γ is the hangup rate. The principal theoretical result of this paper is Theorem 1 in Section IV. This result gives formulas for the generating functions

$$\psi_n(z) = \sum_{k \geqq 0} z^k \Pr\{N(k) = n\}$$

for an arbitrary initial distribution of $N(0)$. These formulas depend only on the numbers $a_1, \cdots, a_N$ defined previously, so the entire Markov process $N(k)$ depends only on these numbers. Theorem 1 determines, in principle, the transition probabilities of $N(k)$ purely in terms of $a_1, \cdots, a_N$, and so provides a complete description of the statistical variations of the traffic found by arriving customers. For $N(0) = 0$, the formulas were obtained by Pollaczek;[3] the formulas to be given coincide with those of Pollaczek in this case.

In Section V the limiting probabilities

$$p_n = \lim_{k\to\infty} \Pr\{N(k) = n\},$$

already considered by Palm, Pollaczek and Takács, are briefly discussed. The quantity p_n is the equilibrium chance that an arriving customer find

n trunks busy; in particular, p_N is the probability of loss. It should be kept in mind that p_n is not the probability that, if we inspect the trunk group at a random moment in equilibrium, we will find n trunks busy; the moments of inspection must be those immediately preceding arrivals. In Section V, also, various moments (such as the ordinary, binomial and factorial) and the variance of the limit distribution $\{p_n\}$ are presented. Curves of the probability of loss, the fraction of trunks found busy by an arrival and the variance of $\{p_n\}$ are plotted as functions of the offered erlangs for three choices of the interarrival distribution $A(u)$.

Sections VI and VII discuss bounds for, and approximations to, the probability p_N of loss. The results of Section VI are general; those of Section VII are restricted to the case of regular arrivals. Consideration of the unrealistic (for telephone trunking) special case of regular arrivals is justified (in Section VIII) by the fact that regular arrivals form a limiting best case.

In Section VIII we treat p_N as a functional of the interarrival distribution $A(u)$. The chief results can be summarized informally as follows:

i. For a fixed mean interarrival time and a fixed hang-up rate, the minimum loss is achieved when arrivals are regular.

ii. Arriving customers can, without changing either their mean arrival rate or their hang-up rate, still make the telephone company give them arbitrarily bad service (high loss) by a proper choice of $A(u)$.

iii. The maximum number of erlangs that N trunks can carry at a fixed loss probability p [the maximum being over $A(u)$ that achieve p], is a number depending only on N and p.

Section IX is a brief discussion of $\Pr\{N(k) = N\}$, the chance that the kth arrival suffers loss, as a function of k. The case $N = 2$ is described in detail, and curves are included for one choice of $A(u)$.

Finally, Section X is devoted to the mean value $E\{N(k)\}$ of $N(k)$ as a function of k, and to the covariance function of $N(k)$ defined as

$$R(n) = \lim_{k\to\infty} E\{N(k)N(k+n)\} - E^2\{N(k)\}.$$

General formulas for both $E\{N(k)\}$ and $R(n)$ are derived, together with a recurrence relation for the latter to facilitate computation. The chief practical application of the covariance function is to theoretical estimates of sampling error in traffic measurement. Discussions of the use of our results to estimate sampling error in certain possible kinds of switch counting are given, together with some curves of the covariance. We stress that our results are for a finite, not an infinite, number of trunks. In particular, we show that a natural exponential approximation to the covariance, valid for $N = \infty$, can be several times too large for small N.

III. SUMMARY OF PRINCIPAL NOTATIONS AND DEFINITIONS

'E' is used to denote mathematical expectation

N = number of trunks in the group

γ = hang-up rate = (mean holding time)$^{-1}$

$A(u)$ = $\Pr\{\text{interarrival time} \leqq u\}$

μ_i = ith ordinary moment of the interarrival distribution $A(u)$

$$a_n = \int_0^\infty e^{-n\gamma u}\, dA(u),\ n = 1, 2, \cdots, N$$

$N(t)$ = number of trunks busy at t

t_k = epoch of the kth arrival

$N(k)$ = $N(t_k - 0)$ = number of trunks found busy by the kth arrival

$p_n = \lim_{k\to\infty} \Pr\{N(k) = n\}$ = equilibrium probability of finding n trunks busy

p_N = equilibrium probability of loss

$b_n = \sum_{m=n}^{N} \binom{m}{n} p_m$ = nth binomial moment of the distribution $\{p_m\}$

$M_{(n)} n! b_n = \sum n(n-1) \cdots (n-m+1) p_m$ = nth factorial moment of $\{p_m\}$

$m_n = \sum_m m^n\, p_m$ = nth ordinary moment of $\{p_m\}$

$\sigma^2 = m_2 - m_1^2$ = variance of $\{p_m\}$

$$P_u(x) = 1 + (x-1)e^{-\gamma u}$$

$$E\{x^{N(k)}\} = \sum_{m=0}^{N} x^m \Pr\{k\text{th call find } m \text{ trunks busy}\}$$

$$\varphi(x, z) = \sum_k z^k E\{x^{N(k)}\}$$

$$\psi_n(z) = \sum_k z^k \Pr\{N(k) = n\}$$

$$b_n(z) = \sum_{m=n}^{N} \binom{m}{n} \psi_m(z) = (n!)^{-1} \times \text{factorial moment generating function}$$

$$k_n = \sum_{m=n}^{N} \binom{m}{n} \Pr\{N(0) = m\} = n\text{th binomial moment of initial distribution}$$

$$D(a_1, a_2, \cdots, a_N, z) = \sum_{0}^{N} \binom{N}{i} (1 - za_1) \cdots (1 - za_i)\, a_{i+1} \cdots a_N\, z^{N-i}$$

$$L_k^{(N)} = \prod_{m=1}^{N} (1 - a_k + a_{k+m})$$

$$U_k^{(N)} = \prod_{m=o}^{N-1} (1 - a_{k+m} + a_{k+N})$$

$$f(x_1, x_2, \cdots, x_N) = 1 + \binom{N}{1}\frac{1 - x_1}{x_1} + \cdots + \binom{N}{N}\frac{(1-x_1)\cdots(1-x_N)}{x_1 \cdots x_N}$$

$$R(n) = \lim_{k\to\infty} E\{N(k)N(k + n)\} - E^2\{N(k)\} = \text{covariance function of } N(k)$$

$$Q_k = \sum_{m=0}^{N} mp_m \Pr\{N(k) = N \mid N(0) = m\}$$

IV. DERIVATION OF GENERATING FUNCTIONS

The behavior of a trunk group with (a) independent holding times, (b) independent interarrivals and (c) N trunks with lost calls cleared has been studied by Pollaczek[3], who derived the generating functions

$$\sum_k z^k \Pr\{k\text{th arrival finds } n \text{ trunks busy}\},$$

on the condition that the first arrival found all trunks idle.

Palm and Takács derived the limit probabilities

$$p_n = \lim_{k\to\infty} \Pr\{k\text{th arrival finds } n \text{ trunks busy}\}$$

for the case of exponential holding times, to which we are also limiting ourselves here. Takács used the equilibrium equations for the same Markov process $N(k)$ as we have introduced. We shall show that his functional equation approach can be used to generalize Pollaczek's results, and to obtain further formulas of practical importance in traffic engineering.

We let $P_u(x) = 1 + (x - 1)e^{-\gamma u}$. Then, by the argument of Takács,[4]

$$E\{x^{N(k+1)} \mid N(k)\} = \int_0^\infty [P_u(x)]^{1+N(k)-\delta_{N,N(k)}} \, dA(u),$$

with the δ symbol indicating that lost calls are cleared. Hence

$$E\{x^{N(k+1)}\} = \int_0^\infty \Big[\sum_{n<N} \Pr\{N(k) = n\} P_u^{1+n}(x) + \Pr\{N(k) = N\} P_u^{N}(x) \Big] dA(u).$$

Let

$$\varphi(x,z) = \sum_{k \geqq 0} z^k E\{x^{N(k)}\},$$

$$\psi_n(z) = \sum_{k \geqq 0} z^k \Pr\{N(k) = n\}, \qquad n = 0, 1, \cdots, N.$$

Then φ satisfies the functional equation

$$\varphi(x, z) = E\{x^{N(0)}\} + z \int_0^\infty \Big\{ \varphi[P_u(x), z] P_u(x) - \psi_N(z)[P_u^{N+1}(x) - P_u^{N}(x)] \Big\} dA(u). \tag{1}$$

This is a discrete time-dependent analog of Takács' functional equation. To solve it, set $x = 1 + w$ and define the functions b_n by

$$b_N(z) = \sum_{m=n}^{N} \binom{m}{n} \psi_m(z), \qquad n = 0, 1, \cdots, N.$$

Note that

$$b_n(z) = \psi_N(z), \tag{2}$$

$$\varphi(x, z) = \sum_{n=0}^{N} x^n \psi_n(z). \tag{3}$$

If we now equate coefficients of like powers of w in the functional equation (1), we obtain the following recurrence for the functions $b_n(z)$:

$$b_n(z) = z a_n \Big[b_n(z) + b_{n-1}(z) - \binom{N}{n-1} \psi_N(z) \Big] + k_n, \quad n \geqq 1, \tag{4}$$

where

$$k_n = \sum_{m=n}^{N} \binom{m}{n} \Pr\{N(0) = m\},$$

$$a_n = \int_0^\infty e^{-n\gamma u} \, dA(u).$$

The terms k_n are the binomial moments of the distribution of $N(0)$, and represent initial conditions. Since

$$\sum_{n=0}^{N} \Pr\{N(k) = n\} = 1$$

for each $k \geqq 0$, we find $b_0(z) = (1 - z)^{-1}$.

The solution of the recurrence (4) is

$$b_n(z) = \prod_0^N \frac{za_j}{1 - za_j} \cdot \left\{(1 - z)^{-1} - \sum_{j=1}^{n} \left[\binom{N}{j - 1} b_N(z) - \frac{k_j}{za_j}\right] \prod_0^{j-1} \frac{1 - za_i}{za_i}\right\}, \tag{5}$$

where the first term of the products is always taken to be 1. From this and (2) one can determine $b_N(z)$ and hence all the $\psi_n(z)$. The complete result is

Theorem 1: *The generating function* $\psi_n(z)$ *of* $\Pr\{N(k) = n\}$, *defined by*

$$\psi_n(z) = \sum_{k \geqq 0} z^k \Pr\{N(k) = n\}$$

is given by the formula

$$\psi_n(z) = \sum_{j=0}^{N-n} (-1)^j \binom{n + j}{n} b_{n+j}(z),$$

where the $b_n(z)$ *are solutions of* (4). *In particular, the generating function of the probabilities that the kth arrival find all N trunks busy is*

$$\psi_N(z) = b_N(z) = \sum_{k \geqq 0} z^k \Pr\{N(k) = N\}$$

$$= (1 - z)^{-1} \frac{k_0 + \dfrac{k_1(1 - z)}{za_1} + \cdots + \dfrac{k_N(1 - z)(1 - za_1) \cdots (1 - za_{N-1})}{z^N a_1 a_2 \cdots a_N}}{1 + \dbinom{N}{1}\dfrac{1 - za_1}{za_1} + \cdots + \dbinom{N}{N}\dfrac{(1 - za_1)\cdots(1-za_N)}{z^N a_1 a_2 \cdots a_N}}.$$

This reduces to Pollaczek's result (Ref. 3, p. 1470) when the system starts empty with $N(0) = 0$, since $k_0 \equiv 1$, and $N(0) = 0$ implies that $k_i = 0$ for $i > 0$. Let us set

$$D_N(x_1, x_2, \cdots x_N, z) = \sum_{j=0}^{N} \binom{N}{j} (1 - zx_1) \cdots (1 - zx_j) x_{j+1} \cdots x_N z^{N-j}.$$

In this notation we can write

$$D_N(a_1, a_2, \cdots a_N, z) = a_1 a_2 \cdots a_N z^N \text{ [denominator of } \psi_N(z)].$$

Lemma 1: *The functions* $D_N(x_1, x_2, \cdots, x_N, z)$ *satisfy the recurrence relations*

$$D_{N+1}(x_k, \cdots, x_{k+N}, z) = z x_{k+N} D_N(x_k, \cdots, x_{k+N-1}, z) + (1 - z x_k) D_N(x_{k+1}, \cdots, x_{k+N}, z).$$

Proof of this is from the formula

$$\binom{N+1}{i} = \binom{N}{i} + \binom{N}{i-1}.$$

V. THE STATIONARY DISTRIBUTION

In the terminology of Feller,[7] the variates $N(k)$ form an aperiodic, irreducible Markov chain; hence the limits

$$p_n = \lim_{k\to\infty} \Pr\{N(k) = n\}$$

exist, and can be evaluated from the generating functions $\psi_n(z)$ by Abel's theorem. The result is

Theorem 2: *The stationary distribution of* $N(k)$ *is* $\{p_n\}$, *given by*

$$p_n = \sum_{j=0}^{N-n} (-1)^j \binom{n+j}{n} b_{n+j},$$

with $b_0 = 1$, *and*

$$b_n = \prod_0^n \frac{a_j}{1 - a_j} \left\{1 - p_N \sum_{m=1}^{n} \binom{N}{m-1} \prod_0^{m-1} \frac{1 - a_i}{a_i}\right\}, \tag{6}$$

$$p_N = \text{probability of loss} = \frac{a_1 a_2 \cdots a_N}{D_N(a_1, a_2, \cdots, a_N, 1)}$$
$$= \left\{1 + \binom{N}{1}\frac{1 - a_1}{a_1} + \cdots + \binom{N}{N}\frac{(1 - a_1)\cdots(1 - a_N)}{a_1 a_2 \cdots a_N}\right\}^{-1}. \tag{7}$$

Theorem 2, and the loss formula (7) are due to Palm[1] and Pollaczek;[3] these results have been rederived independently by L. Takács, H. Scarf, the present author — and doubtless many others.

The quantities b_n of Theorem 2 are the binomial moments of $\{p_n\}$, defined as

$$b_n = \sum_{m=n}^{N} \binom{m}{n} p_m,$$

and they satisfy the recurrence

$$b_0 = 1,$$

$$b_n = a_n \left[b_n + b_{n-1} - p_N \binom{N}{n-1} \right], \qquad n > 0,$$

which can be solved to give formulas (6) and (7). The factorial moments $M_{(n)}$ are then given by

$$M_{(n)} = n!\, b_n = \sum_{m=n}^{N} m(m-1) \cdots (m-n+1) p_m ,$$

and they satisfy the recurrence

$$M_{(0)} = 1,$$

$$M_{(n)} = a_n[M_{(n)} + M_{(n-1)} - np_N N(N-1) \cdots (N-n+2)], \qquad n \geqq 1.$$

In Fig. 1, the probability p_N of loss has been plotted as a function of the average offered load, a, in erlangs, for three separate choices of the interarrival distribution $A(u)$, for values of N from 1 to 8. The choices have been intentionally made so that the crucial quantities a_n depend on γ and $A(u)$ only via the offered load, a. The choices are as follows:

i. Poisson arrivals are represented in Fig. 1 by a dashed line. In this case, $a_n = a/(a + n)$.

ii. Suppose that the times between successive arrivals are uniformly distributed in the interval $(\mu_1 - b, \mu_1 + b)$ for $0 < b \leqq \mu_1$. The mean interarrival time is μ_1, and a simple calculation gives

$$a_n = e^{-n\gamma\mu_1} \frac{\sinh n\gamma b}{n\gamma b} . \tag{8}$$

We choose $b = \mu_1$; then a_n depends only on $\gamma\mu_1 = a^{-1}$, and

$$a_n = e^{-n/a} \frac{\sinh n/a}{n/a} .$$

This choice of $A(u)$ we shall call "uniformly distributed interarrivals;" it is represented in Fig. 1 by alternating long and short dashes.

iii. Regular arrivals are represented in Fig. 1 by a solid line. For regular arrivals, $a_n = e^{-n/a}$, which is the limiting form of (8) as b tends to zero.

The curve for regular arrivals ($a_n = e^{-n/a}$) always falls below the curves

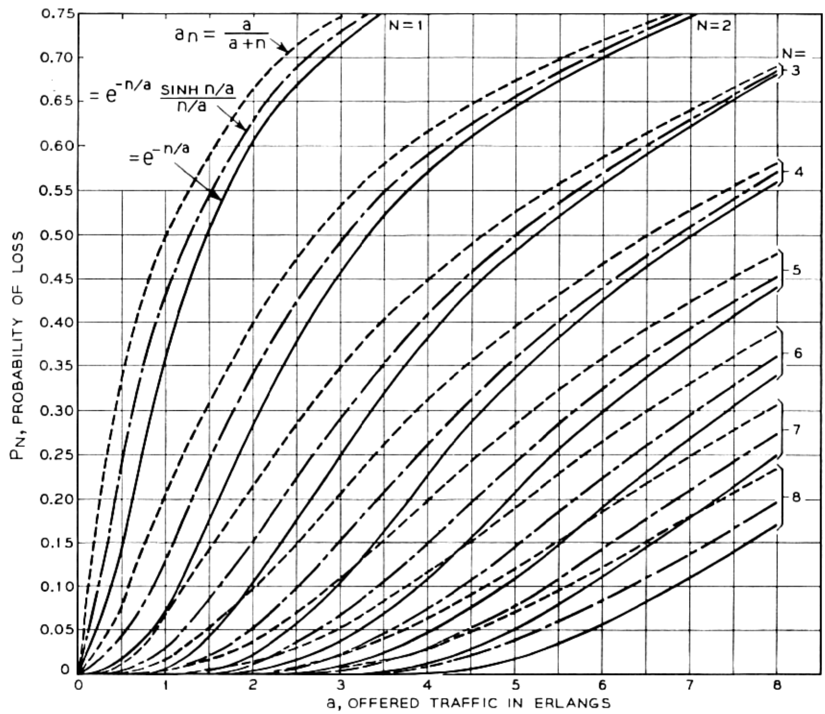

Fig. 1 — The probability of loss: (i) Poisson arrivals, $a_n = a/(a + n)$, dashed line; (ii) uniformly distributed interarrivals, $a_n = e^{-n/a} (\sinh n/a)/(n/a)$, long-and-short dashed line; (iii) regular arrivals, $a_n = e^{-n/a}$, solid line.

for the other two choices. This is a consequence of Theorem 9 of Section VIII, according to which regular arrivals form a limiting best case, for which p_N assumes its lower bound for fixed offered traffic a. On the other hand, the curve for Poisson arrivals, although always above the curves for the other two choices in Fig. 1, is by no means the limiting worst case, since there is none. For Theorem 10 of Section VIII says that, for given $\epsilon > 0$ and offered traffic a, we can always find an interarrival distribution $A(u)$ for which $p_N > 1 - \epsilon$.

The differences in p_N for the various choices of $A(u)$ in Fig. 1 are possibly explainable by considering the amount of mass that $A(u)$ concentrates in the neighborhood of 0. For regular arrivals there is no mass, so that the system always has a "breathing spell" before the next arrival. For uniformly distributed interarrivals, there is always mass in a neighborhood of zero, but the density at 0 is no larger than anywhere else. For Poisson arrivals, however, not only is there mass in any neigh-

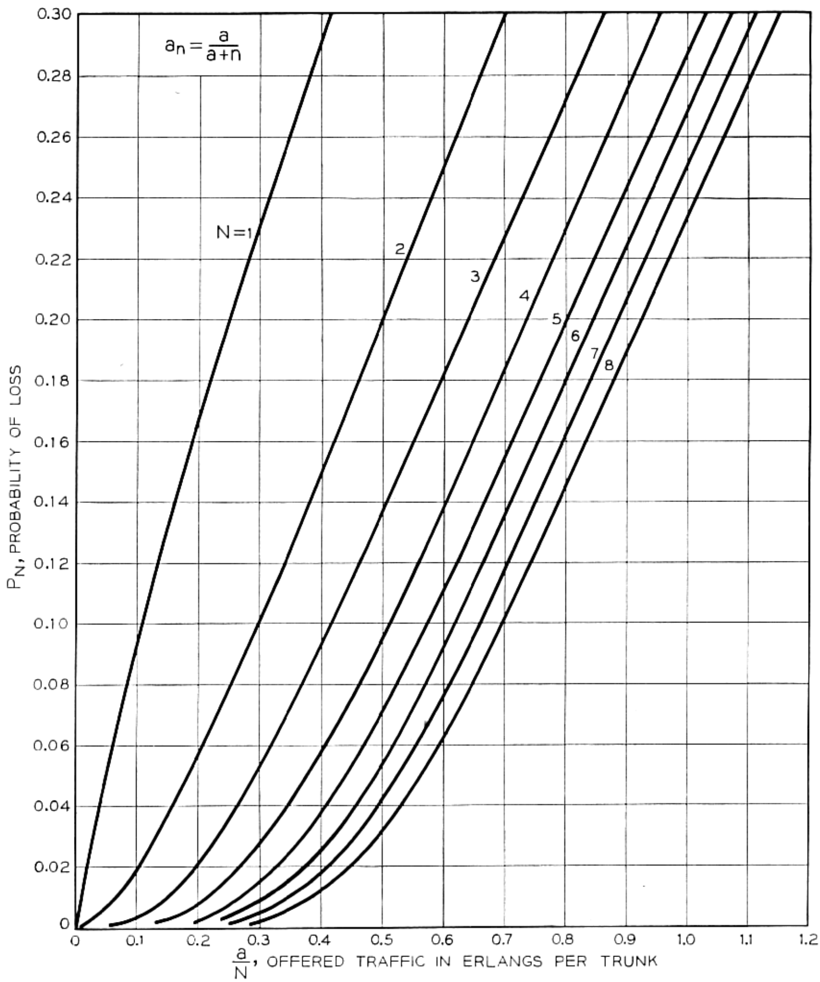

Fig. 2 — The probability of loss as a function of load per trunk for Poisson arrivals, $a_n = a/(a + n)$.

borhood of 0, but the density is a maximum at 0, so that the damaging short interarrivals are, in a sense, the most likely.

From Theorem 13 and the Palm formula (7) it can be verified that, as $a \to \infty$, the curves for the different choices of $A(u)$ must approach each other and 1. But for small values of a there are substantial differences among them. For this reason, they have been replotted in the separate Figs. 2, 3 and 4 as functions of a/N, the offered load per trunk.

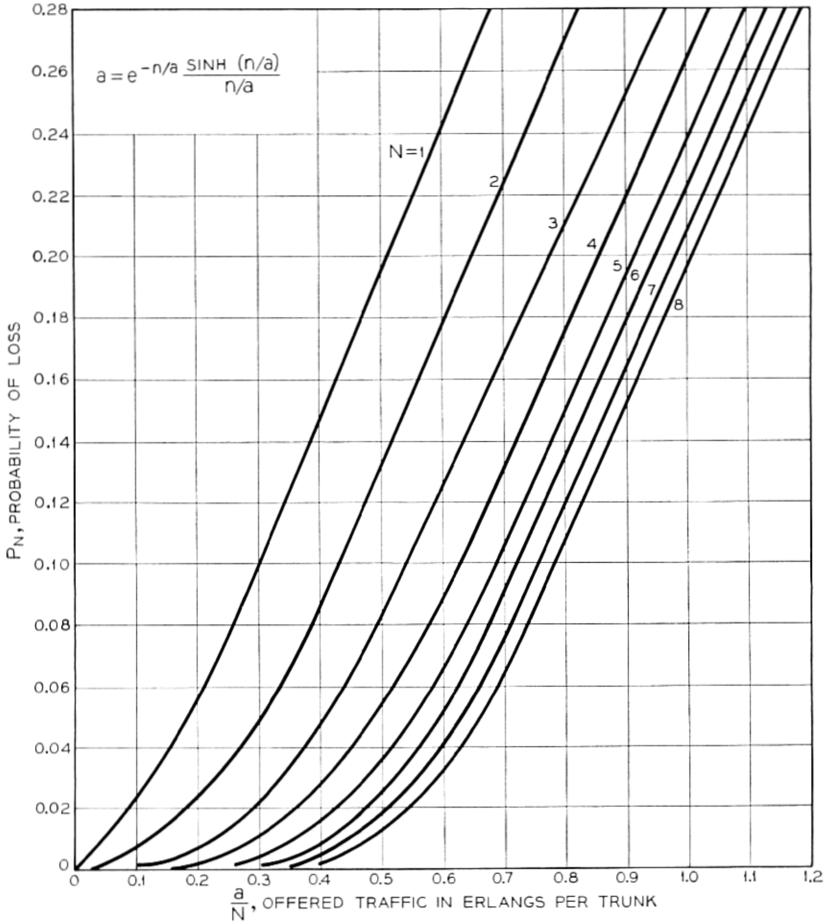

Fig. 3 — The probability of loss as a function of load per trunk for uniformly distributed interarrivals, $a_n = e^{-n/a}$ (sinh $n/a)/(n/a)$.

The first two ordinary moments m_1 and m_2 of $\{p_n\}$ are respectively given by

$$m_1 = M_{(1)} = b_1 = \sum_{n=0}^{N} np_n = \frac{a_1(1 - p_N)}{1 - a_1},$$

$$m_2 = M_{(2)} + M_{(1)} = 2b_2 + b_1 = \sum_{n=0}^{N} n^2 p_n$$

$$= \frac{a_1 a_2(1 - p_N)}{(1 - a_1)(1 - a_2)} - \frac{2a_2 N p_N - m_1}{1 - a_2}, \qquad \text{for } N > 1,$$

$$= a_1, \qquad \text{for } N = 1.$$

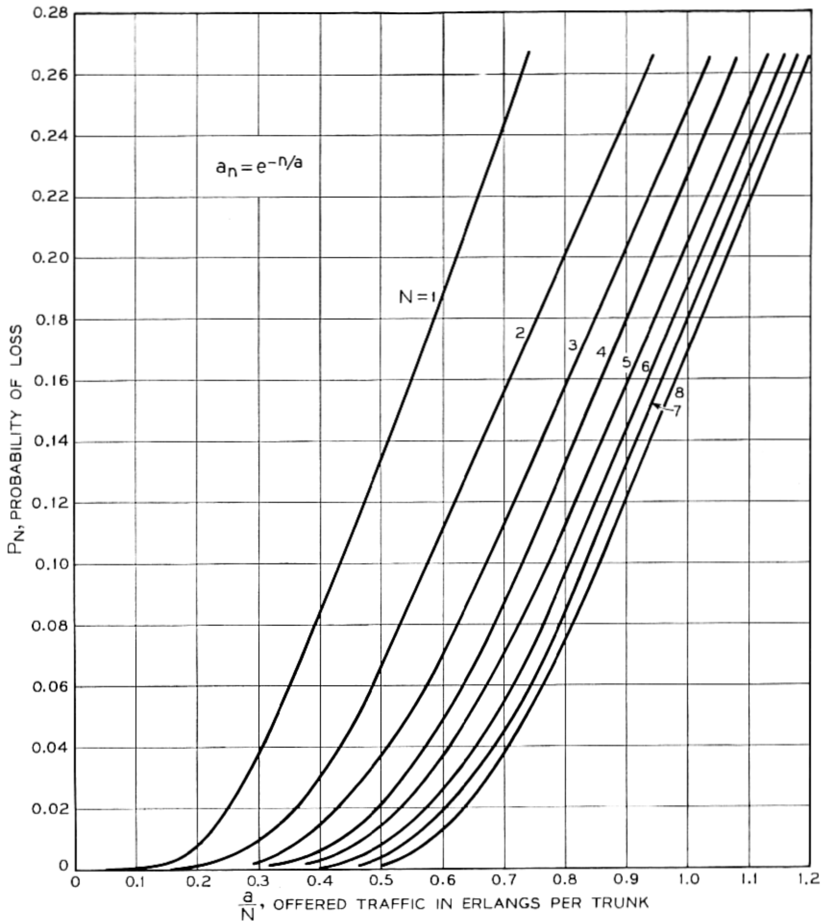

Fig. 4 — The probability of loss as a function of load per trunk for regular arrivals, $a_n = e^{-n/a}$.

The variance associated with $\{p_n\}$ is then

$$\sigma^2 = R(0) = m_2 - m_1^2 = 2b_2 + b_1 - b_1^2,$$

where $R(n)$ is the covariance function.

Because of the bias introduced by defining $N(k)$ to be the number of busy trunks found by the kth arriving customer, it is not in general true that m_1 equals $\lim E\{N(t)\}$ as t tends to ∞, even when this limit exists. In Fig. 5, the ratio

$$\frac{m_1}{N} = \text{fraction trunks found busy}$$

$$= \frac{\text{expected number found busy by an arrival}}{\text{number of trunks}}$$

is plotted as a function of offered load a for Poisson arrivals. In Figs. 6 and 7 the same ratio is plotted for uniformly distributed interarrivals, and regular arrivals, respectively.

In Figs. 8, 9 and 10, the variance σ^2 of $N(k)$ in equilibrium is shown plotted against the offered load a for Poisson arrivals, uniformly distributed interarrivals and regular arrivals, respectively. The variance is also the value of the covariance function $R(n)$ for $n = 0$. In all cases, as the load a increases, the variance increases to a unique maximum, and then decreases to zero.

VI. BOUNDS FOR, AND APPROXIMATIONS TO, p_N FOR GIVEN $a_1, \cdots, a_N$

This section is devoted to inequalities which may be useful in estimating the loss probability p_N without too much computation. Since $1 > a_1 > \cdots > a_N$, we have

$$\frac{1 - a_n}{a_n} < \frac{1 - a_{n+1}}{a_{n+1}},$$

so that, from (7), we find

$$\sum_0^N \binom{N}{j} \left(\frac{1 - a_1}{a_1}\right)^j \leqq p_N^{-1} \leqq \sum_0^N \binom{N}{j} \left(\frac{1 - a_N}{a_N}\right)^j.$$

This proves:

Theorem 3: The probability p_N of loss satisfies $(a_N)^N \leqq p_N \leqq (a_1)^N$.

To obtain a sharper result, write

$$p_N^{-1} = (a_1 a_2 \cdots a_N)^{-1} \sum_0^N \binom{N}{j} (1 - a_1) \cdots (1 - a_j) a_{j+1} \cdots a_N .$$

Then, in view of $1 > a_1 > \cdots > a_N$,

$$\sum \binom{N}{j} (1 - a_1)^j (a_N)^{N-j} \leqq \frac{a_1 a_2 \cdots a_N}{p_N} \leqq \sum \binom{N}{j} (1 - a_N)^j (a_1)^{N-j}.$$

From this we conclude:

Theorem 4: The probability p_N of loss satisfies

$$(1 - a_1 + a_N)^{-N} \leqq \frac{p_N}{a_1 a_2 \cdots a_N} \leqq (1 + a_1 - a_N)^{-N}.$$

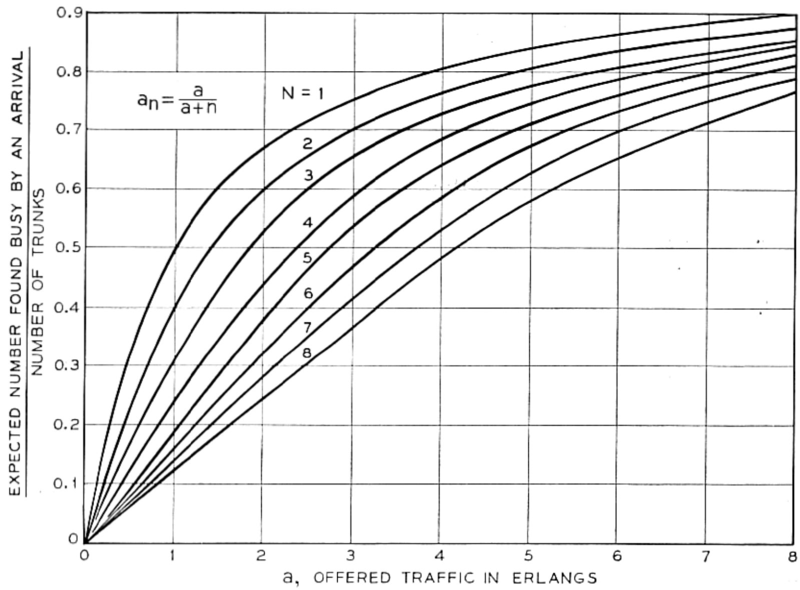

Fig. 5 — $\lim_{k\to\infty} E\{N(k)\}/N = m_1/N$ as a function of offered traffic a for Poisson arrivals.

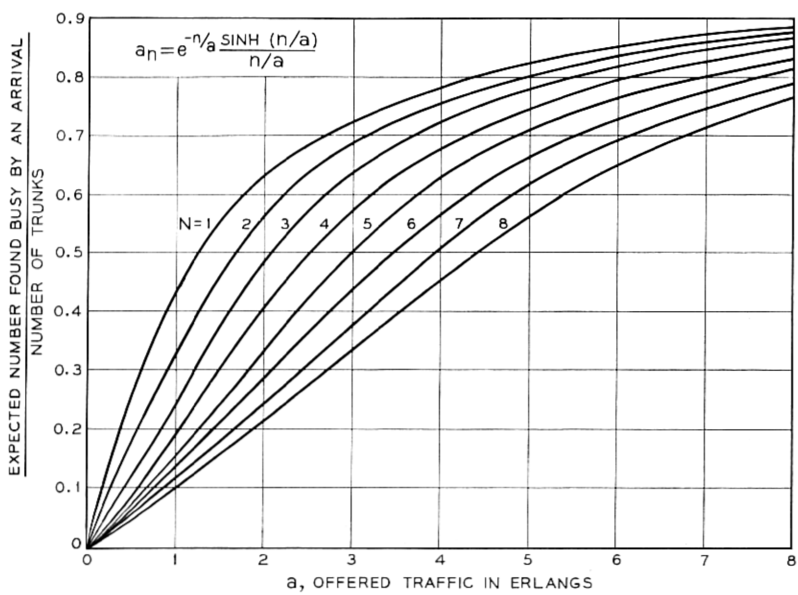

Fig. 6 — $\lim_{k\to\infty} E\{N(k)\}/N = m_1/N$ as a function of offered traffic a for uniformly distributed interarrivals.

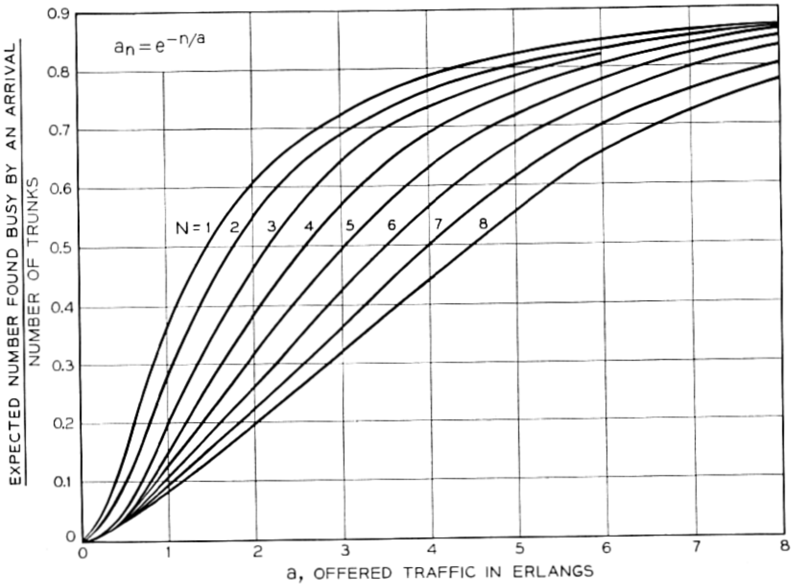

Fig. 7 — $\lim_{k\to\infty} E\{N(k)\}/N = m_1/N$ as a function of offered traffic a for regular arrivals.

This result suggests that, if $a_1 - a_N$ is sufficiently small, then the product $a_1a_2 \cdots a_N$ can serve as an approximation to p_N. There are cases, to be exemplified later, in which this is a good approximation. However, the next theorem shows that the product $a_1a_2 \cdots a_N$ always *underestimates* the loss.

Theorem 5: *For* $N = 1$, $p_N = a_1$; *for* $N \geqq 2$, $p_N > a_1a_2 \cdots a_N$.* To prove this, we write p_N in the notation of Lemma 1 as

$$p_N = \frac{a_1a_2 \cdots a_N}{D_N(a_1, \cdots, a_N, 1)},$$

so that it suffices to prove that $D_N(a_1, \cdots, a_N, 1) < 1$. We shall actually prove the stronger result that $D_N(a_k, \cdots, a_{k+N-1}, 1) < 1$ for $k \geqq 1$. First we note

$$\begin{aligned} D_2(a_k, a_{k+1}, 1) &= a_ka_{k+1} + 2(1 - a_k)a_{k+1} + (1 - a_k)(1 - a_{k+1}) \\ &= 1 - a_k + a_{k+1} < 1. \end{aligned}$$

Now, because $1 > a_1 > \cdots > a_k > \cdots$, we find

$$\frac{D_N(a_k, \cdots, a_{k+N-1}, 1)}{a_ka_{k+1} \cdots a_{k+N-1}} < \frac{D_N(a_{k+1}, \cdots, a_{k+N}, 1)}{a_{k+1}a_{k+2} \cdots a_{k+N}}.$$

* A. J. Goldstein has pointed out that Theorem 4 implies directly that $p_N > a_1a_2 \cdots a_N$ for $N \geqq 2$.

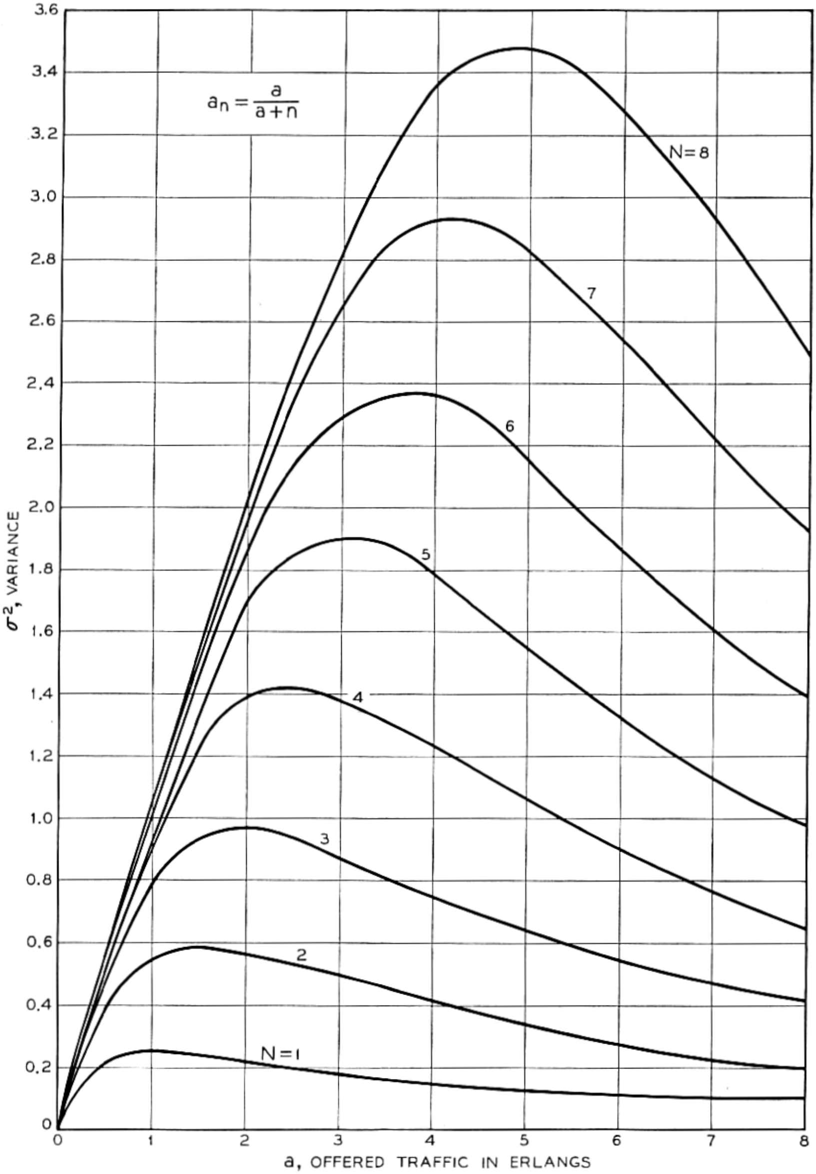

Fig. 8 — The variance σ^2 [= $R(O)$] of $N(k)$ in equilibrium for Poisson arrivals.

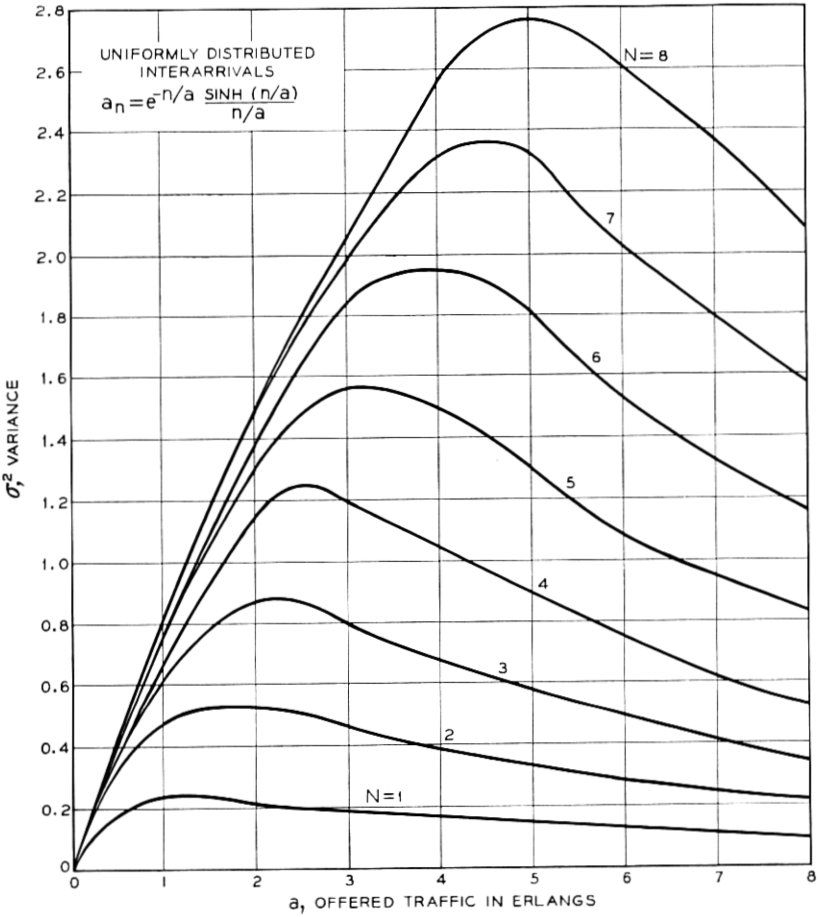

Fig. 9 — The variance σ^2 [$= R(O)$] of $N(k)$ in equilibrium for uniformly distributed interarrivals.

Therefore, the recurrence of Lemma 1 gives, for $z = 1$,

$$D_{N+1}(a_k, \cdots, a_{k+N}, 1) < D_N(a_{k+1}, \cdots, a_{k+N}, 1) < 1,$$

and the result follows by induction.

We now discuss the approximation $p_N \sim a_1 a_2 \cdots a_N$. Since $1 > a_1 > a_N$, two cases in which $a_1 - a_N$ is small are as follows: (a) a_1 is close to 0 and p_N is very small; (b) a_N is close to 1 and p_N is very high. The quantity $a_1 - a_N$ determines the excellence of the approximation, as meas-

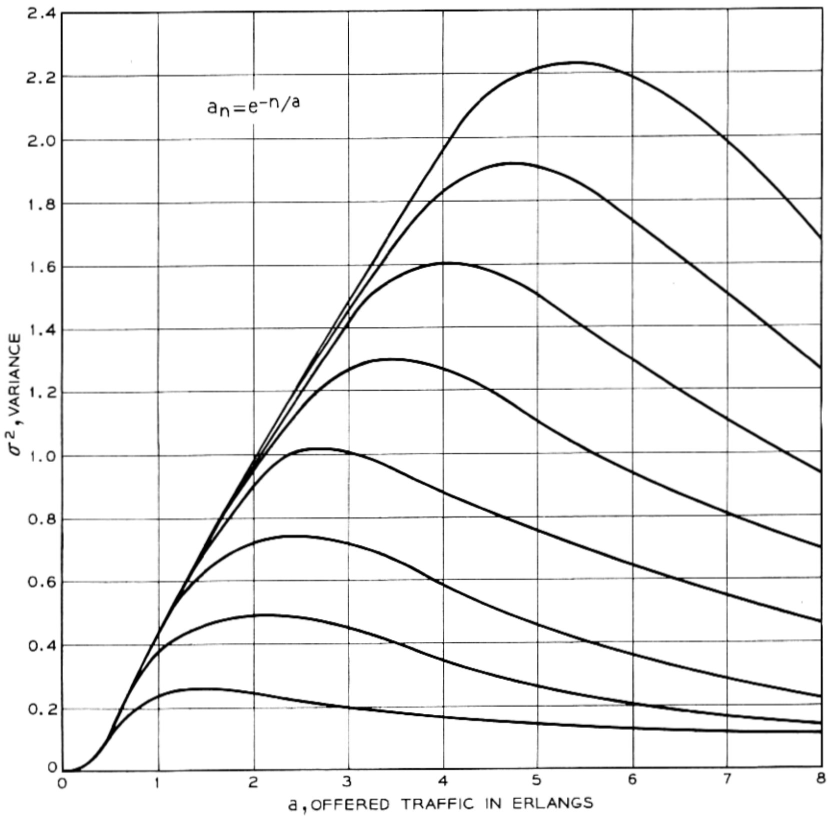

Fig. 10 — The variance σ^2 [$= R(O)$] of $N(k)$ in equilibrium for regular arrivals.

ured by Theorem 4. The value of $a_1 - a_N$ may be estimated from below in terms of a_1 alone by the inequality $a_1 - a_N \geqq a_1 - a_1^N$. From Theorems 4 and 5 we see that

$$(1 - a_1 + a_N)^N \leqq r = \frac{a_1 a_2 \cdots a_N}{p_N} < 1,$$

and this inequality indicates the values of $a_1 - a_N$ for which $p_N \sim a_1 a_2 \cdots a_N$ is justified.

To put the matter more intuitively, we note that a_1 is the chance that a conversation, in progress at one arrival epoch, is still in progress at the next arrival epoch, i.e.,

$$a_1 = \Pr\{\text{holding time} > \text{interarrival time}\}.$$

Similarly, if $h_1, \cdots, h_N$ are N (independent) holding times,

$$a_N = \Pr\left\{\min_{1 \leqq j \leqq N} h_j > \text{interarrival time}\right\}.$$

So the approximation is likely to be good at least when the chance that one holding time exceeds an interarrival time is not very different from the chance that each of N holding times exceeds an interarrival time (the same one for all N). As a tentative conclusion we may say that $p_N \sim a_1 a_2 \cdots a_N$ is good when the loss is very high or very low.

The ratio $r = a_1 a_2 \cdots a_N / p_N$ has been plotted as a function of the average offered traffic a in Figs. 11, 12 and 13 for Poisson arrivals, uniformly distributed interarrivals and regular arrivals, respectively. The curves bear out the conclusions of the previous paragraph, that the

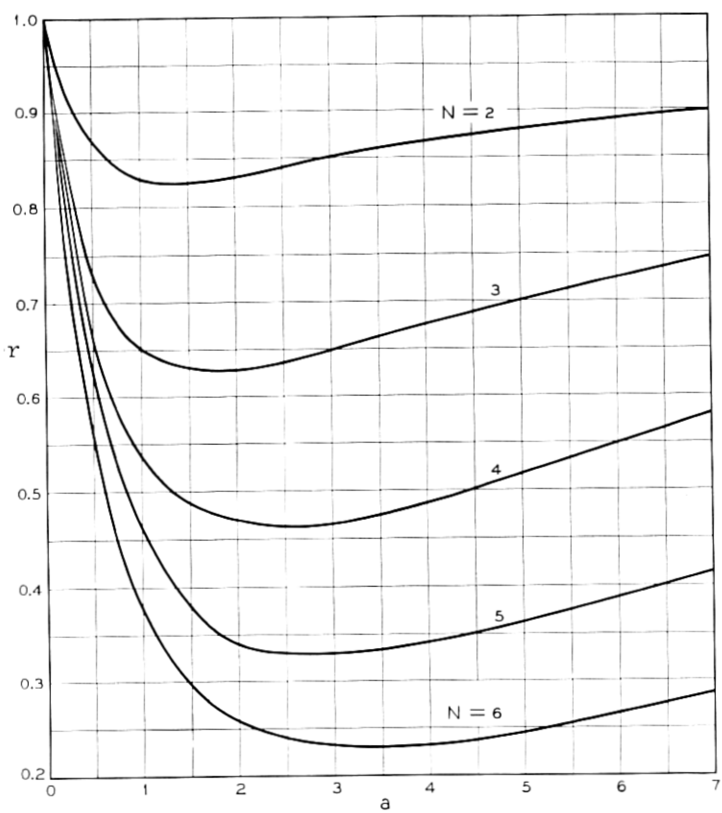

Fig. 11 — The ratio $r = (a_1 a_2 \cdots a_N)/p_N$ as a function of traffic a for Poisson arrivals.

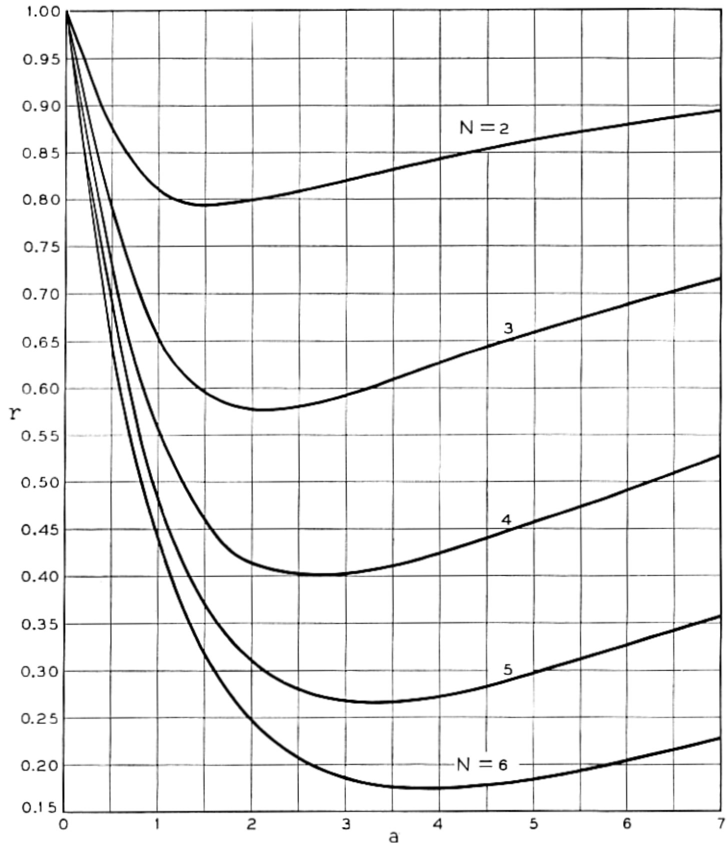

Fig. 12 — The ratio $r = (a_1a_2 \cdots a_N)/p_N$ as a function of traffic a for uniformly distributed interarrivals.

approximation $p_N \sim a_1a_2 \cdots a_N$ is good for low and high traffic. Fig. 14 shows a detail of r for very low traffic, for all cases at once.

Lemma 2: For $m, k \geqq 1$, $a_{k+1} + a_{k+m} \leqq a_k + a_{k+m+1}$.

Proof: the case $m = 1$ holds by convexity; for the same reason,

$$a_k + a_{k+2} \geqq 2a_{k+1} .$$

Assume that the lemma holds for a given m and all $k \geqq 1$. Then

$$a_{k+2} + a_{k+1+m} \leqq \frac{a_k + a_{k+2}}{2} + a_{k+2+m} ,$$

$$a_{k+1} + a_{k+m+1} \leqq a_{k+1} + \frac{a_k - a_{k+2}}{2} + a_{k+m+2} .$$

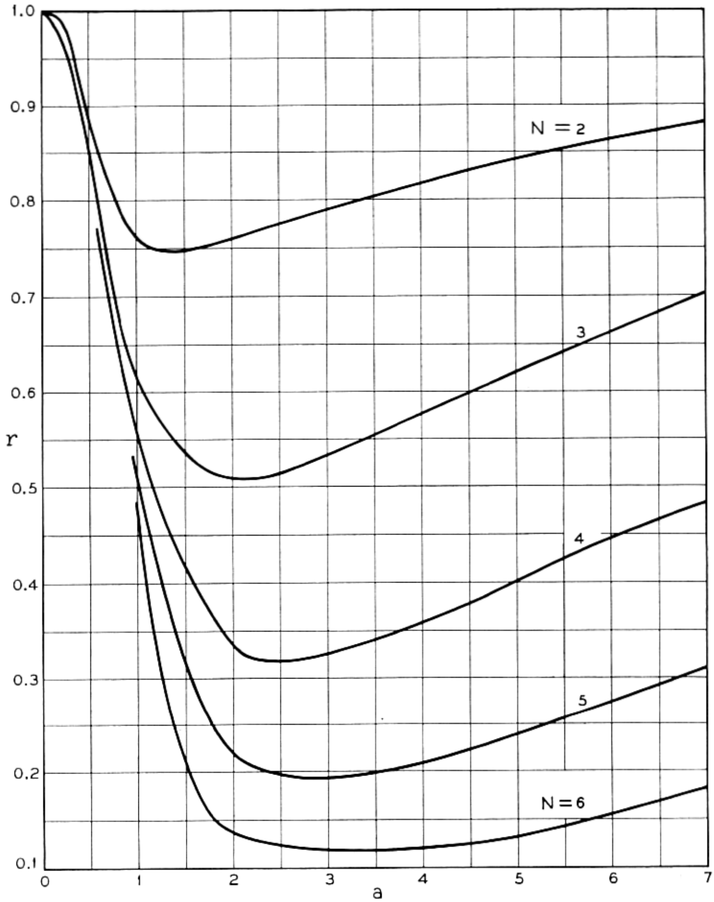

Fig. 13 — The ratio $r = (a_1 a_2 \cdots a_N)/p_N$ as a function of traffic a for regular arrivals.

But $a_k + a_{k+2} \geqq 2a_{k+1}$ implies

$$a_{k+1} + \frac{a_k - a_{k+2}}{2} \leqq a_k,$$

so the lemma follows by induction.

Theorem 6: *Let*

$$L_k^{(N)} = \prod_{m=1}^{N} [1 - a_k + a_{k+m}],$$

$$U_k^{(N)} = \prod_{j=0}^{N-1} [1 - a_{k+j} + a_{k+N}].$$

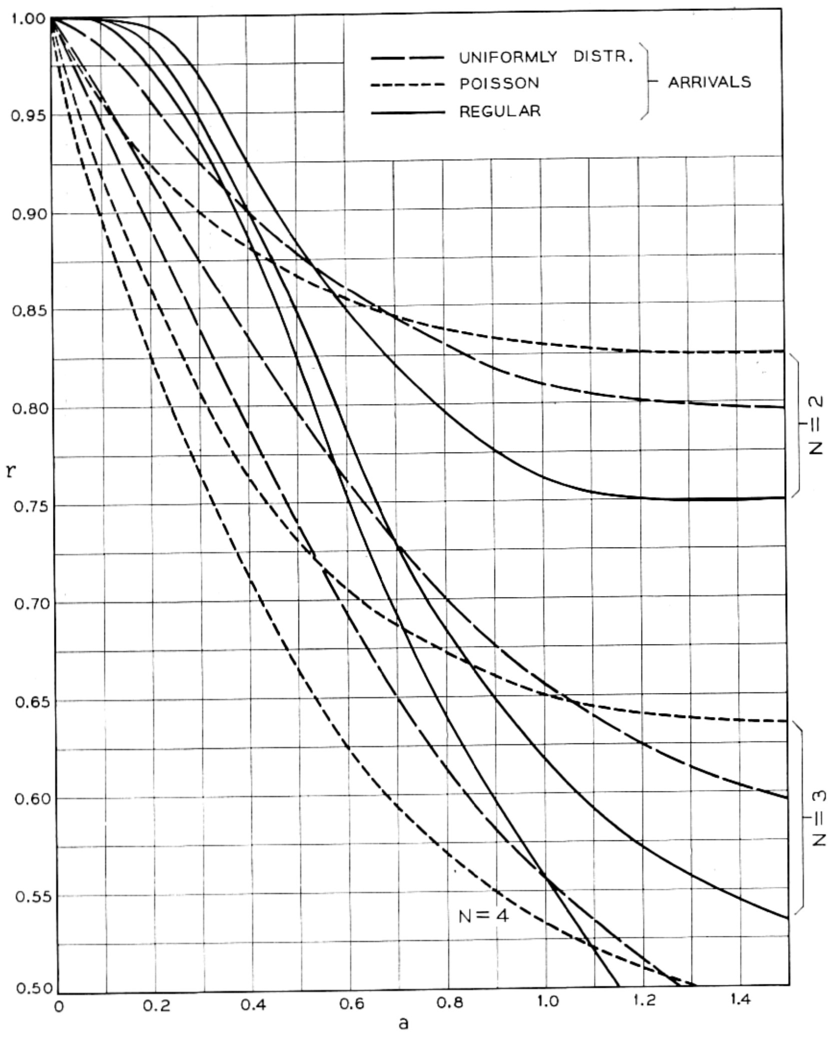

Fig. 14 — Detail of $r = (a_1 a_2 \cdots a_N)/p_N$ for low traffic a for Poisson arrivals, uniformly distributed interarrivals and regular arrivals.

Then

$$L_k^{(N-1)} \leqq D_N(a_k\,,\,\cdots\,,\,a_{k+N-1}\,,\,1) \leqq U_k^{(N-1)}$$

and, also, the chance of loss satisfies

$$\frac{a_1a_2\cdots a_N}{U_1^{(N-1)}} \leqq p_N = \frac{a_1a_2\cdots a_N}{D_N(a_1\,,a_2\,,\cdots,a_N\,,1)} \leqq \frac{a_1a_2\cdots a_N}{L_1^{(N-1)}}.$$

Proof: For $N = 2$, we have for $k \geqq 1$:

$$D_2(a_k\,,\,a_{k+1}\,,\,1) = 1 - a_k + a_{k+1} = U_k^{(1)} = L_k^{(1)}.$$

Now, assume that for all $k \geqq 1$

$$L_k^{(N-1)} \leqq D_N(a_k\,,\,\cdots\,,\,a_{k+N-1}\,,\,1) \leqq U_k^{(N-1)}.$$

Then, by Lemma 1,

$$a_{k+N}L_k^{(N-1)} + (1 - a_k)L_{k+1}^{(N-1)} \leqq D_{N+1}(a_k\,,\,\cdots\,,\,a_{k+N}\,,\,1)$$
$$\leqq a_{k+N}U_k^{(N-1)} + (1 - a_k)U_{k+1}^{(N-1)}.$$

By convexity and Lemma 2,

$$L_{k+1}^{(N-1)} \geqq L_k^{(N-1)},$$
$$U_{k+1}^{(N-1)} \geqq U_k^{(N-1)}.$$

Therefore

$$L_k^{(N-1)}(1 - a_k + a_{k+N}) \leqq D_{N+1}(a_k\,,\,\cdots\,,a_{k+N}\,,1)$$
$$\leqq (1 - a_k + a_{k+N})U_{k+1}^{(N-1)}.$$

But

$$(1 - a_k + a_{k+N})L_k^{(N-1)} = L_k^{(N)},$$
$$(1 - a_k + a_{k+N})U_{k+1}^{(N-1)} = U_k^{(N)},$$

so the theorem follows by induction.

VII. BOUNDS AND APPROXIMATIONS WHEN ARRIVALS ARE REGULAR

In telephony, it is unrealistic to expect regular arrivals. Nevertheless, the results of Section VIII indicate that regularity of arrivals represents in a definite sense a limiting best case, for which the loss assumes a lower bound. For this reason we devote some effort to approximating the loss p_N in this case.

For regular arrivals the loss p_N is given by

$$(p_N)^{-1} = \sum_j \binom{N}{j}\frac{1-x}{x}\,\frac{1-x^2}{x^2}\cdots\frac{1-x^j}{x^j},$$

where $x = \exp\{-(1/a)\}$ and a = offered erlangs. A simple procedure for obtaining an upper bound on p_N is as follows: we note that, since $x < 1$,

$$(p_N)^{-1} = \sum_j \binom{N}{j} (1 - x)^j (1 + x^{-1})(1 + x^{-1} + x^{-2}) \cdots \left(\sum_0^{j-1} x^{-1}\right)$$

$$\geqq \sum_j \binom{N}{j} (1 - x)^j j! = N!(1 - x)^N \sum_j \frac{(1 - x)^{j-N}}{(N - j)!}.$$

The term on the right of the last inequality is seen to be the reciprocal of $B[N, 1/(1 - x)]$, where $B(c, a)$ is the classical Erlang B function; that is,

$$B(c, a) = \frac{\dfrac{a^{c/N}}{(c/N)!}}{\displaystyle\sum_{j=0}^{c/N} \frac{a^j}{j!}} = \frac{e^{-a}}{1 - P(c + 1, a)},$$

where $P(c, a)$ is the cumulative term $\sum_{n \geq c} a^n e^{-a}/n!$ of the Poisson distribution. This proves:

Theorem 7: If arrivals are regular and a erlangs are offered, then $p_N \leqq B(N, \eta)$, *where B is Erlang's function, and* $\eta = (1 - x)^{-1} = (1 - e^{-1/a})^{-1}$.

From Theorem 9 of Section VIII we know that $p_N \leqq B(N, a)$; that is, we overestimate the loss for regular arrivals if we pretend that arrivals are Poisson. Let us therefore see whether the bound of Theorem 7 is better than $B(N, a)$. Let $a = (1 - \zeta)^{-1}$, so that $\eta = (1 - e^{\zeta-1})^{-1}$. Now ζ is tangent to $e^{\zeta-1}$ at $\zeta = 1$, i.e., at $a = \infty$, and $e^{\zeta-1}$ is convex; hence $e^{\zeta-1} \geqq \zeta$, and $1 - \zeta \geqq 1 - e^{\zeta-1}$, so that

$$a = (1 - \zeta)^{-1} < (1 - e^{\zeta-1})^{-1} = \eta$$

for finite a. Since B is monotone increasing in the offered erlangs we conclude that $B(N, a) < B(N, \eta)$. Thus the bound of Theorem 7 is nowhere as good as the overestimate $B(N, a)$ for p_N.

However, there is a systematic way of obtaining a useful upper bound on p_N for regular arrivals. This bound again has the functional form of Erlang's formula $B(N, \eta)$. However, η, instead of being chosen equal to a, is chosen to correspond to a Poisson process, which gives the right value of a_1, $\exp\{-(1/a)\}$, and involves fewer offered erlangs $\eta < a$. Now

$$a_1 = \begin{cases} e^{-1/a} \text{ for regular arrivals at } a \text{ erlangs} \\ \eta/(1 + \eta) \text{ for Poisson arrivals at } \eta \text{ erlangs.} \end{cases}$$

So η erlangs will give the right value of a_1 if and only if

$$\eta = \frac{y}{1 - y} \quad \text{for} \quad y = e^{-1/a}.$$

We first show that, if η is defined in this way, then $\eta < a$. For $u > 0$, we have $u + 1 < e^u$, so that for $u = a^{-1}$ we find

$$e^{-1/a} < a(1 - e^{-1/a}),$$

$$\eta = \frac{y}{1 - y} < a \quad \text{for} \quad y = e^{-1/a}.$$

For this choice of η, then, $B(N, \eta) < B(N, a)$. Now, from formula (7), it is apparent that if the a_i are replaced term by term with quantities a_i', with $a_i \leqq a_i'$, the result will be $\geqq p_N$. We choose

$$a_i' = \eta/(\eta + i), \qquad i = 1, 2, \cdots, N.$$

The a_i' correspond to Poisson arrivals with η erlangs offered. To obtain a bound it remains to be shown that, for $i = 2, 3, \cdots, N$,

$$a_i = e^{-i/a} \leqq a_i' = \frac{\eta}{\eta + i}.$$

This is equivalent to

$$y + i = iy \leqq y^{1-i}, \quad \text{for} \quad y = e^{-1/a},$$

which is seen to be true because $y + i - iy$ is tangent to y^{1-i} at $y = 1$. The result of replacing a_i by the chosen a_i' is just $B(N, \eta)$. This proves:

Theorem 8: *If arrivals are regular and a erlangs are offered, then* $p_N \leqq B(N, \eta) < B(N, a)$, *where B is Erlang's function, and*

$$\eta = \frac{y}{1 - y} = \frac{e^{(-1/a)}}{1 - e^{(-1/a)}}.$$

This result suggests use of $B(N, \eta)$ as an approximation to p_N. Two numerical cases illustrate this approximation:

i. $N = 8$, 8 erlangs are offered; then $y = e^{-0.125}$ and $\eta = 0.747$. We find $p_N = 0.17$, $B(N, \eta) = 0.20$, $B(N, a) = 0.235$.

ii. $N = 5$, 8 erlangs are offered; again, $\eta = 0.747$, and $p_N = 0.437$, $B(N, \eta) = 0.450$, $B(N, a) = 0.478$.

VIII. THE LOSS AS A FUNCTIONAL OF $A(u)$

For each N, and each hang-up rate γ, the loss p_N can be regarded as a mapping from the set of distributions $A(u)$ of positive variates to the interval (0, 1). We write $p_N(A)$ in this section for the loss resulting from the interarrival distribution $A(u)$, and we study the loss as a functional of $A(u)$.

First, it is instructive to keep the mean interarrival time μ_1 fixed and to vary the interarrival distribution $A(u)$. Since $e^{-n\gamma u}$ is convex in u, we find

$$e^{-n\gamma\mu_1} \leqq \int_0^\infty e^{-n\gamma u}\, dA(u) = a_n ,$$

and hence

$$\frac{1 - e^{-n\gamma\mu_1}}{e^{-n\gamma\mu_1}} \geqq \frac{1 - a_n}{a_n}.$$

But $e^{-n\gamma\mu_1} = a_n$ for the case where arrivals are regular, and μ_1 apart. This proves:

Theorem 9: *If* γ, μ_1 are positive constants, then

$$\inf_A \left\{ p_N(A) \,\middle|\, \int u\, dA(u) = \mu_1 \right\}$$

is achieved for the unit step distribution

$$A(u) = \begin{cases} 1, & u \geqq \mu_1 , \\ 0, & u < \mu_1 . \end{cases}$$

Thus, the probability of loss assumes a minimum, for fixed γ and μ_1, when the arrivals are regularly spaced at epochs μ_1 apart.

We next show that, if the mean interarrival time μ_1 and the hang-up rate γ are kept fixed, then the probability of loss can still be made arbitrarily close to unity by a proper choice of $A(u)$.

Theorem 10: *If* γ, μ_1 *are positive constants, then*

$$\sup_A \left\{ p_N(A) \,\middle|\, \int u\, dA(u) = \mu_1 \right\} = 1.$$

To prove this, let $1 > \epsilon > 0$ be given, and consider those distributions which have a mass $(1 - p)$ at $y_0 > 0$, and a mass p at $y_1 > 0$. For such an $A(u)$ we have

$$a_n = (1 - p)e^{-n\gamma y_0} + pe^{-n\gamma y_1}.$$

Let $q = (1 - p) \exp\{-N\gamma y_0\}$, so that, for each n,

$$\frac{1 - q}{q} \geqq \frac{1 - a_n}{a_n}.$$

Then, since $a_N < a_{N-1} < \cdots < a_1 < 1$, we find from Palm's formula (7) that

$$p_N(A) \geqq \left(1 + \frac{1-q}{q}\right)^{-N}.$$

We can now choose p and y_0 so small that $p_N(A) \geqq 1 - \epsilon$, independently of y_1, which can then be chosen to satisfy $\mu_1 = y_0(1-p) + y_1 p$. This proves the theorem.

It is natural to use

$$\frac{1}{\mu_1} = \frac{1}{\int_0^\infty u\,dA(u)}$$

as a measure of the calling rate, and to use

$$r_N(A) = \frac{1 - p_N(A)}{\mu_1} = \text{fraction served times calling rate}$$

as a measure of the rate of service, the rate at which calls are actually being completed. Suppose now that we are willing to tolerate a probability p of loss. Can we find an interarrival distribution $A(u)$ which achieves p and for which the rate of service is a maximum for a given hang-up rate γ? To answer this question, define the function

$$f(x_1, x_2, \cdots, x_N) = 1 + \binom{N}{1}\frac{1-x_1}{x_1} + \cdots + \binom{N}{N}\frac{(1-x_1)\cdots(1-x_N)}{x_1 \ldots x_N},$$

so that $p_N(A) = [f(a_1, a_2, \cdots, a_N)]^{-1}$.

Theorem 11: *If* $\gamma > 0$ *and* $0 < p < 1$, *then*

$$\sup_A\{r_N(A) \mid p_N(A) = p\} = \frac{\gamma(1-p)}{-\log x},$$

where x is the unique solution of the equation $f(x, x^2, \cdots, x^N) = p^{-1}$ in the unit interval. The supremum is achieved by the unit step distribution $A(u)$ defined by

$$A(u) = \begin{cases} 1, & u \geqq -\gamma^{-1}\log x, \\ 0, & u < -\gamma^{-1}\log x. \end{cases} \tag{9}$$

The function $f(x, x^2, \cdots, x^N)$ is monotone, decreasing from ∞ to 1 in the unit interval. Since f is continuous, and $0 < p < 1$, there exists a solution x of the equation $f(x, x^2, \cdots, x^N) = p^{-1}$. Obviously, for $A(u)$ defined by (9), we have $p_N(A) = p$. Now let $B(u)$ be any other interarrival distribution with a finite mean, so that the service rate $r_N(B)$

exists. Suppose that $B(u)$ achieves the probability p of loss; i.e., that $p_N(B) = p$. We show that

$$\int_0^\infty u \, dB(u) \geqq -\gamma^{-1} \log x.$$

For, suppose the contrary and set

$$y = \exp\left\{-\gamma \int_0^\infty u \, dB(u)\right\};$$

then, by Theorem 9, $[f(y, y^2, \cdots, y^N)]^{-1} \leqq p$, and

$$\int_0^\infty u \, dB(u) < -\gamma^{-1} \log x$$

implies $y > x$, so that $[f(x, x^2, \cdots, x^N)]^{-1} < [f(y, y^2, \cdots, y^N)]^{-1} \leqq p$, which is impossible. This proves that

$$\inf_B \left\{\int u \, dB(u) \,\middle|\, p_N(B) = p\right\} = -\gamma^{-1} \log x,$$

and also Theorem 11. Note that the supremum in Theorem 11 is a linear function of the hang-up rate, γ.

Let $N(t)$ be the number of trunks busy at time t, and let $E\{N(t)\}$ be its average. It is not always true that $\lim E\{N(t)\}$ exists as $t \to \infty$. However, if $A(u)$ is not a lattice distribution, then

$$\lim_{t\to\infty} E\{N(t)\} = \frac{1 - p_N(A)}{\gamma\mu_1},$$

where μ_1 may be ∞. This limit is the number of erlangs carried by the trunk group in equilibrium (see Takács[4]). Now a lattice distribution can be approximated arbitrarily closely by absolutely continuous distributions. Thus, an immediate consequence of Theorem 11 is:

Theorem 12: *If* $0 < p < 1$, *and* x *is as in Theorem* 11, *then*

$$\sup_A \lim_{t\to\infty} E\{N(t)\} = \frac{1 - p}{-\log x},$$

where the supremum is taken over $A(u)$ *such that* $p_N(A) = p$ *and such that* $\lim E\{N(t)\}$ *as* $t \to \infty$ *exists.*

This theorem means, intuitively, that the maximum number of erlangs that N trunks can carry at a fixed loss probability p [the maximum being over the appropriate $A(u)$ which achieve a loss p] is a number depending only on N and p.

It may also be of interest sometimes to know what is the least probability of loss incurred by offering a traffic of a erlangs to N trunks, with $A(u)$ being varied, and γ as well. The answer is given by:

Theorem 13: *If* $a > 0$, *then*

$$\inf_{A,\gamma} \left\{ p_N(A) \,\middle|\, \int \gamma u \, dA(u) = a^{-1} \right\} = [f(x, x^2, \cdots, x^N)]^{-1},$$

where $x = e^{-1/a}$, *and the inf is achieved by any unit step distribution* $A(u)$ *and* $\gamma > 0$ *such that*

$$A(u) = \begin{cases} 1, & u \geqq (a\gamma)^{-1}, \\ 0, & u < (a\gamma)^{-1}. \end{cases}$$

The proof is essentially that of Theorem 9, and is omitted.

IX. $\Pr\{N(k) = N)$ AS A FUNCTION OF k

The time-dependent behavior of the process $N(k)$ is only touched on here, since a complete treatment requires the detailed investigation of the roots of the polynomial $D_N(a_1, a_2, \cdots, a_N, z)$ occurring in the generating function $\psi_N(z)$. Such a study is still incomplete.

Nevertheless, some hints of the rate of approach to the limit p_N can be obtained from Theorem 1 and $\psi_N(z)$ as they stand. For instance, if $N(0) = 0$, then

$$\psi_N(z) = \frac{(1 - z)^{-1} a_1 a_2 \cdots a_N z^N}{\sum_{i=0}^{N} \binom{N}{i} (1 - a_1 z) \cdots (1 - a_i z) a_{i+1} \cdots a_N z^{N-i}}.$$

From this it can be seen directly that

$$\Pr\{N(k) = N \mid N(0) = 0)\} =$$

$$\begin{cases} 0 & \text{for } k < N, \\ a_1 a_2 \cdots a_N & \text{for } k = N, \\ a_1 a_2 \cdots a_N \left[1 + \sum_{j=1}^{N-1} (a_j - a_N) \right] & \text{for } k = n + 1. \end{cases} \tag{10}$$

More terms may be computed from the generating function, but the labor involved increases rapidly. It is to be noted that (10), together with Theorem 5, suggests that the approach to p_N is monotone; also, the first nonzero term is the approximating product $a_1 a_2 \cdots a_N$ discussed in Section VI.

For $N = 2$ trunks, it is possible to discuss $\Pr\{N(k) = N \mid N(0)\}$ in a particularly simple way. The results are given here, together with a numerical illustration, for the light they shed on the time development of the process. From Theorem 1 we find that

$$\sum_k z^k \Pr\{N(k) = 2 \mid N(0) = 0\} = \frac{a_1 a_2 z^2}{(1 - z)(1 - za_1 + za_2)},$$

so that

$$\Pr\{N(k) = 2 \mid N(0) = 0\} = \begin{cases} 0 & \text{for } k = 0, 1 \\ \dfrac{a_1 a_2}{1 - a_1 + a_2}\,[1 - (a_1 - a_2)^{k-1}] & \text{for } k \geqq 2. \end{cases}$$

Here p_N is $a_1 a_2/(1 - a_1 + a_2)$, and is approached exponentially.

Similarly, the generating function of $\Pr\{N(k) = 2 \mid N(0) = 1\}$ is

$$\frac{a_1 a_2 z^2}{(1 - z)(1 - za_1 + za_2)} + \frac{za_2}{1 - za_1 + za_2},$$

so that

$$\Pr\{N(k) = 2 \mid N(0) = 1\} = \begin{cases} 0 & \text{for } k = 0 \\ a_2 & \text{for } k = 1 \\ \dfrac{a_1 a_2}{1 - a_1 + a_2}\,[1 - (a_1 - a_2)^{k-1}] + a_2(a_1 - a_2)^{k-1} & \text{for } k \geqq 2. \end{cases}$$

Finally, the generating function of $\Pr\{N(k) = 2 \mid N(0) = 2\}$ is

$$\frac{a_1 a_2 z^2}{(1 - z)(1 - za_1 + za_1)} + \frac{za_2}{1 - za_1 + za_2} + 1,$$

from which we find

$$\Pr\{N(k) = 2 \mid N(0) = 2\} = \begin{cases} 1 & \text{for } k = 0 \\ a_2 & \text{for } k = 1 \\ \dfrac{a_1 a_2}{1 - a_1 + a_2}\,[1 - (a_1 - a_2)^{k-1}] + a_2(a_1 - a_2)^{k-1} & \text{for } k \geqq 2. \end{cases}$$

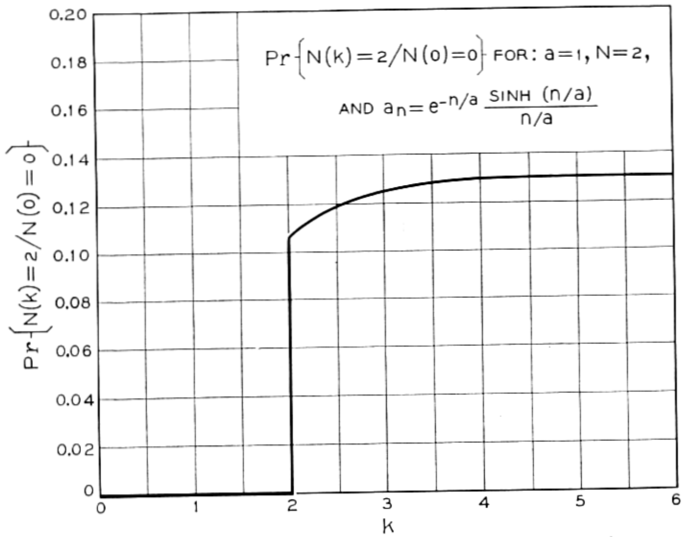

Fig. 15 — $\Pr\{N(k) = 2|N(O) = 0\}$ for $a = 1$ erlang, $N = 2$ trunks and uniformly distributed interarrivals.

This agrees with the previous conditional probability for $k \geqq 1$, as it should.

The three conditional probabilities $\Pr\{N(k) = 2 \mid N(0) = m\}$ for $m = 0, 1, 2$ have been plotted as functions of k for uniformly distributed interarrivals in Figs. 15, 16 and 17, respectively. The probabilities have been drawn continuously, but of course the functions are only defined for integers k. The example chosen exhibits a very rapid approach to equilibrium in terms of numbers of arriving calls, since the third arriving call finds essentially the equilibrium situation.

X. THE EXPECTATION OF $N(k)$ AND THE COVARIANCE

The next result gives a formula for the mean value $E\{N(k)\}$ in terms of the initial value $E\{N(0)\}$, and the probabilities $\Pr\{N(j) = N\}$ for $j \leqq k - 1$.

Theorem 14: *The mean value of $N(k)$ is*

$$E\{N(k)\} = \frac{a_1(1 - a_1^{\,k})}{1 - a_1} + a_1^{\,k} E\{N(0)\} - \sum_{j=0}^{k} a_1^{\,j+1} \Pr\{N(k - j - 1) = N\}.$$

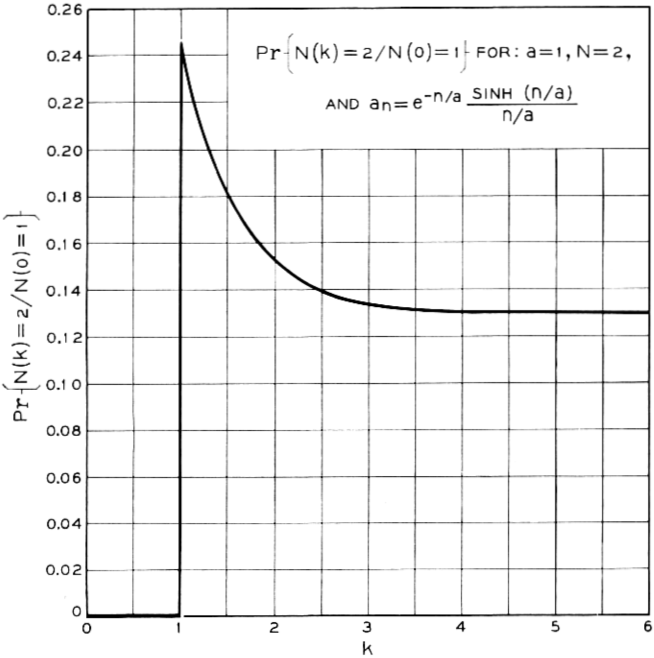

Fig. 16 — $\Pr\{N(k) = 2|N(O) = 1\}$ for $a = 1$ erlang, $N = 2$ trunks and uniformly distributed interarrivals.

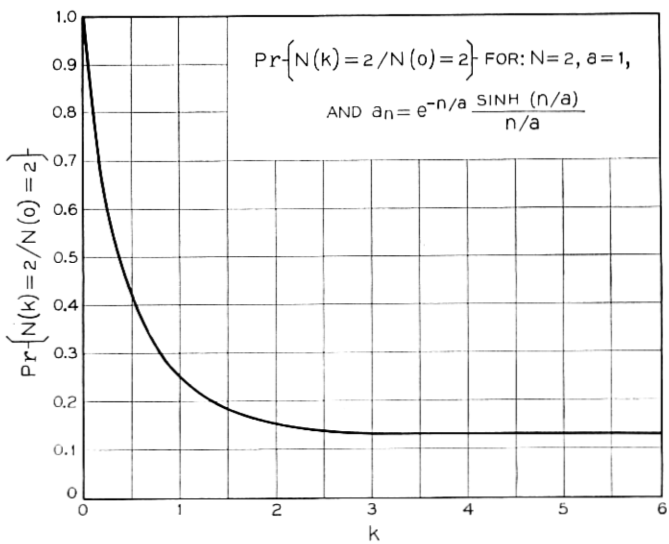

Fig. 17 — $\Pr\{N(k) = 2|N(O) = 2\}$ for $a = 1$ erlang, $N = 2$ trunks and uniformly distributed interarrivals.

To prove this we first obtain the generating function $b_1(z)$, either by differentiation from (1) or directly from the recurrences (4). We find

$$\sum_{k \geqq 0} z^k E\{N(k)\} = b_1(z) = \frac{za_1[(1 - z)^{-1} - \psi_N(z)] + E\{N(0)\}}{1 - za_1},$$

and this gives Theorem 14 upon expansion in powers of z.

We define the covariance function $R(n)$ of the random process $N(k)$ by

$$R(n) = \lim_{k \to \infty} E\{N(k)N(k + n)\} - E^2\{N(k)\}.$$

From Theorem 14 we can derive a formula for the covariance function $R(n)$.

Theorem 15: *If $\psi_{n,N}(z)$ is $\psi_N(z)$ for the initial condition* $N(0) = n$, $|z| < 1$, *$\{p_m\}$ is the stationary distribution of $N(k)$, and* $m_i = \sum_m m^i p_m$ *for* $i = 1, 2$, *then*

$$\sum_{n \geqq 0}^{\infty} z^n R(n) = \sum_m p_m m \left\{ \frac{za_1[(1 - z)^{-1} - \psi_{m,N}(z)] + m}{1 - za_1} \right\} - (1 - z)^{-1} \left(\frac{a_1 - a_1 p_N}{1 - a_1} \right)^2,$$

and

$$R(n) = R(-n) = \frac{m_1(a_1 - a_1^{n+1})}{1 - a_1} + a_1^n m_2 - m_1^2 - \sum_m m p_m \sum_{j=0}^{n-1} a_1^{j+1} \Pr\{N(n - j - 1) = N \mid N(0) = m\}.$$

Before developing the results of Theorem 15 into a form useful for computation, we shall sketch the reasons for interest in the covariance function $R(n)$. The function expresses quantitatively the cohesiveness of the process, the extent to which $N(k + n)$ and $N(k)$ are correlated. Besides this theoretical role, the covariance is involved in the practical matter of evaluating (theoretically) the sampling error in a certain kind of switch count (traffic measurement.) For a concrete example, suppose that

$$S = \sum_1^n N(k)$$

is used to estimate the average traffic encountered by arriving customers. Here n is the number of successive observations of the random process $N(k)$. The variance of S is

$$\begin{aligned}\text{var}\{S\} &= E\left\{\sum_i \sum_j N(i)N(j)\right\} - n^2 m_1^2 \\ &= \sum_i \sum_j \text{cov}\{N(i), N(j)\} \\ &= \sum_i \sum_j R(i - j) \\ &= nR(0) + 2\sum_{j=1}^{n-1} (n - j)R(j),\end{aligned}$$

where we have assumed that the observations began in a condition of equilibrium. Thus var$\{S\}$ can be expressed in terms of the covariance function $R(n)$.

The formula for $R(n)$ can be made more useful for computation by turning it into a recurrence relation for successive values of a certain linear function of $R(n)$. We define auxiliary quantities Q_k by

$$Q_k = \sum_{m=0}^{N} mp_m \Pr\{N(k) = N \mid N(0) = m\} \tag{11}$$

and note that

$$R(n) + m_1^2 - \frac{m_1 a_1}{1 - a_1} = a_1^n \left(m_2 - \frac{m_1 a_1}{1 - a_1}\right) - \sum_{j=0}^{n-1} a_1^{j+1} Q_{n-j-1} .$$

Hence also

$$\begin{aligned}R(n + 1) + m_1^2 - \frac{m_1 a_1}{1 - a_1} \\ &= a_1 \left\{a_1^n \left(m_2 - \frac{m_1 a_1}{1 - a_1}\right) - Q_n - \sum_{j=0}^{n-1} a_1^{j+1} Q_{n-j-1}\right\} \qquad (12) \\ &= a_1 \left\{R(n) + m_1^2 - \frac{m_1 a_1}{1 - a_1} - Q_n\right\}\end{aligned}$$

Thus, if the Q_k are known, the $R(n)$ may be calculated by a simple recursive procedure from $R(0)$, which is the variance. The calculation of the Q_k is simplified by the fact that, for small k, (a region of principal interest), many terms of the sum defining Q_k are 0. For example, if $0 \leq m < N - k$, the conditional probability $\Pr\{N(-k) = N \mid N(0) = m\}$ is 0, since it is not possible for the kth man to find all trunks busy if the 0th man found fewer than $N - k$ busy. The first few correction

TABLE I. — $\Pr\{N(k) = N \mid N(0) = m\}$.

m	k			
	0	1	2	3
N	1	a_N	$Na_Na_{N-1} - (N-1)a_N^2$	$(N^2 - N + 1)a_N^3 + \left(N - 2N^2 + \frac{N^2 - N}{2}\right)a_{N-1}a_N^2 + Na_{N-1}^2a_N + \frac{N(N-1)}{2}a_{N-2}a_{N-1}a_N$
N-1	0	a_N	$Na_{N-1}a_N - (N-1)a_N^2$	same as above
N-2	0	0	$a_{N-1}a_N$	$a_{N-1}[a_{N-1} + (N-1)(a_{N-2} - a_N)]$
N-3	0	0	0	$a_{N-2}\, a_{N-1}\, a_N$

terms Q_k as defined above may be computed (by summation) for $k = 0, 1, 2, 3$ from Table I, which shows $\Pr\{N(k) = N \mid N(0) = m\}$, valid for $m \geqq 0$:

Curves of the covariance function $R(n)$ for $n = 1, 2$ and 3 are plotted as functions of the offered traffic a for trunk group sizes $N = 2, \cdots, 8$, as follows: in Figs. 18 through 20 for Poisson arrivals; in Figs. 21 through 23 for uniformly distributed interarrivals; and in Figs. 24 through 26 for regular arrivals. The curve for $N = 1$ is not shown in any of Figs. 18 through 26 because, in this case, $R(n) = 0$ for $|n| > 0$ (see below).

The following conclusions seem to be reasonable after examination of the curves:

i. $R(n)$ is nonnegative and monotone decreasing in $|n|$.

ii. For n and traffic a fixed, the covariance $R(n)$ for Poisson arrivals exceeds the covariance $R(n)$ for both the other two interarrival distributions (uniform and fixed) we have considered. Similarly, the covariance $R(n)$ for regular arrivals falls below the value of $R(n)$ for both Poisson arrivals and uniform interarrivals. We conjecture that $R(n)$ for regular arrivals is less than or equal to $R(n)$ for any other distribution of interarrivals, for the same traffic.

A particularly simple but important case arises when $N = 1$; the case is simple because $R(n) = 0$ except for $n = 0$; the case is important, not because groups consisting of a single trunk are common (they are not), but because the case $N = 1$ corresponds to making a measurement only on the first trunk of a group (of arbitrary size) in which the trunks are tried in a fixed order. For $N = 1$ it is easy to see (from Theorem 1) that

$$\begin{aligned} \Pr\{N(k) = 1 \mid N(0)\} &= \delta_{N(0),1} \quad \text{for } k = 0, \\ &= a_1 \quad\quad \text{otherwise,} \end{aligned}$$

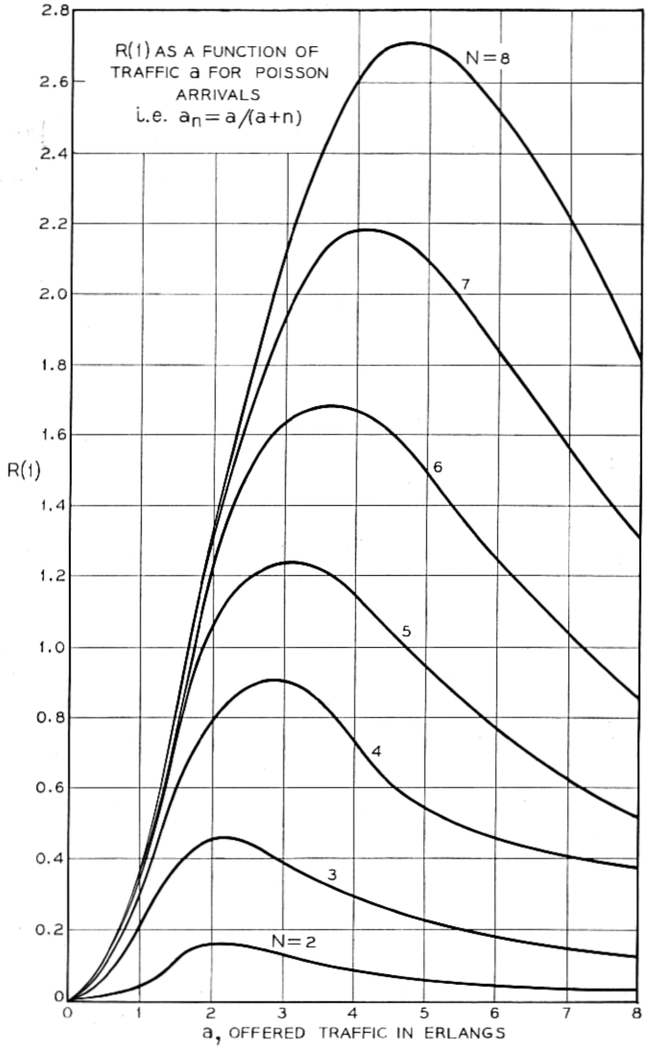

Fig. 18 — The covariance value $R(1)$ as a function of traffic a for Poisson arrivals.

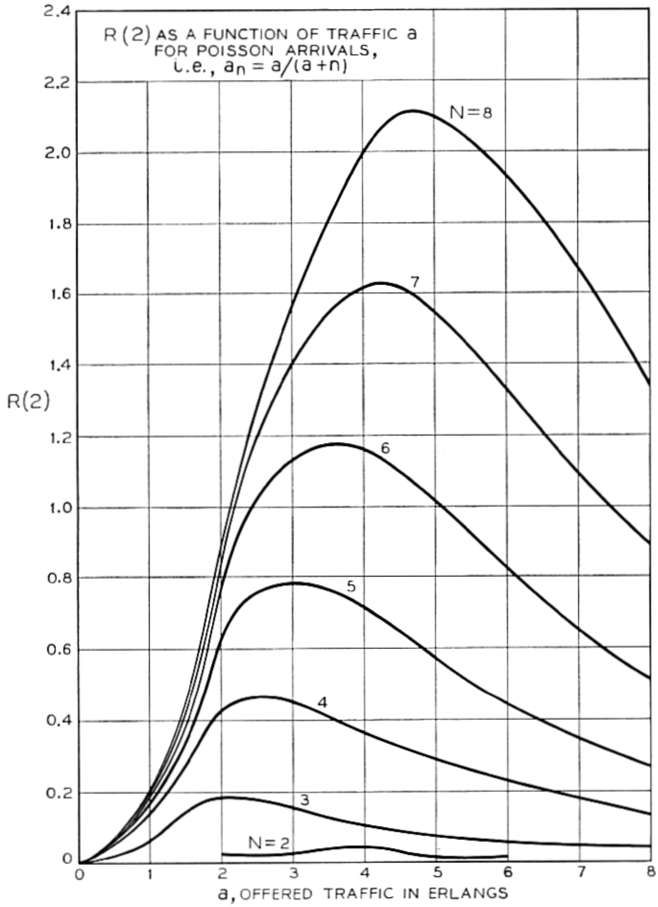

Fig. 19 — The covariance value $R(2)$ as a function of traffic a for Poisson arrivals.

so that $N(k)$ is *independent* of $N(0)$ for $k > 0$. Thus, in this case,

$$R(0) = \text{var}\,\{N(k)\} = a_1 - a_1^2,$$

$$R(n) = 0, \qquad \text{for } n \neq 0,$$

$$E\{S/n\} = E\{N(k)\} = a_1,$$

$$\text{var}\{S/n\} = \frac{a_1 - a_1^2}{n},$$

so that S/n is a consistent and unbiased estimator for a_1. It is to be

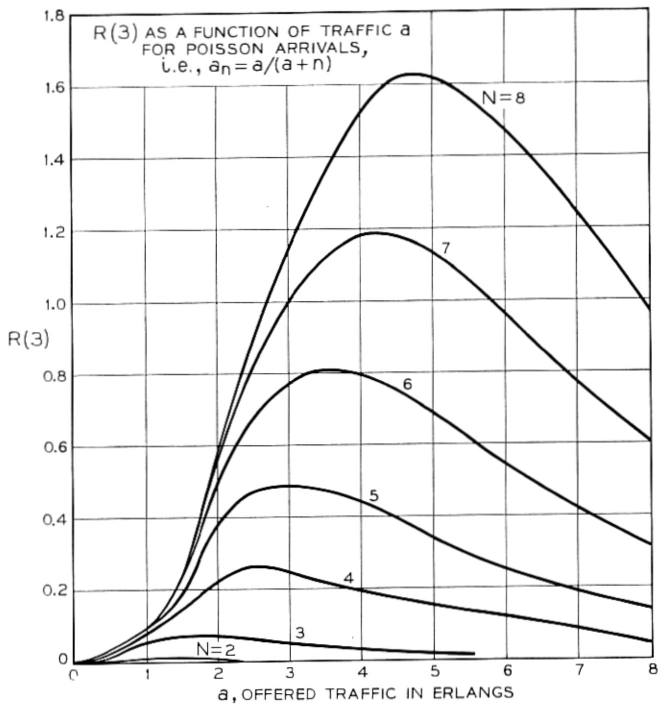

Fig. 20 — The covariance value $R(3)$ as a function of traffic a for Poisson arrivals.

emphasized that in this case S is the sum of n independent identically distributed random variables, each equal to 1 with probability a_1, and to 0 with probability $1 - a_1$. Thus, S has a binomial distribution with "success" parameter a_1.

The method of traffic measurement (on a group) outlined in the preceding paragraph has the disadvantage that it collects information very slowly. But it is relatively cheap, since all that has to be recorded is whether the first trunk is busy at arrival epochs or not, and it has the additional advantage that its statistical theory is relatively simple and has been well developed in the literature. It must be kept in mind that the sampling error estimates we develop are limited to measurements made at epochs just preceding arrivals.

Often the traffic engineer needs to estimate the *load offered* to a group, rather than the *load carried* by it. The use of S to estimate a_1 tells him what fraction of the time the first trunk is busy. However, there are cases in which the knowledge of a_1 determines the offered load. This

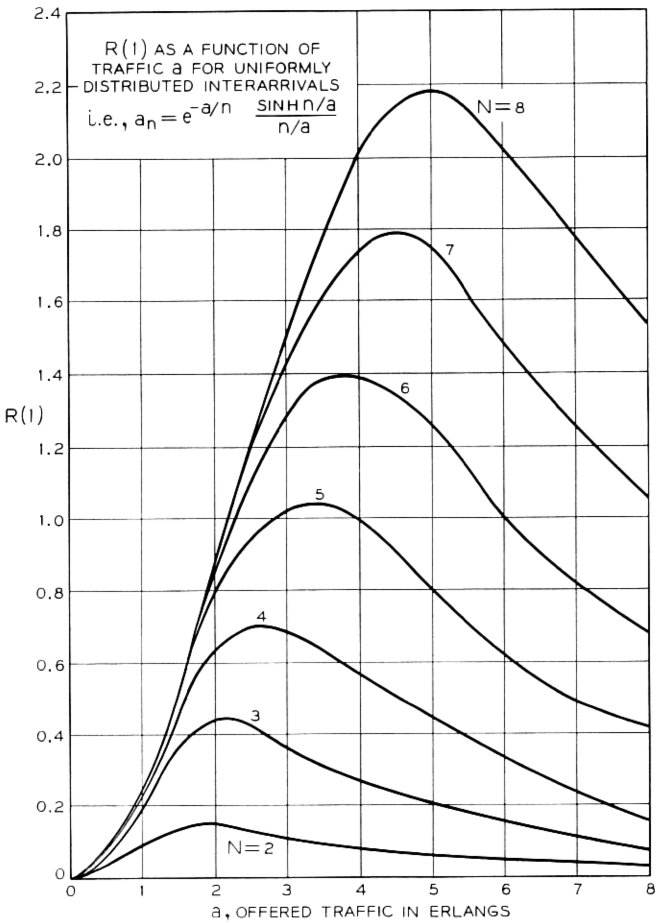

Fig. 21 — The covariance value $R(1)$ as a function of traffic a for uniformly distributed interarrivals.

occurs, in fact, whenever a_1 is a monotone function of the offered load a only. For example, when arrivals are Poisson, we have $a_1 = a/(1 + a)$, so it is reasonable to use $S/(n - S)$ as an estimator of the offered load a. When arrivals are regular, $a_1 = e^{-1/a}$, so a reasonable estimate of a is $1/(\log n - \log S)$.

In the Poisson example, this method of estimating a can be evaluated readily if we estimate a^{-1} instead by means of $(n + 1)/(S + 1) - 1$, whose stochastic limit is obviously a^{-1}.

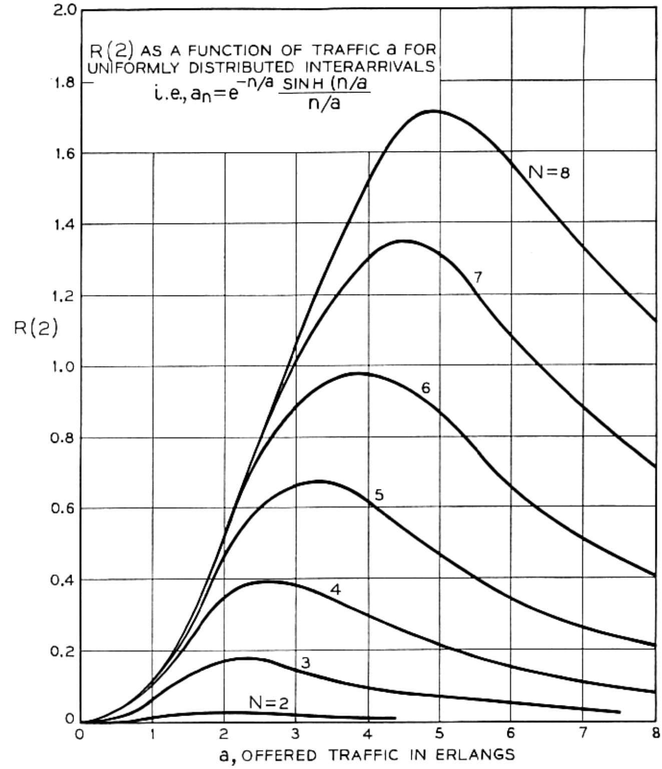

Fig. 22 — The covariance value $R(2)$ as a function of traffic a for uniformly distributed interarrivals.

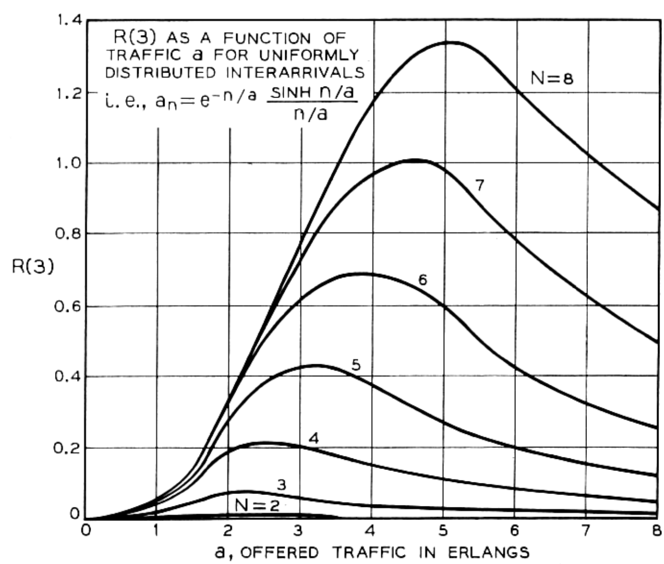

Fig. 23 — The covariance value $R(3)$ as a function of traffic a for uniformly distributed interarrivals.

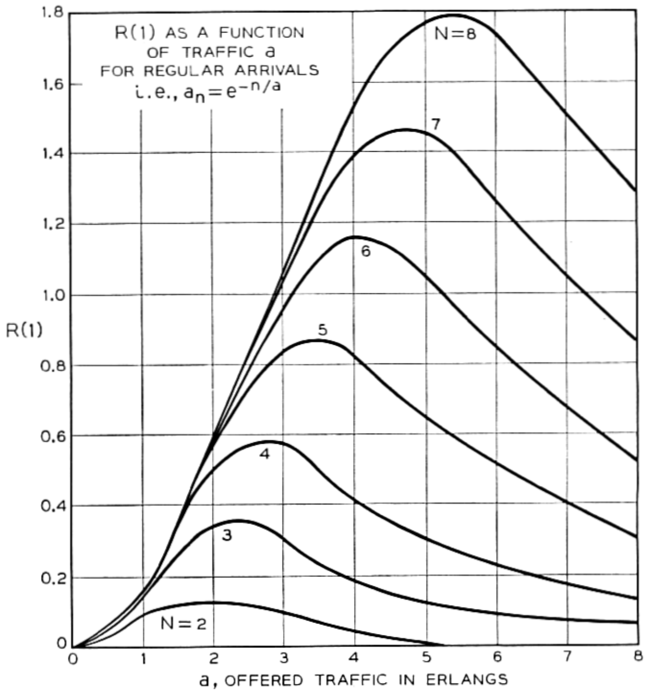

Fig. 24 — The covariance value $R(1)$ as a function of traffic a for regular arrivals.

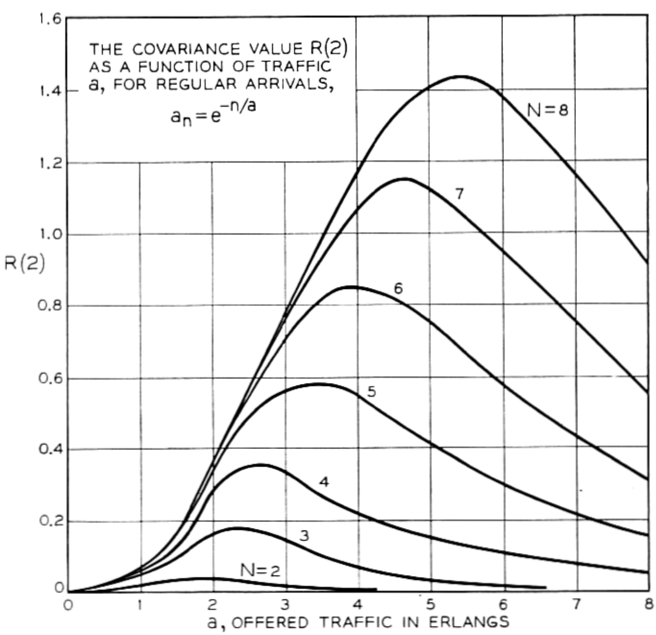

Fig. 25 — The covariance value $R(2)$ as a function of traffic a for regular arrivals.

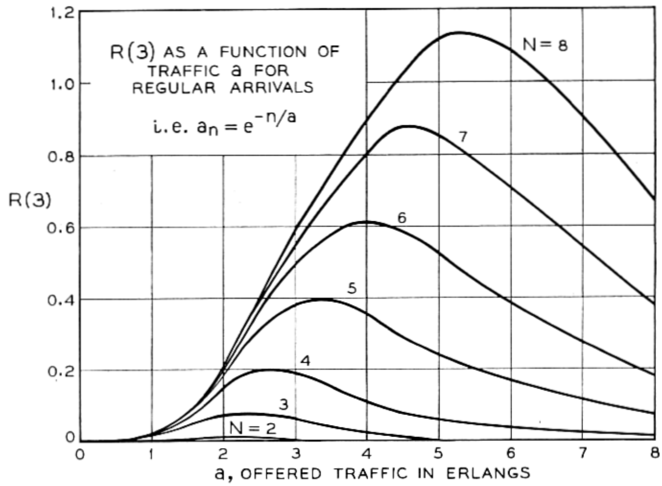

Fig. 26 — The covariance value $R(3)$ as a function of traffic a for regular arrivals.

The generating function of S is

$$E\{x^S\} = [1 + (x - 1)a_1]^n = \sum_{j=0}^{n} x^j \Pr\{S = j\}.$$

Hence

$$E\{S + 1)^{-1}\} = \int_0^1 E\{x^S\}\, dx = \frac{1 - (1 - a_1)^{n+1}}{(n + 1)a_1},$$

$$E\left\{\frac{n + 1}{S + 1} - 1\right\} = \frac{1 - a_1}{a_1}[1 - (1 - a_1)^n].$$

There seems to be no simple formula for the second moment of this estimator, nor for that of n/S. However, noting that

$$\frac{(n + 1)^2}{(S + 1)(S + 2)} \leqq \frac{(n + 1)^2}{(S + 1)^2},$$

we can verify (by the same method as above) that

$$E\{(S^2 + 3S + 2)^{-1}\} = \int_0^1 \int_0^y E\{x^S\}\, dx\, dy = \frac{1 + a_1(1 - a_1)^{n+1}}{(n + 1)(n + 2)a_1^2},$$

$$E\left\{\frac{(n + 1)^2}{(S + 1)(S + 2)}\right\} = \frac{n + 1}{n + 2}\,\frac{1 + a_1(1 - a_1)^{n+1}}{a_1^2},$$

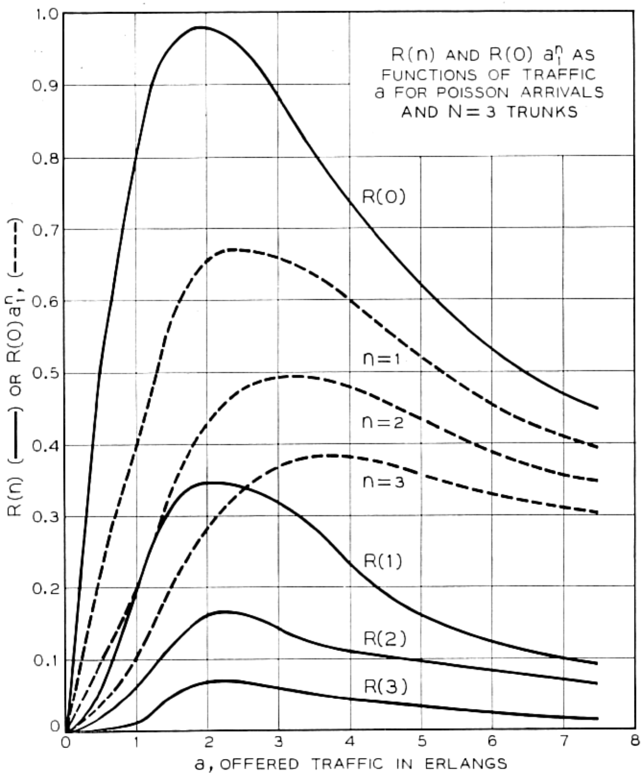

Fig. 27 — $R(n)$ and $R(O)a_1^n$ as functions of traffic a for Poisson arrivals and $N = 3$ trunks.

and so conclude that

$$\operatorname{var}\left\{\frac{n+1}{S+1} - 1\right\} \geqq \frac{n+1}{n+2} \frac{1 + a_1(1 - a_1)^{n+1}}{a_1^2} - \frac{[1 - (1 - a_1)^{n+1}]^2}{a_1^2}.$$

This lower bound is likely to be very close to the variance on the left for large n, so that, in this region,

$$\operatorname{var}\left\{\frac{n+1}{S+1} - 1\right\} \sim \frac{(a_1 + 2)(1 - a_1)^{n+1} - (1 - a_1)^{2n+2}}{a_1^2}.$$

It can easily be shown (by the methods of Section IV) that, if $N = \infty$; i.e., if the trunk group is unlimited in size, the covariance function is exponential in character:

$$R(n) = R(0)a_1^n.$$

This suggests that, in some cases, $R(n) \sim R(0)a_1^n$ is a good approximation to the covariance for $N < \infty$. This approximation is equivalent to ignoring the correction terms Q_k in the recurrence relation (12) for the covariance. Since the sign of the Q_k in (12) is negative, it is clear that the approximation is an *overestimate*.

The covariance $R(n)$ for $n = 0, 1, 2, 3$ and the overestimate $R(0)a_1^n$ for $R(n)$ have been plotted together in Figs. 27, 28 and 29 for 3, 5 and 8 trunks, respectively, and Poisson arrivals. The curves suggest the following conclusions:

i. The approximation $R(n) \sim R(0)a_1^n$ is likely to be good if the load per trunk a/N is low.

ii. If the load per trunk a/N is high, e.g., $a/N = 1$, the approximation $R(n) \sim R(0)a_1^n$ may give a figure for the covariance (between

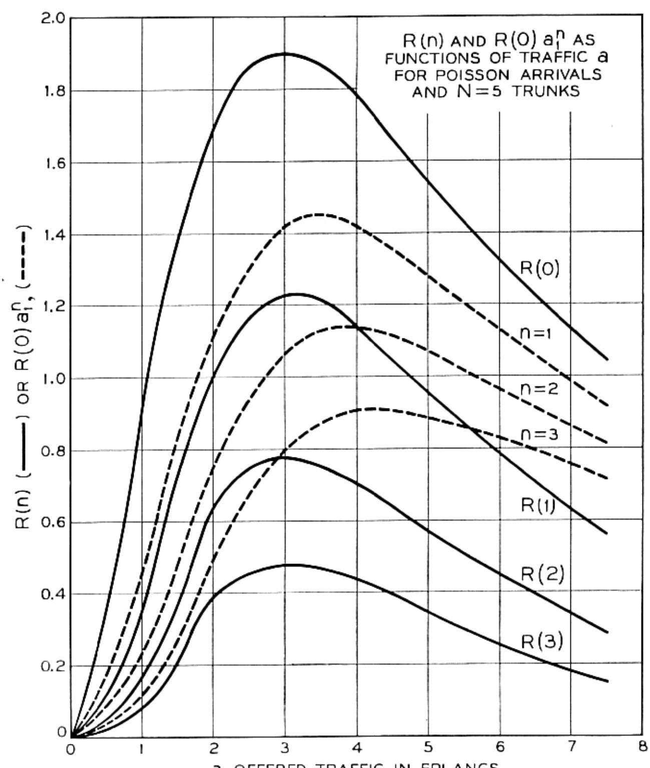

Fig. 28 — $R(n)$ and $R(O)a_1^n$ as functions of traffic a for Poisson arrivals and $N = 5$ trunks.

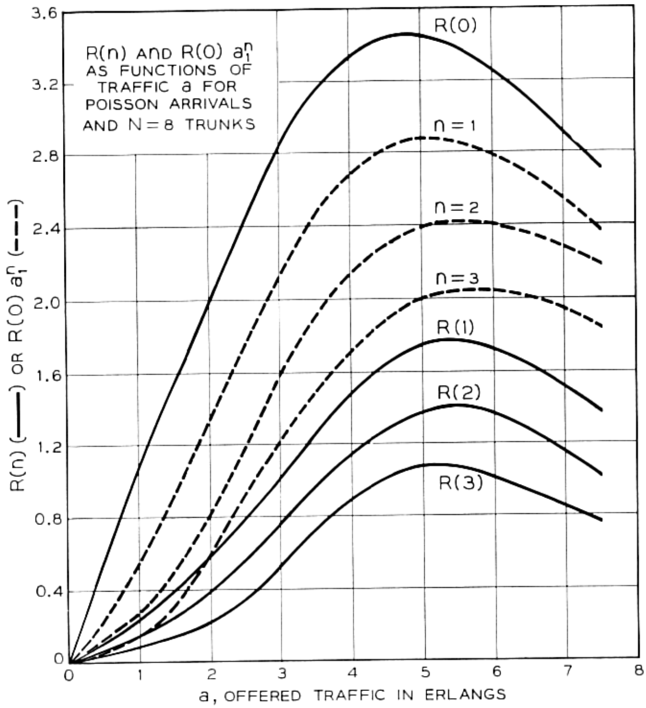

Fig. 29 — $R(n)$ and $R(O)a_1^n$ as functions of traffic a for Poisson arrivals and $N = 8$ trunks.

separate observations of $N(k)$) that is several times the actual value. This effect seems to increase with the separation, n.

iii. Variances, such as that of

$$S = \sum_1^n N(k),$$

computed on the basis of the approximation $R(n) \sim R(0)a_1^n$ are *overestimates*, so that use of this approximation in estimating sampling error is *conservative*.

iv. The value of a at which $R(n)$ has its (apparently unique) maximum seems to be the same for all n, depending only on N, the size of the group.

XI. ACKNOWLEDGMENT

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REFERENCES

1. Palm, C., Intensitätsschwankungen im Fernsprechverkehr, Ericsson Technics, **44,** 1943.
2. Feller, W., On the Theory of Stochastic Processes with Particular Reference to Applications, *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability*, Univ. of California Press, Berkeley, Calif., 1949.
3. Pollaczek, F., Genéralization de la théorie probabiliste des systémes téléphoniques sans dispositif d'attente, Comptes Rendus, **236,** 1953, pp. 1469–1470.
4. Takács, L., On the Generalization of Erlang's Formula, Acta Math. Acad. Sci. Hungar., **7,** 1956, pp. 419–433.
5. Takács, L., On a Probability Problem Concerning Telephone Traffic, Acta Math. Acad. Sci. Hungar, **8,** 1957, pp. 319–324.
6. Cohen, J. W., The Full Availability Group of Trunks with an Arbitrary Distribution of the Interarrival Times and a Negative Exponential Holding-Time Distribution, Phillips Report, 1956, pp. 1–10.
7. Feller, W., *An Introduction to Probability Theory and Its Applications*, John Wiley & Sons, New York, 1950.