Space-Charge Wave Excitation in Solid-Cylindrical Brillouin Beams

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The voltage and current modulation of ideal cylindrical electron beams in Brillouin flow, as well as beams in zero magnetic field, are studied by means of Laplace transforms. With a large-diameter beam of this class, suddenly accelerated from a temperature-limited cathode and without transverse velocities, the minimum noise figure of an amplifier is found to be smaller than it would be for a narrow, essentially one-dimensional (filament or sheet) beam, or for a confined-flow beam with the same diameter, longitudinal velocity and direct current.

Certain space-charge wave solutions obtained in field analyses of beams from shielded diodes, which have never been detected experimentally, are found to be nonexistent in the sense that no phenomenon taking place in a vacuum tube excites them.

I. INTRODUCTION

When a beam only partly fills the space within a concentric drift tube, the field patterns of the modes derived by small-signal slow-wave analysis are not orthogonal to one another. This makes it difficult to find the amplitude of any single mode excited by an arbitrary initial disturbance. The cases of ion-neutralized beams in the absence of a magnetic field and of Brillouin flow are even more difficult, for in these cases infinite groups of modes assume the same phase velocity and degenerate into a wave of arbitrary transverse distribution, which, we shall show, cannot be excited at all.

In treating the excitation of a confined-flow beam, Scotto and Parzen[1] have circumvented such difficulties by means of a Laplace transform procedure. More recently, Bresler, Joshi and Marcuvitz[2] have succeeded in formulating a complete set of orthogonal modes for such unidirectional electron beams, at the cost of some increased complexity in description.

In this paper, a technique similar to that of Scotto and Parzen will be employed to solve several problems in the excitation of a solid-cylindrical beam, focused in ideal Brillouin flow. The method consists of transforming the exciting current or voltage with respect to the axial coordinate z, and finding the beam response by means of a transfer function which satisfies the transverse boundary conditions. The relative amplitudes of each of the various modes could be found, in this way, by using transfer functions evaluated in terms of each such mode. Here, only the fundamental mode, having axial symmetry, will be considered. The solutions so obtained will also apply to the beam in zero magnetic field, as the mode patterns are the same in both cases.[3]

The first problem treated, of field modulation by means of an annular gap in a concentric drift tube, will illustrate the general technique. The remaining three calculations deal with different aspects of the problem of noise excitation of a finite-diameter beam in a shielded diode, in which the effect of transverse electron motions is disregarded. These calculations show that the "noisiness" of such a beam falls to half that for a narrow beam or a one-dimensional beam as the diameter is increased (as βb is made larger). An additional calculation shows that certain space-charge waves obtained in field analyses of such beams,[4, 5] which are independent of transverse boundary conditions, cannot be excited and therefore do not exist.

The prospects of producing low-noise amplifiers with large-diameter beams in Brillouin flow are not very good, because of large transverse electron excursions near the cathode. However, it is possible that a similar noise-reduction mechanism may be present in confined-flow beams abruptly hollowed-out (relative to the cathode surface) close to the cathode. The extremely low noise figures reported[6, 7] for TWT amplifiers using beams of this sort are chiefly due to other noise-reduction processes,[8, 9] but the effect of large beam size may perhaps be important at higher frequencies.

II. MODULATING VOLTAGE ACROSS GAP IN DRIFT TUBE

At the input plane, $z = 0$, an ac voltage V is impressed across a very short gap in a drift tube of radius a, concentric with and enclosing a Brillouin-flow beam of radius b. The response is sought in the form of the total current in the drift tube to the right of this plane, $i_t(z, a)$. Polar cylindrical coordinates (r, θ, z) and MKS units will be employed, consistent with the notation of Ref. 5, in which axial-symmetric space-charge waves in beams of this type are described. All of the ac quantities associated with any such wave are assumed to propagate as

$$\exp (j\omega t) \cdot \exp (-j\beta z) \tag{1}$$

with the time variation suppressed.

As the amplitudes of all ac quantities are zero to the left of the input plane, it is convenient to use the Laplace transform pairs in the form[10]

$$F(\beta) = \int_0^\infty f(z) \exp(j\beta z)\, dz, \tag{2}$$

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\beta) \exp(-j\beta z)\, d\beta. \tag{3}$$

The integration contour for the inverse transform (3) is along the real axis of β, indented above any poles on that axis, and enclosing the third and fourth quadrants of the complex β-plane. When $F(\beta)$ has n simple, distinct poles within this contour, the last integral can be evaluated by means of Cauchy's residue theorem, for $z > 0$, as

$$f(z) = -j \sum_n [(\beta - \beta_n)F(\beta) \exp(-j\beta z)]_{\beta=\beta_n}. \tag{4}$$

Accordingly, the transform of the impressed field, in a gap of nominal (but negligible) width d, is

$$A(\beta) = \int_0^d (-V/d) \exp(j\beta z)\, dz \cong -V. \tag{5}$$

The response current is found by multiplying this quantity by a transfer function $Y(\beta)$ to obtain the transform of that current, and then its inverse transform. Any transfer function relating two ac quantities with the same (z, t) variation will, in general, be a function of the propagation constant β and the transverse properties of the electron beam and its cylindrical enclosure. The transfer function $Y(\beta)$ relating the ac amplitudes $i_t(z, a)$ and $E_z(z, a)$ will therefore be the same as that relating their transforms $\hat{\imath}_t(\beta, a)$ and $\hat{E}_z(\beta, a)$. In the present instance, the z-component of the field equation for curl $\mathbf{H}$ provides the desired relation defining $Y(\beta)$:

$$\hat{\imath}_t(\beta, a) = 2\pi a \hat{H}_\theta = 2\pi a \hat{E}_z Y(\beta), \tag{6}$$

and the response current is given by

$$\begin{aligned} i_t(z, a) &= -aV \int_{-\infty}^{\infty} Y(\beta) \exp(j\beta z)\, d\beta \\ &= j2\pi a V \sum_n [(\beta - \beta_n) Y(\beta) \exp(-j\beta z)]_{\beta=\beta_n}. \end{aligned} \tag{7}$$

The boundary equations at the surface of a drifting Brillouin-flow beam[5] must be solved in terms of $Y(\beta)$, rather than of the infinite admittance of a conducting wall. The axial electric field inside and outside of the beam, respectively, can be written

$$E_z = AI_{0r} \qquad (0 \leqq r \leqq b), \tag{8}$$

$$E_z = BI_{0r} + CK_{0r} \qquad (b \leqq r \leqq a), \tag{9}$$

where A, B, C are arbitrary constants, and I and K stand for the modified Bessel functions of the first and second kinds, respectively, the first subscript denoting the order number and the second the radius. The propagation factor, exp $j(\omega t - \beta z)$, as well as the argument (βr) are omitted for brevity, and will be omitted elsewhere when they are unambiguous.

The surface ripple due to ac radial electron motions can be represented by a surface charge density,

$$\sigma = -R\epsilon E_r = -\frac{j\epsilon R}{\beta}\frac{\partial E_z}{\partial r}, \tag{10}$$

evaluated in terms of the fields just inside of the beam, where R is the square of the reciprocal of the space-charge reduction factor, p, defined in terms of the radian plasma frequency ω_p, the excitation frequency ω and the beam drift velocity u, as follows:

$$R = \frac{1}{p^2} = \frac{{\omega_p}^2}{(\omega - \beta u)^2} = \frac{{\beta_p}^2}{(\beta - \beta_e)^2}. \tag{11}$$

In the last expression, $\beta_p = \omega_p/u$ and $\beta_e = \omega/u$.

The boundary equations at $r = b$ can then be written

$$AI_{0b} = BI_{0b} + CK_{0b}\,, \tag{12}$$

$$A(1 - R)I_{1b} = BI_{1b} - CK_{1b}\,, \tag{13}$$

and the admittance function is

$$\begin{aligned} Y(\beta) &= \left(\frac{H_\theta}{E_z}\right)_{r=a} = \frac{\omega\epsilon}{\beta}\left(\frac{E_r}{E_z}\right)_{r=a} \\ &= \frac{j\omega\epsilon}{\beta}\frac{I_{1a}}{I_{0a}}\left[\frac{(B/C) - (K_{1a}/I_{1a})}{(B/C) + (K_{0a}/I_{0a})}\right]. \end{aligned} \tag{14}$$

Substitution here of (B/C), found by solving the two boundary equations, yields:

$$Y(\beta) = \frac{j\omega\epsilon}{\beta}\frac{I_{1a}}{I_{0a}}\frac{p^2 - wn}{p^2 - wm}, \tag{15}$$

where

$$w = \beta b I_{1b} K_{0b}, \tag{16}$$

$$n = 1 + \frac{I_{0b} K_{1a}}{K_{0b} I_{1a}}, \tag{17}$$

$$m = 1 - \frac{I_{0b} K_{0a}}{K_{0b} I_{0a}}. \tag{18}$$

By writing

$$\frac{p^2 - wn}{p^2 - wm} = \frac{(\beta - \beta_e)^2 - \beta_p^2 wn}{(\beta - \beta_e)^2 - \beta_p^2 wm} \tag{19}$$

it is readily verified that, along the real β-axis, the only poles of $Y(\beta)$ are

$$\beta_{1,2} = \beta_e \pm \beta_p \sqrt{wm} = \beta_e \pm \beta_q, \tag{20}$$

in terms of which

$$p^2 - wm = (\beta - \beta_1)(\beta - \beta_2). \tag{21}$$

In addition, as the integration contour encloses the third and fourth quadrants of the β-plane, the term $I_0(\beta a)$ contributes poles at each of the zeros of $J_0(x)$ along the negative imaginary β-axis. For each root x_n, the pole is $\beta = (jx_n/a)$, so that the corresponding residue contains a factor $\exp(-x_n z/a)$. Since such terms decay rapidly with distance z from the input plane, and we are solely interested in propagating waves, they will not be considered further.

If the changes in w, m and n due to changes of β are neglected, by evaluating all Bessel-function arguments at β_e, the expression for the current response reduces to the following:

$$\begin{aligned} i_t(z, a) &\doteq -\pi\omega\epsilon \frac{\beta_p}{\beta_e} aV \frac{I_{1a}}{I_{0a}} \frac{w(m - n)}{\sqrt{wm}} [\exp(-j\beta_1 z) - \exp(-j\beta_2 z)] \\ &\doteq \frac{-jV 2\pi\epsilon\omega_q \sin \beta_q z \exp(-j\beta_e z)}{\beta_e I_{0a}^2 \left(\dfrac{K_{0b}}{I_{0b}} - \dfrac{K_{0a}}{I_{0a}}\right)}. \end{aligned} \tag{22}$$

In klystron theory, it is customary to write this quantity in another form, by introducing the dc beam current I_0 and voltage V_0. For a beam with negligible potential depression,

$$\frac{I_0}{V_0} = \frac{2\pi\epsilon\omega_p^2 b^2}{u}. \tag{23}$$

With this and the reduction factor $p = \sqrt{wm}$, we obtain

$$i_t(z, a) \doteq \frac{-jVI_0}{V_0} \left(\frac{I_{0b}I_{1b}}{I_{0a}^2}\right)_{\beta=\beta_e} \frac{\sin \beta_q z \exp(-j\beta_e z)}{\beta_q b}. \tag{24}$$

Beck[11] has treated this problem in a slightly different way, by introducing several additional approximations. His result consists of the above expression, followed by a smaller second term. The present derivation shows that this latter term should be simply zero.

III. MODULATION BY INJECTED FILAMENT OF NOISE CURRENT

The response of a one-dimensional beam to injected noise current has been computed by one of the authors[10] with the Laplace transform technique described above. Within the framework of its assumptions, this computation led to results in agreement with the work of Rack, Llewellyn and Peterson,[12] thereby establishing its validity as an alternative procedure. It is now proposed to extend this treatment to the noise excitation of a finite-diameter beam in Brillouin flow or in zero magnetic field, and with an infinitely remote outer conducting tube. The treatment will be for a source of electrons with no transverse velocities. This may be unrealistic, but it is not unphysical, for such a source can be approximated by collimating the electron flow from a cathode by means of an array of holes, such as a thick hexagonal grid. First, the response will be found to a slender filament of noise charge injected at the axis of this beam, and later on the response will be calculated for noise-charge modulation over the entire beam area. Comparison of the results with those for the one-dimensional beam should reveal the effect of beam diameter on its noisiness.

The approximations used in the one-dimensional computation[10] are to be adopted here as well, and the reader is referred to Ref. 10 for a detailed discussion of their meaning. Effects due to the multivelocity nature of the beam and the inertial effects of a space-charge cloud during acceleration are avoided by assuming the beam to be abruptly accelerated from a temperature-limited cathode. The modes of propagation of the beam are assumed to be indistinguishable from those for a beam without thermal velocities. Excitation of Landau-type damped plasma oscillations,[13] which tend to decelerate fast-entering charges, is neglected.

The noise excitation due to injected charge in each velocity class is calculated in a narrow frequency band, and its mean square summed over all velocity classes, restricted to a small spread about the mean beam velocity. The beam is thus regarded as a linear impedance through which the exciting charges flow. The entering charges are treated as current filaments with discrete velocities, which are modulated by the noise field due to all the other charges, but have no separate identities with respect to entering times.

The Brillouin beam is taken to have radius b, and to be drifting in free space. In a narrow frequency band about ω, the injected filament with velocity v can be regarded as a circular electron stream of radius δ, carrying a convection current

$$i_1 = i_0 S(z) \exp(-j\gamma z), \tag{25}$$

$$\gamma = \frac{\omega}{v}, \tag{26}$$

where S is the unit step function, and z is measured from the entering plane. This current corresponds to an ac charge density

$$\rho_1 = \frac{\gamma}{\omega} \frac{i_1}{\pi\delta^2}. \tag{27}$$

The total charge density ρ_t at the input plane must satisfy Poisson's equation

$$\rho_t = \rho + \rho_1 = \epsilon \operatorname{div} \mathbf{E}, \tag{28}$$

where ρ is the induced charge density in the driven beam, consistent with the dynamics and charge-conservation equations for axial-symmetric space-charge waves in Brillouin-flow beams:

$$\rho = R\epsilon \operatorname{div} \mathbf{E}. \tag{29}$$

Thus the total charge density at the input plane is related to the injected charge density ρ_1 as follows:

$$\rho_t = \frac{\rho_1}{1 - R} = \frac{\gamma i_1}{\omega(\pi\delta^2)(1 - R)}. \tag{30}$$

Outside of the radius δ the charge density is zero, and the axial electric field up to the rim of the beam can be written:

$$E_{z1} = A\, I_{0r} + B\, K_{0r}, \tag{31}$$

omitting the propagation factor $\exp(-j\beta z)$ for brevity. In terms of these constants, the total charge per unit length within the very small radius δ is then

$$q_t = j2\pi\delta\epsilon(AI_{1\delta} - BK_{1\delta}) \cong -\frac{j2\pi\epsilon B}{\beta} \tag{32}$$

for $\beta\delta \ll 1$.

In the unbounded space outside of the beam, the longitudinal electric field must have the form

$$E_{z2} = C\, K_{0r}. \tag{33}$$

Taking the surface charge of the beam into account, the boundary equations at radius b are

$$A\, I_{0b} + B\, K_{0b} = C\, K_{0b}\,, \tag{34}$$

$$(1 - R)(A\, I_{1b} - B\, K_{1b}) = -C\, K_{1b}\,. \tag{35}$$

The total current inside of a cylinder of radius $r > b$ is

$$i_t(r) = 2\pi r H_\theta = \frac{j2\pi r\omega\epsilon}{\beta^2}\frac{\partial E_{z2}}{\partial r} = -\frac{j2\pi r\omega\epsilon}{\beta} C K_{1r}\,. \tag{36}$$

To obtain the transfer function needed in this problem, a relation between the injected current i_1 and the total induced current $i_t(r)$, or between their transforms $\hat{\imath}_1$ and $\hat{\imath}_t(r)$, the boundary equations must be solved for the constant C, as follows:

$$B = \frac{j\beta q_t}{2\pi\epsilon} = \frac{j\gamma\beta i_1}{2\pi\omega\epsilon(1 - R)}\,, \tag{37}$$

$$C = \frac{B(1 - R)}{1 - R\beta b I_{1b} K_{0b}} = \frac{j\gamma\beta i_1}{2\pi\omega\epsilon(1 - R\beta b I_{1b} K_{0b})}\,, \tag{38}$$

$$i_t(r) = \frac{\gamma r K_{1r} i_1}{1 - R\beta b I_{1b} K_{0b}} = F(\gamma, \beta) i_1 \tag{39}$$

and

$$\hat{\imath}_t(\beta, r) = F(\gamma, \beta)\hat{\imath}_1(\beta), \tag{40}$$

where

$$\hat{\imath}_1(\beta) = i_0 \int_0^\infty \exp j(\beta - \gamma) z\, dz = \frac{j i_0}{\beta - \gamma}\,. \tag{41}$$

The response current within the radius r is thus

$$i_t(r, z) = -\frac{i_0}{2\pi j}\int_{-\infty}^{\infty} \frac{F(\gamma, \beta) \exp(-j\beta z)\, d\beta}{\beta - \gamma} \tag{42}$$

$$= i_0 \sum_n \left[\frac{(\beta - \beta_n) F(\gamma, \beta) \exp(-j\beta z)}{\beta - \gamma}\right]_{\beta=\beta_n}. \tag{43}$$

The integrand

$$\frac{F(\gamma, \beta)}{\beta - \gamma} = \frac{(\gamma r)(\beta - \beta_e)^2 K_{1r}}{(\beta - \gamma)[(\beta - \beta_e)^2 - \beta_p{}^2 \beta b I_{1b} K_{0b}]} \tag{44}$$

has four poles:

$$\beta_{1,2} = \beta_e \pm \beta_p(\beta b I_{1b} K_{0b})^{1/2} = \beta_e \pm \beta_q \tag{45}$$

and

$$\beta_3 = \gamma, \qquad \beta_4 = 0. \tag{46}$$

The pole of $K_1(\beta r)$ at $\beta = 0$ contributes a residue $-i_0$, which serves to make $i_t(r, z)$ zero at zero frequency. This is consistent with the formulation of the problem, in which the dc component of the entering charge is neglected, and the beam itself manifests its dc current only in the plasma wave number. However, as the calculation is only valid for slow-wave propagating modes ($\beta > k$), this residue will be disregarded.

As before, the resultant expression is simplified by neglecting the small rate of change of the Bessel functions with β, replacing β by β_e where this error is small. With the time factor suppressed, the result is

$$\begin{aligned} i_t(r, z) = {} & i_0\gamma r K_1(\beta_e r)\left[\frac{\beta_q \exp(-j\beta_1 z)}{2(\beta_1 - \gamma)} - \frac{\beta_q \exp(-j\beta_2 z)}{2(\beta_2 - \gamma)}\right] \\ & + i_0\gamma r K_1(\gamma r)\left[\frac{(\gamma - \beta_e)^2 \exp(-j\gamma z)}{(\gamma - \beta_e)^2 - \beta_q^2}\right]. \end{aligned} \tag{47}$$

The assumption of small velocity spread in the entering charges, centered about the mean velocity u of the beam, permits the definition of a small quantity associated with each value of v:

$$\epsilon = \frac{v - u}{v} \ll 1, \tag{48}$$

$$(\gamma - \beta_e)^2 = (-\epsilon\beta_e)^2 \approx 0, \tag{49}$$

such that only terms up to first order in ϵ need be retained, to a good approximation. The expression for total current response then reduces to

$$i_t(r, z) \cong i_0(\gamma r)K_1(\beta_e r) \exp(-j\beta_e z)\left(\cos \beta_q z + j\epsilon \frac{\omega}{\omega_q} \sin \beta_q z\right). \tag{50}$$

The total current in the drifting beam, $i_t(b)$, is related to the total convection current, $i_c(b)$, by the ratio:[5]

$$\frac{i_c(b)}{i_t(b)} = \frac{R}{R - 1} = \frac{1}{1 - \beta b I_{1b} K_{0b}} = \frac{1}{\beta b I_{0b} K_{1b}}. \tag{51}$$

Thus,

$$i_c(b, z) = \frac{i_0 \gamma b \exp(-j\beta_e z)}{\beta_e b I_{0b}} \left(\cos \beta_q z + j\epsilon \frac{\omega}{\omega_q} \sin \beta_q z \right), \tag{52}$$

the argument of the Bessel function understood to be $(\beta_e b)$ here.

The beam responses due to electrons in different velocity ranges are assumed to add in a mean square manner. In each velocity class, the impressed current has only shot noise. Thus, using the subscript n for each velocity class, the mean square impressed current in each class is

$$i_n^2 = 2eI_n \Delta f, \tag{53}$$

where e is the electronic charge, Δf the bandwidth about $f = \omega/2\pi$, and I_n the direct current in the nth velocity class. The mean square convection current response in the beam, due to i_n, is

$$| i_c^2 |_n = \frac{2eI_n \Delta f}{I_{0b}^2} (\cos^2 \beta_q z + \epsilon_n^2 \sin^2 \beta_q z), \tag{54}$$

where ϵ_n is associated with v_n as in (48) and, approximately,

$$\overline{\gamma_n^2} \cong \beta^2 \cong \beta_e^2. \tag{55}$$

The total mean square convection current is then

$$| i_c^2 | = \frac{2eI_0 \Delta f}{I_{0b}^2} (\cos^2 \beta_q z + \overline{\epsilon^2} \sin^2 \beta_q z), \tag{56}$$

where I_0 is the total direct current in the injected filament, and

$$\overline{\epsilon^2} = \frac{\sum_n I_n \epsilon_n^2}{I_0} \cong \frac{1}{u^2} \sum_n \frac{I_n}{I_0} (v_n - u)^2, \tag{57}$$

assuming that

$$\epsilon_n^2 = \frac{(v_n - u)^2}{(v_n)^2} \cong \frac{(v_n - u)^2}{u^2}, \tag{58}$$

where u is the average velocity, given by

$$u = \frac{1}{I_0} \sum_n I_n v_n . \tag{59}$$

The expression for $| i_c^2 |$ in the finite beam is the same as that previously obtained in the one-dimensional analysis,[10] except for the presence of β_q in place of β_p within the brackets, and the term I_{0b}^2 in the denominator. Thus the maximum value of $| i_c^2 |$ is less than the total impressed shot-noise current by the factor $1/I_{0b}{}^2$, which is smaller, the larger the beam diameter.

IV. NOISE-CURRENT MODULATION OVER ENTIRE BEAM AREA

The beam of the previous section is now supposed to be uniformly modulated by impressed noise current over its entire area, subject to all of the assumptions and conditions stipulated earlier. Since the space-charge mode of interest has axial symmetry, the contribution to the total induced current by any entering charge filament is independent of its angular position. The elementary areas of excitation can be taken to be thin rings (r to $r + \delta r$), for which the transfer function relating the induced to the exciting current is the same for noise-current modulation in each velocity class as for coherent rings of injected charge, of the same velocity.

The rms charge in a ring of current with velocity v is related to the rms current in the nth velocity class by

$$dq_n = \frac{di_n}{v}, \tag{60}$$

where

$$di_n = (\overline{di_n{}^2})^{1/2} = [(J_n 2\pi r\delta r)(2e\Delta f)]^{1/2}, \tag{61}$$

J_n being the portion of the uniform current density with this velocity. As in the previous section, the total ring of charge at the input plane is related to this current element by

$$dq_t = \frac{dq_n}{1 - R} = \frac{\gamma\, di_n}{\omega(1 - R)} \tag{62}$$

and

$$di_n = |\, di_n \,| \exp{(-j\gamma z)}, \tag{63}$$

where, as before, $\gamma = \omega/v$.

To evaluate the transfer function giving the current within some radius a, outside of the beam (radius b), the cross section is divided into three regions, separated by the rings of charge at radii r and b:

$$E_{z1} = AI_{0r'} \qquad (0 \leqq r' \leqq r_-), \tag{64}$$

$$E_{z2} = BI_{0r'} + CK_{0r'} \qquad (r_+ \leqq r' \leqq b_-), \tag{65}$$

$$E_{z3} = DK_{0r'} \qquad (r' \geqq b_+). \tag{66}$$

The first expression holds inside of the injected charge ring; the second between that radius, r, and the beam boundary, b; and the last in free space outside of the beam.

The boundary equations at r and b, respectively, are:

$$AI_{0r} = BI_{0r} + CK_{0r}\,, \tag{67}$$

$$AI_{1r} - \frac{j\,dq_t}{2\pi r\epsilon} = BI_{1r} - CK_{1r}\,, \tag{68}$$

$$BI_{0b} + CK_{0b} = DK_{0b}, \tag{69}$$

$$(1 - R)[BI_{1b} - CK_{1b}] = DK_{1b}. \tag{70}$$

The total current within radius a, due to the injected charge ring at r, is

$$di_t(a) = \frac{j2\pi a\omega\epsilon}{\beta^2}\frac{\partial E_{z3}}{\partial r} = -\frac{j2\pi a\omega\epsilon}{\beta}DK_{1a}\,, \tag{71}$$

where

$$D = \frac{j\beta rI_{0r}\,dq_t}{2\pi\epsilon r\left[1 + \left(\frac{R}{1 - R}\right)\beta bI_{0b}K_{1b}\right]}. \tag{72}$$

Thus, we obtain the transfer function $F(\gamma, \beta)$ relating the transform of the total induced current $d\hat{\imath}_t(\beta, a)$ to that of the injected current ring $d\hat{\imath}_n(\beta, r)$:

$$d\hat{\imath}_t(\beta, a) = \frac{\gamma aK_{1a}I_{0r}(\beta - \beta_e)^2\,d\hat{\imath}_n}{(\beta - \beta_1)(\beta - \beta_2)} = F(\gamma, \beta)\,d\hat{\imath}_n(\beta, r), \tag{73}$$

where

$$\beta_{1,2} = \beta_e \pm \beta_p(\beta bI_{1b}K_{0b})^{1/2} = \beta_e \pm \beta_q\,. \tag{74}$$

The inverse transform of $d\hat{\imath}_t(\beta, a)$, describing the total current in the propagating wave, is evaluated as before with the approximations

$$\gamma \cong \beta \cong \beta_e \tag{75}$$

in terms that are not sensitive to changes in β:

$$di_t(z, a) = |\,di_n\,|\int_{-\infty}^{\infty}\frac{F(\gamma, \beta)\exp(-j\beta z)\,d\beta}{\beta - \gamma}, \tag{76}$$

$$= |\,di_n\,|\sum_n\left[\frac{(\beta - \beta_n)F(\gamma, \beta)\exp(-j\beta z)}{\beta - \gamma}\right]_{\beta=\beta_n}, \tag{77}$$

$$\cong |\,di_n\,|\,(\beta_e a)K_{1a}I_{0r}\exp(-j\beta_e z)\left[\cos\beta_q z + j\epsilon\frac{\omega}{\omega_q}\sin\beta_q z\right]. \tag{78}$$

Following the same summation procedure as in the case of the single injected noise filament, the total mean square current due to charge rings at r in all of the velocity classes is

$$| di_t^2(z, a) | = (2eJ_0\Delta f)(\beta_e a)^2 K_{1a}^2 I_{0r}^2 2\pi r\delta r \cdot \left[\cos^2 \beta_q z + \overline{\epsilon^2}\left(\frac{\omega}{\omega_q}\right)^2 \sin^2 \beta_q z\right], \tag{79}$$

where

$$J_0 = \sum_n J_n \tag{80}$$

is the total direct current density. The square of the total response current is found by integrating this quantity over the beam radius:

$$| i_t^2(z, a) | = (2eI_0\Delta f)(\beta_e a K_{1a})^2 (I_{0b}^2 - I_{1b}^2) \cdot \left[\cos^2 \beta_q z + \overline{\epsilon^2}\left(\frac{\omega}{\omega_q}\right)^2 \sin^2 \beta_q z\right], \tag{81}$$

where I_0 is the total direct current.

The mean square noise convection current in the drifting beam is consequently

$$| i_c^2(z) | = (2eI_0\Delta f)\left(\frac{I_{0b}^2 - I_{1b}^2}{I_{0b}^2}\right)\left[\cos^2 \beta_q z + \overline{\epsilon^2}\left(\frac{\omega}{\omega_q}\right)^2 \sin^2 \beta_q z\right] \tag{82}$$

The noise convection current at the maxima and minima of this standing wave are, respectively,

$$| i_c^2 |_{\max} = 2eI_0\Delta f\left[1 - \left(\frac{I_{1b}}{I_{0b}}\right)^2\right], \tag{83}$$

$$| i_c^2 |_{\min} = 2eI_0\Delta f\left[1 - \left(\frac{I_{1b}}{I_{0b}}\right)^2\right]\overline{\epsilon^2}\left(\frac{\omega}{\omega_q}\right)^2. \tag{84}$$

The product of maximum and minimum rms amplitudes of the noise convection current can therefore be written in the form

$$\frac{| i_{\max} i_{\min} |_B}{2eI_0\Delta f} = \left(1 - \frac{I_{1b}^2}{I_{0b}^2}\right)\frac{\omega}{\omega_q}(\overline{\epsilon^2})^{1/2}, \tag{85}$$

where the subscript B stands for the Brillouin-flow beam. If all of the electrons are accelerated by the same dc voltage V_0, such that $(eV_0/kT_c) \gg 1$, where T_c is the cathode temperature,

$$\overline{\epsilon^2} = \tfrac{1}{2}(kT_c/eV_0), \tag{86}$$

and

$$\frac{| i_{\max} i_{\min} |_B}{2eI_0\Delta f} = \left(1 - \frac{I_{1b}^2}{I_{0b}^2}\right)\frac{\omega}{2\omega_q}\frac{kT_c}{eV_0}. \tag{87}$$

By comparison, the result of the same analysis applied to the one-dimensional beam,[10] which is identical with that obtained by the Rack-Llewellyn-Peterson method,[12] is

$$\frac{|\, i_{\max} i_{\min} \,|_T}{2eI_0\Delta f} = \frac{\omega}{2\omega_q}\frac{kT_c}{eV_0}, \tag{88}$$

the subscript T standing for the "thin" beam; or

$$\frac{|\, i_{\max} i_{\min} \,|_B}{|\, i_{\min} i_{\max} \,|_T} = 1 - \frac{I_{1b}^2}{I_{0b}^2} \tag{89}$$

if the two types of beam are compared on the basis of the same I_0, V_0, T_c and ω/ω_q. This ratio is less than unity for finite βb.

Although the "noisiness" $|\, i_{\max}\, i_{\min} \,|$ of a thin beam is a measure of the least attainable noise figure of any amplifier using that beam,[14, 15, 16, 17] it does not follow from this result that the Brillouin-flow beam is necessarily less noisy than a thin beam with the same direct current and voltage. For instance, in a thin beam the shot-noise current is $2eI_0\Delta f$ and all of I_0 is effective in interaction with the longitudinal RF field of an amplifier circuit. In the Brillouin-flow beam, however, the RF field has both longitudinal and transverse components, and varies in intensity over the beam cross section. The effective part of the total beam current, therefore, may be less than I_0.

In single-velocity thin-beam theory, the kinetic power P_k accounts for virtually all of the power transported by the space-charge waves, and may be defined by[15]

$$\mathrm{Re}(P_k) = \tfrac{1}{2}K(i_f^2 - i_s^2), \tag{90}$$

where i_f and i_s are the convection currents in the "fast" and "slow" traveling waves, respectively, and

$$K = 2\frac{\omega_q}{\omega}\frac{V_0}{I_0}. \tag{91}$$

In terms of K the noise-current expression for the thin beam may be rewritten as

$$P_s = \tfrac{1}{2}K\,|\, i_{\max} i_{\min} \,| = kT_c\Delta f. \tag{92}$$

This noise quantity has the dimensions of power; we may call it noisiness. It is invariant in all beam transformations not involving loss of RF power.[14] The minimum attainable noise figure F_T of any amplifier depending on RF interaction between a circuit and the slow space-charge wave has been shown to be[15, 16]

$$F_T = 1 + P_s/(kT\Delta f) = 1 + T_c/T, \tag{93}$$

where T is the ambient temperature. This summary of thin-beam theory applies to a thin hollow beam as well as a filamentary beam, as in both such beams the RF field acts equally on all of the direct current I_0.

From this it follows that the minimum noise figure F_B of any amplifier using the ideal Brillouin beam we have discussed can be evaluated by finding the noise kinetic power of an equivalent thin beam. Both beams will be equivalent with respect to interaction with any external RF circuit if both produce the same fields (or wave admittances) in free space just outside of the thick beam, at $r = b$.

Just outside of the Brillouin beam, with current I_0 and voltage V_0, the TM wave admittance looking into the beam is[5]

$$Y = \frac{H_\theta}{E_z} = \frac{j\omega\epsilon}{\beta}\left[1 - \frac{\omega_p^2}{(\omega - \beta u)^2}\right]\frac{I_{1b}}{I_{0b}}. \tag{94}$$

The portion Y_d of Y due to displacement current i_d in the volume occupied by the beam is given by the same expression, with $\omega_p^2 = 0$:

$$Y_d = \left(\frac{i_d}{2\pi b E_z}\right)_{r=b} = \frac{j\omega\epsilon}{\beta}\frac{I_{1b}}{I_{0b}}. \tag{95}$$

The remainder of the total admittance is due to the convection current i_c in the beam:

$$Y_e = \left(\frac{i_c}{2\pi b E_z}\right)_{r=b} = -\frac{j\omega\epsilon}{\beta}\frac{\omega_p^2}{(\omega - \beta u)^2}\frac{I_{1b}}{I_{0b}}. \tag{96}$$

The equivalent beam is chosen to be a thin hollow beam, of the same radius b as the Brillouin beam, with current i_0 not yet specified, and the same voltage V_0. We can take the ac convection current of the thin beam as equal to the total convection current of the Brillouin flow beam, because it can be shown that the total convection current of the Brillouin beam is equal to the surface current to within a small fraction ω_q/ω.

The relation between total convection current i_c and longitudinal field E_z in this hollow beam is

$$\left(\frac{i_c}{E_z}\right)_{r=b} = -\frac{j\beta_e}{(\beta_e - \beta)^2}\left(\frac{i_0}{2V_0}\right). \tag{97}$$

Its electronic admittance in space just outside of this beam is

$$Y_e = \left(\frac{H_\theta}{E_z}\right)_{r=b} = -\frac{j\beta_e}{2\pi b(\beta_e - \beta)^2}\left(\frac{i_0}{2V_0}\right). \tag{98}$$

Near $\beta_e u = \omega$, the admittance Y_d due to displacement current in the space inside of this cylinder is the same as for the Brillouin-flow beam:

$$Y_d = \frac{j\omega\epsilon}{\beta}\frac{I_{1b}}{I_{0b}}. \tag{99}$$

The two beams will then be equivalent if their electronic admittances are the same at $r = b$:

$$-\frac{j\beta_e}{2\pi b(\beta_e - \beta)^2}\left(\frac{i_0}{2V_0}\right) = -\frac{j\omega\epsilon}{\beta}\frac{\omega_p^{\ 2}}{(\omega - \beta u)^2}\frac{I_{1b}}{I_{0b}}, \tag{100}$$

$$\frac{2V_0}{i_0} = \frac{\beta u}{2\pi b\epsilon\omega_p^2}\frac{I_{0b}}{I_{1b}}. \tag{101}$$

As this expression changes relatively slowly with βb, the admittances of the thin hollow beam and of the Brillouin beam vary in essentially the same way with β. This approximation, therefore, is fairly good over a small range of β about ω/u.

The noisiness of the equivalent hollow beam is

$$P_s = \tfrac{1}{2}K \mid i_{\max} i_{\min} \mid , \tag{92}$$

where

$$K = \frac{2\omega_q}{\omega}\frac{V_0}{i_0} = \frac{\omega_q}{\omega}\frac{\beta u}{2\pi b\epsilon\omega_p^2}\frac{I_{0b}}{I_{1b}} \tag{102}$$

and

$$\mid i_{\max} i_{\min} \mid = \left(1 - \frac{I_{1b}^{\ 2}}{I_{0b}^{\ 2}}\right)\frac{\omega I_0(kT_c\Delta f)}{\omega_q V_0}, \tag{103}$$

as found above for the thick beam. Since the direct current density and longitudinal velocity of this beam are constant over its cross section,

$$\frac{I_0}{2V_0} = \frac{e}{m}\frac{I_0}{u^2} = \frac{\pi b^2}{u}\omega_p^{\ 2}\epsilon. \tag{104}$$

With these substitutions, the expression for noisiness P_s in the Brillouin-flow beam reduces to

$$P_s = (I_{0b}^{\ 2} - I_{1b}^{\ 2})\left(\frac{\beta b}{2I_{1b}I_{0b}}\right)kT_c\Delta f. \tag{105}$$

Another way to state this result is to express the minimum attainable noise figure F_B of the Brillouin-flow beam in terms of that of the thin beam (whose noisiness is $kT_c\Delta f$):

$$\frac{F_B - 1}{F_T - 1} = (I_{0b}^{\ 2} - I_{1b}^{\ 2})\left(\frac{\beta b}{2I_{1b}I_{0b}}\right). \tag{106}$$

This ratio, plotted in Fig. 1, varies rather slowly from unity at $\beta b = 0$, to one-half at $\beta b \rightarrow \infty$. With $F_T = 4$, corresponding to about 6 db, the predicted value[16] for a univelocity thin beam, the least noise figure of the infinitely broad beam, for example, would be 4 db.

We should, of course, recall that this result applies for the unusual but not unphysical case of a beam with no transverse velocities.

Haus[18] has demonstrated formally that an amplifier with a thick beam in confined flow cannot have a lower noise figure than one with a thin beam, when the input conditions are full shot-noise current and the Rack equivalent velocity fluctuations. His proof depends on expansion of the excitation in terms of a complete orthogonal set of functions at the input plane. In the absence of mode coupling in the acceleration region, each mode can be treated as though it were along a single thin beam, independent of the other modes. The opposite point of view has been advanced by Beam and Bloom[19] and by Paschke.[20] They have argued, essentially, that a lower noise figure can be obtained with a thicker beam (in confined flow), because the field of the RF circuit couples less effectively to the beam interior than to its surface, whereas

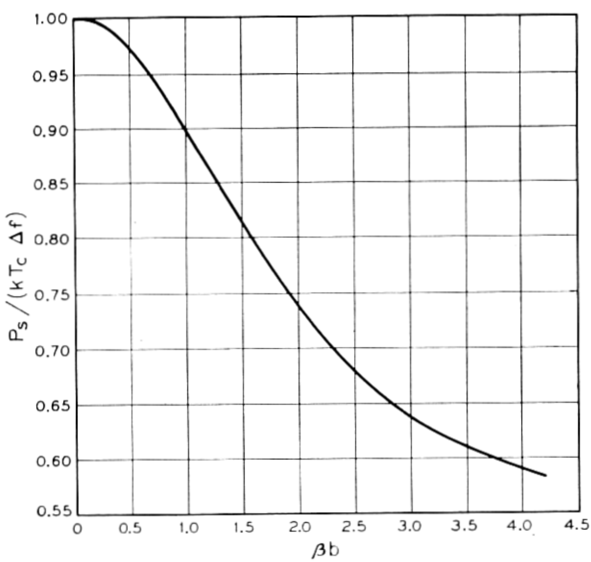

Fig. 1. — The ratio of the noisiness P_s of an idealized Brillouin-flow beam to that of an equivalent thin hollow beam in confined flow, as a function of the product of propagation constant β and beam radius b. The ordinates also represent the ratio $(F_B - 1)/(F_T - 1)$, where F_B and F_T are the minimum noise figures attainable with the two types of electron beam, respectively, when they are abruptly accelerated from temperature-limited cathodes [see (106)].

the noise excitation is uniform over the entire cross section. This argument, however, assumes that the circuit field in the presence of the beam is the same as in its absence—an assumption open to question.

In connection with the fact that we have found a noisiness less than that prescribed by Haus, we can only note that, for the beam with zero magnetic field and for the Brillouin flow beam, in whose interior the ac space-charge density is zero, the set of propagating space-charge modes is incomplete. (There are no slow space-charge modes with radial periodicity.) It may be that the axial-symmetric mode fails to propagate all of the axial-symmetric noise excitation and the higher-order modes fail to carry all of the excitation with angular periodicity. Recent calculations by Bobroff and Haus[21] point to the same conclusions—that the space-charge wave modes in such beams do not form a complete set, and therefore that an arbitrary initial excitation cannot be expanded in terms of these modes.

The noisiness of beams produced by shielded guns is actually much greater than that calculated for the idealized beam, because of the transverse thermal electron velocities near the cathode, neglected in the calculation. Their principal effect, as Beam has shown,[22] is to increase the velocity fluctuations near the potential minimum due to "mixing" of electrons from different parts of the cathode. The increase in noisiness due to this effect probably outweighs any possible decrease due to increase in beam diameter. However, the noise reduction mechanism described by the calculations may perhaps play a role in low-noise beams of a special type.

Noise figures considerably less than the 6-db minimum for an abruptly accelerated thin beam have been observed by a number of workers. Using a hollow confined-flow beam in a backward-wave amplifier, Currie and Forster[6] have measured a noise figure of less than 4 db. More recently, St. John and Caulton[7] have attained a noise figure of 4.5 db with a fairly conventional gun and, by using a solid-circular gun similar in cross section to that of Currie and Forster's annular gun, they attained a 3.5-db noise figure at microwave frequencies. Noise reduction due to a gradual acceleration allowing drifting[9] has been put forward as a plausible explanation of such low noise figures.

It should be noted, however, that in both instances the beams were found to have current density profiles sharply peaked at the surface, so as to resemble to some degree the case of Brillouin flow, in which the ac current is at the surface of the beam. Their low noisiness, therefore, might, at least in part, have been due to the noise-reduction mechanism described by the calculations of this paper.

V. SPACE-CHARGE WAVES INDEPENDENT OF BOUNDARY CONDITIONS

In analyses of slow-wave propagation along electron beams produced by magnetically shielded guns,[4, 5] two pairs of space-charge waves are found. In one of these, the field distributions and propagation constants depend in the usual way on the transverse boundary conditions. The waves of the second pair, however, are not accompanied by any field outside of the beam; they have never been detected experimentally and they are not found when magnetic flux, however slight, threads the cathode.[23, 24] These very singular waves appear to have first been described in 1946 by Feenberg and Feldman.[4]

For simplicity, the properties of such waves will be examined in the case of axial-symmetric fields in a Brillouin-flow beam.[5] At the surface of this beam, the boundary conditions are (i) that E_z be continuous, and (ii) that

$$[(1 - R)E_r]^{\text{beam}} = [E_r]^{\text{space}}, \tag{107}$$

where $R = \omega_p^2/\omega_q^2$ as defined in (11). For these waves, $R = 1$. It follows that the fields are zero outside of the beam, and E_z is zero at the common boundary. The waves, therefore, cannot be excited by fields outside of the beam.

Within the beam, if excited somehow, they would propagate with arbitrary radial field distribution and the longitudinal propagation constants

$$\beta_{1,2} = \beta_e \pm \beta_p , \tag{108}$$

which are characteristic of waves with *purely longitudinal fields*. (In ordinary space-charge waves, the plasma oscillation frequency is reduced, because of transverse fields coupling the current filaments to one another and to other currents.) However, if E_r were zero everywhere inside of the beam, E_z would also be zero, as it is zero at the boundary. This leads one to suspect that these waves do not really exist at all.

It was shown that, when a Brillouin-flow beam is current-modulated, the total charge density ρ_t at any point in the excitation plane is related to the injected charge density ρ_1, and to that induced in the smoothed-out beam, ρ, by the equations

$$\rho_t = \rho + \rho_1 = \epsilon \operatorname{div} \mathbf{E}, \tag{28}$$

$$\rho = R\epsilon \operatorname{div} \mathbf{E}. \tag{29}$$

When $R = 1$, therefore, the initial conditions are $\rho_t = \rho$ for all values of the injected charge ρ_1. This means that the $R = 1$ modes cannot be excited by charge modulation or, since the charge-injection velocity is arbitrary, by either current or velocity modulation within the beam. As, in addition, they cannot be excited by external voltage modulation, the $R = 1$ modes are physically nonexistent.

REFERENCES

1. Scotto, M. and Parzen,P., Excitation of Space-Charge Waves in Drift Tubes, J. Appl. Phys, **27,** April 1956, p. 275.
2. Bresler, A. D., Joshi, G. H. and Marcuvitz, N., Orthogonality Properties for Modes in Passive and Active Uniform Waveguides, J. Appl. Phys., **29,** May 1958, p. 794.
3. Ramo, S., Space Charge and Field Waves in an Electron Beam, Phys. Rev., **56,** August 1, 1939, p. 276.
4. Feenberg, E. and Feldman, D., Theory of Small-Signal Bunching in a Parallel Electron Beam of Rectangular Cross Section, J. Appl. Phys., **17,** December 1946, p. 1025.
5. Rigrod, W. W. and Lewis, J. A., Wave Propagation Along a Magnetically Focused Cylindrical Electron Beam, B.S.T.J., **33,** March 1954, p. 399.
6. Currie, M. R. and Forster, D. C., Low-Noise Tunable Preamplifiers for Microwave Receivers, Proc. I.R.E., **46,** March 1958, p. 570; Currie, M. R., Letter to the Editor, Proc. I.R.E., **46,** May 1958, p. 911.
7. St. John, G. E. and Caulton, M., S-Band TWT with Noise Figure Below 4 db, Proc. I.R.E., **46,** May 1958, p. 911.
8. Siegman, A. E., Analysis of Multivelocity Electron Beams by the Density-Function Method, J. Appl. Phys., **28,** October 1957, p. 1132.
9. Siegman, A. E., Watkins, D. A. and Shieh, H., Density-Function Calculations of Noise Propagation on an Accelerated Multivelocity Electron Beam, J. Appl. Phys., **28,** October 1957, p. 1138.
10. Pierce, J. R., A New Method of Calculating Microwave Noise in Electron Streams, Proc. I.R.E., **40,** December 1952, p. 1675.
11. Beck, A. H. W., High-Order Space-Charge Waves in Klystrons, J. of Elect., **2,** March 1957, p. 489.
12. Pierce, J. R., *Traveling-Wave Tubes*, D. Van Nostrand Co., New York, 1950, Ch. 10.
13. Landau, L., On the Vibrations of the Electronic Plasma, J. Phys. (U.S.S.R.), **10,** 1946, p. 25.
14. Pierce, J. R., A Theorem Concerning Noise in Electron Streams, J. Appl. Phys., **25,** August 1954, p. 931.
15. Haus, H. A. and Robinson, F. N. H., The Minimum Noise Figure of Microwave Beam Amplifiers, Proc. I.R.E., **43,** August 1955, p. 981.
16. Pierce, J. R. and Danielson, W. E., Minimum Noise Figure of Traveling-Wave Tubes with Uniform Helices, J. Appl. Phys., **25,** September 1954, p. 1163.
17. Bloom, S. and Peter, R. W., A Minimum Noise Figure for the Traveling-Wave Tube, R.C.A. Rev., **15,** June 1954, p. 252.
18. Haus, H. A., Limiting Noise Figure of a Microwave Tube with a Beam of Finite Diameter, Quart. Prog. Rep., M. I. T. Res. Lab. of Elect., **32,** January 15, 1957.
19. Beam, W. R. and Bloom, S., Minimum Noise Figure of Traveling-Wave Tubes, Including Higher Space-Charge Wave Modes, I.R.E.-A.I.E.E. Electron Tube Research Conf., Boulder, Colo., June 1956.
20. Paschke, F., Die Wechselseitigkeit der Kopplung in Wanderfeldröhren, Arch. Elek.Über., **11,** April 1957, p. 137.
21. Bobroff, D. L. and Haus, H. A., Uniqueness and Orthogonality of Small Signal Solutions in Electron Beams, to be published.
22. Beam, W. R., Noise Wave Excitation at the Cathode of a Microwave Beam Amplifer, Trans. I.R.E., **ED-4,** July 1957, p. 226.
23. Brewer, G. R., Some Effects of Magnetic Field Strength on Space-Charge Wave Propagation, Proc. I.R.E., **44,** July 1956, p. 896.
24. Rigrod, W. W., this issue, pp. 119–139.