Nonuniformities in Laminated Transmission Lines

By GORDON RAISBECK

(Manuscript received April 4, 1958)

The effect on transmission properties of certain nonuniformities in laminated transmission lines has been calculated by a perturbation method. Particular cases have been calculated, including the effect of varying radius of curvature in a cylindrical Clogston line, the effect of systematic variation in effective dielectric constant and the effect of random variation in layer thickness where the nonuniformities are known only through statistical properties. In one Clogston line where measurements of nonuniformity have been made the method predicts a transmission impairment in substantial agreement with observations.

I. INTRODUCTION

The remainder of this paper is divided into seven parts. Section II contains an outline of the notation used and then a discussion of a new formula for the losses in a parallel-plane laminated transmission line* due to irregularities in the laminations. The section concludes with an outline of a procedure for calculating losses in a given line due to known irregularities.

Section III shows the derivation of the formula by the application of a perturbation procedure to a differential equation derived by Morgan.[3]† The procedure leads to an expansion of the attenuation in a

* The fundamental notions about laminated conductors, often called "Clogston conductors" or "Clogston lines", are given in a paper by Clogston.[1] Further details and embodiments are also shown in U. S. Patents 2,769,147 (A. M. Clogston and H. S. Black), 2,769,148 (A. M. Clogston), 2,769,149 (J. G. Kreer) and 2,769,150 (H. S. Black and S. P. Morgan). Experimental results were described in a report by Black, Mallinckrodt and Morgan.[2] The most thorough mathematical treatment published to date is an exhaustive and detailed analysis by Morgan.[3] Vaage[4] has reduced some of the results to terms more familiar to transmission engineers. King and Morgan[5] give a lucid retrospective glance over the whole subject, and present a series of charts and formulas which enable one easily to make quantitative estimates of transmission parameters of interest. This paper is probably the easiest place to start a study of laminated conductors, and provides more than enough background for reading the present paper.

† Ref. 3 is referred to hereafter simply as Morgan.

power series whose argument is the magnitude of the irregularities in the laminations and whose coefficients depend on the distribution of the irregularities. The expansion is carried as far as the first nonvanishing term.

Section IV compares the magnitude of allowable irregularities computed with the new formula to values computed by an exact method for the two extreme cases described by Morgan. The results correspond within five per cent.

Section V shows that the effect of finite lamination thickness can be correctly computed by considering the finite laminae as perturbations of a uniform medium. It is proved heuristically that the effects of nonuniformity and of finite lamination thickness are independent and additive.

Section VI applies the formula of Section III to random irregularities. It is shown that the formula is directly applicable when the irregularities are specified by the absolute value of their Fourier spectra (i.e., by their autocorrelation), and two special cases are worked out in detail.

Section VII applies the formula to the measured irregularities in an experimental cylindrical laminated line. The correspondence between predicted and measured losses is satisfactory, and demonstrates that the method described, despite drastic simplifying assumptions, gives results agreeing quantitatively with experiment.

Section VIII shows how to compute the effect of curvature of laminae in a coaxial Clogston line. The results are precisely those predicted by Morgan. They give independent support to his formulas, which are based on certain plausible physical assumptions and approximations.

II. DESCRIPTION OF RESULTS

The notation used by Morgan will be adopted throughout. In particular, we shall deal repeatedly with thin laminae of a conductor and of an insulating dielectric, with physical properties identified as follows:

$$\begin{aligned} &\text{conductor thickness} = t_1 , \\ &\text{conductor magnetic permeability} = \mu_1 , \\ &\text{conductor conductivity} = g_1 , \\ &\text{insulator thickness} = t_2 , \\ &\text{insulator dielectric permittivity} = \epsilon_2 , \\ &\text{insulator magnetic permeability} = \mu_2 . \end{aligned} \tag{1}$$

Sometimes the properties of all layers of one material will be assumed identical, but at other times, as shown by the context, we shall assume

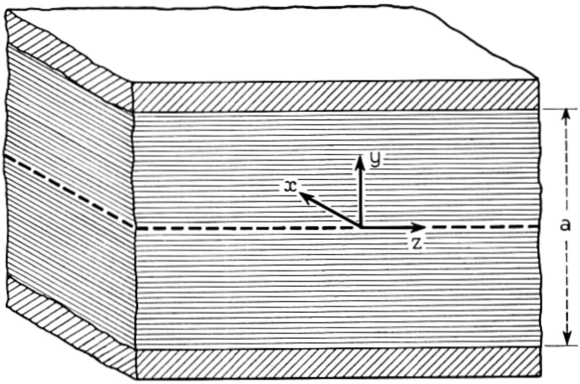

Fig. 1 — A schematic section of a parallel-plane Clogston 2 laminated transmission line between two bounding surfaces, showing the orientation of a coordinate system.

that the properties vary from layer to layer about their nominal values. In general, we shall assume

$$\mu_1 = \mu_2 = \mu_0 \tag{2}$$

because this seems to be sufficiently general to include most cases of interest.

Consider a parallel-plane Clogston 2 transmission line bounded by infinite-impedance sheets at $y = \pm\frac{1}{2}a$ (see Fig. 1). Near any given point the average electrical constants of the stack are* (see Fig. 2)

$$\begin{aligned} \bar{\epsilon} &= \frac{\epsilon_2}{1 - \theta}, \\ \bar{\mu} &= \theta\mu_1 + (1 - \theta)\mu_2, \\ \bar{g} &= \theta g_1, \end{aligned} \tag{3}$$

where the subscript "1" refers to the conductor material, the subscript "2" to the dielectric material and

$$\theta = \frac{t_1}{t_1 + t_2} \tag{4}$$

is the fraction of the cross section of the line made up of the conducting material.

Assume that $\bar{\epsilon}$, $\bar{\mu}$ and $\bar{g}$ are not quite constant, but vary about average values thus:

* Morgan,[3] Equation 90.

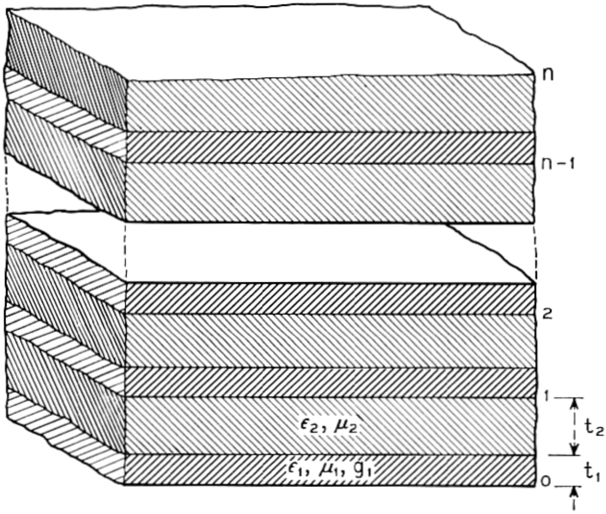

Fig. 2 — Detailed schematic section showing the individual layers in a laminated transmission line.

$$
\begin{aligned}
\bar{\epsilon} &= \bar{\epsilon}_0 + \Delta\bar{\epsilon}, \\
\bar{\mu} &= \bar{\mu}_0 + \Delta\bar{\mu}, \\
\bar{g} &= \bar{g}_0 + \Delta\bar{g},
\end{aligned}
\tag{5}
$$

where the average values are characterized by a subscript "0", and $\Delta\bar{\epsilon}$, $\Delta\bar{\mu}$ and $\Delta\bar{g}$ have average value zero.

Following Morgan, let

$$\xi = \frac{y}{a} + \frac{1}{2}. \tag{6}$$

This normalizes the thickness, so that in terms of ξ the line is bounded by the planes $\xi = 0$, $\xi = 1$. Then let

$$w(\xi) = H_x(y), \tag{7}$$

the x-component of the magnetic field of a wave traveling down the line in the z-direction. Also let

$$
\begin{aligned}
\frac{\Delta\bar{\mu}}{\bar{\mu}_0} + \frac{\Delta\bar{\epsilon}}{\bar{\epsilon}_0} &= \frac{C}{\omega\bar{\mu}_0\bar{g}_0 a^2}\,\varphi(y) \\
&= \frac{C}{\omega\bar{\mu}_0\bar{g}_0 a^2}\,f(\xi).
\end{aligned}
\tag{8}
$$

The new function $f(\xi)$ is a dimensionless function which is a measure of the deviation of $\bar{\mu}$ and $\bar{\epsilon}$ from their average values $\bar{\mu}_0$ and $\bar{\epsilon}_0$. The constant C is a dimensionless parameter. (It is convenient to assume that $f(\xi)$ has fixed magnitude in some sense but a variable shape, and that all variation in the magnitude of the irregularities of the line is due to C. Actually, all that is specified is the product $Cf(\xi)$, which may be factored in any way you please.) It is shown by Morgan that

$$\frac{d^2w}{d\xi^2} + [\Lambda - iCf(\xi)]w(\xi) = 0, \tag{9}$$

with the boundary conditions

$$w(0) = w(1) = 0, \tag{10}$$

and also that

$$\gamma = \alpha + j\beta = i\omega\sqrt{\bar{\mu}\bar{\epsilon}_0}(1 + \Lambda/i\omega\bar{\mu}_0\bar{g}_0a^2)^{1/2}, \tag{11}$$

or approximately

$$\begin{aligned} \alpha = \operatorname{Re} \gamma &= \operatorname{Re} \frac{\Lambda}{2\sqrt{\bar{\mu}_0/\bar{\epsilon}_0}\bar{g}_0a^2}, \\ \beta = \operatorname{Im} \gamma &= \omega\sqrt{\bar{\mu}_0\bar{\epsilon}_0} + \operatorname{Im} \frac{\Lambda}{2\sqrt{\bar{\mu}_0/\bar{\epsilon}_0}g_0a^2}, \end{aligned} \tag{12}$$

where Λ is an eigenvalue of the differential equation. If the stack is perfectly uniform, $\Delta\bar{\mu}$ and $\Delta\bar{\epsilon}$ are zero, and the eigenvalues are

$$\Lambda = \pi^2, 4\pi^2, 9\pi^2, \cdots \tag{13}$$

corresponding to the eigenfunctions

$$w = \sqrt{2} \sin \pi\xi, \sqrt{2} \sin 2\pi\xi, \sqrt{2} \sin 3\pi\xi, \cdots . \tag{14}$$

These eigenfunctions show that (as is well known from Morgan) the magnetic field strengths in the various modes in a uniform line vary sinusoidally across the stack, the variation going through one, two, three, $\cdots$ half sine waves and always vanishing at the surfaces.

Since C varies continuously, we expect the eigenvalues and eigenfunctions to vary continuously in a manner depending on $f(\xi)$. If C is small, i.e., if the irregularity of the stack is small, the eigenvalues can be expanded in a power series in C. The analysis is in Section III of this paper. The first three terms of the series for the lowest eigenvalue are

$$\Lambda_1 = \pi^2 + iCa_1 + C^2 \sum_{m=2}^{\infty} \frac{a_m^{\ 2}}{\pi^2m^2 - \pi^2} + O(C^3), \tag{15}$$

where

$$a_m = 2 \int_0^1 f(x) \sin \pi x \sin m\pi x \, dx; \tag{16}$$

that is, the coefficients a_m are the coefficients of the Fourier sine series for

$$f(x)\sqrt{2} \sin \pi x = \sum_1^\infty a_m \sqrt{2} \sin \pi m x. \tag{17}$$

From the above expression one can conclude that

$$\alpha = \alpha_0 \left(1 + \frac{C^2}{\pi^2} \sum_2^\infty \frac{{a_m}^2}{\pi^2 m^2 - \pi^2}\right), \tag{18}$$

where α_0 is the attenuation which the line would have if it had no irregularities.

An alternative approach is derived from the theory of operators. The result is

$$\begin{aligned}\Lambda_1 &= \pi^2 + iC \int_0^1 f(x)\sqrt{2} \sin \pi x \sqrt{2} \sin m\pi x \, dx \\ &+ C^2 \int_0^1 \int_0^1 k(x, y) f(x) \sqrt{2} \sin \pi x \, f(y) \sqrt{2} \sin \pi y \, dx \, dy + O(C^3),\end{aligned} \tag{19}$$

where $k(x, y)$ is the reduced resolvent kernel:

$$\begin{aligned} k(x, y) &= \sum_{m=2}^\infty \frac{\sqrt{2} \sin m\pi x \sqrt{2} \sin m\pi y}{m^2\pi^2 - \pi^2} \\ &= \frac{1}{\pi^2} [\sin \pi x \sin \pi y - 2\pi x \cos \pi x \sin \pi y \\ &\qquad -2\pi y \cos \pi y \sin \pi x + 2\pi \sin \pi x \cos \pi y], \qquad x \leqq y, \\ k(y, x) &= k(x, y). \end{aligned} \tag{20}$$

In this particular case it is not hard to show that the results are identical.

For practical results, it is convenient to let

$$\frac{\Delta\bar{\epsilon}}{\bar{\epsilon}_0} + \frac{\Delta\bar{\mu}}{\bar{\mu}_0} = A\varphi(y) = Af(\xi) \tag{21}$$

and to assume

$$\int_0^1 f^2(\xi) \, d\xi = 1. \tag{22}$$

Note that

$$A = \left\langle \frac{\Delta \bar{\epsilon}}{\bar{\epsilon}_0} + \frac{\Delta \bar{\mu}}{\bar{\mu}_0} \right\rangle, \tag{23}$$

where the angular brackets denote the root mean square value. As a convenience, we make the definition

$$\Sigma = \sum_2^\infty \frac{a_m^{\ 2}}{\pi^2 m^2 - \pi^2} = 2 \int_0^1 \int_0^1 k(x, y) f(x) f(y) \sin \pi x \sin \pi y \, dx \, dy, \tag{24}$$

where a_m and k are defined as above.

The expression for attenuation can be written

$$\alpha = \alpha_0 \left(1 + \frac{f^2}{f_i^2} \right), \tag{25}$$

where

$$f_i = \frac{1}{2A \bar{\mu}_0 \bar{g}_0 a^2 \Sigma^{1/2}}. \tag{26}$$

In a line having finite laminations,

$$\alpha_0 = \alpha_{00} \left[1 + \frac{f^2}{(f_2')^2} \right] \tag{27}$$

(see Fig. 3), except for very low or very high frequencies, where

$$f_2' = \frac{\sqrt{3}}{2\mu_1 g_1 t_1 T_1}. \tag{28}$$

Heuristic reasoning and experimental evidence are given in a later section to show that the effects of finite laminations and of nonuniformity are additive, at least if both are small. Hence we can write

$$\begin{aligned} \alpha &= \alpha_{00} \left[1 + \frac{f^2}{f_i^2} + \frac{f^2}{(f_2')^2} \right] \\ &= \alpha_{00} \left[1 + \frac{f^2}{(f_2')^2} (1 + b^2) \right] \\ &= \alpha_{00} \left[1 + \frac{f^2}{(f_{i2})^2} \right], \end{aligned} \tag{29}$$

where

$$b = \frac{f_2'}{f_{i2}} \tag{30}$$

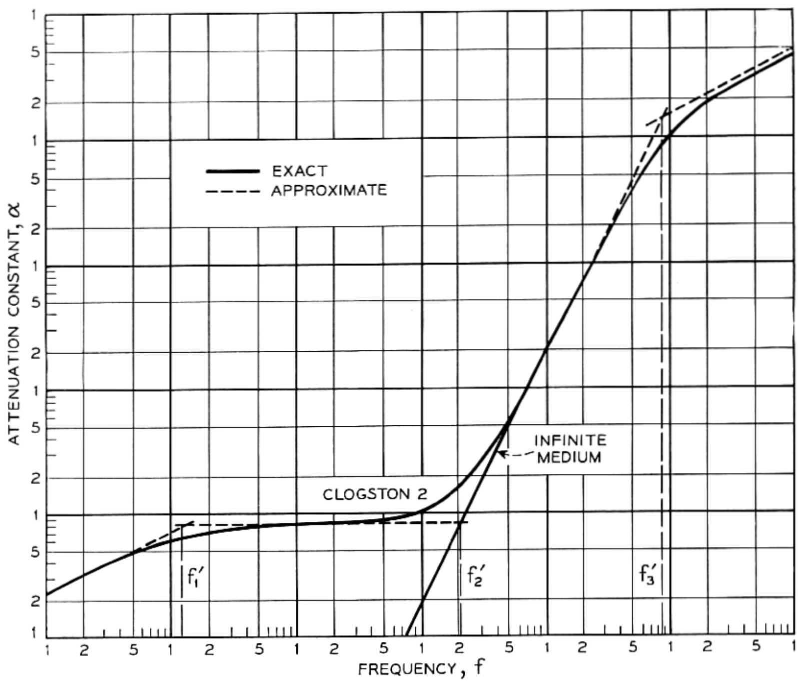

Fig. 3 — Attenuation in a plane Clogston 2 laminated transmission line, and in an infinite laminated medium, on a log-log scale as a function of frequency.

and α_{00} is the low-frequency attenuation of the uniform line. The particular reason for this choice of variable is the simplification of various formulae. If we assume

$$\mu_1 = \mu_2 = \bar{\mu}, \tag{31}$$

as is ordinarily the case when no ferromagnetic materials are involved,

$$b = \frac{\sqrt{3}\,nA\sqrt{\Sigma}}{\theta}, \tag{32}$$

where n is the number of layers in the stack. This expression is simple, and separates the various parameters that enter into the problem: n and θ depend only on the geometry of the line, A on the magnitude of the irregularities and Σ on the distribution of the irregularities. The whole effect of the irregularities can be summed up by saying that the effective bandwidth of the line is reduced by a factor

$$\frac{f_{i2}}{f_2'} = \frac{1}{\sqrt{1 + b^2}}. \tag{33}$$

It is clear that such quantities as conductivity, dielectric constant, dimensions and so on do not appear in the expression for b. Hence, although the cutoff frequency of the line is a complex function of all these quantities, the relative bandwidth of the line compared with that of a similar line having no irregularities depends only on n, θ, the magnitude of the irregularities A and the quantity Σ derived from the distribution of the irregularities.

If it is assumed that $\mu_1 = \mu_2$, then it is logical to assume that $\Delta\bar{\mu} = 0$. In this case

$$Af(\xi) = \frac{\Delta\epsilon}{\bar{\epsilon}_0}. \tag{34}$$

If it is further assumed* that the local variations are due to variation in thickness t_1 and t_2, it is easy to show from the formula for $\bar{\epsilon}$ that

$$\frac{\Delta\bar{\epsilon}}{\bar{\epsilon}_0} = \theta\left(\frac{\Delta t_1}{t_1} - \frac{\Delta t_2}{t_2}\right), \tag{35}$$

where t_1 and t_2 are nominal values, and Δt_1 and Δt_2 are deviations of the actual from the nominal values, of the thicknesses of conductor material and dielectric material, and

$$\theta = \frac{t_1}{t_1 + t_2}. \tag{36}$$

As a practical matter, the computation of the loss of bandwidth due to irregularities can be carried out as follows. First, determine the fractional irregularities in the line

$$\frac{\Delta t_1}{t_1} \tag{37}$$

and

$$\frac{\Delta t_2}{t_2} \tag{38}$$

* At present, it seems likely that the effect of irregularity in ϵ_2 will be small compared to the effect of thickness irregularities in practical cases.

as functions of $\xi = y/a + 1/2$. Then determine

$$A = \left\langle \frac{\Delta\bar{\epsilon}}{\bar{\epsilon}_0} \right\rangle = \theta \left[\int_0^1 \left(\frac{\Delta t_1}{t_1} - \frac{\Delta t_2}{t_2} \right)^2 d\xi \right]^{1/2} \tag{39}$$

and

$$f(\xi) = \frac{\theta}{A} \left(\frac{\Delta t_1}{t_1} - \frac{\Delta t_2}{t_2} \right). \tag{40}$$

Now,

$$\int_0^1 f^2(\xi)\, d\xi = 1 \tag{41}$$

and

$$A f(\xi) = \frac{\Delta\bar{\epsilon}}{\bar{\epsilon}_0}. \tag{42}$$

Then, using either the Fourier coefficients a_m (16) or the reduced resolvent k (20), find the sum Σ defined in (24). Finally, form the quantity b (32). Then the reduction in bandwidth of the line compared to a similar line with no irregularities is given by (33).

The perturbation method can also be applied to finding the effect of finite lamination thickness. When applied to the above case, it gives Morgan's result

$$\gamma^2 = -\omega^2\mu\bar{\epsilon} \left[1 + \frac{\pi^2}{i\omega\mu\bar{g}a^2} - \frac{i\omega\mu\bar{g}\theta^2 a^2}{12n^2} \right]. \tag{43}$$

When applied to a cylindrical line of outer radius b and inner radius zero, it gives, in agreement with Morgan's equation 486,

$$\gamma^2 = -\omega^2\mu\bar{\epsilon} \left[1 + \frac{(3.8317)^2}{i\omega\mu\bar{g}b^2} - \frac{i\omega\mu\bar{g}\theta^2 b^2}{12n^2} \right]. \tag{44}$$

The only difference between (43) and (44) is that the first root π of

$$\sin x = 0 \tag{45}$$

s replaced by the first root 3.8317 of

$$J_1(x) = 0. \tag{46}$$

As a result, the low-frequency attenuation of a solid cylindrical Clogston line of radius b is higher than that of a flat line of thickness b by a factor $(3.8317/\pi)^2 = 1.4876$, but the high-frequency attenuation is exactly the same! The crossover frequency f_{2i} is shifted up by a factor $3.8317/\pi = 1.2197$.

III. EVALUATION OF ATTENUATION IN THE NONUNIFORM LINE

The expansion of Λ_1, and hence of α, in terms of powers of C is accomplished as follows.† The governing equation is equation 532 of Morgan (Ref. 3, p. 131):

$$\frac{d^2w}{d\xi^2} + [\Lambda - jCf(\xi)]w(\xi) = 0, \tag{47}$$

subject to the boundary conditions

$$w(0) = 0, \qquad w(1) = 0. \tag{48}$$

Here Λ is the eigenvalue from which the propagation constant of the line can be derived by (11), and $w(\xi)$ is the corresponding eigenfunction which tells, according to (7), how the amplitude of the transverse component of the magnetic field varies across the stack. Equation (47) can be considered a perturbation of the equation

$$\frac{d^2w}{d\xi^2} + \Lambda w = 0, \tag{49}$$

subject to the same boundary conditions. Suppose this equation has as its eigenvalues

$$\Lambda_1, \Lambda_2, \cdots, \tag{50}$$

and as its corresponding normalized eigenfunctions

$$w_1, w_2, \cdots. \tag{51}$$

In fact,

$$\begin{aligned} \Lambda_n &= \pi^2 n^2, \\ w_n &= \sqrt{2} \sin \pi n \xi. \end{aligned} \tag{52}$$

Now suppose the perturbed equation

$$\frac{d^2w^*}{d\xi^2} + [\Lambda^* - jCf(\xi)]w^* = 0, \tag{53}$$

† This method is given by Courant and Hilbert.[6] It was pointed out to the author by S. P. Morgan.

where $f(\xi)$ is a given function of ξ, and C is a (small) parameter, has as its eigenvalues

$$\Lambda_1^*, \Lambda_2^*, \cdots \tag{54}$$

and as corresponding eigenfunctions

$$w_1^*, w_2^*, \cdots . \tag{55}$$

Following Courant and Hilbert,[6] suppose there exists an expansion

$$\begin{aligned} w_n^* &= w_n + Cx_n + C^2 y_n + \cdots , \\ \Lambda_n^* &= \Lambda_n + C\mu_n + C^2 \nu_n + \cdots . \end{aligned} \tag{56}$$

Using the notation of Courant and Hilbert, let

$$\begin{aligned} d_{nl} &= \int_0^1 j f(\xi) w_n(\xi) w_l(\xi)\, d\xi, \\ a_{nn} &= 0, \\ a_{nl} &= \frac{d_{nl}}{\Lambda_n - \Lambda_l}, \qquad n \neq l. \end{aligned} \tag{57}$$

Then, as shown in Courant and Hilbert, the series expansions to terms of the second degree are:

$$\begin{aligned} w_n^* &= w_n + C \sum_{j=1}^{\infty} \frac{d_{nj}}{\Lambda_n - \Lambda_j} w_j \\ &\quad + C^2 \left[\sum_{j=1}^{\infty} \frac{w_j}{\Lambda_n - \Lambda_j} \left(\sum_{k=1}^{\infty} a_{nk}\, d_{kj} - \mu_n a_{nj} \right) - \tfrac{1}{2} w_n \sum_{k=1}^{\infty} a_{nk}^2 \right] \\ \Lambda_n^* &= \Lambda_n + C\, d_{nn} + C^2 \sum_{j=1}^{\infty} a_{nj}\, d_{jn} + \cdots . \end{aligned} \tag{58}$$

By following the same method, further terms of the power series could be found.

An equivalent exposition in terms of linear operator is given by Kato.[7] He discusses the behavior of eigenvalues of the equation

$$(H_0 + kH^{(1)} - \lambda_k)\varphi = 0 \tag{59}$$

and shows that, formally,

$$\lambda_k = \lambda_0 + k(H^{(1)}\varphi_0 , \varphi_0) + k^2[-(SH^{(1)}\varphi_0 , H^{(1)}\varphi_0)] + O(k^3), \tag{60}$$

and a corresponding equation for φ_K. The relation of his notation to ours is given below. Note that H_0, $H^{(1)}$ and S represent operators, not functions, and that (,) is a functional of the variables it surrounds.

$$
\begin{aligned}
\epsilon &\leftrightarrow -iC, \\
\lambda_0 &\leftrightarrow \Lambda_1, \\
\lambda_k &\leftrightarrow \Lambda_1^*, \\
\varphi_0 &\leftrightarrow w_1, \\
\varphi_k &\leftrightarrow w_1^*, \\
H_0 u &= \frac{d^2u}{dx^2}, \\
H^{(1)}u &= f(x)u(x), \\
(u, v) &= \int_0^1 u(x)v(x)\,dx, \\
Su &= \int_0^1 k(x, y)u(y)\,dy,
\end{aligned}
\tag{61}
$$

where k is the resolvent kernel

$$
k(x, y) = \sum_2^\infty \frac{w_m(x)w_m(y)}{\Lambda_m - \Lambda_1}. \tag{62}
$$

The resulting expression for Λ_1^* is

$$
\begin{aligned}
\Lambda_1^* &= \Lambda_1 + jC\int_0^1 f(x)w_1^2(x)\,dx \\
&\quad + C^2\int_0^1\int_0^1 k(x, y)f(x)f(y)w_1(x)w_1(y)\,dx\,dy + O(C^3) \\
&= \Lambda_1 + 2jC\int_0^1 f(x)\sin^2 \pi x\,dx \\
&\quad + 2C^2\int_0^1\int_0^1 k(x, y)f(x)f(y)\sin \pi x \sin \pi y\,dx\,dy + O(C^3),
\end{aligned}
\tag{63}
$$

as stated before.

The advantage of these expressions is that the function $f(\xi)$ need not be specified beforehand. In fact, as we shall see later, it need not even be exactly specified. We shall presently apply these expressions to cases where $f(\xi)$ is a random function with a specified spectrum.

IV. NUMERICAL RESULTS — COMPARISON WITH MORGAN

As a preliminary to further numerical results, we can check the formulas just derived with exact numerical results derived in Morgan for certain special cases. For example, in Fig. 22 of Ref. 3 an irregularity is described for which

$$\begin{aligned} f(\xi) &= -1, & 0 \leqq \xi \leqq \tfrac{1}{2}, \\ f(\xi) &= +1, & \tfrac{1}{2} < \xi \leqq 1. \end{aligned} \tag{64}$$

In the case of the lowest eigenvalue

$$\begin{aligned} a_m &= 0, && m \text{ odd} \\ &= 4\int_0^{1/2} \sin \pi\xi \sin \pi m\xi \, d\xi && m \text{ even} \\ &= \frac{4m(-1)^{(m/2)+1}}{\pi(m^2-1)} && m \text{ even,} \\ \Sigma &= \sum_2^\infty \frac{16m^2}{\pi^4(m^2-1)^3} \\ &= 0.0329, \\ \alpha &= \alpha_0\left[1 + \frac{C^2}{\pi^2}\Sigma\right] \\ &= \alpha_0[1 + 0.00333C^2]. \end{aligned} \tag{65}$$

For the second eigenvalue

$$\alpha = \alpha_0[4 - 0.0019C^2], \tag{66}$$

where α_0 is still the low frequency attenuation corresponding to the lowest eigenvalue. Fig. 4 shows a comparison of these curves with the curves published in Ref. 3, Fig. 22. The value of C which makes α twice α_0 is

$$C = (0.00333)^{-1/2} = 17.3. \tag{67}$$

This compares well with Morgan's precise value 16.5. Similarly, if

$$Cf(\xi) = -C \cos 6\pi\xi, \tag{68}$$

as in Ref. 3, Fig. 30,

$$Cf(\xi) \sin \pi\xi = -\frac{C}{2} \sin 7\pi\xi + \frac{C}{2} \sin 5\pi\xi,$$

$$C^2\Sigma = \frac{C^2}{\pi_2}\left(\frac{1/4}{49-1} + \frac{1/4}{25-1}\right) = \frac{C^2}{64\pi^2}, \tag{69}$$

$$\alpha = \alpha_0\left(1 + \frac{C^2}{64\pi^4}\right).$$

On a graph, this is indistinguishable from the curve in Ref. 3, Fig. 30. The value of C for which the attenuation is doubled is

$$C = 8\pi^2 = 78.9, \tag{70}$$

the same as Morgan's result to three significant figures. These two results, representing both extremes in the computation reported in Morgan, are sufficiently close to give us a good deal of confidence in this method.

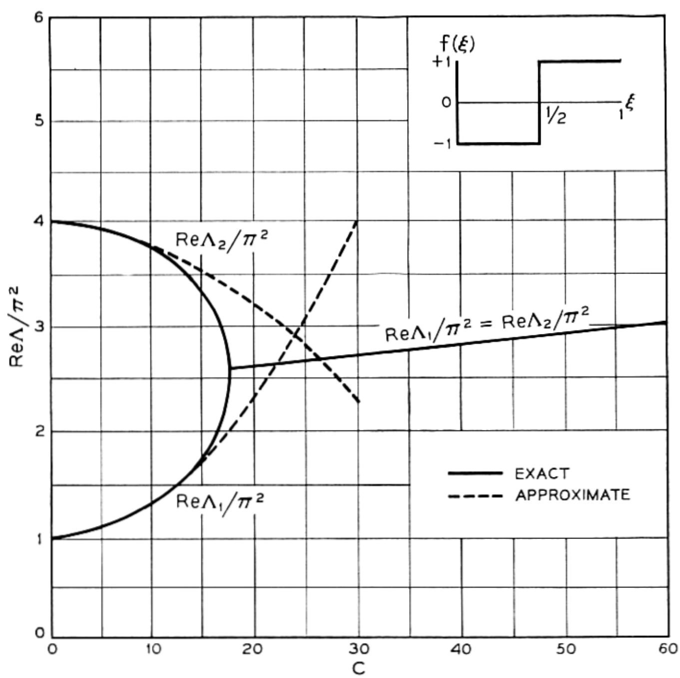

Fig. 4 — Eigenvalues for a nonuniform laminated stack whose average properties are constant except for a single symmetric step-discontinuity: approximate values computed from two terms of the perturbation-method power series compared with exact values derived by Morgan.

The following relation can be used to simplify numerical computation. If $f(x)$ is expanded in a Fourier cosine series

$$f(x) = \frac{b_0}{2} + \sum_1^\infty b_m \cos m\pi x, \tag{71}$$

then

$$\begin{aligned} f(x) \sin \pi x &= \frac{b_0}{2} \sin \pi x + \sum_1^\infty b_m (\cos m\pi x \sin \pi x) \\ &= \frac{b_0}{2} \sin \pi x + \sum_1^\infty \frac{b_m}{2} [\sin (m+1)\pi x - \sin (m-1)\pi x] \\ &= \sum_0^\infty \frac{b_{m-1} - b_{m+1}}{2} \sin m\pi x \\ &= \sum_0^\infty a_m \sin m\pi x, \end{aligned} \tag{72}$$

where

$$a_m = \frac{b_{m-1} - b_{m+1}}{2}, \tag{73}$$

and we assume that b_{-1} (otherwise undefined) equals zero.

If b_{m-1} and b_{m+1} are random variables with Gaussian distribution about zero and rms values $\langle b_{m-1} \rangle$ and $\langle b_{m+1} \rangle$, then a_m is also a random variable with Gaussian distribution about zero, and its rms value is

$$\langle a_m \rangle = \tfrac{1}{2}\sqrt{\langle b_{m-1} \rangle^2 + \langle b_{m+1} \rangle^2}. \tag{74}$$

V. THE EFFECT OF FINITE LAMINATIONS

Following Morgan, equations 2 and 3 [but supposing that $(g + j\omega\epsilon)$ is a function of y, as it is indeed if the medium has finite laminations], we arrive at the equation

$$\frac{1}{g + j\omega\epsilon} \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial H_x}{\partial y} \frac{\partial}{\partial y} \left(\frac{1}{g + j\omega\epsilon} \right) - \left(j\omega\mu - \frac{\gamma^2}{g + j\omega\epsilon} \right) H_x = 0. \tag{75}$$

Now let

$$\begin{aligned} \bar{f} &= \int_{-a/2}^{+a/2} (g + j\omega\epsilon)\, dy, \\ u &= \frac{1}{\bar{f}} \int_{-a/2}^{y} (g + j\omega\epsilon)\, dy, \\ H_x(y) &= H(u). \end{aligned} \tag{76}$$

Then we find

$$\frac{\partial^2 H}{\partial u^2} - \bar{f}^2 \left[\frac{j\omega\mu}{g + j\omega\epsilon} - \frac{\gamma^2}{(g + j\omega\epsilon)^2}\right] H = 0, \tag{77}$$

subject to the boundary conditions

$$H(0) = 0, \qquad H(1) = 0. \tag{78}$$

We can regard this as a perturbation of

$$\frac{\partial^2 H}{\partial u^2} + k^2 H = 0, \tag{79}$$

where

$$\begin{aligned} k^2 &= -\bar{f}^2 \int_0^1 \left[\frac{j\omega\mu}{g + j\omega\epsilon} - \frac{\gamma^2}{(g + j\omega\epsilon)^2}\right] du \\ &= -\bar{f} \int_{-a/2}^{a/2} \left(j\omega\mu - \frac{\gamma^2}{g + j\omega\epsilon}\right) dy. \end{aligned} \tag{80}$$

To a very good degree of approximation,

$$\begin{aligned} \bar{f} &= a\theta g_1 = a\bar{g}, \\ k^2 &= -j\omega\mu\bar{g}a^2 + \frac{\gamma^2 a^2 \bar{g}}{j\omega\bar{\epsilon}}. \end{aligned} \tag{81}$$

The equations for $\bar{f}$ and k^2 provide some insight into the source of the effective or average values $\bar{\epsilon}$, $\bar{g}$ and $\bar{\mu}$. Inasmuch as ϵ and g always occur in the combination $g + j\omega\epsilon$, it is hard to see why two different "average" values arise. Now it is clear that $\bar{\epsilon}$ is the harmonic mean value and $\bar{g}$ the arithmetic mean value. Because of the overwhelmingly large ratio of g_1 to $j\omega\epsilon_2$, the larger dominates the arithmetic mean and the smaller the harmonic mean. If we had assumed μ_1 and μ_2 different, $\bar{\mu}$ would turn out to be the arithmetic mean, but, inasmuch as they are of the same general order of magnitude, both μ_1 and μ_2 appear in the resulting formula for $\bar{\mu}$.

The eigenvalues of the wave equation above are (Ref. 3, equation 537)

$$k^2 = n^2\pi^2 \tag{82}$$

and the corresponding values of γ are

$$\gamma = j\omega\sqrt{\mu\bar{\epsilon}}\left[1 + \frac{n^2\pi^2}{i\omega\mu\bar{g}a^2}\right]^{1/2} \tag{83}$$

in agreement with Morgan's equation 534.

In terms of the variable u, the function

$$-jCf(u) = -\bar{f}^2 \left[\frac{j\omega\mu}{g + j\omega\epsilon} - \frac{\gamma^2}{(g + j\omega\epsilon)^2} \right] + j\omega\mu\bar{g}a^2 - \frac{\gamma^2 a^2 \bar{g}}{j\omega\bar{\epsilon}} \quad (84)$$

alternates periodically between two values. It is interesting to note, however, that the "electrical thickness" of a dielectric layer, i.e., its thickness in terms of u, is

$$\delta u = \frac{1}{\bar{f}} \int_{y_0}^{y_0+t_2} j\omega\epsilon_2 \, dy = \frac{j\omega\epsilon t_2}{a\bar{g}}, \quad (85)$$

which is infinitesimal if $\bar{g}$ has any reasonable value. Hence the function $f(u)$ in question really has the appearance of a sequence of sharply spaced spikes, having

$$\begin{aligned} \text{spike width} &= \frac{j\omega\epsilon t_2}{a\bar{g}}, \\ \text{spike height} &= -a^2\bar{g}^2 \left(\frac{u}{\epsilon} + \frac{\gamma^2}{\omega^2\epsilon^2} \right), \\ \text{spike spacing} &= \frac{1}{n}. \end{aligned} \quad (86)$$

Using the approximation

$$\gamma^2 = -\omega^2\mu\epsilon$$

and defining spike strength S as the product of spike height and spike length, we find

$$\begin{aligned} S &= \frac{j\omega\epsilon t_2}{a\bar{g}} (-a^2\bar{g}^2) \left(\frac{\mu}{\epsilon} - \frac{\omega^2\mu\bar{\epsilon}}{\omega^2\epsilon} \right) \\ &= \frac{j\omega\mu\bar{g}\theta a^2}{m}. \end{aligned} \quad (87)$$

Now it happens that the Fourier coefficients a_m can be evaluated explicitly in this case, and this procedure leads fairly easily to the results in Morgan's equations 455, 459 and 460. The computation succeeds only because $f(u)$ is in this case a supremely simple function. When $f(u)$ is not quite so regular, it is easier to return to the double integral formulation. The more general procedure will be carried out here for two reasons. First, it shows that the effects of local variations of layer thickness and dielectric constant are independent of the effect of finite lamination thickness; i.e., the two effects can be computed separately and added. Second, it shows that the result is not restricted to flat lines, but is immediately applicable to other cases which can be be set up as self-adjoint differential equations, including the cylindrical case.

The central idea is to observe that, when $f(u)$ is a function consisting of a sequence of (nearly) equally spaced pulses of (nearly) equal amplitudes, then

$$\int_0^1\int_0^1 f(u)f(v)k(u,v)w_1(u)w_1(v)\,du\,dv \cong \int_0^1\int_0^1 f(u)f(v)\,du\,dv \int_0^1\int_0^1 k(u,v)w_1(u)w_1(v)\,du\,dv. \tag{88}$$

The expression on the left is a finite sum which approximates, by the two-dimensional analog of the trapezoid rule, the integral on the right. The remainder of the process consists in evaluating the difference between the two, i.e., the error in the approximation. It is not surprising that the result depends heavily on the special characteristics of the reduced resolvent kernel $k(u, v)$, which depend in turn on its relation to Green's function. Let

$$k(u,v)w_1(u)w_1(v) = h(u,v). \tag{89}$$

Then we need in particular:

$$\int_0^1\int_0^1 h(u,v)\,du\,dv = 0, \tag{90}$$

$$\int_0^1 h(u,v)\,dv = 0, \tag{91}$$

$$\left.\frac{\partial h}{\partial u}\right|_{u=v+0} - \left.\frac{\partial h}{\partial u}\right|_{u=v-0} = w_1^2(v), \tag{92}$$

$$\int_0^1 w_1^2(v)\,dv = 1, \tag{93}$$

$$\begin{aligned} h(u,v) &= O(u^2) \\ &= O(v^2) \\ &= O(1-u)^2 \\ &= O(1-v)^2 \end{aligned} \tag{94}$$

in the neighborhood of the boundaries of the unit square.

First note that, because of the pulse character of f, the integral can be transformed into a sum:

$$\iint C^2 f(u) f(v) h(u, v)\, du\, dv = \sum_i \sum_j S^2 h(u_i, v_j), \tag{95}$$

where

$$\begin{aligned} u_i &= u_0 + i/n, \\ v_j &= v_0 + j/n, \\ u_0 &= v_0 . \end{aligned} \tag{96}$$

We can approximate each term of the sum by an integral thus by setting

$$\begin{aligned} h(u_i + x, v_j + y) = h(u_i, v_j) &+ x \frac{\partial h}{\partial u} + y \frac{\partial h}{\partial v} \\ &+ \frac{x^2}{2} \frac{\partial^2 h}{\partial u^2} + xy \frac{\partial^2 h}{\partial u\, \partial v} + \frac{y^2}{2} \frac{\partial^2 h}{\partial v^2} + O(n^{-3}), \\ &\quad | x | \leqq \frac{1}{2n}, \qquad | y | \leqq \frac{1}{2n}. \end{aligned} \tag{97}$$

Integrating directly, we get, for $i \neq j$:

$$\begin{aligned} &\int_{-1/2n}^{1/2n} \int_{-1/2n}^{1/2n} h(u_i + x, v_j + y)\, dx\, dy \\ &\qquad = n^{-2} h(u_i v_j) + \frac{1}{24} n^{-4} \frac{\partial^2 h}{\partial u^2} + \frac{1}{24} n^{-4} \frac{\partial^2 h}{\partial v^2} + O(n^{-5}) \end{aligned} \tag{98}$$

and, for $i = j$:

$$\begin{aligned} \iint = n^{-2} h(u_j, v_j) &- \frac{1}{12} n^{-3} \left(\frac{\partial h^-}{\partial u} - \frac{\partial h^+}{\partial u} \right) \\ &- \frac{1}{12} n^{-3} \left(\frac{\partial h^-}{\partial u} + \frac{\partial h^+}{\partial u} \right) + O(n^{-4}). \end{aligned} \tag{99}$$

Now, in performing the indicated double summation, note that

$$\sum_{i=1}^{j-1} \frac{\partial^2 h}{\partial u^2} (u_i, v_j) = n \left(\frac{\partial h}{\partial u} \bigg|_{u=v-0} - \frac{\partial h}{\partial u} \bigg|_{u=0} \right) + O(1) \tag{100}$$

and, hence, using (92) and (93),

$$\begin{aligned} \left(\sum_{i=1}^{j-1} + \sum_{j+1}^{n} \right) \frac{\partial^2 h}{\partial u^2} (u_i, v_j) &= -n w^2(v_j) + O(1) \\ \sum_{i \neq j}\sum \frac{\partial^2 h}{\partial u^2} &= -n^2 + O(n) \end{aligned} \tag{101}$$

and, similarly,

$$\sum_{i \neq j}\sum \frac{\partial^2 h}{\partial v^2} = -n^2 + O(n). \tag{102}$$

In the terms involving $i = j$, using (92) and (93) again, we find

$$\sum_{i=1}^{n} w^2(u_i) = -n + O(1). \tag{103}$$

Now, taking all terms and summing, we get

$$\begin{aligned} \int_{u=u_0-1/2n}^{1+u_0-1/2n} \int_{v=v_0-1/2n}^{1+v_0-1/2n} & h(u, v)\, du\, dv \\ &= n^{-2} \sum\sum h(u_i, v_j) + \frac{1}{24} n^{-4}[-2n^2 + O(n)] \\ &\quad - \frac{1}{12} n^{-3}[-2n + O(1)] \\ &= n^{-2} \sum\sum h(u_i, v_j) + \frac{n^{-2}}{12} + O(n^{-3}). \end{aligned} \tag{104}$$

But the integral, by direct integration using (91) and (94), is $O(n^{-3})$. Hence,

$$\sum\sum h(u_j, v_j) = -\frac{1}{12} + O(n^{-1}),$$

$$\begin{aligned} \iint C^2 f(u) f(v) h(u, v)\, du\, dv &= S^2 \sum\sum h(u_i, v_j) \\ &= -\frac{S^2}{12} + O(n^{-1} S^2), \end{aligned} \tag{106}$$

$$C^2\Sigma = \frac{\omega^2 \mu^2 \bar{g}^2 \theta^2 a^4}{12n^2} + O(n^{-3}),$$

$$\gamma^2 = -\omega^2 \mu \bar{\epsilon} \left(1 + \frac{\pi^2}{i\omega\mu\bar{g}a^2} - \frac{i\omega\mu\bar{g}\theta^2 a^2}{12n^2}\right).$$

This is the same as Morgan's formula 455, except that, in his case, the number of layers is $2n$ rather than n. In the appropriate frequency range, (including Morgan's "low" and "high", but not his "very low" and "very high" frequencies)

$$\begin{aligned} \alpha &= \frac{\pi^2}{2\sqrt{\mu/\bar{\epsilon}}\,\bar{g}a^2} + \frac{\omega^2 \mu^2 \bar{g} t_1^2}{24\sqrt{\mu/\epsilon}}, \\ \beta &= \omega\sqrt{\bar{\mu}/\epsilon}, \end{aligned} \tag{107}$$

as already found by Morgan (Ref. 3, equations 459 and 460). To simplify the arithmetic, it has been assumed that $\mu_1 = \mu_2 = \bar{\mu}$, and the computation has been restricted to the lowest-order mode. Generalization to remove these restrictions alters nothing.

By this time the reader, exhausted or bored according as he has or has not attempted to bridge the gaps in the above computation, may reasonably ask: why compute the effect of finite lamination thickness by such a laborious method when the matrix method of Morgan yields the same result so simply? The answer is that the matrix method fails* if the layers are not identical, whereas the perturbation method does not. The exercise in this section was designed simply to show that when both methods apply, they agree.

Two important cases arise where the layers are not identical. The first is a cylindrical laminated line, where the radii of the layers gradually increases from center to outside layer. This case is discussed in Section VIII. The second is a laminated line in which the lamination are finite but not perfectly regular. In this case one would expect some contribution from irregularity and some from finite thickness, and may ask how they combine. From the nature of the computation for the uniform case, it is easy to guess that the contributions are independent and additive. We can replace the integral to be approximated by one including the effects of irregularity, which does not vanish identically, but the contribution due to finite granularity of the approximating sum remains unchanged. Alternatively, we can actually compute the effect of irregularities as a finite sum rather than an integral, taking one point corresponding to each layer. This was what was in fact done in the experimental case presented in Section VII. In this case, the sum computed can be compared term-by-term with the sum for the ideal case above, and the difference turns out to be precisely the finite analog of the integrals at the end of Section IV. The results can be presented in several forms, but the conclusion is always the same: that to a first order of approximation the contribution to the quantity Σ (24) due to finite layer thickness is independent of the contribution due to irregularities in the layers.

VI. NUMERICAL RESULTS — RANDOM VARIATIONS

The expression for b derived in Sections II and III does not depend explicitly on $f(\xi)$, but only on the squares of the coefficients of its Fourier

* But see Hayashi and U-O.[8] Here the matrix computation is carried out by an approximation method which appears to be valid if the radius of the innermost layer is not too small compared to the total thickness of the stack.

series, or on what is known to communications engineers and statisticians as its power spectrum. Consequently, the expression is especially adapted for use when f is a random function known only by its average power spectrum. The functions of this class have been used widely in discussions of electrical noise and other random processes, and their introduction into this problem should not come as a surprise.

Suppose that $\bar{\epsilon}$ varies randomly from layer to layer, i.e., that

$$\bar{\epsilon} = \bar{\epsilon}_0 + u(\xi), \tag{108}$$

where $u(\xi)$ is a random variable whose value in every layer is independent of its value in every other layer. Then $f(\xi)$ is a random function having a flat spectrum, and can be represented as

$$f(\xi) = \sum_{m=1}^{n} b_m \cos m\pi\xi, \tag{109}$$

where

$$\langle b_m \rangle = k, \tag{110}$$

k being a constant. (The angular brackets indicate, as before, rms values.)

Inasmuch as the number of layers is finite, it seems reasonable that the series should be terminated after a finite number of terms. Because of the strong convergence factor $1/(m^2 - 1)$ the exact termination point is not very important. Inasmuch as $f(\xi)$ has only n degrees of freedom, we have terminated the series after n terms. In the practical cases worked out, we used the formulation having a definite integral, which was integrated point by point, using one data-point per layer. This was precisely the number of measured data available (i.e., one conductor-to-conductor capacitance measurement through each insulating layer) so there was no choice about how many terms to use anyway.

The function $f(\xi)$ was defined so that $\langle f(\xi) \rangle$ is unity. It follows from Parseval's theorem that

$$\int_0^1 f^2(\xi)\, d\xi = 1 = \tfrac{1}{2} \sum_1^n b_m^{\,2} = \frac{n}{2} k^2, \tag{111}$$

and hence that

$$k = \sqrt{\frac{2}{n}}. \tag{112}$$

It follows that

$$\begin{aligned}
\langle a_m \rangle &= \tfrac{1}{2}\sqrt{\langle b_{m+1}\rangle^2 + \langle b_{m-1}\rangle^2} \\
&= n^{-(1/2)}, \\
\Sigma &= \sum_{m=2}^{n} \frac{1}{\pi^2 n(m^2 - 1)} \\
&= \frac{3n^2 - n - 2}{(4n^3 + 4n^2)\pi^2} \\
&\cong \frac{3}{4n\pi^2},
\end{aligned} \tag{113}$$

and

$$\begin{aligned}
b &= \frac{3\sqrt{n}}{2\pi\theta}\left\langle \frac{\Delta\bar{\epsilon}}{\bar{\epsilon}_0} \right\rangle \\
&= \frac{3\sqrt{n}}{2\pi}\left\langle \frac{\Delta t_2}{t_2} - \frac{\Delta t_1}{t_1} \right\rangle.
\end{aligned} \tag{114}$$

From this quantity b it is easy to compute the reduction in bandwidth $(1 + b^2)^{-1/2}$, as a function of n and the mean square fractional variations of t_1 and t_2. The reduction in bandwidth is plotted as a function of the number of layers n for several values of

$$\left\langle \frac{\Delta t_2}{t_2} - \frac{\Delta t_1}{t_1} \right\rangle$$

in Fig. 5. Notice that, if the variations in t_1 and t_2 are independent,

$$\left\langle \frac{\Delta t_2}{t_2} - \frac{\Delta t_1}{t_1} \right\rangle = \left\langle \frac{\Delta t_2}{t_2} \right\rangle + \left\langle \frac{\Delta t_2}{t_1} \right\rangle. \tag{115}$$

However, if t_1 and t_2 are the thicknesses of layers successively formed about the same core, one can imagine that they would not be uncorrelated. In fact, it is plausible to believe that when one is too small, the other will be too large. In any case, however, the expression at the left can be no greater than twice the expression at the right.

As an alternative to controlling the effective dielectric constant of each layer independently, one might control the average dielectric constant of the incomplete stack, as it is built up layer by layer. In this case, one would expect the average dielectric constant to vary in a random manner about its nominal value. Suppose, for example, that the elastance per unit length of the stack is measured after each layer is added, that the next layer is added to bring the average dielectric

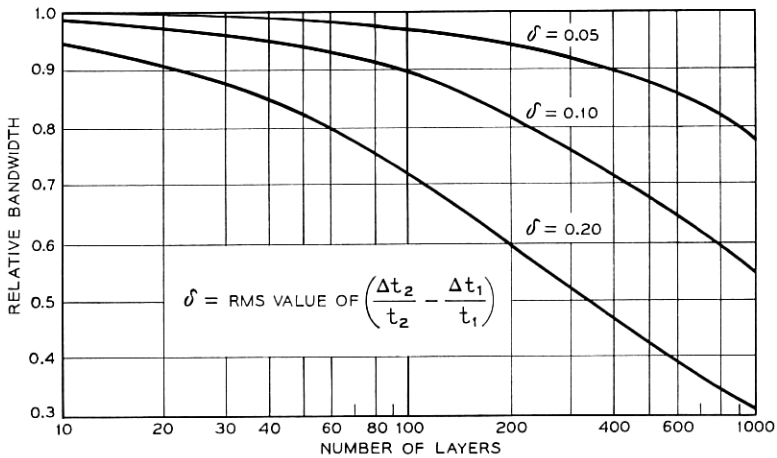

Fig. 5 — Bandwidth degradation in a laminated transmission line due to random irregularities of a certain type plotted as a function of number of laminae for several amounts of irregularity.

constant to its nominal value and that the error (i.e., the measure of the amount by which we fail to bring the average dielectric constant to its nominal value) varies randomly from layer to layer. Then, except for a multiplicative constant,

$$S(\xi) = \int_0^{\xi} \frac{d\xi}{\bar{\epsilon}} = \frac{\xi}{\bar{\epsilon}_0} + u(\xi), \tag{116}$$

where $S(\xi)$ is the elastance of a unit length measured from one side to the mth layer, $m = n\xi$ and $u(\xi)$ is a random function with a flat spectrum. Then

$$\frac{\Delta\bar{\epsilon}}{\bar{\epsilon}_0} = \frac{\bar{\epsilon} - \bar{\epsilon}_0}{\bar{\epsilon}_0} = -\bar{\epsilon}\,\frac{du}{d\xi} \cong -\bar{\epsilon}_0\,\frac{du}{d\xi}. \tag{117}$$

Now $f(\xi)$ is proportional to $-(du/d\xi)$, and hence has a spectrum whose amplitude increases with frequency, i.e.,

$$\langle b_m \rangle = km. \tag{118}$$

Proceeding as before,

$$\begin{aligned} \int_0^1 f^2(\xi)\, d\xi &= \tfrac{1}{2} \sum_1^n b_m^{\ 2} \\ &= \frac{k^2}{2} \sum_1^n m^2 \\ &\cong \frac{k^2 n^3}{6}, \end{aligned} \tag{119}$$

or

$$k \cong \sqrt{\frac{6}{n^3}}. \tag{120}$$

Then

$$\begin{aligned} \langle a_m \rangle &= \tfrac{1}{2} \sqrt{\langle b_{m-1} \rangle^2 + \langle b_{m+1} \rangle^2} \\ &\cong m \sqrt{\frac{3}{n^3}} \end{aligned} \tag{121}$$

and

$$\begin{aligned} \Sigma &= \sum_{m=2}^n \frac{a_m^{\ 2}}{\pi^2 m^2 - \pi^2} \\ &\cong \sum_{m=2}^n \frac{3m^2}{\pi^2 n^3 (m^2 - 1)} \\ &\cong \frac{3}{n^2 \pi^2}. \end{aligned} \tag{122}$$

Hence,

$$\begin{aligned} b &= \frac{3}{\pi\theta} \frac{\Delta\bar{\epsilon}}{\bar{\epsilon}_0} \\ &= \frac{3}{\pi} \left\langle \frac{\Delta t_2}{t_2} - \frac{\Delta t_1}{t_1} \right\rangle. \end{aligned} \tag{123}$$

Notice that, for the same rms deviations in t_1 and t_2, b is smaller by a factor $2/\sqrt{n}$. This is, of course, due to the fact that the variations in successive layers are no longer uncorrelated, but are adjusted to make the variations cancel over several layers. The variations of average dielectric constant having long wavelength are reduced, and the only remaining variations left have short wavelength. However, in the expression for Σ the contribution of each component is weighted with the factor $1/(m^2 - 1)$, and hence the contribution of the components having short wavelengths is reduced. The over-all result is that the line is not degraded so much by this kind of variation. The relation in bandwidth for the same values of the parameters as were used in Fig. 5 is independent of n, and has the values 0.983, 0.995 and 0.999 for $\delta = 0.20$, 0.10 and 0.05, respectively.

One might ask how the way in which one aims for the nominal dimensions can make a difference, if the errors in film thickness are the same in both cases. The fact can be made plausible by the following argument: in the first case a layer is laid in ignorance of what has gone before, while in the second case the layers that have gone before are studied and an attempt is made to compensate for past errors. As an example, one might examine the following: suppose one is making a ruler by marking off successive inches with a pair of dividers. First one might set the dividers and mark off, say, 36 nominally equal spaces. As an alternative, one might measure the result after each step, and adjust the dividers so that the measured distance plus the distance laid down for the new step should be as near as possible to an integral number of inches. There is no doubt that, even if the precision of each individual inch is the same in both cases, the latter process will make a ruler having a more uniform scale. This example is more than an illustration; it is a good analog of the two processes of laminated line construction which are described above.

This conclusion is of great practical importance. In dealing with layers with a thickness smaller than 0.001 inch, a variation of 0.05 in relative thickness corresponds to a thickness change of one wavelength of visible light. There is not much hope of producing laminated lines having hundreds of layers if this kind of precision must be maintained. On the other hand, the measurement of capacitance to a high degree of accuracy is quite reasonable, and a feedback mechanism to translate that measurement into an objective for the thickness of the next layer is quite conceivable. With this kind of servo control in the fabrication process, laminated lines having any number of layers could be constructed using layers no more regular than were those in the experiment described in Section VII, with degradation of bandwidth no more than 2 per cent. Thus, irregularities can be conquered by feedback, and the laminated line made of less-than-perfect materials can be rescued and brought back to the realm of the practical.

VII. COMPARISON WITH EXPERIMENT

In 1951 an experimental 100-layer Clogston 2 cable was fabricated. A description of this cable and of measurements made on it has never been published, largely because measured transmission properties could not be reconciled quantitatively with the theory available at the time. The unresolved differences motivated the research which culminated in the results of the present paper. A laboratory report on the fabrication

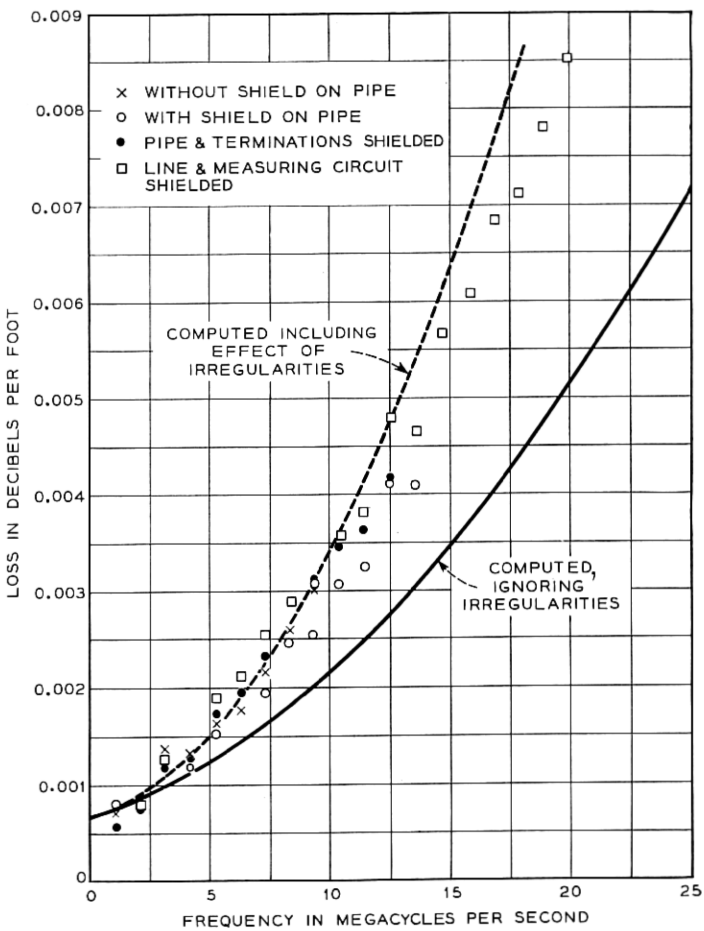

Fig. 6 — Computed and measured losses in a 276-foot terminated laminated transmission line. The measurements were made with several different electrical shielding means to exclude certain possible spurious effects.

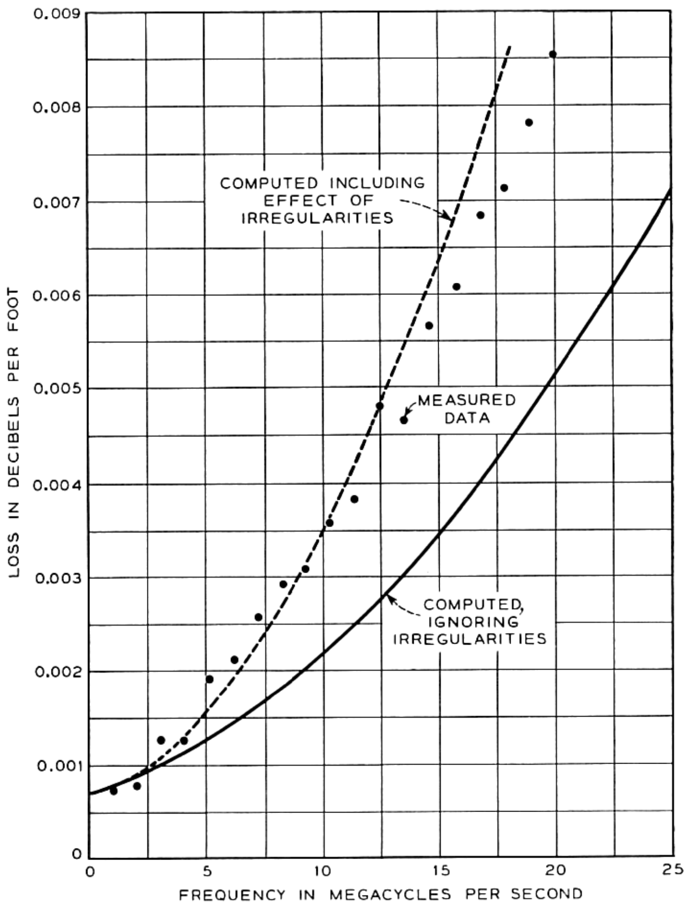

Fig. 7 — Computed and measured losses in a 276-foot terminated laminated transmission line. The line and the measurement circuit were completely shielded electrically, resulting in less scatter than in the previous figure.

and measurement of this cable is being published as a companion paper[9] to the present theoretical study.

The cable was a 276-foot laminated conductor assembled by hand around a $\frac{7}{8}$-inch conducting core. The laminations consisted of 100 concentric layers of 0.00025-inch aluminum foil separated by 99 polystyrene cylindrical insulators each 1.35 mils thick. After fabrication the cable was shielded by a wrap of 0.010-inch aluminum foil.

The Clogston 2 dominant mode was propagated. Measurements were

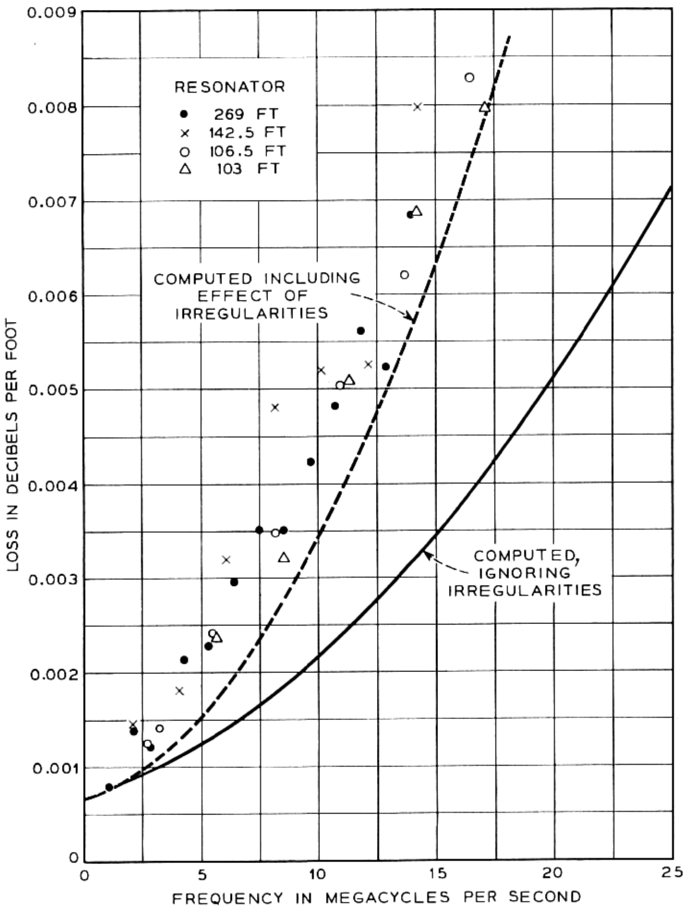

Fig. 8 — Computed and measured losses in several lengths of a laminated transmission line measured as a resonator.

made of the mode pattern and the attenuation as a function of frequency up to 25 mc. Measured attenuation was compared with attenuation computed for a uniform laminated line, and exceeded the theoretical values by a factor of two, as shown in Figs. 6, 7 and 8. At the time, the discrepancy was attributed to lack of uniformity in the laminae. However, no quantitative theory of the effect of irregularities was available and definite association of the discrepancy with this cause was not possible.

At the time the 100-layer Clogston line was tested, a wise decision was made to measure the capacitance between consecutive layers of conductor. The thickness of the dielectric layer is easily derived by assuming that the capacitance is proportional to area and inversely proportional to thickness. Because of the fact that the conducting layers each consist of a single sheet of relatively firm metal, while the dielectric layers are built up of several thicknesses of relatively soft dielectric, it is reasonable to assume that the principal effect of irregularity of layer thickness can be traced to variation in dielectric thickness.

With the assumption that the principal effect of irregularity is due to variations in dielectric thickness, it is not difficult to show that

$$\frac{\bar{\epsilon}_m}{\bar{\epsilon}_0} = (1 - \theta) + \theta \frac{C_m}{C_{\text{av}}}, \tag{124}$$

where

$$\begin{aligned} \bar{\epsilon}_m &= \text{the effective dielectric constant at layer } m, \\ \bar{\epsilon}_0 &= \text{the average dielectric constant}, \\ C_m &= \text{the capacitance per unit area across the layer } m, \\ C_{\text{av}} &= \text{the average capacitance per unit area, averaged over all} \\ &\quad\ \text{the dielectric layers.} \end{aligned} \tag{125}$$

The value C_m was derived from measurements of capacitance between adjacent conducting layers, which were measured separately for two different lengths of the completed cable. To simplify the computation, the small variation in area from layer to layer was ignored. The reduction in bandwidth b was determined by (32),

$$b = \frac{\sqrt{3} n A \sqrt{\Sigma}}{\theta}, \tag{126}$$

where

$$\begin{aligned} A^2\Sigma &= \frac{\theta^2}{\pi^2 n^2 C_{\text{av}}^2} \left(\tfrac{1}{2} I_n^2 - \frac{2\pi}{n} I_n' I_n + 2\pi I_n'' \right), \\ I_n &= \sum_1^n C_m \sin^2 (\pi m/n), \\ I_n' &= \sum_1^n m C_m \sin (\pi m/n) \cos (\pi m/n), \\ I_n'' &= \sum_1^n C_m \sin (\pi m/n) \cos (\pi m/n) I_m . \end{aligned} \tag{127}$$

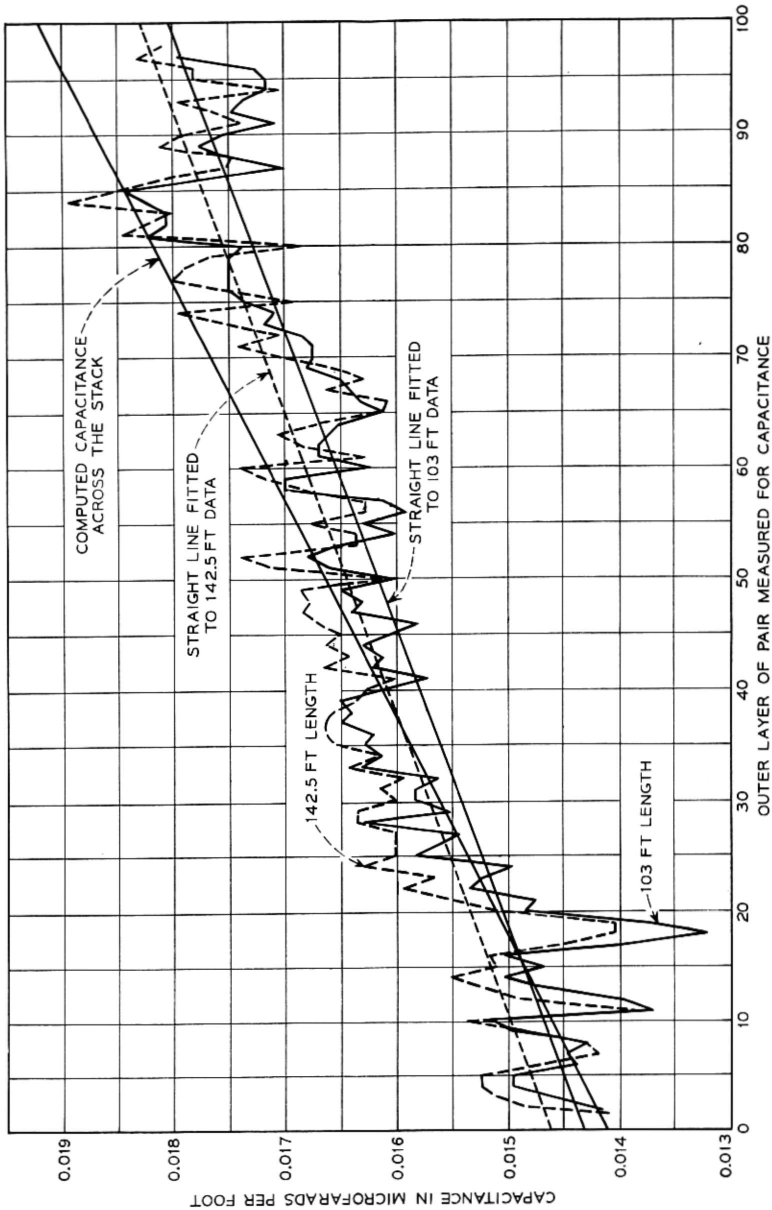

Fig. 9 — Measured capacitances between adjacent layers in two sections of a laminated transmission line.

This is the finite analog of the integral (24)

$$
\begin{aligned}
A^2\Sigma = A^2 \int_0^1 \int_0^1 f(x)f(y)h(x, y)\,dx\,dx &= \left| \frac{1}{\pi} \int_0^1 Af(x) \sin^2 \pi x\,dx \right|^2 \\
&- \frac{4}{\pi} \int_0^1 Af(x) \sin^2 \pi x\,dx \int_0^1 xAf(x) \sin \pi x \cos \pi x\,dx \qquad (128) \\
&+ \frac{4}{\pi} \int_0^1 Af(x) \sin \pi x \cos \pi x\,dx \int_0^x Af(y) \sin^2 \pi y\,dy,
\end{aligned}
$$

where $h(x, y)$ is the function defined in (89). The reduction to iterated integrals made numerical computation much simpler. In actual computation, C_m was used rather than $\Delta C_m = C_m - C_{\text{av}}$, because the nature of the computation is such that addition of a constant to the values of ΔC_m does not alter the result.

The computation was carried out for two sets of data, measured on two lengths of the original 100-layer Clogston cable. The data, shown in graphical form in Fig. 9, are in the form of measured values of capacitance between consecutive layers of conductor. Only 98 values were recorded in one set and 97 in the other, because the outer layers were apparently not rigid enough to give reproducible capacitance measurements. To simplify the computation, only 97 values were used (i.e., $n = 97$ was assumed). Inasmuch as the values near the surface would have been weighted with a factor $\sin n\pi/100$, the resulting error is likely to be small. In one case, where the capacitance measurement was indeterminate because of a short circuit, the mean of the two adjacent measurements was used to avoid introducing an apparent gross discontinuity.

The computation resulted in two different values of b:

$$b = 1.345 \text{ for a 145.5-foot length,}$$

$$b = 1.296 \text{ for a 103-foot length.}$$

The corresponding reductions in bandwidth are

$$
\begin{aligned}
\frac{f_{i2}}{f_2'} = \frac{1}{\sqrt{1 + b^2}} &= 0.597 \text{ for length 145.5 feet,} \qquad (129) \\
&= 0.611 \text{ for length 103 feet.}
\end{aligned}
$$

These differ by about 2 per cent. It would be misleading to assume, because of this agreement, that the result is accurate within 2 per cent: on the basis of these two results only, there is still a chance of one in 20 that a large sample of similar measurements might have a standard deviation as high as 50 per cent.[10]

The correction was applied to results computed earlier by Morgan, for a perfectly uniform line. The results are shown as dashed lines on the figures.* It is clear that the correction accounts for substantially all of the difference between the measured data and the curve computed for a uniform line. The corrected curves predict an attenuation somewhat lower than that measured by resonance measurements and slightly higher than that measured by direct transmission measurements.

In comparing the new computed results with experiment, the following three facts must be borne in mind:

i. The theory of losses due to irregularities is not known to be accurate beyond a point where loss is about twice the low frequency loss (say 0.002 db/ft on the three figures). Hence, the calculated values are of doubtful validity in the upper three-quarters of each graph in Figs. 6, 7 and 8.

ii. Variations in conductor thickness have been assumed to be negligible.

iii. Longitudinal variations in the line have been ignored.

In view of the good correlation between measurement and theory, we can tentatively draw the following conclusions:

i. The theory of losses in Clogston 2 lines is consistent with the experiments. This also provides an experimental confirmation of the work of S. P. Morgan on which it was founded.

ii. We know enough about the effect of irregularities to predict in an intelligent and sufficiently accurate way the losses to be expected from manufacturing irregularities in Clogston 2 lines, provided we can make sufficiently accurate estimates of manufacturing tolerances.

VIII. CYLINDRICAL CLOGSTON LINES WITH LAYERS OF FINITE THICKNESS

Let us now apply the methods of Sections II, III, and V to a structure having finite laminations and cylindrical (rather than planar) symmetry. We shall study the properties of a Clogston 2 line which is completely filled with laminated layers; i.e., the radius of the inner core is zero.

The equation to be solved, with its boundary condition, is (from Ref.

---

* Actually, Morgan's curves include a linear term accounting for dielectric loss. The curve computed for the uniform line is approximately $\alpha = 0.00065[1 + (f/9.372)^2] + 0.000076f$, where α is in db/ft and f in mc. The dashed curve is $\alpha = 0.00065[1 + (f/5.659)^2] + 0.000076f$. Thus $f_{i2}/f_2 = 5.659/9.372$, corresponding to a value $b = 1.32$ in (129).

3, equations 27, 28 and 29

$$\frac{d}{d\rho}\left[\frac{1}{\rho(g + j\omega\epsilon)}\frac{d}{d\rho}[\rho H_\varphi]\right] + \left[\frac{\gamma^2}{g + j\omega\epsilon} - j\omega\mu_0\right]H_\varphi = 0, \quad (130)$$

$$H_\varphi(0) = H_\varphi(b) = 0,$$

where b is the radius of the outer sheath. The real interest in the equation is the determination of the eigenvalues γ.

In order to transform the equation to a self-adjoint equation with no first-order term, define a new independent variable u by

$$u^2 = \frac{2}{k}\int_0^\rho r(g + j\omega\epsilon)\, dr \quad (131)$$

and a new dependent variable, W, by

$$W = \frac{f}{\sqrt{u}} H_\varphi . \quad (132)$$

If we choose

$$k = 2\int_0^b r(g + j\omega\epsilon)\, dr \cong \theta b^2 g, \quad (133)$$

then the boundary conditions on W are

$$W(0) = W(1) = 0 \quad (134)$$

and the equation is transformed to

$$\frac{d^2W}{du^2} + \left\{k^2\frac{u^2}{\rho^2}\left[\frac{\gamma^2}{(g + i\omega\epsilon)^2} - \frac{j\omega\mu_0}{g + i\omega\epsilon}\right] - \frac{3}{4u^2}\right\}W = 0. \quad (135)$$

This can be considered a perturbation of the equation

$$\frac{d^2W}{du^2} + \left(\Lambda - \frac{3}{4u^2}\right)W = 0, \qquad W(0) = W(1) = 0, \quad (136)$$

where Λ is the average value

$$\Lambda = k^2\int_0^1 \frac{u^2}{\rho^2}\left[\frac{\gamma^2}{(g + j\omega\epsilon)^2} - \frac{j\omega\mu_0}{g + j\omega\epsilon}\right] du. \quad (137)$$

Write

$$jCf(u) = \Lambda - k^2\frac{u^2}{\rho^2}\left[\frac{\gamma^2}{(g + i\omega\epsilon)^2} - \frac{j\omega\mu_0}{g + i\omega\epsilon}\right] \quad (138)$$

and the original equation becomes

$$\frac{d^2W}{du^2} - \frac{3}{4u^2} W + [\Lambda - jCf(u)]W = 0. \tag{139}$$

Having once obtained the quantities Λ_n , $n = 1, 2, \cdots$, we can get γ_n from the relation (137).

The eigenfunctions of the unperturbed equations are

$$w_n = \sqrt{2u} \frac{J_1(\sqrt{\lambda_n}u)}{J_1'(\sqrt{\lambda_n})} \tag{140}$$

and the eigenvalues are determined from

$$\begin{aligned} J_1(\sqrt{\lambda_n}) &= 0: \\ \sqrt{\lambda_1} &= 3.8317, \\ \sqrt{\lambda_2} &= 7.0156, \\ \sqrt{\lambda_n} &\cong n\pi + \tfrac{1}{4}. \end{aligned} \tag{141}$$

From the technique in Courant and Hilbert,[6] we can write a second-order approximation to the first eigenvalue (the one we desire) as follows:

$$\Lambda_1 = \lambda_1 + 2Ca_1 + C^2 \sum_2^\infty \frac{{a_m}^2}{\lambda_m - \lambda_1}, \tag{142}$$

where

$$a_n = 2 \int_0^1 uf(u) \frac{J_1(\sqrt{\lambda_1}u)}{J_1'(\sqrt{\lambda_1})} \frac{J_1(\sqrt{\lambda_n}u)}{J_1'(\sqrt{\lambda_n})} du. \tag{143}$$

Proceeding by either route used before, we find

$$\Sigma = 2 \int_0^1 \int_0^1 \sqrt{uv} f(u) f(v) \frac{J_1(\sqrt{\lambda_1}u) J_1(\sqrt{\lambda_1}v)}{[J_1(\sqrt{\lambda_1})]^2} k(u, v)\, du\, dv. \tag{144}$$

The actual value of k is not important for the present study, but is given here as a matter of record:

$$\begin{aligned} k(v, u) = k(u, v) &= \frac{\pi\sqrt{uv}}{2J_1'(l)} [uJ_1'(ul)J_1(lv)Y_1(l) \\ &+ vJ_1'(lv)J_1(lu)Y_1(l) + J_1(lu)J_1(lv)Y_1'(l)] - \frac{\pi}{2}\sqrt{uv}J_1(lv)Y_1(lv) \\ &- \sqrt{uv}\frac{J_1(lu)J_1(lv)}{l^2[J_1'(l)]^2}, \qquad u \leqq v, \qquad l = \sqrt{\lambda_1}. \end{aligned} \tag{145}$$

Now let us re-examine (138) in detail, bearing in mind (131) and (133).

The thickness of the nth conducting layer is determined in terms of u as follows:

$$\begin{aligned} u_0^2 - u_i^2 &= \frac{2}{k}\int_{\rho_i}^{\rho_0} rg\,dr \\ &= \frac{2}{k}g\,\frac{\rho_0^2 - \rho_i^2}{2} = \frac{1}{\theta b^2}(\rho_0^2 - \rho_i^2). \end{aligned} \tag{146}$$

If the layers are uniform,

$$\begin{aligned} \rho_i &= \frac{m - c}{n}\,b, \\ \rho_0 &= \frac{m - c + \theta}{n}\,b, \end{aligned} \tag{147}$$

where the constant c is unity if the center is a conducting rod of radius equal to a layer, θ if the center is a dielectric rod of radius equal to a dielectric layer and intermediate otherwise. It is arithmetically convenient to choose a fractional value for c. Squaring and subtracting, we get

$$\rho_0^2 - \rho_i^2 = \frac{b^2}{n^2}(\theta)(2m - 2c + \theta). \tag{148}$$

The total through the first n layers is

$$\begin{aligned} u_{0,m}^2 &= \sum_{j=1}^{m} \frac{b^2\theta}{n^2}\,\frac{1}{\theta b^2}(-2c + \theta + 2m) \\ &= \frac{1}{n^2}[(-2c + \theta)m + 2m(m + 1)] \\ &= \frac{m^2}{n^2} + \frac{m}{n^2}(\theta - 2c + 1). \end{aligned} \tag{149}$$

Choosing

$$c = \frac{\theta + 1}{2}, \tag{150}$$

we find

$$u_{0,m} = \frac{m}{n}. \tag{151}$$

This corresponds physically to putting one-half layer of metal, i.e., a wire of radius $\theta b/2n$ at the center.

Under these conditions, the thickness of each metal layer is, in terms of u, precisely $1/n$.

The pulse strength is

$$\int f(u)\,du \tag{152}$$

integrated through one dielectric layer. Recalling (151) and (133), and assuming the unperturbed value of the propagation constant

$$\gamma^2 = -\omega^2\mu\bar{\epsilon}, \tag{153}$$

we get, after some tedious calculation

$$S = j\omega\epsilon\theta b^2 g_1 \left[\frac{\mu}{\epsilon}\left(\frac{\bar{\epsilon}}{\epsilon} - 1\right)\right] \frac{1-\theta}{n} [1 + \mathrm{O}(m^{-2})]. \tag{154}$$

Recalling (3), we finally get

$$S = \frac{j\omega\mu\theta b^2 \bar{g}}{n} [1 + \mathrm{O}(m^{-2})]. \tag{155}$$

Thus we see that the pulse strength is approximately the same as before [see (87)], but is slightly altered in the neighborhood of the center.

If we now evaluate the double sum, we find

$$\begin{aligned} \sum_i\sum_j S_iS_jh(u_i, v_j) &= \sum\sum S^2(1 + \delta_i + \delta_j + \delta_i\delta_j)h_{ij} \\ &= S^2 \sum\sum h_{ij} + S^2 \sum_i \delta_i \sum_j h_{ij} \\ &\quad + S^2 \sum_j \delta_j \sum_i h_{ij} + S^2 \sum\sum \delta_i\delta_j h_{ij}, \end{aligned} \tag{156}$$

where

$$\frac{S_i}{S} - 1 = \delta_i = \mathrm{O}(m^{-2}). \tag{157}$$

Now

$$\begin{aligned} \delta_i h_{ij} &= \mathrm{O}(n^{-2}), \\ \sum_j h_{ij} &= 0 + \mathrm{O}(n^{-1}), \\ \delta_i\delta_j h_{ij} &= \mathrm{O}(n^{-4}), \end{aligned} \tag{158}$$

and hence

$$\sum_i\sum_j S_iS_jh(u_i, v_j) = S^2 \sum\sum h_{ij} + \mathrm{O}(S^2n^{-1}). \tag{159}$$

Therefore, to the order of precision attained in the plane case, the result is the same. However, the unperturbed eigenvalues are not the numbers $n^2\pi^2$, but rather the squares of the roots of $J_1(x) = 0$ (141). The first root, in particular, is 3.8317. The analogs of (106) and (107) for a cylinder of radius b are:

$$\gamma^2 = -\omega^2\mu\bar{\epsilon}\left[1 + \frac{(3.8317)^2}{j\omega\mu\bar{g}b^2} - \frac{j\omega\mu\bar{g}\theta^2 b^2}{12n^2}\right],$$

$$\alpha = \frac{(3.8317)^2}{2\sqrt{\bar{\mu}/\bar{\epsilon}}\,\bar{g}b^2} + \frac{\omega^2\mu^2\bar{g}t_1^2}{24\sqrt{\bar{\mu}/\bar{\epsilon}}}, \tag{160}$$

$$\beta = \omega\sqrt{\bar{\mu}/\bar{\epsilon}}.$$

The second of these is equivalent to Morgan's equation 486 (remembering that Morgan calls the number of layers in his cable $2n$ rather than n). In presenting this equation, Morgan emphasized that it depended on certain physical assumptions and approximations; in view of the present result, these assumptions and approximations seem amply justified. The principle adopted by Morgan of assuming the current distribution of an unperturbed mode in a system and computing losses consequent from such currents is often the easiest to use and sometimes the only known way to compute attenuation. It is highly gratifying to see an application in a case quite different from the ordinary where the result is confirmed by an alternative method. Our confidence in this method is thus increased.

A careful analysis of the method used in the present paper would probably show that the particular choice of physical configuration at the origin is immaterial. We can conclude, in agreement with Morgan's equations 484 and 485, that the result is valid also for a Clogston 2 line bounded inside with a finite cylinder, provided that appropriate eigenvalue of the unperturbed problem is used. As the ratio of inside to outside diameter increases towards unity, these eigenvalues rapidly approach the values $n^2\pi^2$, which are those of the plane configuration.

IX. ACKNOWLEDGMENTS

The help of S. P. Morgan, J. M. McKenna, H. P. Kramer, Mrs. R. A. King and the computing group in the Mathematical Research Department of Bell Telephone Laboratories is gratefully acknowledged.

REFERENCES

1. Clogston, A. M., Reduction of Skin Effect Losses by the Use of Laminated Conductors, B.S.T.J., **30,** July 1951, p. 491.

2. Black, H. S., Mallinckrodt, C. O. and Morgan, S. P., Experimental Verifications of the Theory of Laminated Conductors, Proc. I.R.E., **40,** August 1952, p. 902.
3. Morgan, S. P., Mathematical Theory of Laminated Transmission Lines, B.S.T.J., **31,** September 1952, p. 883; November 1952, p. 1121.
4. Vaage, E. F., Transmission Properties of Laminated Clogston-Type Conductors, B.S.T.J., **32,** May 1953, p. 695.
5. King, Mrs. R. A. and Morgan, S. P., Transmission Formulas and Charts for Laminated Coaxial Cables, Proc. I.R.E., **42,** August 1954, p. 1250.
6. Courant, R. and Hilbert, D. *Methods of Mathematical Physics*, Interscience, New York, 1953, Vol. 1, Chs. 5 and 13.
7. Kato, T., On the Convergence of the Perturbation Method, J. Fac. Sci. Univ. Tokyo, **6,** March 31, 1951.
8. Hayashi, S. and U-O, Koji, The Attenuation Characteristics of the Coaxial Laminated Cable, Technical Report No. 40, The Engineering Research Institute, Kyoto Univ., Kyoto, Japan, December 1957.
9. King, Mrs. R. A., this issue, p. 517.
10. Kaplan, E. L., private communication.