# A Method of Computing Bivariate Normal Probabilities

## With an Application to Handling Errors in Testing and Measuring

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*Charts and formulas are presented from which bivariate normal probabilities may be computed. Formulas involving the bivariate normal are given for the solution of a problem of handling errors in testing and measuring. These formulas include, in addition to previously published cases, two new cases. In one, the product is not necessarily centered relative to two-sided specification limits; in the other, one-sided specification limits are considered.*

### I. INTRODUCTION

Many manufactured products are 100 per cent tested with the idea of insuring that each unit of product meets the performance specifications. The general procedure is to set the test specification limits, which are the limits used for product unit acceptance, at or arbitrarily near the performance specification limits. These are established by engineering requirements and require a test-set accuracy and precision of a specified amount with respect to the performance specification limits or some nominal value. Eagle[1] and many others have studied the problem of locating test specification limits with respect to performance specification limits under various conditions of test-set precision when testing itself is subject to random error. He pointed out that, when random errors of testing exist, two types of errors or mistakes can occur which should be taken into account in setting test specification limits. The first error, called consumer's loss (CL), is defined as the probability that nonconforming product units will be accepted. The second error, called pro-

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ducer's loss (PL),* is defined as the probability that conforming product units will be rejected.

The problem of locating test specification limits with respect to performance specification limits may be considered heuristically as in Fig. 1, where A and D are performance specification limits, and B and C are test specification limits. Consider a product unit which just fails to meet the lower performance specification limit and therefore is nonconforming and should not be accepted. The chance that a product unit of this value will be accepted by the test set is shown by the shaded area under the test-set-error distribution curve to the right of B, the lower test specification limit. This then is a part of the consumer's loss (CL), and the summation of similar considerations for all product units outside the performance specification limits constitutes the CL. Consider now a product unit with value at C, well within the performance specification limits, which should be accepted. The shaded area under the test-set-error distribution curve to the right of C, the upper test specification limit, shows that there is a 50 per cent chance of the test set rejecting a product unit of this value, and thus this is a part of the producer's loss (PL). The summation of similar considerations for all product units inside the performance specification limits constitutes PL. Different settings of the test specification limits, B and C, with respect to the performance specification limits, A and D, and/or different spreads (standard deviations) of the product and test-set distributions would lead to different producer's and consumer's losses.

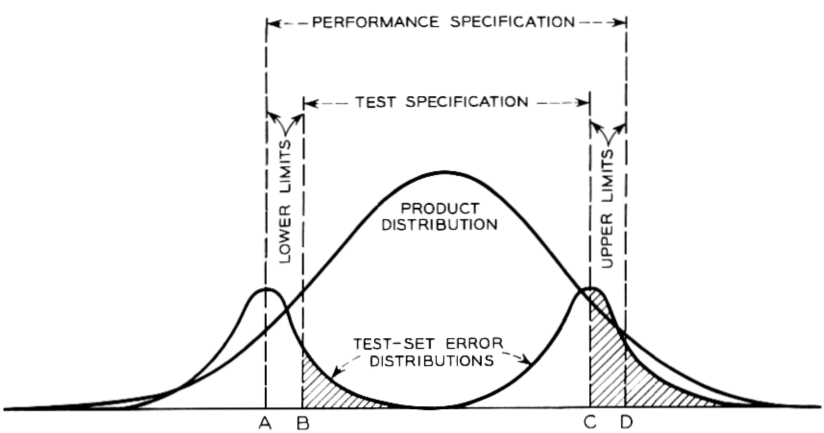


Fig. 1 — Diagram showing effect of test-set errors on test acceptance limits.

* Consumer's loss and producer's loss are called consumer's risk and producer's risk by Grubbs and Coon[2] and others. The terminology used here is an effort to avoid confusion with present standard statistical quality control terminology.

Eagle presented graphs of PL and CL which assume that the product distribution and test-set-error distribution are both normal (Gaussian) and also that the mean of the product distribution is midway between the performance specification limits and also midway between the test specification limits. Thus, these two sets of limits are symmetrical with respect to the mean of the product distribution. He also presented formulas based on Pearson's tables[3] for computing PL and CL. Pearson's tables are now out of print and are not readily available to many who may wish to make these calculations. However, the National Bureau of Standards[4] plans to reissue Pearson's tables with some extensions.

Hayes[5] and Wiesen and Clark[6] have presented additional graphs of PL and CL for the same conditions as those considered by Eagle. In the present paper, the solution is given in formula form without the product distribution centering requirement. Also, one-sided test and performance specification limits are considered.

Grubbs and Coon[2] also considered the problem of setting the performance and test specification limits for a centered product distribution. They found the locations which minimized the sum CL plus PL, and also the locations which minimized total cost when the consumer's loss was subject to a given cost and the producer's loss subject to another cost. In this paper, formulas are given for the same cost assumptions, but for both one-sided and not necessarily centered two-sided performance and test specification limits.

Tingey and Merrill[7] considered the problem of minimizing the total cost when the cost to the consumer of accepting nonconforming product units varied with the degree of nonconformance. They presented a table of constants for constructing test specification limits under these circumstances. Formulas (20) and (21) are given below for computing the total cost under these assumptions.

Eagle[1] and Owen[8,9] showed that the bivariate normal distribution underlies the above problems. Consequently, a method for computing bivariate normal probabilities is first considered here. Although this method is discussed relative to the present problem, the method is perfectly general and can be used wherever bivariate normal probabilities are required.

## II. COMPUTATION OF BIVARIATE NORMAL PROBABILITIES FROM THE CHARTS

Let

$$M(h,k;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_h^\infty \int_k^\infty \exp\left[-\frac{1}{2}\left(\frac{x^2 - 2\rho xy + y^2}{1-\rho^2}\right)\right] dx\, dy. \quad (1)$$

$M(h, k; \rho)$ is the probability that a normal random variable $X$ with mean zero and variance one is greater than $h$, and another normal random variable $Y$ with mean zero and variance one is greater than $k$, where $\rho$ is the correlation between $X$ and $Y$; that is, $\Pr(X > h, Y > k) = M(h, k; \rho)$.

It is convenient also to have a functional notation for the univariate normal integral. To this end, define

$$G(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{h} \exp\left(\frac{-x^2}{2}\right) dx.$$

Volumes of the bivariate normal over other rectangular regions may be expressed in terms of the $M$- and $G$-functions. In terms of standardized variables (zero means and unit variances),

$$\begin{aligned} \Pr(X < h, Y < k) &= M(-h, -k; \rho) \\ &= G(h) + G(k) + M(h, k; \rho) - 1, \end{aligned} \tag{2}$$

$$\begin{aligned} \Pr(X < h, Y > k) &= M(-h, k; -\rho) = G(-k) - M(h, k; \rho) \\ &= G(h) - M(-h, -k; \rho) \end{aligned} \tag{3}$$

$$\begin{aligned} \Pr(X > h, Y < k) &= M(h, -k; -\rho) = G(-h) - M(h, k; \rho) \\ &= G(k) - M(-h, -k; \rho). \end{aligned} \tag{4}$$

Ref. 8, which is an elaboration of Ref. 9, shows that

$$\begin{aligned} M(h, k; \rho) = M&\left(h, 0; \frac{(\rho h - k)(\operatorname{sgn} h)}{\sqrt{h^2 - 2\rho hk + k^2}}\right) \\ &+ M\left(k, 0; \frac{(\rho k - h)(\operatorname{sgn} k)}{\sqrt{h^2 - 2\rho hk + k^2}}\right) - \begin{cases} 0 \\ \frac{1}{2}, \end{cases} \end{aligned} \tag{5}$$

where the upper choice is made if $hk > 0$ or if $hk = 0$ but $h + k \geqq 0$, and the lower choice is made otherwise; and $\operatorname{sgn} h = +1$ if $h \geqq 0$, and $\operatorname{sgn} h = -1$ if $h < 0$. Note that sgn $h$ as used in (5) is a multiplicative factor; i.e., it affects the sign of the quantity it operates on but not the absolute value of the quantity.

Equation (5) means that bivariate normal probabilities with any limits can be computed from a table of the bivariate normal probabilities where one of the limits is zero. Figs. 2 through 5 are graphs of these probabilities, i.e., of $M(h, 0; \rho)$. Figs. 4 and 5 can be obtained by rotating Figs. 2 and 3, respectively, 180 degrees and replacing $h$ by $-h$ and $\rho$ by $-\rho$. However, to avoid confusion both figures are included. The following examples illustrate the use of these graphs.

*Example 1:* The following probabilities may be read from Figs. 2 through 5:

$$M(1, 0; 0.5) = 0.127,$$

$$M(1, 0; -0.5) = 0.031,$$

$$M(-1, 0; 0.5) = 0.469,$$

$$M(-1, 0; -0.5) = 0.373.$$

*Example 2:* Find $M(1.96, 1; 0.56)$.

*Solution:*

$$\sqrt{h^2 - 2\rho hk + k^2} = \sqrt{2.6464} = 1.6268.$$

$$\begin{aligned} M(1.96, 1; 0.56) &= M(1.96, 0; 0.060) + M(1, 0; -0.861) \\ &= 0.014 + 0.002 \\ &= 0.016. \end{aligned}$$

The value computed by more elaborate methods[8,9] is 0.016.

*Example 3:* Find $M(0.4, 0.1; -0.5)$.

*Solution:*

$$\sqrt{h^2 - 2\rho hk + k^2} = \sqrt{21}.$$

$$\begin{aligned} M(0.4, 0.1; -0.5) &= M(0.4, 0; -0.655) + M(0.1, 0; -0.982) \\ &= 0.069 + 0.015 \\ &= 0.084. \end{aligned}$$

The value read from Pearson's tables[3] is 0.0836.

*Example 4:* Find $M(1, -2; 0.7)$.

*Solution:*

$$\sqrt{h^2 - 2\rho hk + k^2} = 2.793.$$

$$\begin{aligned} M(1, -2; 0.7) &= M(1, 0; 0.967) + M(-2, 0; 0.859) - \tfrac{1}{2} \\ &= 0.159 + 0.500 - 0.500 \\ &= 0.159. \end{aligned}$$

The value computed from Pearson's tables[3] is 0.1587.

*Example 5:* Find $M(1, -2; -0.5)$.

*Solution:*

$$\sqrt{h^2 - 2\rho hk + k^2} = 1.732.$$

$$\begin{aligned} \mathrm{M}(1, -2; -0.5) &= M(1, 0; 0.866) + M(-2, 0; 0) - \tfrac{1}{2} \\ &= 0.157 + 0.489 - 0.500 \\ &= 0.146. \end{aligned}$$

The value computed from Pearson's tables[3] is 0.1454.

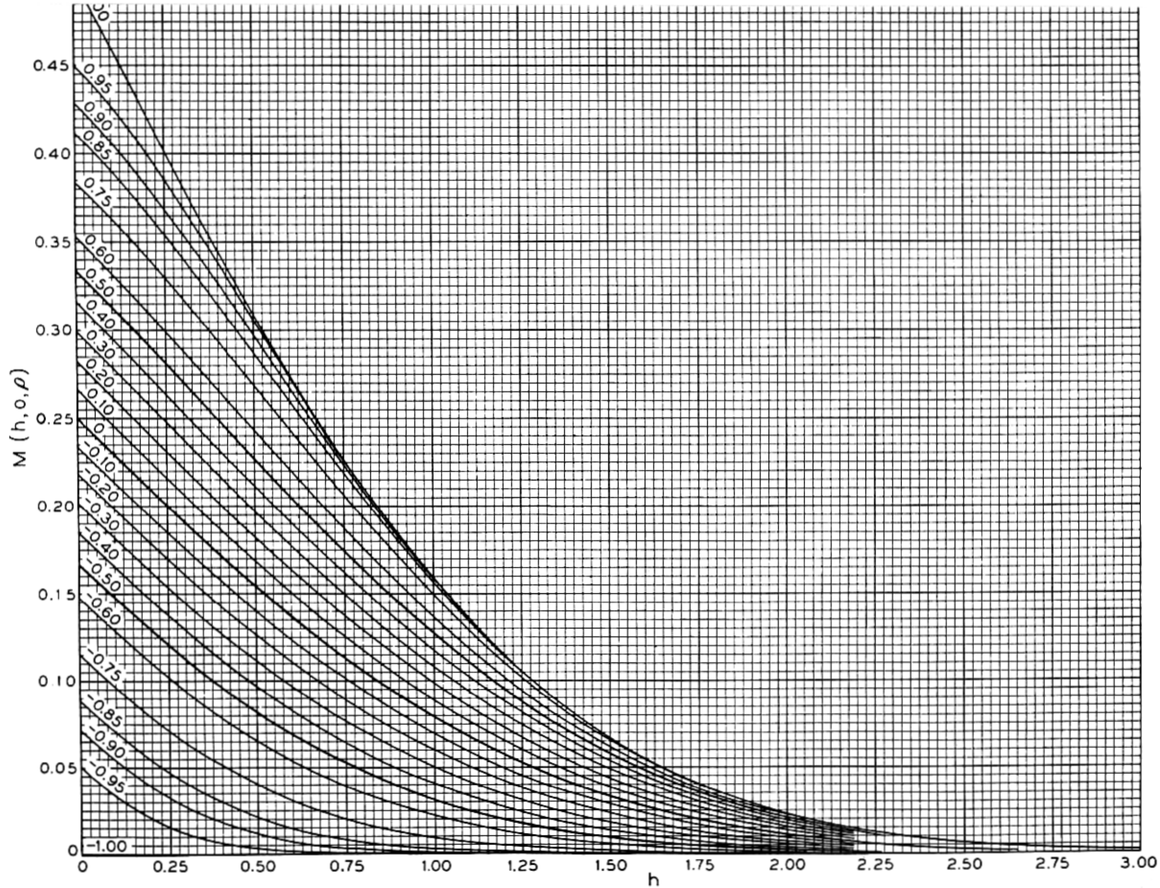


Fig. 2 — Bivariate normal probabilities for positive values of the parameter $h$. This chart gives the values of $M(h, 0; \rho) = \Pr(X > h, Y > 0; \text{correlation} = \rho)$, where $X$ and $Y$ are standardized normal variates and the number associated with each curve indicates the value of $\rho$ for that curve.

To use this chart, first locate the required value of $h$ on the abscissa and erect a perpendicular to the horizontal axis at that point. Where this perpendicular intersects the curve having the proper correlation, draw a

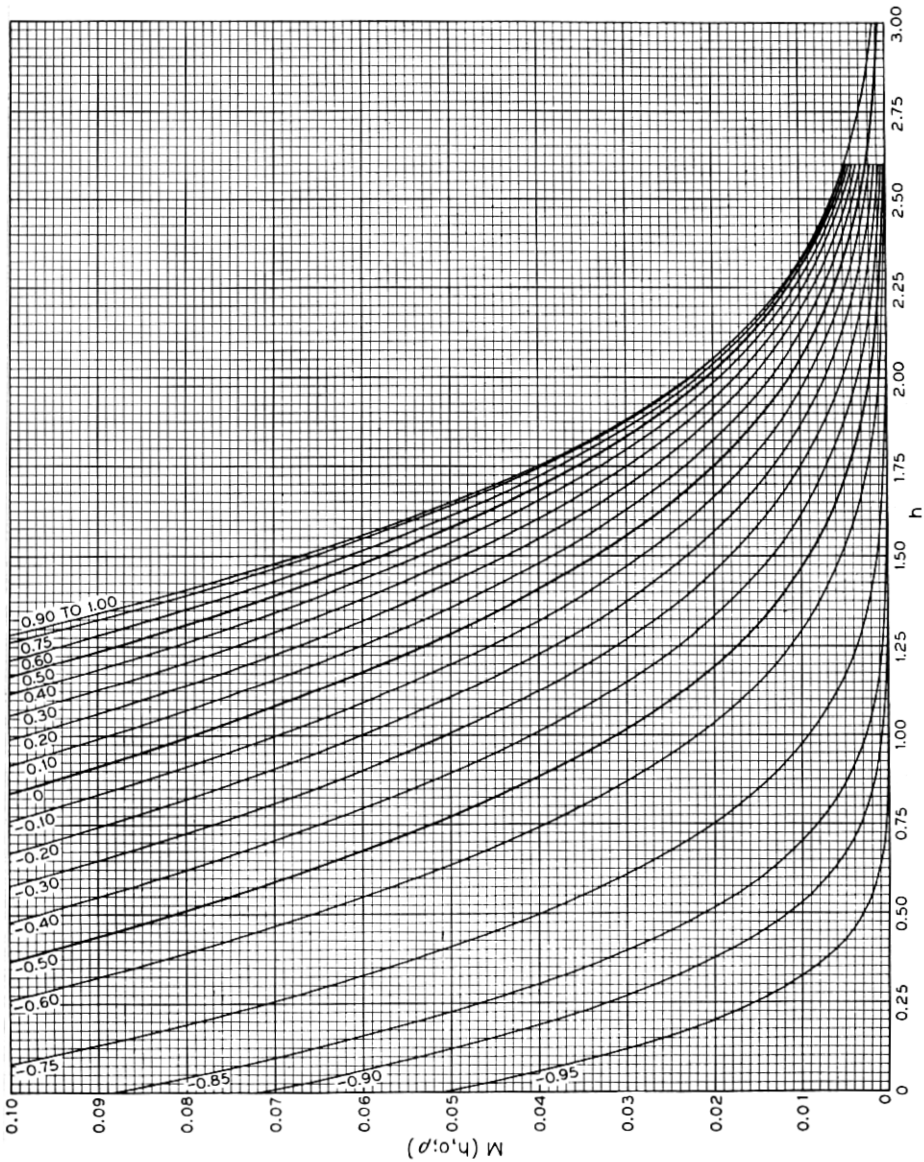


**Fig. 3** — Bivariate normal probabilities for positive values of the parameter $h$ and for values of $M(h, 0; \rho)$ from 0 to 0.10.

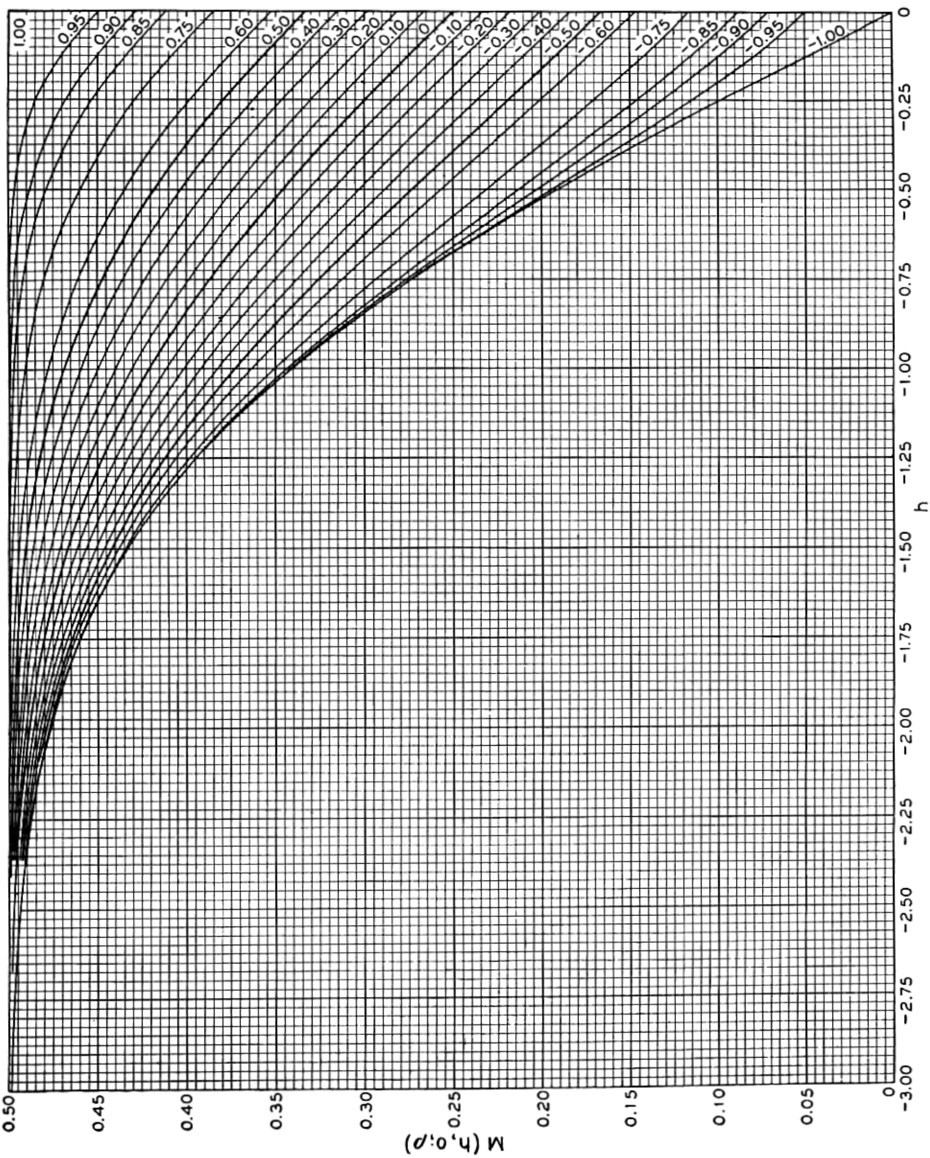


Fig. 4 — Bivariate normal probabilities for negative values of the parameter $h$. Note that, for $\rho = 1.00$, $M(h, 0; \rho) = 0.50$ for negative values of $h$. Values of $M(h, 0; \rho)$ between 0.40 and 0.50 may be read more ...

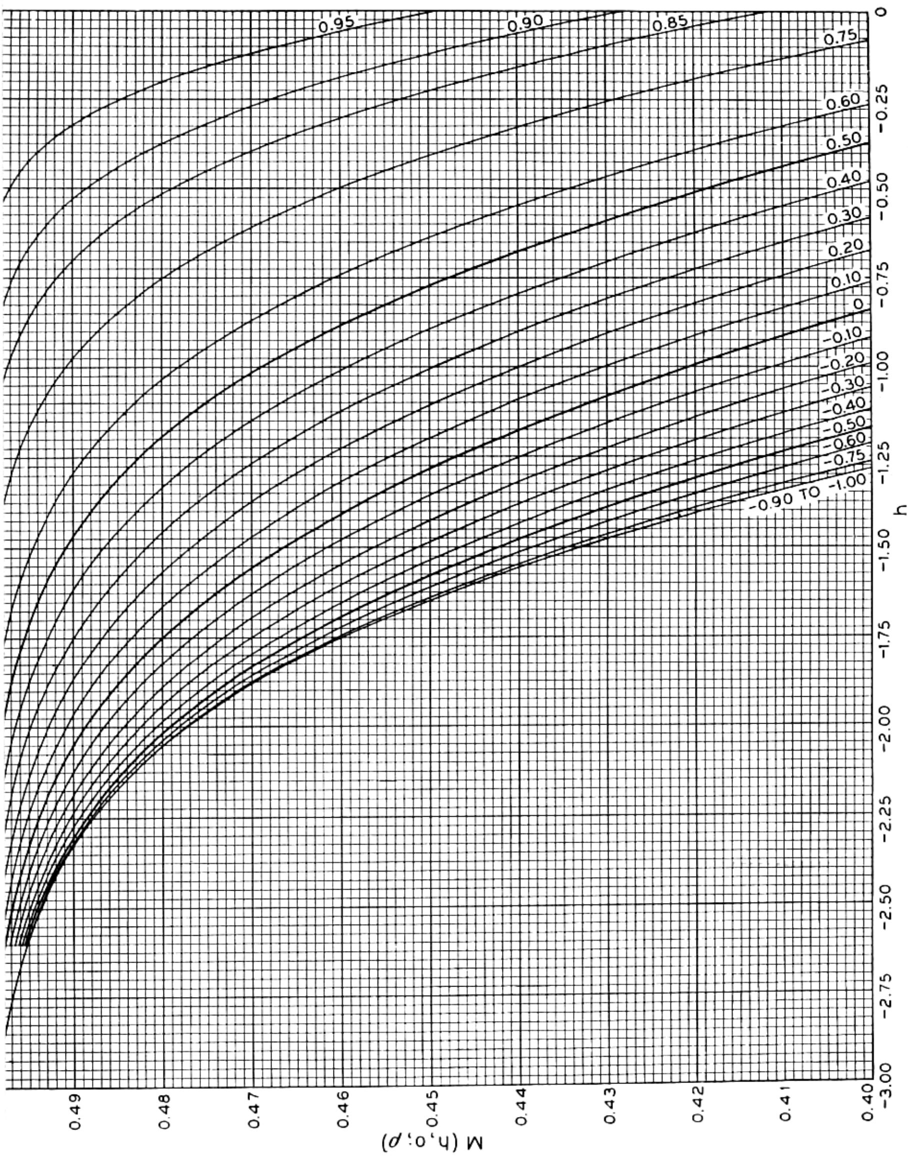


Fig. 5 — Bivariate normal probabilities for negative values of the parameter $h$ and for values of $M(h, 0; \rho)$ from 0.40 to 0.50.

## III. ERRORS IN TESTING AND MEASURING

Before giving the formulas for the problem of errors in testing and measuring, let the following quantities be defined.

Let $P$ = a random variable (normally distributed with mean $\mu$ and standard deviation $\sigma_p$) which describes the true product values.

Let $T$ = a random error of measurement (test-set error) normally distributed with mean $\lambda$ and standard deviation $\sigma_t$.

The term $\lambda$ is called the bias of the measuring instrument. In Refs. 1, 2, 5 and 6, $\lambda$ was taken to be equal to zero. This is not necessary, as only a slight change in the definition of the test specification limits takes care of this. (See the definition of $b_1$ and $b_2$ below.) Of course, in most cases where $\lambda$ is known, the test instrument would be recalibrated and the bias eliminated. There may be situations, however, where this would not be practical and it really is no problem to carry $\lambda$ as an extra parameter. Also, in assessing the problem of an unknown bias, it is convenient to have the parameter $\lambda$ available (see Example 8 below).

Let $S$ = the observed measurement = $P + T$, so that, if $T$ and $P$ are taken to be independent, $S$ is normal with mean $\mu + \lambda$ and variance $\sigma_s^2 = \sigma_p^2 + \sigma_t^2$. Also, $P$ and $S$ are correlated, and it will be shown that the correlation is equal to $\sigma_p/\sigma_s$.

The expectation or mean of a random variable $X$ is indicated symbolically by $\mathrm{E}(X)$. The correlation between two variables $X$ and $Y$ is defined as

$$\frac{\mathrm{E}[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y}.$$

For $P$ and $S$, then, the numerator of the correlation is

$$\begin{aligned}\mathrm{E}[(P - \mu)\,(S - \mu - \lambda)] &= \mathrm{E}[(P - \mu)\,(P + T - \mu - \lambda)] \\ &= \mathrm{E}[P^2 + PT - P\mu - P\lambda - \mu P - \mu T + \mu^2 + \mu\lambda].\end{aligned}$$

Now, $\mathrm{E}(X^2) = \sigma_X^2 + \mu^2$; $\mathrm{E}(\text{constant}) = \text{constant}$; $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$, if $X$ and $Y$ are independent; $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$ and $\mathrm{E}(X) = \mu_X$. The correlation between $P$ and $S$ is then

$$\rho_{ps} = \frac{\sigma_p^2 + \mu^2 + \mu\lambda - \mu^2 - \mu\lambda - \mu^2 - \mu\lambda + \mu^2 + \mu\lambda}{\sigma_p \sigma_s} = \frac{\sigma_p}{\sigma_s}.$$

Let $k_1$ and $k_2$ be defined so that the performance specification limits are at $\mu + k_1\sigma_p$ and $\mu - k_2\sigma_p$. For a lower performance specification limit only, take $k_1 = \infty$; and for an upper performance specification limit only, take $k_2 = \infty$. In Refs. 1, 2, 5 and 6, the $k$'s were taken to be equal which meant the product distribution was assumed to be centered with respect to the performance specification limits.

Let $b_1$ and $b_2$ be defined so that, when the test specification limits are placed at $\mu + \lambda + k_1\sigma_p - b_1\sigma_t$ and at $\mu + \lambda - k_2\sigma_p + b_2\sigma_t$, either a desired consumer's loss or a desired producer's loss is not exceeded, or some combination of these losses is not exceeded.

In this paper, $\mu$, $\lambda$, $\sigma_p$ and $\sigma_t$ are assumed to be known. When any of them are unknown, experiments must be run to establish their values. In this connection, the reader may wish to read a paper by Grubbs,[10] which gives methods for estimating $\sigma_p$ and $\sigma_t$ when the variance of each observation is a linear function of $\sigma_p^2$ and $\sigma_t^2$. Fig. 6 shows diagrammatically the relationship between $\mu$, $\lambda$, $\sigma_p$, $\sigma_t$, $b_1$, $b_2$, $k_1$ and $k_2$.

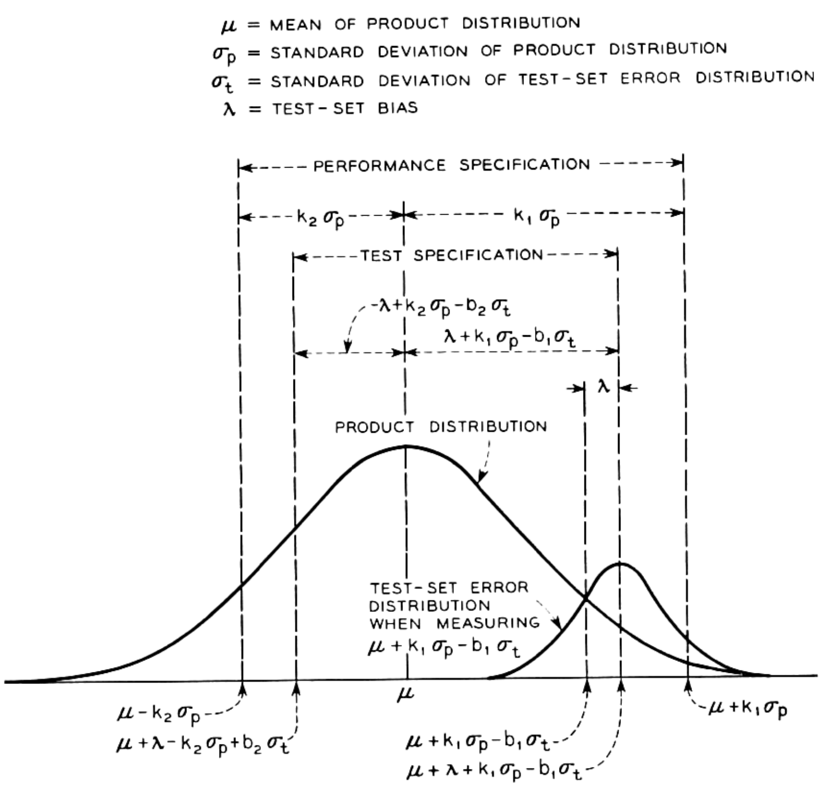


Fig. 6 — Diagram showing various constants defined for errors in the testing-and-measuring problem.

From the definition of consumer's loss (CL) it is clear that

$$\begin{aligned} \mathrm{CL} = \Pr[&P > \mu + k_1\sigma_p \quad \text{or} \quad P < \mu - k_2\sigma_p \\ &\text{and} \quad \mu + \lambda - (k_2\sigma_p - b_2\sigma_t) < S < \mu + \lambda + (k_1\sigma_p - b_1\sigma_t)]. \end{aligned} \tag{6}$$

Since the correlation between $P$ and $S$ is given by

$$\rho_{ps} = \rho = \sigma_p/\sigma_s = \sigma_p/\sqrt{\sigma_p^2 + \sigma_t^2},$$

and $P$ and $S$ are assumed to be bivariate normal, the above probability can be expressed in terms of the $M$-function. The first step, as with the univariate normal, is to standardize the variables, i.e., subtract the mean and divide by the standard deviation. In order to save space and give compact formulas, additional notation is introduced. Let

$$q_1 = \frac{k_1\sigma_p - b_1\sigma_t}{\sqrt{\sigma_p^2 + \sigma_t^2}} \quad \text{and} \quad q_2 = \frac{k_2\sigma_p - b_2\sigma_t}{\sqrt{\sigma_p^2 + \sigma_t^2}}.$$

Then,

$$\begin{aligned} \mathrm{CL} &= \Pr\left(\frac{P - \mu}{\sigma_p} > k_1 \quad \text{or} \quad \frac{P - \mu}{\sigma_p} < -k_2 \right. \\ &\qquad\qquad\qquad \left. \text{and} \quad -q_2 < \frac{S - \mu - \lambda}{\sigma_s} < q_1\right) \\ &= \Pr\left(\frac{P - \mu}{\sigma_p} > k_1 \quad \text{and} \quad \frac{S - \mu - \lambda}{\sigma_s} > -q_2\right) \\ &\qquad - \Pr\left(\frac{P - \mu}{\sigma_p} > k_1 \quad \text{and} \quad \frac{S - \mu - \lambda}{\sigma_s} > q_1\right) \\ &\qquad + \Pr\left(\frac{P - \mu}{\sigma_p} < -k_2 \quad \text{and} \quad \frac{S - \mu - \lambda}{\sigma_s} > -q_2\right) \\ &\qquad - \Pr\left(\frac{P - \mu}{\sigma_p} < -k_2 \quad \text{and} \quad \frac{S - \mu - \lambda}{\sigma_s} > q_1\right). \end{aligned}$$

Now (2), (3) and (5) are used to reduce CL to its most easily used form:

$$\begin{aligned} \mathrm{CL} = M(k_1, -q_2; \rho) &- M(k_1, q_1; \rho) \\ &- M(k_2, q_2; \rho) + M(k_2, -q_1; \rho). \end{aligned} \tag{7}$$

The producer's loss (PL) is given by

$$\begin{aligned} \mathrm{PL} = \Pr[&\mu - k_2\sigma_p < P < \mu + k_1\sigma_p \quad \text{and} \\ &S < \mu + \lambda - (k_2\sigma_p - b_2\sigma_t) \quad \text{or} \quad S > \mu + \lambda + (k_1\sigma_p - b_1\sigma_t)]. \end{aligned} \tag{8}$$

If the same procedure as that with consumer's loss is followed, it can be shown that (8) reduces to

$$\mathrm{PL} = \mathrm{CL} + G(k_1) + G(k_2) - G(q_1) - G(q_2). \qquad (9)$$

For one-sided specification limits the above formulas reduce to:

*For an upper limit only:*

$$\mathrm{CL} = \Pr(P > \mu + k_1\sigma_p \quad \text{and} \quad S < \mu + \lambda + k_1\sigma_p - b_1\sigma_t);$$

hence

$$\mathrm{CL} = G(-k_1) - M(k_1, q_1; \rho) \qquad (10)$$

and

$$\mathrm{PL} = \Pr(P < \mu + k_1\sigma_p \quad \text{and} \quad S > \mu + \lambda + k_1\sigma_p - b_1\sigma_t);$$

and hence,

$$\mathrm{PL} = G(-q_1) - M(k_1, q_1; \rho). \qquad (11)$$

*For a lower limit only:*

$$\mathrm{CL} = \Pr(P < \mu - k_2\sigma_p \quad \text{and} \quad S > \mu + \lambda - k_2\sigma_p + b_2\sigma_t);$$

hence

$$\mathrm{CL} = G(-k_2) - M(k_2, q_2; \rho) \qquad (12)$$

and

$$\mathrm{PL} = \Pr(P > \mu - k_2\sigma_p \quad \text{and} \quad S < \mu + \lambda - k_2\sigma_p + b_2\sigma_t);$$

hence

$$\mathrm{PL} = G(-q_2) - M(k_2, q_2; \rho). \qquad (13)$$

The following examples illustrate the use of (7) through (13).

*Example 6:* Suppose the performance specification limits are at $\mu + 2\sigma_p$ and at $\mu - 3\sigma_p$. Suppose also that the test specification limits have been set at $\mu + \lambda + 2\sigma_p$ and at $\mu + \lambda - 3\sigma_p + \sigma_t$. If $\sigma_t/\sigma_p = 0.5$, determine the producer's and consumer's losses of this procedure. Since the bias is known in this example and has been allowed for in setting the test specification limits, $\lambda$ does not appear in the calculations of PL and CL. (See example 8 for the case where $\lambda$ is not known.)

*Solution:* Here, $k_1 = 2$, $k_2 = 3$, $b_1 = 0$, and $b_2 = 1$. Then,

$$q_1 = \frac{k_1\sigma_p - b_1\sigma_t}{\sqrt{\sigma_p^2 + \sigma_t^2}} = \frac{2 - 0}{\sqrt{1.25}} = 1.789,$$

$$q_2 = \frac{k_2\sigma_p - b_2\sigma_t}{\sqrt{\sigma_p^2 + \sigma_t^2}} = \frac{3 - 0.5}{\sqrt{1.25}} = 2.236$$

and

$$\rho = \frac{\sigma_p}{\sqrt{\sigma_p^2 + \sigma_t^2}} = \frac{1}{\sqrt{1.25}} = 0.894.$$

From (7),

$$\begin{aligned} \text{CL} = {} & M(2, -2.236; 0.894) - M(2, 1.789; 0.894) \\ & -M(3, 2.236; 0.894) + M(3, -1.789; 0.894). \end{aligned}$$

Application of (5) results in

$$\begin{aligned} \text{CL} &= M(2, 0; 0.976) + M(-2.236, 0; 0.970) - 0.5 - M(2, 0; 0) \\ &\quad - M(1.789, 0; -0.447) - M(3, 0; 0.316) - M(2.236, 0; -0.707) \\ &\quad + M(3, 0; 0.958) + M(-1.789, 0; 0.985) - 0.5 \\ &= 0.023 + 0.500 - 0.500 - 0.017 - 0.004 - 0.000 \\ &\quad - 0.000 + 0.000 + 0.500 - 0.500 \\ &= 0.008. \end{aligned}$$

From (7),

$$\begin{aligned} \text{PL} &= 0.008 + G(2) + G(3) - G(1.789) - G(2.236) \\ &= 0.008 + 0.977 + 0.999 - 0.963 - 0.987 = 0.034. \end{aligned}$$

*Example 7:* The circumference of a product has a mean value of 28.5 inches with a standard deviation of 0.5 inch. Only circumferences less than 29 inches are acceptable. The device for measuring the circumference is known to be biased so that on the average it measures 0.1 inch too small with a standard deviation of 0.2 inch. If the upper test specification limit is set at 29 inches, and there is no lower limit, what are the producer's and consumer's losses?

*Solution:* Here, $\mu = 28.5$, $\sigma_p = 0.5$, $\lambda = -0.1$ and $\sigma_t = 0.2$. Hence, for the performance specification limits,

$$\mu + k_1\sigma_p = 29,$$

or

$$28.5 + k_1(0.5) = 29.$$

Solving this equation results in $k_1 = 1$. For the test specification limits,

$$\mu + \lambda + k_1\sigma_p - b_1\sigma_t = 29,$$

or

$$28.5 - 0.1 + 1(0.5) - b_1(0.2) = 29.$$

Solving this equation results in $b_1 = -0.5$. Hence

$$q_1 = \frac{k_1\sigma_p - b_1\sigma_t}{\sqrt{\sigma_p^2 + \sigma_t^2}} = \frac{0.5 + 0.01}{\sqrt{0.29}} = 1.114,$$

and

$$\rho = \frac{\sigma_p}{\sqrt{\sigma_p^2 + \sigma_t^2}} = \frac{0.5}{\sqrt{0.29}} = 0.928\,.$$

From (10),

$$\begin{aligned} \text{CL} &= G(-1) - M(1, 1.114; 0.928) \\ &= 0.159 - M(1, 0; -0.447) - M(1.114, 0; 0.083) \\ &= 0.159 - 0.036 - 0.073 = 0.050. \end{aligned}$$

From (11),

$$\text{PL} = G(-1.114) - M(1, 1.114; 0.928) = 0.133 - 0.109 = 0.024.$$

Thus, for the measuring device considered here, the consumer's loss is 0.050 and the producer's loss is 0.024.

*Example 8*: If the bias $\lambda$ is assumed to be zero when it is actually positive, the effect is to accept more products in the lower tail of the product distribution and to reject more in the upper tail. The effect on CL and PL is the same as if $k_1\sigma_p$ had been decreased by $\lambda$ and $k_2\sigma_p$ increased by $\lambda$, i.e., as if both test specification limits had been decreased by the quantity $\lambda$.

As an example, the performance specification limits for a product are at $\mu + k_1\sigma_p = 90$ and $\mu - k_2\sigma_p = 80$. Also, $\mu = 85$, $\sigma_t = 1$ and $\sigma_p = 2$. Then $k_1 = k_2 = 2.5$, and Table I of Grubbs and Coon[2] can be used to find $b_1 = b_2$ so that the consumer and the producer accept equal losses. The table gives $b_1 = b_2 = -0.5902$ and results in a consumer's loss and producer's loss each equal to 0.0061 or a total loss of 0.0122. Now suppose that the tester is biased one unit high, i.e., $\lambda = 1$. What effect does this have on the producer's loss and the consumer's loss?

*Solution:* The test limits were set at

$$\mu + k_1\sigma_p - b_1\sigma_t = 90.5902$$

and

$$\mu - k_2\sigma_p + b_2\sigma_t = 79.4098,$$

whereas, if the bias had been taken into account, they should have been set at

$$\mu + \lambda + k_1\sigma_p - b_1\sigma_t = 91.5902$$

and

$$\mu + \lambda - k_2\sigma_p + b_2\sigma_t = 80.4098$$

to maintain the equality of PL and CL. To find the effect of this error, it is only necessary to decrease $k_1\sigma_p$ by $\lambda$; i.e., redefine $k_1$ so that $k_1\sigma_p = 5 - \lambda = 4$ and hence take $k_1 = 2$, and redefine $k_2$ so that $k_2\sigma_p = 5 + \lambda = 6$ and hence take $k_2 = 3$. Now the formulas for CL and PL [(7) and (9)] are used with the following parameters:

$$k_1 = 2, \qquad k_2 = 3,$$

$$b_1 = -0.5902, \qquad b_2 = -0.5902,$$

$$\sigma_p = 2, \qquad \sigma_t = 1.$$

$$\begin{aligned}
\text{CL} &= M(2, -2.947; 0.894) - M(2, 2.053; 0.894) - M(3, 2.947; 0.894) \\
&\quad + M(3, -2.053; 0.894) \\
&= M(2,0;0.982) + M(-2.947,0;0.962) - 0.500 - M(2,0;-0.281) \\
&\quad - M(2.053,0;-0.180) - M(3,0;-0.193) - M(2.947,0;-0.267) \\
&\quad + M(3, 0; 0.962) + M(-2.053, 0; 0.982) - 0.500 \\
&= 0.023 + 0.500 - 0.500 - 0.006 - 0.007 - 0.000 - 0.000 + 0.001 \\
&\quad + 0.500 - 0.500 \\
&= 0.011.
\end{aligned}$$

$$\begin{aligned}
\text{PL} &= 0.011 + G(2) + G(3) - G(2.053) - G(2.947) \\
&= 0.011 + 0.977 + 0.999 - 0.980 - 0.998 = 0.009.
\end{aligned}$$

Hence, because of the bias in the test set, the consumer's loss is increased from 0.006 to 0.011, and the producer's loss is increased from 0.006 to

0.009. If the same test specification limits are used over a period of time, it would probably be worthwhile to compute the consumer's loss and producer's loss for a whole range of biases. These could be plotted on a graph showing the assumed bias on the abscissa and the producer's or consumer's loss on the ordinate.

## IV. SPECIAL CRITERIA FOR DETERMINING SPECIFICATION LIMITS

Following Grubbs and Coon,[2] if the condition is that the producer and consumer accept the same or equal losses in rejecting a conforming product unit and in accepting a nonconforming product unit, the solution is obtained by setting CL = PL in (9) for two-sided test and performance specification limits, or setting CL = PL in (10) and (11) or (12) and (13) for one-sided test and performance specification limits. This results in $q_1 = k_1$ and $q_2 = k_2$ or

$$b_1 = \frac{k_1\sigma_p - k_1\sqrt{\sigma_p^2 + \sigma_t^2}}{\sigma_t} \tag{14}$$

and

$$b_2 = \frac{k_2\sigma_p - k_2\sqrt{\sigma_p^2 + \sigma_t^2}}{\sigma_t}. \tag{15}$$

Equations (14) and (15) may be solved for $b_1$ and $b_2$, respectively; or, if $r$ is set equal to $\sigma_p/\sigma_t$, the value of $b$ satisfying (14) and (15) may be read from Table I (p.16) of Ref. 2. The values for consumer's loss and producer's loss, however, will have to be calculated from (7), (9), (10), (11), (12) and (13).

Another criterion discussed by Grubbs and Coon is to assume that the sum of the consumer's and producer's losses is to be a minimum. Then the $b$'s for two-sided test specification limits are obtained by solving the equations

$$G\left(\frac{-k_2 - \rho q_1}{\sqrt{1-\rho^2}}\right) + G\left(\frac{-k_1 + \rho q_1}{\sqrt{1-\rho^2}}\right) = \frac{1}{2}$$

and

$$G\left(\frac{-k_1 - \rho q_2}{\sqrt{1-\rho^2}}\right) + G\left(\frac{-k_2 + \rho q_2}{\sqrt{1-\rho^2}}\right) = \frac{1}{2}.$$

Again following Grubbs and Coon, if $k_1 \geqq 1.5$, $k_2 \geqq 1.5$, $\sigma_p \geqq \sigma_t$ and the $b$'s are negative or small positive, then the first integral in each of the above equations is nearly zero, so that approximate solutions to the above equations are $k_1 \cong \rho q_1$ and $k_2 \cong \rho q_2$, or

$$b_1 \cong -\frac{k_1 \sigma_t}{\sigma_p} \tag{16}$$

and

$$b_2 \cong -\frac{k_2 \sigma_t}{\sigma_p}. \tag{17}$$

The $b$ values can be read from Table II (p. 16) of Ref. 2, but the consumer's and producer's losses have to be computed from the formulas given in this paper. Equations (16) and (17) for the $b$'s are exact for one-sided test specification limits.

Let $C_{cl}$ be the cost of accepting a nonconforming product unit, and $C_{pl}$ be the cost of rejecting a conforming product unit. Then the values for $b_1$ and $b_2$ that minimize the total cost are those that satisfy the equations

$$G\left(\frac{-k_2 - \rho q_1}{\sqrt{1 - \rho^2}}\right) + G\left(\frac{-k_1 + \rho q_1}{\sqrt{1 - \rho^2}}\right) = \frac{C_{pl}}{C_{cl} + C_{pl}},$$

and

$$G\left(\frac{-k_1 - \rho q_2}{\sqrt{1 - \rho^2}}\right) + G\left(\frac{-k_2 + \rho q_2}{\sqrt{1 - \rho^2}}\right) = \frac{C_{pl}}{C_{cl} + C_{pl}}.$$

Again, if $k_1 \geqq 1.5$, $k_2 \geqq 1.5$, $\sigma_p \geqq \sigma_t$ and the $b$'s are negative or small positive, then the first integral in each of the above equations is nearly zero and the $b$'s may be obtained approximately from the equations

$$G\left(\frac{-k_1\sigma_t - b_1\sigma_p}{\sqrt{\sigma_p^2 + \sigma_t^2}}\right) \cong \frac{C_{pl}}{C_{cl} + C_{pl}} \tag{18}$$

and

$$G\left(\frac{-k_2\sigma_t - b_2\sigma_p}{\sqrt{\sigma_p^2 + \sigma_t^2}}\right) \cong \frac{C_{pl}}{C_{cl} + C_{pl}}. \tag{19}$$

In this case the $b$'s may be obtained approximately (for two-sided test specification limits) by setting the limit of the integral to that deviate of a univariate normal which corresponds to the fraction

$$\frac{C_{pl}}{C_{cl} + C_{pl}}.$$

Tingey and Merrill[7] also consider this problem, and they allow the cost of accepting a nonconforming item to be different for the two tails of the product distribution. They give a short table of values of $b_1$ and $b_2$ (their Table II) under these conditions. Note that the subscripts 1 and 2 must be interchanged to enter their table with the formulas given here.

For a one-sided test specification limit the $b$ is obtained exactly by (18) and (19). The corresponding producer's and consumer's losses may then be calculated from (10), (11), (12) and (13) given in this paper.

As mentioned in the introduction, Tingey and Merrill[7] consider the case where the producer's loss is constant but the consumer's loss varies with the degree of nonconformance. They give a table of values of $b_1$ and $b_2$ which will minimize the total cost under these conditions. Note that they use the subscript 1 for the lower tail of the product distribution and 2 for the upper tail, while the reverse has been used here. Hence, if the formulas in this paper are being used, interchange $b_1$ with $b_2$ and $k_1$ with $k_2$ when entering Tingey and Merrill's table. They do not give values for the total loss, but their formula for the total loss may be reduced to a computing form as follows:

Let $C_{cl}{}^u$ = cost to the consumer of accepting a nonconforming product unit from the upper tail only of the product distribution. Then, if the cost to the consumer varies with the degree of nonconformance, this is defined mathematically to be

$$C_{cl}{}^u = \frac{P - \mu - k_1\sigma_p}{\sigma_p} C^u,$$

where $C^u$ is the unit cost associated with the acceptance of nonconforming product in the upper tail of the product distribution. Then, in the notation used here, Tingey and Merrill's formula can be reduced to

$$\begin{aligned} \frac{C_{cl}{}^u}{C^u} = & -k_1 M(k_1, -q_2; \rho) + k_1 M(k_1, q_1; \rho) \\ & + \frac{\rho}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}q_2{}^2) G\left(\frac{-k_1 - \rho q_2}{\sqrt{1-\rho^2}}\right) \\ & - \frac{\rho}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}q_1{}^2) G\left(\frac{-k_1 + \rho q_1}{\sqrt{1-\rho^2}}\right) \\ & - \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}k_1{}^2) G\left(\frac{-q_1 + \rho k_1}{\sqrt{1-\rho^2}}\right) \\ & + \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}k_1{}^2) G\left(\frac{q_2 + \rho k_1}{\sqrt{1-\rho^2}}\right). \end{aligned} \tag{20}$$

Hence, with the aid of Figs. 2, 3, 4 and 5 and a table of the univariate normal distribution function and its derivative, it is possible to compute $C_{cl}{}^u$. A similar formula for the consumer's cost for the lower tail of the product distribution may be obtained by interchanging $k_1$ with $k_2$ and interchanging $q_1$ with $q_2$.

If the cost per product unit to the producer of rejecting conforming product is constant, the total cost to the producer may be found by multiplying the PL obtained from (9) [where the CL in (9) is computed from (7)] by the per product unit cost. Then the total cost to both producer and consumer is obtained by adding (a) the consumer's cost for the upper tail of the product distribution, (b) the consumer's cost for the lower tail of the product distribution and (c) the producer's cost. It is this total cost which is minimized by the values of $b_1$ and $b_2$ found in Tingey and Merrill's Table I.

Consider now the case where the producer's cost of rejecting a conforming product unit is proportional to the degree of nonconformance, but only the upper tail of the product distribution is to be considered. The consumer's cost can be obtained by putting $q_2$ equal to infinity in (20). The result is

$$\begin{aligned} \frac{C_{cl}^{\;u}}{C^u} = & -k_1G(-k_1) + k_1M(k_1, q_1; \rho) \\ & - \frac{\rho}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}q_1^2\right) G\left(\frac{-k_1 + \rho q_1}{\sqrt{1-\rho^2}}\right) \\ & + \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}k_1^2\right) G\left(\frac{q_1 - \rho k_1}{\sqrt{1-\rho^2}}\right). \end{aligned} \tag{21}$$

As before, a similar formula holds if a lower tail only is considered and may be obtained by replacing $k_1$ with $k_2$ and $q_1$ with $q_2$.

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