Synthesis of Active *RC* Networks

By B. K. KINARIWALA

(Manuscript received May 28, 1959)

A basic theorem is derived for RC networks containing active elements. It is shown that no more than one active element, embedded in a passive RC network, is needed to realize any driving-point function. Sufficiency of only one active element is shown by developing a synthesis method. A synthesis technique for n-port passive RC networks is developed in order to establish the sufficiency proof of the basic theorem. A more practical method of realizing driving-point functions, using active RC networks, termed the "cascade" method, is also presented. This method is applied to the design of a tenth-order Tchebycheff parameter filter.

I. INTRODUCTION

Passive filters using only resistive and capacitive elements are attractive for reasons of size, cost and reliability. Their use has been limited because the network complexity of *RC* filters (due to restrictions on impedance functions realizable with *R*'s and *C*'s only) is greater than that of equivalent *RLC* filters. This defect can be overcome by using active elements in addition to passive *RC* networks. Active *RC* networks are particularly attractive for achieving exacting performance at low frequencies, where it is not practical to use either inductors or crystals.

The practicality of active networks is a direct result of the availability of precision resistors and capacitors of small size having small drifts and low temperature coefficients, as well as the development of reliable junction transistors. In fact, it is not unusual to find that the active element's drift with time and temperature is no worse than that of a passive element.

There are several techniques available for synthesis of transfer functions by active *RC* networks.[1,2,3,4,5] The active element used is either a stabilized high-gain feedback amplifier or a negative-impedance converter. With the feedback amplifier, one *RC* network is used to produce the zeros of the transfer function and another *RC* network produces the poles of the function. The number of passive elements required in this case tends to be rather large. The use of a converter usually leads to bridged–T, twin–T or parallel ladder networks when there are finite frequencies of infinite attenuation. There are several modifications of these basic synthesis methods which yield simpler networks. However, the emphasis in all of these methods is on the realization of transfer functions.

This paper presents some theoretical and practical results in the synthesis of active *RC* driving-point functions. Obviously, driving-point impedances can be used to yield any desired transfer characteristic. A theoretical investigation is first undertaken to show that no more than one active element (embedded in an *RC* network) is required to yield any desired driving-point function. The proof of this basic theorem is presented in Section II. Sufficiency of only one active element is shown by an actual synthesis method. This method involves realization of n-port passive *RC* networks. The results used in Section II for this purpose are drawn from material presented in the Appendices.

A more practical method of synthesizing driving-point functions is presented in Section III. This is termed the "cascade" method, since it involves cascading of a passive *RC* network with another passive *RC* network through a negative impedance converter. Network functions of rank 2 lead to particularly simple structures. Such functions are discussed in Section IV. The results obtained in this section are applied to the design of a tenth-order Tchebycheff parameter filter in Section V. The concluding section summarizes the results and discusses the outstanding problems as well as suggests some new avenues of approach.

II. BASIC THEOREM

In this section it will be proved that any driving-point immittance function* can be realized by a transformerless *RC* structure in which is embedded only one active element. The active element may be represented as a "controlled" source or as an amplifier. In any case, the active element is a two-port transducer. These two ports may be considered as external ports which are connected through an active element.

The remaining network is now purely passive. Thus, a driving-point immittance can be represented as a three-port passive *RC* network with two of the ports connected through an active element (Fig. 1). The proof of the basic theorem is then accomplished in two parts:

---

* In this paper, an immittance function is assumed expressible as a ratio of two polynomials in the complex frequency variable, $p = \sigma + j\omega$. The only restriction on these polynomials, except where other restrictions are specifically introduced, is that they have real coefficients.

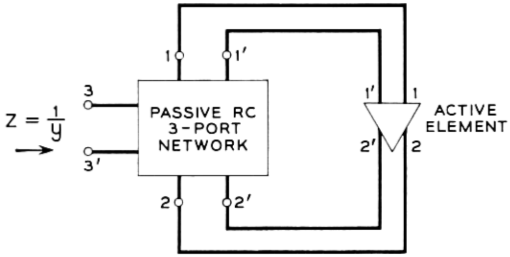

Fig. 1 — Realization of driving-point function with one active element.

i. Determination of the three-port *RC* matrix from the specified immittance function.

ii. Realization of the three-port *RC* network without transformers.

2.1 *Determination of Three-Port RC Matrix*

A driving-point admittance function $y(p)$, where $p = \sigma + j\omega$, is the complex frequency variable, is to be realized by the structure shown in Fig. 1. The admittance function is chosen because it leads to a three-port short-circuit admittance matrix, Y, which is convenient to use in the realization of the desired network.

In the figure, y is the admittance as seen from the terminals 3-3′, μ is the gain of the active element and T_{12} is the current transfer function of the passive *RC* network between terminals 1-1′ and 2-2′. Then, by Blackman's equation,[6]

$$y = y_{33} \left(\frac{1 - \mu T_{12}{}^{0}}{1 - \mu T_{12}{}^{\infty}} \right), \tag{1}$$

where y_{33} is the admittance seen at 3-3′ when the active element is placed in a reference condition of zero forward transmission, $\mu T_{12}{}^{0}$ is the loop-current transmission when zero admittance is introduced between terminals 3-3′ and $\mu T_{12}{}^{\infty}$ is the loop current transmission when infinite admittance is introduced between 3-3′.

For convenience and without loss of generality, it is assumed that the amplifier has zero input and output impedances. The amplifier is thus a current-controlled voltage source (Fig. 2). Thus, y_{33} is the short-circuit admittance at port 3-3′ and $\mu T_{12}{}^{\infty}$ is the loop current transmission when all ports are short-circuited, where

$$\mu T_{12}{}^{\infty} = -R_m y_{12} , \tag{2}$$

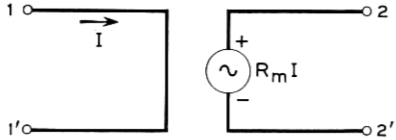

Fig. 2 — Current-controlled voltage source.

and $\mu T_{12}{}^0$, the loop current transmission when port 3-3′ is open-circuited, may be written as

$$\mu T_{12}{}^0 = R_m \left[y_{13} \left(\frac{y_{23}}{y_{33}} \right) - y_{12} \right]. \tag{3}$$

Substituting (2) and (3) in (1),

$$y = y_{33} \left[\frac{1 - R_m \left(\dfrac{y_{13} y_{23}}{y_{33}} - y_{12} \right)}{1 + R_m y_{12}} \right]. \tag{4}$$

Let y be the admittance given as a ratio of two polynomials. Express the numerator and denominator as differences of two polynomials:

$$y = \frac{N}{D} = \frac{N_1 - N_2}{D_1 - D_2}, \tag{5}$$

The polynomials N_1 and D_1 must have only negative real roots. Further, N_1 and D_1 are so chosen that N_1/D_1 is an RC driving-point admittance. The reasons for these requirements will become clear later. It is also desired that the degree of N_2 does not exceed that of N_1 by more than one and that the degree of D_2 does not exceed that of D_1 by more than one. In Appendix A it is shown that, given N and D, it is always possible to find the polynomials N_1, N_2, D_1 and D_2 satisfying these requirements.

It is now possible to rewrite (5),

$$y = \frac{N_1}{D_1} \left(\frac{1 - \dfrac{N_2}{N_1}}{1 - \dfrac{D_2}{D_1}} \right) \tag{6}$$

and, comparing with (4),

$$y_{33} = \frac{N_1}{D_1},$$

$$
\begin{aligned}
-R_m y_{12} &= \frac{D_2}{D_1}, \\
R_m\left(\frac{y_{13}\,y_{23}}{y_{33}} - y_{12}\right) &= \frac{N_2}{N_1}
\end{aligned}
\tag{7}
$$

or

$$-R_m\, y_{13}\, y_{23} = \frac{N_1 D_2 - N_2 D_1}{D_1^2}.$$

The previously mentioned requirements on the polynomials N_1, D_1, etc. are seen to be necessary if the identifications in (7) are to be made. The first equation in (7) gives the requirements on N_1/D_1, the second equation gives those on the degrees of D_2 and D_1, and so on. Since these requirements are already met, one obtains from (7) some of the elements of the short-circuit admittance matrix Y of the three-port *RC* network; y_{33} is completely determined, whereas y_{12} and the product $y_{13}y_{23}$ are determined within a constant multiplier.

Equation (7) gives all the constraints on the elements of the Y matrix imposed by the admittance function. Other elements of the matrix can be chosen arbitrarily. These remaining elements of the Y matrix, R_m and the identification of y_{13} and y_{23}, must be so determined that Y is realizable without transformers. In order to determine these quantities, it is necessary to have a knowledge of the conditions under which the network can be realized without transformers.

2.2 *Realization of Three-Port RC Networks*

Since synthesis of three-port networks is a special case of general n-port synthesis, a synthesis technique for n-port *RC* networks is developed below. The results obtained are then applied to the three-port case.

2.2.1 *Synthesis of n-Port RC Networks*

The synthesis method consists of generating an auxiliary $(m + n)$-port* matrix representing a purely resistive network. When this network is terminated at its m-ports in appropriate capacitors, the desired n-port *RC* network is obtained (Fig. 3). It is assumed that the terminations are unit capacitors. If desired, the values of these capacitors are changed by a simple scaling of impedances at the appropriate ports.

---

* m represents the number of capacitors in the *RC* network.

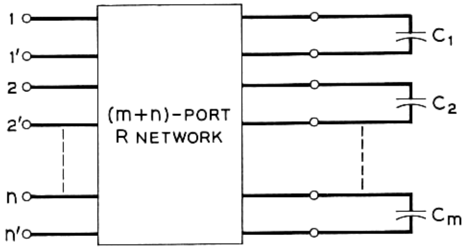

Fig. 3 — $(m + n)$-port resistance network terminated in m capacitors.

The short-circuit admittance matrix Y of the RC network may be expressed as

$$Y = Kp + K_\infty - \sum_\nu K_\nu \left(\frac{1}{p + \sigma_\nu}\right), \tag{8a}$$

where K, K_∞ and K_ν are $(n \times n)$ matrices of residues. It is shown in Appendix B that it is preferable, though not necessary, to have $K \equiv 0$; i.e., the matrix Y has no pole at $p = \infty$. Furthermore, it is always possible to determine Y from the given driving-point admittance such that there is no pole at $p = \infty$ (see Appendix A). Consequently, it is convenient to consider

$$Y = K_\infty - \sum_\nu K_\nu \left(\frac{1}{p + \sigma_\nu}\right). \tag{8b}$$

Let the short-circuit admittance matrix for the purely resistive $(m + n)$-port be

$$G = \begin{array}{cc} \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right] & \begin{array}{c} n \\ m \end{array} \\ \begin{array}{cc} n & m \end{array} & \end{array}, \tag{9}$$

where A_{ij} are matrices of appropriate order. It is desired to determine G such that, when the m-ports are terminated in unit capacitors, the matrix Y is obtained. It is shown in Appendix B that

$$\begin{aligned} A_{11} &= K_\infty, \\ A_{12} &= M, \qquad A_{21} = \bar{M} = \text{transpose of } M \\ A_{22} &= S. \end{aligned} \tag{10}$$

where

$$M = [\mathrm{M}_1 , M_2 , \quad], \tag{11}$$

and M_ν is defined by

$$K_\nu = M_\nu \bar{M}_\nu ; \tag{12}$$

M_ν has n rows and m_ν columns. Thus, M has n rows and $m = \sum_\nu m_\nu$ columns. It is seen from (9) that m determines the number of capacitors. S is a diagonal matrix defined by

$$S = \begin{bmatrix} \sigma_1 I_1 & 0 & 0 & 0 & & & 0 \\ 0 & \sigma_2 I_2 & 0 & 0 & & & 0 \\ 0 & 0 & \sigma_3 I_3 & 0 & & & 0 \\ 0 & 0 & 0 & & & & 0 \\ & & & & & & 0 \\ & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \end{bmatrix}, \tag{13}$$

where I_j is a unit matrix of order m_j .

If it is desirable to obtain a network with a minimum number of capacitors, it is necessary to obtain a minimum m. This is accomplished by having each m_ν a minimum number. A minimum m_ν is obtained if

$$M_\nu = [M_{\nu 1} , M_{\nu 2} , M_{\nu 3} , \cdots], \tag{14}$$

where $M_{\nu i}$ are determined by expressing the quadratic form of K_ν as sum of squares,

$$\bar{X}\, K_\nu X = \sum_i (X M_{\nu i})^2. \tag{15}$$

Each $M_{\nu i}$ is a column vector. It is seen that, if K_ν is of order n and rank δ_ν , then M_ν has n rows and $m_\nu = \delta_\nu$ columns. The matrix M is then obtained by using (11). The minimum number m of capacitors is seen to be the sum of the ranks of matrices of residues in the finite poles.

The $(m + n)$-port G matrix is thus determined by (9) and (10). The $(m + n)$-port resistance network must be terminated at its m-ports in unit capacitors to obtain the desired n-port RC network.

If the matrix G can now be realized without ideal transformers, then terminating the resistance network in capacitors at appropriate ports will yield the RC network without transformers. Synthesis of resistance networks is considered below.

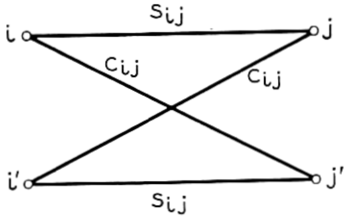

Fig. 4 — A typical unit in the realization of n-port resistance network.

2.2.2 *Synthesis of n-Port R Networks*

It is generally known and easy to show that a resistance network need have no internal nodes.* It follows that an n-port R network can always be realized as a completely connected graph having only the $2n$ external nodes. Necessary and sufficient conditions for the general n-port R network are not yet obtained. However, conditions for a restricted class of networks are obtained here.

Consider a symmetrical balanced structure constructed by introducing symmetrical lattices between each set of node-pairs (Fig. 4). If all node-pairs are short-circuited and a one-volt signal is introduced between one pair of terminals (say i-i'), then from the balanced structure it is seen that the current drawn by the network is determined solely by structures directly connected to terminals i-i'. Short-circuit admittance seen at i-i' is the sum of the short-circuit admittances of the lattices connected directly to i-i'. Furthermore, the short-circuit transfer admittances between terminals i-i' and j-j', for all j, are determined only by the lattice structure between each set of node-pairs. Thus,

$$g_{ii} = \sum_{j \neq i} A_{ij}\,, \tag{16}$$

where A_{ij} is the short-circuit admittance of the lattice between node-pairs i-i' and j-j', and g_{ii} is the short-circuit admittance seen at i-i'. Also,

$$g_{ij} = B_{ij}\,,$$

where B_{ij} is the short-circuit transfer admittance of the lattice between node-pairs i-i' and j-j', and g_{ij} is the short-circuit transfer admittance between terminals i-i' and j-j'.

Since for any lattice $A_{ij} \geqq | B_{ij} |$, (16) may be written as

$$g_{ii} \geqq \sum_{j \neq i} | g_{ij} |. \tag{18}$$

---

* One way to show this is by repeated application of the results of Rosen.[7]

The conditions of (18) are shown to be necessary for the structure of Fig. 4. If any additional branches were placed between terminals i and i', the conditions of (18) would still be valid.

Sufficiency of the above conditions is proved by realization of the network as follows. If S_{ij} and C_{ij} are series and cross-arm admittances respectively of the i–j lattice,

$$\left.\begin{aligned} A_{ij} &= \tfrac{1}{2}\,(S_{ij} + C_{ij}) \\ g_{ij} &= B_{ij} = \tfrac{1}{2}\,(-S_{ij} + C_{ij}) \end{aligned}\right\}. \tag{19}$$

Let $A_{ij} = | B_{ij} | = | g_{ij} |$. Then,

$$\left.\begin{aligned} C_{ij} &= | g_{ij} | + g_{ij} \\ S_{ij} &= | g_{ij} | - g_{ij} \end{aligned}\right\}. \tag{20}$$

It is seen that C_{ij} and S_{ij} are non-negative, and therefore realizable. Any excess admittance determined by

$$g_{ii} - \sum_{j \neq i} | g_{ij} |$$

is then introduced between terminals i and i'. This proves the sufficiency of conditions (18). A matrix G that satisfies the conditions of (18) will be called a "dominant diagonal" or D matrix.

Necessary and sufficient conditions for the existence of balanced R network are that the short-circuit admittance matrix be a D matrix.

2.2.3 *Three-Port RC Networks*

In the case of three-ports for active networks under consideration, it will be shown that it is possible to find a matrix G which is a D matrix. It is also desirable to obtain the three-port RC network with a minimum number of capacitors.

If the case $n = 3$ is considered and the minimum number of capacitors is desired, then $m_\nu = 1$ and K_ν should be expressible as

$$K_\nu = M_\nu \tilde{M}_\nu\,, \tag{21}$$

where M_ν is a matrix with three rows and one column. If $K_\nu = \| k_{ij}{}^{(\nu)} \|$ it is shown in Appendix C that, from (7),

$$k_{13}{}^{(\nu)} k_{23}{}^{(\nu)} - k_{12}{}^{(\nu)} k_{33}{}^{(\nu)} = 0, \qquad \text{for all } \nu. \tag{22}$$

Furthermore, since y_{11} and y_{22} are not specified, they may be chosen so that the K_ν , for all ν, are positive semidefinite matrices of rank 1. Hence, the representation in (21) is possible with

$$K_\nu = \begin{bmatrix} \dfrac{k_{13}^{(\nu)2}}{k_{33}^{(\nu)}}, & k_{12}^{(\nu)}, & k_{13}^{(\nu)} \\ k_{12}^{(\nu)}, & \dfrac{k_{23}^{(\nu)2}}{k_{33}^{(\nu)}}, & k_{23}^{(\nu)} \\ k_{13}^{(\nu)}, & k_{23}^{(\nu)} & k_{33}^{(\nu)} \end{bmatrix}, \tag{23}$$

$$\bar{M}_\nu = \pm \frac{1}{\sqrt{k_{33}^{(\nu)}}} [k_{13}^{(\nu)}, k_{23}^{(\nu)}, k_{33}^{(\nu)}],$$

and relation (22) is assumed to be valid.

The elements of interest in G are

$$G = \left[\begin{array}{ccccc|c} k_{11}^{\infty} & k_{12}^{\infty} & k_{13}^{\infty} & & & \\ k_{21}^{\infty} & k_{22}^{\infty} & k_{23}^{\infty} & & & \\ k_{31}^{\infty} & k_{32}^{\infty} & k_{33}^{\infty} & \pm\sqrt{k_{33}^{(1)}} & \pm\sqrt{k_{33}^{(2)}} & \sqrt{k_{33}^{(m)}} \\ & & & \sigma_1 & 0 & 0 \\ & & & 0 & \sigma_2 & 0 \\ \hline & & & 0 & 0 & \sigma_m \end{array}\right].$$

The first two rows of G can be made to satisfy the dominant diagonal conditions, since the diagonal terms may be chosen arbitrarily. The rows having σ_ν for diagonal terms can also be made to satisfy the same conditions by multiplying appropriate row and column by $\sqrt{C_\nu}$. This effectively changes the termination from unit capacitor to one having value C_ν. To make the third row satisfy the same conditions, R_m must be chosen large enough, thereby reducing the magnitudes of the residues of y_{13} and y_{23} [see (7)]. Further, the freedom in the choice of P_1 and Q_1 [see (5)] can be used to make the diagonal term in the third row greater than the sum of magnitudes of the remaining terms.

It is seen that the three-port RC matrices arising from the active networks under discussion can be realized without transformers.

A much simpler way of proving the basic theorem becomes obvious at this point. From discussion of n-port R networks it is seen that n-port RC networks are realizable if the matrices of residues in various poles and $Y(0)$ are all D matrices. For example, the Y matrix may be written as

$$Y = K_0 + \sum_\nu \frac{p}{p + \sigma_\nu} K_\nu', \tag{25}$$

where $K_0 = Y(0)$ and $K'_\nu = K_\nu/\sigma_\nu$. If K_0 is a D matrix, it is realized by resistors. Each remaining term $p(K'_\nu)/p + \sigma_\nu$ is realized in the same manner as the R networks, except that each branch would be a resistor and a capacitor in series. The networks are all connected in parallel to obtain the n-port RC network.

From (7), it is seen that y_{11} and y_{22} are unconstrained and hence can be chosen such that the matrices K_0 and K'_ν have the diagonal terms in the first and second rows greater than the sum of the magnitudes of the off-diagonal terms of the corresponding rows. The third rows can be made to satisfy these conditions by an appropriate choice of R_m. A large enough value of R_m can always be chosen to make the residues of y_{13} and y_{23} sufficiently small [see (7)].

Thus, there are two alternative ways of proving the basic theorem. Comparing the two as synthesis methods, the second way is much simpler, but uses a rather large number of resistors and capacitors. The first method uses a minimum number of capacitors but still a large number of resistors. It also offers a choice as to the number of capacitors desired. However, it is not a practical method because of the large number of resistors required. The first method could turn out to be practical if it were possible to realize the R networks with much fewer resistors.

A much more practical way of designing active RC impedances is next undertaken.

III. CASCADE SYNTHESIS

In active networks, negative impedances are admissible and can be incorporated into a realization technique with relative ease. A possible way of realizing a negative impedance is with the help of a negative-impedance converter.[8] A negative-impedance converter is an active two-port which presents at either of its ports the negative of the impedance connected at the other port. Consider an impedance function

$$Z(p) = \frac{N}{D} = \frac{N_1 - N_2}{D_1 - D_2}, \tag{26}$$

where N and D are polynomials in p, whose degrees do not differ by more than one.

If it is possible to break up N and D into N_1, N_2 and D_1, D_2 as in (26) such that N_1, N_2, D_1 and D_2 all have only real and negative roots, then it might be possible to realize $Z(p)$ as a two-terminal-pair RC network terminated by either a negative resistance or a negative RC impedance. (Refer to Fig. 5.)

It is shown in Appendix A that any polynomial having real coefficients

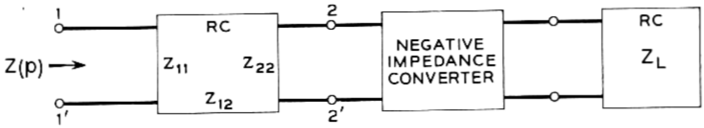

Fig. 5 — Realization of driving-point function by cascade method.

can be easily expressed as a difference of two polynomials each of which has only negative real roots. It is easy to show that an RC network terminated in a negative resistance cannot produce complex poles of the driving-point impedance function. The poles of $Z(p)$ are the zeros of $(z_{22} - R)$, where z_{22} is the open-circuit impedance of the RC two-port and $-R$ is the termination at end 2. It should be obvious that any resistance (positive or negative) added to an RC impedance merely shifts the zeros along the σ-axis without moving them off the axis. If the termination is $-Z_L$, a negative RC impedance, then the poles of $Z(p)$ are the zeros of $(z_{22} - Z_L)$. It is seen from Appendix A that a difference of two RC impedances can indeed produce the desired complex zeros.

3.1 *Realization Problem*

Consider an impedance function

$$Z(p) = \frac{N}{D} = \frac{\frac{N}{B}}{\frac{D}{B}}, \tag{27}$$

where B is a polynomial having only negative real roots. Then, as per Appendix A,

$$Z(p) = \frac{\frac{P_1}{P_2} - \frac{P_3}{P_4}}{\frac{Q_1}{P_2} - \frac{Q_3}{P_4}}, \qquad \text{where } P_2P_4 = B \tag{28a}$$

$$= \frac{P_1}{Q_1} \frac{\frac{P_3}{P_1} - \frac{P_4}{P_2}}{\frac{Q_3}{Q_1} - \frac{P_4}{P_2}}. \tag{28b}$$

For an RC two-port terminated in a negative impedance,

$$Z(p) = z_{11} \frac{\dfrac{1}{y_{22}} - Z_L}{z_{22} - Z_L}, \tag{29}$$

where z_{11}, z_{22} are the open-circuit impedances of the *RC* two-port, y_{22} is its short-circuit admittance and $-Z_L$ is the terminating impedance. Comparing (28) and (29):

$$\begin{aligned} z_{11} &= \frac{P_1}{Q_1}, \\ z_{22} &= \frac{Q_3}{Q_1}, \\ \frac{1}{y_{22}} &= \frac{P_3}{P_1}, \\ Z_L &= \frac{P_4}{P_2}. \end{aligned} \tag{30}$$

Without loss of generality, we will assume that $Z(p)$ has only complex zeros and poles. It should be noted that P_1, P_3, Q_1, Q_3 have only negative real roots. As shown in Appendix A, Z_L is necessarily an *RC* impedance function and $-Z_L$ is realized by means of a negative-impedance converter. It remains to be shown that z_{11}, z_{22}, y_{22} represent a two-port *RC* network. From (30),

$$\frac{1}{y_{22}} = \frac{P_3}{P_1} = \frac{z_{11}z_{22} - z_{12}^2}{z_{11}}$$

and

$$z_{12} = \frac{\sqrt{P_1Q_3 - P_3Q_1}}{Q_1}. \tag{31}$$

Since z_{12} must be a rational function $(P_1Q_3 - P_3Q_1)$ must be a perfect square.

Then, to sum up, the necessary and sufficient conditions for physical realizability are that:

i. z_{11} and z_{22} represent *RC* driving-point impedances;
ii. $(P_1Q_3 - P_3Q_1)$ be a perfect square;
iii. residue condition be satisfied at all the poles.*

---

* For *RC* networks, the statement includes the point $p = \infty$, where the functions may be nonzero.

The condition iii is always satisfied. The residue condition is

$$k_{11}k_{22} - k_{12}^2 \geqq 0, \tag{32}$$

where k_{ij} represents the residue of z_{ij} in a given pole. The residues of z_{11} and z_{22} may be written as

$$\begin{aligned} k_{11}^i &= \frac{P_1(-\sigma_i)}{\dot{Q}_1(-\sigma_i)}, \\ k_{22}^i &= \frac{Q_3(-\sigma_i)}{\dot{Q}_1(-\sigma_i)}, \end{aligned} \tag{33}$$

where σ_i is any root of Q_1 and $\dot{Q}_1$ is the derivative of Q_1 with respect to p. The residue of z_{12} is

$$k_{12}^i = \frac{\sqrt{P_1(-\sigma_i)Q_3(-\sigma_i) - Q_1(-\sigma_i)P_3(-\sigma_i)}}{\dot{Q}_1(-\sigma_i)}. \tag{34}$$

But

$$Q_1(-\sigma_i) \equiv 0,$$

and so

$$k_{11}^i k_{22}^i - (k_{12}^i)^2 \equiv 0. \tag{35}$$

Thus, the residue condition is always satisfied with an equal sign at all the poles. From (30) and (31) it is seen that the requirements at infinity are also met.

In reference to condition i, consider the location of the roots of P_1, P_2, etc. as shown in Fig. 6. As already pointed out, P_4/P_2 is an RC im-

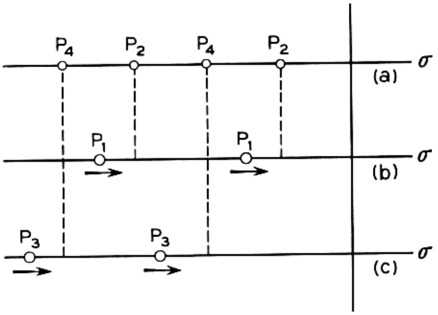

Fig. 6 — Adjustment of the roots of P_3 and P_1 so that P_3/P_1 represents an RC impedance function: (a) roots of P_2 and P_4; (b) roots of P_1; (c) roots of P_3.

pedance function. Hence it will have alternating roots in the correct order. This is shown as (a) in the figure. P_3/P_1 may or may not have the desired alternation. This can be corrected by adding the same constant to P_1/P_2 and P_3/P_4 [see (28a)]. Such addition of a constant will move the roots of P_1 closer to those of P_2 and move the roots of P_3 closer to those of P_4. This can always be done such that P_3/P_1 will have the desired alternation. In a similar manner, Q_3/Q_1 is corrected such that P_1/Q_1 also has the required form. In the case of an impedance Z of rank 2, the above corrections can generally be done so as to make $(P_1Q_3 - P_3Q_1)$ a perfect square. If and when this is not possible surplus factors might have to be added to $Z(p)$ to obtain a rational z_{12}.

It is not at all necessary to make all the ratios $(P_1/Q_1, Q_3/Q_1, P_3/P_1)$ have the desired alternation. It is obvious that, if $z_{11}(P_1/Q_1)$ and $z_{22}(Q_3/Q_1)$ are RC impedances and z_{12} is a rational function satisfying the residue conditions, $1/y_{22}$ or P_3/P_1 will necessarily have the proper form. Furthermore, if z_{12} is a ratio of two polynomials, the residues in its poles (real and negative) will be real and $(k_{12}{}^i)^2$ will always be positive. If z_{22} is an RC impedance function, its residues are necessarily positive. From (35), it is seen that z_{11} will automatically have positive residues; i.e., it must be an RC impedance. Thus, in order to realize the RC two-port one need only make z_{11} or z_{22} an RC impedance and $(P_1Q_3 - P_3Q_1)$ a complete square.

It is shown in Appendix D that, if Z has only complex poles, then z_{22} is always an RC impedance and remains so under the addition and subtraction of a constant in the denominator.* Thus, in such a case all that is needed is to make z_{12} a rational function. An illustrative example is considered to show how z_{12} can be made rational by the addition and subtraction of an appropriate constant in the denominator of Z.

Example:

$$Z(p) = \frac{p^2 + p + 1}{p^2 + p + 2} = \frac{\dfrac{p^2 + p + 1}{(p + 1)(p + 2)}}{\dfrac{p^2 + p + 2}{(p + 1)(p + 2)}}$$

$$= \frac{\dfrac{p + 2}{p + 1} - \dfrac{3}{p + 2}}{\dfrac{p + 3}{p + 1} - \dfrac{4}{p + 2}}$$

---

* This proof was developed by J. M. Sipress.

$$= \frac{p+2}{p+3} \frac{\dfrac{3}{p+2} - \dfrac{p+2}{p+1}}{\dfrac{4}{p+3} - \dfrac{p+2}{p+1}}.$$

In this case, $P_1Q_3 - P_3Q_1$ is to be made a perfect square. It is seen that Q_3/Q_1 has the proper alternation. Now, add a constant K to

$$(p + 3)/(p + 1)$$

and $4/(p + 2)$. Then,

$$\begin{aligned} P_1Q_3 - P_3Q_1 &= (p + 2)[4 + K(p + 2)] - 3[K(p + 1) + p + 3] \\ &= Kp^2 + (K + 1)p + (K - 1). \end{aligned}$$

For this to be a complete square,

$$3K^2 - 6K - 1 = 0$$

and

$$K = 1 + \frac{2}{\sqrt{3}}.$$

The open-circuit impedances of the two-port are

$$z_{11} = \frac{p + 2}{(K + 1)(p + 1) + 2},$$

$$z_{22} = \frac{K(p + 2) + 4}{(K + 1)(p + 1) + 2},$$

$$z_{12} = \sqrt{K} \frac{p + \dfrac{K + 1}{2K}}{(K + 1)(p + 1) + 2}.$$

It is seen that these satisfy all the required conditions. It should be noted that adding the constant K to and subtracting it from the numerator alone instead of the denominator alone of $Z(p)$ will not yield a rational z_{12}. This is an obvious consequence of the fact that the introduction of K in the numerator alone is an attempt to make z_{12} nonvanishing at infinity, whereas z_{22} vanishes at infinity.

Realization of the RC two-port can be easily accomplished when there is only one term in the partial-fraction expansion of each z_{ik}.[9] Only one ideal transformer is present in such a case, and this is removed by means of impedance-scaling of the relevant part of the network. When several poles are present in the impedances z_{ik}, such a realization procedure requires several ideal transformers, which cannot be eliminated in the above manner.

In such cases, one can make z_{11} and z_{12} satisfy the conditions of Fialkow and Gerst[10] by introducing one ideal transformer. The network is then realized without transformers and appropriately scaled to remove the ideal transformer that was introduced above. Such a realization would not give the desired z_{22}. To obtain this desired z_{22}, positive or negative elements are then added in series at the load end of the network. Negative elements are then realized together with the load through the negative-impedance converter.

3.2 *Surplus Factors*

As seen above, it is possible in simple cases to make $(P_1Q_3 - P_3Q_1)$ a perfect square by the use of an appropriate constant. However, this is not always possible. It will now be shown how $(P_1Q_3 - P_3Q_1)$ can always be made a perfect square by introducing surplus factors in the original expression for $Z(p)$. A new formulation is necessary so as to have z_{12} present explicitly in the expressions. Such a formulation can be obtained by writing $Z(p)$ in the following manner:

$$Z(p) = z_{11} - \frac{z_{12}^2}{z_{22} - Z_L} = \frac{N}{D}$$

$$= \frac{P_1}{Q_1} - \frac{\dfrac{g^2}{Q_1^2}}{\dfrac{Q_3}{Q_1} - \dfrac{P_4}{P_2}}$$

[from (30) and letting $z_{12} = g/Q_1$]

$$= \frac{P_1}{Q_1} - \frac{\dfrac{g^2}{Q_1^2}}{\dfrac{-D}{Q_1P_2}}. \tag{36}$$

This equation follows because $(-D = P_2Q_3 - Q_1P_4)$, as can be seen from (28a). Then,

$$Z(p) = \frac{\dfrac{DP_1}{D_1} + \dfrac{g^2P_2}{Q_1}}{D}$$

and

$$N = \frac{DP_1 + g^2P_2}{Q_1}$$

or

$$g^2P_2 = NQ_1 - DP_1 . \tag{37}$$

This equation must be subject to the constraints that $(P_1/Q_1)(z_{11})$ be an RC impedance. A further constraint requires Q_1/P_2 to be also an RC impedance. This is seen to be satisfied in (28a) since roots of P_2 and P_4 alternate. From a different viewpoint, it is seen from (36) that $-D/Q_1P_2$ must give a difference of two RC impedances having Q_1 and P_2 for their denominators. So the roots of Q_1 and P_2 must alternate. The sign of (Q_3/Q_1) is positive, and so the nearest singularity must be that of P_2. Therefore, Q_1/P_2 must be an RC impedance. If (37) is satisfied and P_1/Q_1 and Q_1/P_2 are RC impedances, then z_{11}, z_{22} and Z_L will all be RC impedances. If g is also a polynomial, then the impedance Z is physically realizable. The conditions for physical realizability can then be written as

$$g^2P_2 = NQ_1 - DP_1$$

and (38)

$$\left.\begin{matrix} \dfrac{P_1}{Q_1} \\ \\ \dfrac{Q_1}{P_2} \end{matrix}\right\} \text{are } RC \text{ impedances.}$$

It is important to note that (37) contains N and D, which are already known. Choice of P_2 is arbitrary. Freedom in the choice of P_4 is not explicit in (37). However, this freedom can be assigned to Q_1, since $-D/Q_1P_2$ gives z_{22} and Z_L. Thus, Q_1 and P_2 can be considered to be arbitrary instead of P_2 and P_4. Further, one can consider that P_1 and Q_1 are arbitrary in (37). This follows since, if (38) is satisfied, it does not matter whether P_2 and Q_1 are chosen independently and P_1 is determined from (37) or if P_1 and Q_1 are chosen and P_2 determined from (37).

Assume for the moment that it is somehow possible to find P_1, Q_1, P_2 which satisfy (38). It will be shown that it is possible to make g^2 a complete square by introducing surplus factors. Once this point is clarified, it will be shown how P_1, Q_1 and P_2 can always be chosen to satisfy (38), N and D being still assumed to have only complex conjugate roots.

This restriction will be removed later without altering the arguments presented here. For convenience in discussion, $Z(p)$ is chosen to be of rank $n = 2$. Similar results are obtained for any n.

Consider $Z(p)$ of rank $n = 2$ (i.e., N and D of degree 2). From (28), P_1, Q_1 and P_2 are of degree 1. These are determined such that (38) is satisfied. Now g^2 may or may not be a perfect square. If g^2 is a perfect square, the problem is solved and the network can be realized. If g^2 is not a perfect square, surplus factors M are introduced in $Z(p)$, making $N' = NM$ and $D' = DM$ of fourth degree. New polynomials P_1', Q_1' and P_2' of second degree are required [see (28)] and are somehow determined. Then (37) may be written as

$$g^2 P_2' = M(NQ_1' - DP_1'). \qquad (39)$$

In place of (39), consider

$$(NQ_1' - DP_1') = P_2'R. \qquad (40)$$

If (40) is satisfied subject to the constraints of (38), where R is any remainder polynomial, one merely multiplies both sides of (40), by $M = R$ to give (39). Thus, it is seen how g^2 can be made a complete square by the introduction of an appropriate surplus factor M. All that remains to be shown is that (40) can be satisfied subject to the constraints of (38). This is no different from the problem of finding P_1, Q_1 and P_2 satisfying (38), which was assumed somehow possible. It will now be shown that it *is* possible to find polynomials P_1, Q_1 and P_2 such that

$$(NQ_1 - DP_1) = P_2R, \qquad (40\text{a})$$

subject to the constraints

$$\left.\begin{array}{c}\dfrac{P_1}{Q_1}\\[2ex] \dfrac{Q_1}{P_2}\end{array}\right\} \text{are } RC \text{ impedances.} \qquad (40\text{b})$$

As discussed previously, P_1, Q_1 of appropriate degrees are chosen arbitrarily such that P_1/Q_1 is an RC impedance. The roots of P_1 and Q_1 of second degree are shown in Fig. 7. No singularity is chosen at the origin for reasons that will soon become clear. Symbol I refers to the function NQ_1 and II refers to $(-DP_1)$. The superscripts of I and II represent the corresponding signs of the values of the functions in the designated regions on the real axis. The shaded areas in the figure show regions where

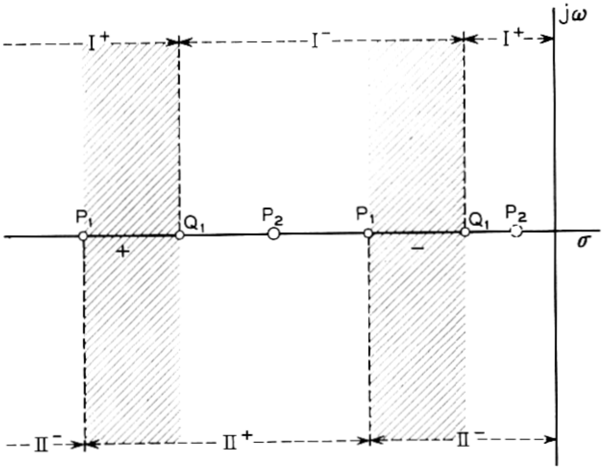

Fig. 7 — Determination of roots of P_2 when roots of P_1 and Q_1 are known.

both functions have the same sign. The two shaded areas shown are of opposite signs. Therefore, there must be at least one zero of the function $(I + II)$ in the region between these two areas. Such a zero in this region is assigned to P_2. Another zero of P_2 is required to the right of all the four roots so as to satisfy (40b). This is easily accomplished by making $z_{11} = Z$ at some point to the right of the other roots. This gives the other root of P_2 shown dotted in the figure. Thus (40a) and (40b) are completely satisfied.

The relevant steps involved for $Z(p)$ of rank n are:

1. Choose P_1 and Q_1 each of degree n, so that P_1/Q_1 is an RC impedance.
2. Evaluate $Z(p)$ at some point on the negative real axis closer to the origin than the root of Q_1 nearest to the origin.
3. Make $Z(p) = z_{11} = P_1/Q_1$ at this point by merely multiplying z_{11} by an appropriate constant.
4. Determine P_2R by finding $(NQ_1 - DP_1)$, and find the roots of P_2R.
5. Assign the appropriate roots to P_2 from steps 3 and 4.
6. Determine R from steps 4 and 5 and find $g^2 = R^2$, $N' = NR$ and $D' = DR$.
7. Express $(-D'/Q_1P_2)$ in partial-fraction form to obtain

$$(Q_3/Q_1 - P_4/P_2).$$

8. All the polynomials are determined and the relevant impedances are obtained from (30) and noting that $z_{12} = g/Q_1$.

Note that z_{22} (i.e., Q_3/Q_1) vanishes at infinity, whereas z_{11} and z_{12} do not. A sufficiently large constant is added to z_{22} and Z_L such that the two-port impedances satisfy the residue condition at infinity.

Example:

$$Z(p) = \frac{N}{D} = \frac{p^2 + p + 1}{p^2 + p + 2}.$$

1. $z_{11}(p) = K_0 \dfrac{(p + 3)(p + 5)}{(p + 2)(p + 4)} = K_0 \dfrac{(p^2 + 8p + 15)}{(p^2 + 6p + 8)}.$

2. $Z(-1) = \frac{1}{2}.$

3. $z_{11}(-1) = K_0 \frac{8}{3} = Z(-1) = \frac{1}{2},$

$$K_0 = \tfrac{3}{16},$$

$$P_1 = 3(p^2 + 8p + 15),$$

$$Q_1 = 16(p^2 + 6p + 8).$$

4. $NQ_1 - DP_1 = P_2R = 13p^4 + 85p^3 + 165p^2 + 131p + 38$

$$\cong (p + 1)(p + 3.89)(13p^2 + 21.38p + 9.76).$$

5. $P_2 = (p + 1)(p + 3.89).$

6. $R = 13p^2 + 21.38p + 9.76,$

$$g^2 = R^2,$$

$$N' = NR,$$

$$D' = DR = (p^2 + p + 2)(13p^2 + 21.38p + 9.76).$$

7. $\dfrac{-D}{Q_1P_2} = -\dfrac{1}{16}\left(13 + \dfrac{0.318}{p + 1} + \dfrac{2716.72}{p + 3.89}\right)$

$$+ \frac{1}{16}\left(\frac{20.104}{p + 2} + \frac{2804.81}{p + 4}\right).$$

8. $z_{11} = \dfrac{1}{16}\left(3 + \dfrac{4.5}{p + 2} + \dfrac{1.5}{p + 4}\right),$

$$z_{12} = \frac{1}{16}\left(13 + \frac{9.5}{p + 2} + \frac{66.12}{p + 4}\right),$$

$$z_{22} = \frac{1}{16}\left(K_1 + \frac{20.1}{p + 2} + \frac{2804.81}{p + 4}\right),$$

$$Z_L = \frac{1}{16}\left(13 + K_1 + \frac{0.318}{p + 1} + \frac{2716.72}{p + 3.89}\right),$$

where K_1 is chosen large enough so that $K_1 \geqq \frac{169}{3}$. The complete network is then realized.

3.3 *Elimination of Redundant Elements*

Introduction of surplus factors as proposed in the last section will introduce redundant elements in the realization. A possible formulation which may avoid the redundant elements is briefly presented here. If the roots of g are given by $p = p_i$, then, from (36),

$$Z(p_i) = z_{11}(p_i)$$

and (41)

$$\left.\frac{d}{dp} Z\right|_{p=p_i} = \left.\frac{d}{dp} z_{11}\right|_{p=p_i}, \qquad \text{for all } i.$$

Equation (41) assures that g is a polynomial. The frequencies p_i may be arbitrarily selected and (41) may then be solved. Then P_2 can be determined from (37) such that it satisfies the last restriction of (38). There is considerable freedom in the solution of (41), and this freedom may be used to satisfy the restrictions on P_2.

The difficulty encountered in the solution of (41) stems from the nature of the simultaneous equations involved. These equations are nonlinear and cumbersome to solve. It is not possible to state whether the method outlined above has a solution or not. It merely indicates a possible direction for further work on this subject.

3.4 *Restrictions on Impedance Functions*

It has been assumed that the driving-point impedances have only complex conjugate poles and zeros. The only use made of this assumption was the consequent positive sign of the impedance function on the negative σ-axis. This fact ensured that, for N/P_2P_4 and D/P_2P_4, the residues in the negative real poles (arbitrarily chosen) had alternating signs in the alternate poles. Thus, the method is applicable to the realization of any driving-point impedance (having poles and zeros anywhere in the complex plane) as long as the impedance function is positive on some interval of the negative σ-axis. If no such interval exists, then the negative of the desired impedance is realized and a second converter is used to obtain the required impedance function.

Functions that are negative on the entire negative real axis fall in two classes: (a) total number of poles and zeros on the positive σ-axis including the origin is even, with the function having a negative multiplier and (b) total number of poles and zeros on the positive σ-axis including the origin is odd, with a positive multiplier. In both these cases, there are no odd-order zeros or poles on the negative σ-axis excluding the origin. A second converter is needed in both the above cases, with the possible exception of the case when there is an odd-order pole at the origin with no other odd-order zeros or poles on the whole σ-axis. In such a case, a simple pole at the origin is removed if the residue in this simple pole is positive, and the remainder function is checked to determine if it is positive or negative on the negative real axis. If it is positive, the realization is carried out on the remainder function and a capacitor corresponding to the removed pole is added in series. If the remainder function turns out to be negative on the negative real axis, a second converter seems to be needed.

In some cases, a pole at the origin can be handled in the following manner:

$$\begin{aligned} Z = \frac{N}{pD} &= \frac{1}{p}\left[\frac{\dfrac{P_1}{P_2} - \dfrac{P_3}{P_4}}{\dfrac{Q_1}{P_2} - \dfrac{Q_3}{P_4}}\right] \\ &= \frac{1}{p}\left[\frac{\dfrac{P_1}{P_2} - \dfrac{P_3}{P_4}}{\left(\dfrac{K}{p} + \dfrac{Q_1}{P_2}\right) - \left(\dfrac{K}{p} + \dfrac{Q_3}{P_4}\right)}\right] \\ &= \frac{1}{p}\left[\frac{\dfrac{P_1}{P_2} - \dfrac{P_3}{P_4}}{\dfrac{Q_1''}{pP_2} - \dfrac{Q_3''}{pP_4}}\right] \\ &= \left[\frac{\dfrac{P_1}{P_2} - \dfrac{P_3}{P_4}}{\dfrac{Q_1''}{P_2} - \dfrac{Q_3''}{P_4}}\right]. \end{aligned} \tag{42}$$

Equation (42) is now used to determine the value of K for which z_{12} is a rational function. In the next section, such a development is applied to the synthesis of functions of rank 2.

The only other restriction on the impedance function is that the degrees of the numerator and denominator polynomials do not differ by more than one. This ensures that the degrees of the polynomials P_1, P_2, Q_1, Q_3, etc., lead to physical expressions for z_{ik}. It should be noted that the impedance function is not required to be positive-real, nor are the poles and zeros required to be in the left-half plane.

IV. NETWORK FUNCTIONS OF RANK 2

The cascade synthesis method will be applied to the realization of transfer functions of rank 2 (i.e., the numerator and denominator polynomials of second degree). Functions of rank 2 are by far the most important functions from sensitivity considerations. Functions with a single pair of complex poles are less sensitive to changes in the converter constant than are functions with more than one pair of complex poles.[3] Consequently, it is preferable to realize a function of rank n in sections of rank 2 and cascading these sections through isolation sections such as emitter followers.

Synthesis of driving-point functions can be adapted in several ways to the realization of transfer functions. Consider the realization of open-circuit voltage transfer function. Only two particularly simple but extremely important cases will be discussed here.

Case 1

In some instances, the input impedance may be chosen to be

$$Z(p) = \frac{N}{D} = \frac{E_0}{I_0} \tag{43}$$

and synthesized so that the resulting network has a shunt resistor R across the input terminals. Conversion from current source to voltage source $E_i = RI_0$ yields the desired transfer function [Fig. 8(a)] within a constant gain multiplier

$$\frac{E_0}{E_i} = \frac{1}{R}\frac{N}{D}. \tag{44}$$

Case 2

In other instances, $Z(p)$ may be chosen such that

$$Z(p) = \frac{E_0}{I_0} = \frac{N}{pD} \tag{45}$$

and synthesized so that the resulting network has a shunt capacitor, C, across the input terminals. A source conversion $E_i = I_0/pC$ yields the

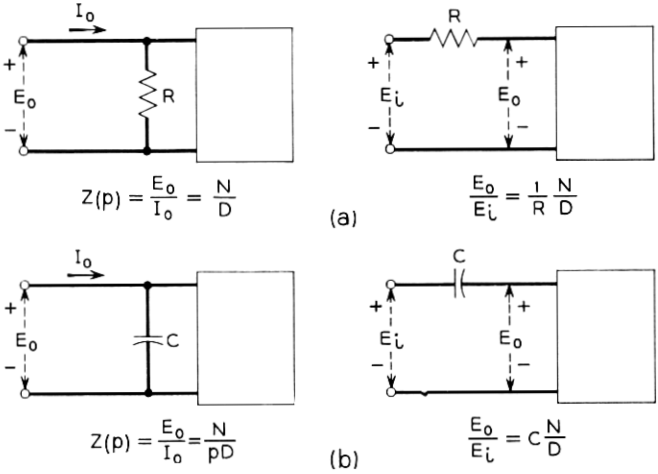

Fig. 8 — Adaptation of driving-point synthesis to voltage transfer function synthesis: (a) case 1; (b) case 2.

desired transfer function [Fig. 8(b)]:

$$\frac{E_0}{E_i} = C\,\frac{N}{D}. \tag{46}$$

The above two cases play an important role in the realization of transfer functions of rank 2. Consider the realization of open-circuit voltage transfer function of the form

$$T(p) = K_0\,\frac{p^2 + ap + b}{p^2 + cp + d}. \tag{47}$$

It is assumed that $T(p)$ has complex conjugate poles and zeros in the left-half plane. Similar developments can be carried out for other cases, but they are not discussed here.

It is shown in Appendix E that d must be greater than b if the rationalization of z_{12} is to be accomplished through the addition and subtraction of a constant, K, in the denominator of (28a). In this case, the driving-point impedance, $Z(p)$, may be synthesized so that the resulting network has a shunt resistor across its input terminals. Consequently, the transfer function of (47) may be realized as discussed above if d is greater than b. The structure is shown in Fig. 9(a).

If d is less than b, the addition and subtraction of a term of the form K/p in the denominator of (28a) will permit the rationalization of z_{12},

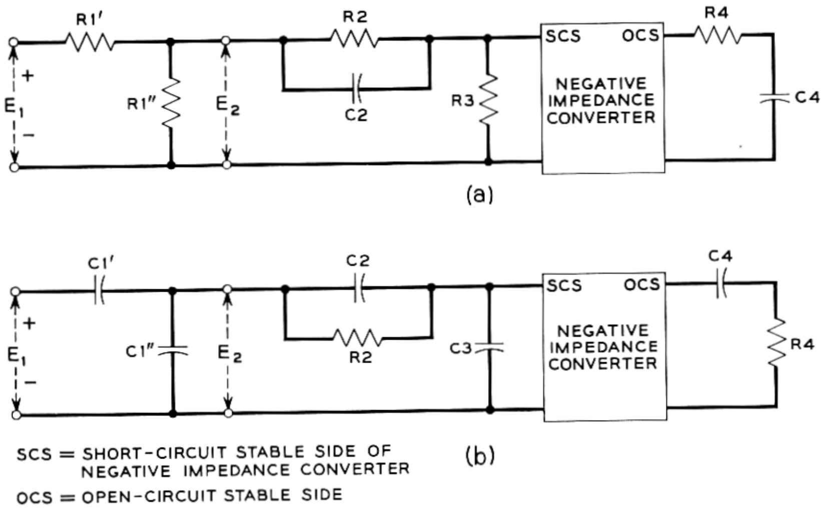

Fig. 9 — Realization of transfer functions of rank 2: (a) $d > b$; (b) $d < b$.

and the driving-point impedance $Z(p) = [T(p)/p]$ may be synthesized so that the resulting network has a shunt capacitor across its input terminals. The proper source conversion yields the desired transfer function for d less than b. The structure for this case is shown in Fig. 9(b).

V. DESIGN EXAMPLE

The results obtained in Section IV for the realization of transfer functions of rank 2 by the cascade method will be applied to the design of a practical filter. Bandpass filters with high selectivity are of great importance in frequency-multiplexing schemes. A typical bandpass filter frequently encountered in such systems is selected as a design example. The over-all transfer function is represented as a product of functions of rank 2, and each of these is realized as discussed in Section IV. Each of the sections is a simple ladder structure regardless of the positions of the zeros and poles.

The specifications on the filter are as follows: the pass band extends from 12.33 to 15.25 kc, and in this range the transmission characteristic must be flat to within ±0.1 db; the stop band is symmetrical; and, starting at 11.50 kc, a minimum of 50 db of rejection must be provided.

The approximation problem is solved through the utilization of a Tchebycheff parameter equal-ripple chraacteristic. The transmission function $T(p)$ requires five pairs of complex poles, four pairs of imaginary zeros and a simple zero at the origin and infinity:

$$T(p) = K \frac{p(p^2 + \omega_{01}^2)(p^2 + \omega_{02}^2)(p^2 + \omega_{03}^2)(p^2 + \omega_{04}^2)}{(p^2 + 2\alpha_1 p + \omega_{n1}^2)(p^2 + 2\alpha_2 p + \omega_{n2}^2)(p^2 + 2\alpha_3 p + \omega_{n3}^2) \cdot (p^2 + 2\alpha_4 p + \omega_{n4}^2)(p^2 + 2\alpha_5 p + \omega_{n5}^2)},$$

where

$$\begin{aligned}
\omega_{01} &= 2\pi \times 10349.7 \text{ rps}, \\
\omega_{02} &= 2\pi \times 11388.7 \text{ rps}, \\
\omega_{03} &= 2\pi \times 16510.4 \text{ rps}, \\
\omega_{04} &= 2\pi \times 18167.9 \text{ rps},
\end{aligned}$$

$$\begin{aligned}
\alpha_1 &= 2\pi \times 143.6 \text{ rps}, & \omega_{n1} &= 2\pi \times 12273.5 \text{ rps}, \\
\alpha_2 &= 2\pi \times 492.1 \text{ rps}, & \omega_{n2} &= 2\pi \times 12695.6 \text{ rps}, \\
\alpha_3 &= 2\pi \times 767.1 \text{ rps}, & \omega_{n3} &= 2\pi \times 13691.0 \text{ rps}, \\
\alpha_4 &= 2\pi \times 573.2 \text{ rps}, & \omega_{n4} &= 2\pi \times 14788.6 \text{ rps}, \\
\alpha_5 &= 2\pi \times 179.3 \text{ rps}, & \omega_{n5} &= 2\pi \times 15318.1 \text{ rps}.
\end{aligned}$$

This transmission characteristic is illustrated in Fig. 10 and the pole-zero configuration is shown in Fig. 11. The transmission function is expressed as a product of second order expressions:

$$\begin{aligned}
T(p) = K &\left(\frac{p^2 + \omega_{04}^2}{p^2 + 2\alpha_1 p + \omega_{n1}^2}\right)\left(\frac{p^2 + \omega_{03}^2}{p^2 + 2\alpha_2 p + \omega_{n2}^2}\right) \\
&\cdot\left(\frac{p}{p^2 + 2\alpha_3 p + \omega_{n3}^2}\right)\left(\frac{p^2 + \omega_{02}^2}{p^2 + 2\alpha_4 p + \omega_{n4}^2}\right) \\
&\cdot\left(\frac{p^2 + \omega_{01}^2}{p^2 + 2\alpha_5 p + \omega_{n5}^2}\right).
\end{aligned}$$

Each factor is selected so as to obtain a reasonably low sensitivity to changes in element values of the network section. Each of the second-order expressions is then realized in a separate section as a voltage transfer ratio. The last two factors belong to Case 1 and are realized to yield structures of the form shown in Fig. 9(a). The first two factors belong to Case 2 and are realized to yield structures of the form shown in Fig. 9(b). The middle factor has a zero at the origin and would either require the use of surplus factors, or perhaps two converters, as discussed above. However, the Q of the poles of this factor is rather low. Consequently, for the sake of convenience, this factor is realized with a simple series *RLC* circuit with a ferrite coil. Design of one section is considered below as an illustration.

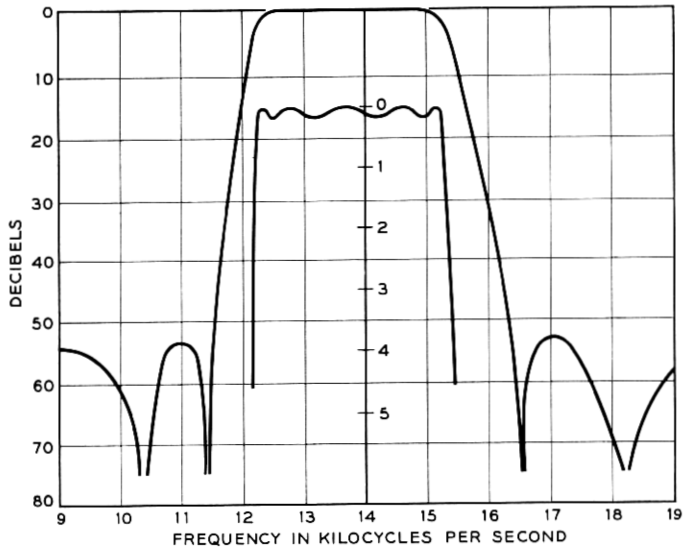

Fig. 10 — Design example — filter characteristic.

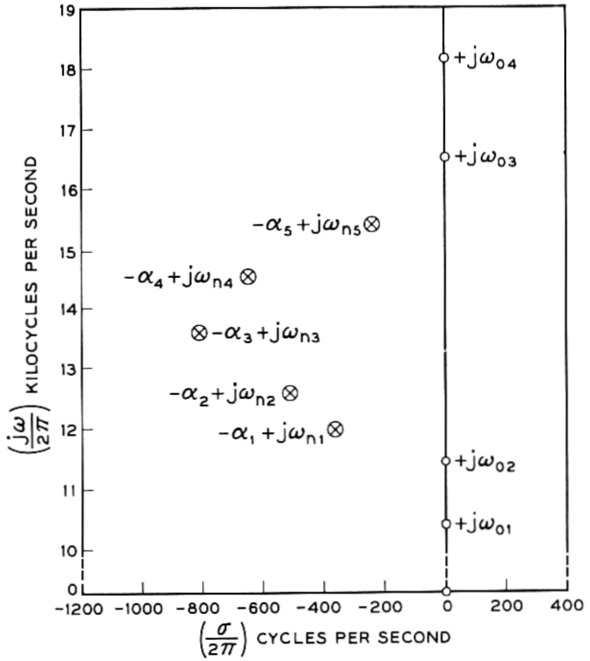

Fig. 11 — Design example — pole-zero configuration (not drawn to scale; pole and zero positions normalized with respect to 2τ).

Consider the factor

$$\frac{p^2 + \omega_{01}{}^2}{p^2 + 2\alpha_5 p + \omega_{n5}{}^2}$$

normalized with respect to frequency with a scale factor of 10^5:

$$Z(p) = \frac{p^2 + 0.423}{p^2 + 0.023p + 0.926}.$$

Choose

$$\sigma_1 = 0,$$
$$\sigma_2 = \sqrt{0.423} = 0.650.$$

Then

$$P_2 = p,$$
$$P_4 = p + 0.650,$$
$$P_1 = p + 0.650,$$
$$P_3 = 2 \times 0.650 = 1.30,$$
$$Q_1' = (1 + K)p + \frac{0.926}{0.650},$$
$$Q_3' = Kp + \frac{(1 + K)0.423 - 0.023 \times 0.65 + 0.926}{0.650}.$$

The value of K for $(P_1Q_3' - Q_1'P_3)$ to be a complete square is found to be $K = 1.455$. Impedances z_{11}, z_{12}, z_{22} and Z_L are now obtained and the network realized. In order to ensure that there are no transformers present, it is sufficient to make at $p = \infty$

$$z_{22}(\infty) = z_{12}(\infty) = z_{11}(\infty).$$

The above requirement for no transformers is obvious in view of the structure of Fig. 9(a). This requirement is satisfied very simply by the appropriate scaling of the impedances involved. Consider

$$Z = z_{11} - \frac{K^* z_{12}{}^2}{K^* z_{22} - K^* Z_L}.$$

Here a scale factor K^* is introduced to satisfy the above condition of no transformers.

The new impedances are z_{11}, $\sqrt{K^*}\, z_{12}$, $K^* z_{22}$ and $K^* Z_L$. It should be observed that $K^* = 1/K$. The network is now realized to yield the desired impedance function. The transfer function is obtained by the source conversion; either the whole or part of the shunt resistor may be used for this purpose. The actual values of the elements are obtained by introducing appropriately the frequency scale factor of 10^5. An impedance scale factor of 5×10^3 is then introduced to obtain element values which are practical. The element values of all the four active sections are given in Tables I and II. Table III gives the values for the series *RLC* circuit.

It is indeed possible to prepare tables which directly yield the element values in terms of the coefficients of the numerator and denominator polynomials. Such tables have been prepared and used to design filter sections with a considerable saving in time.

TABLE I — ELEMENT VALUES IN OHMS AND MICROMICROFARADS FOR TWO SECTIONS OF THE DESIGN EXAMPLE — $d > b$

$T(p)$	$K_1 \dfrac{p^2 + \omega_{01}^2}{p^2 + 2\alpha_5 p + \omega_{n5}^2}$	$K_2 \dfrac{p^2 + \omega_{02}^2}{p^2 + 2\alpha_4 p + \omega_{n4}^2}$
$R1'$	36755	4166
$R1''$	2661	2528
$R2$	17314	17020
$R3$	11412	8442
$R4$	3410	2790
$C2$	2227	2466
$C4$	4453	4934

TABLE II — ELEMENT VALUES IN OHMS AND MICROMICROFARADS FOR TWO SECTIONS OF THE DESIGN EXAMPLE — $d < b$

$T(p)$	$K_3 \dfrac{p^2 + \omega_{03}^2}{p^2 + 2\alpha_2 p + \omega_{n2}^2}$	$K_4 \dfrac{p^2 + \omega_{04}^2}{p^2 + 2\alpha_1 p + \omega_{n1}^2}$
$R2$	7350	6601
$R4$	3650	3275
$C1'$	1769	247.4
$C1''$	2915	3416
$C3$	873	797
$C2$	433.5	525
$C4$	2613	2644

TABLE III — ELEMENT VALUES IN OHMS, MILLIHENRIES AND MICROMICROFARADS

$T(p)$	$K_5 \dfrac{p}{p^2 + 2\alpha_3 p + \omega_{n3}^2}$
R	375
L	52.25
C	2438.20

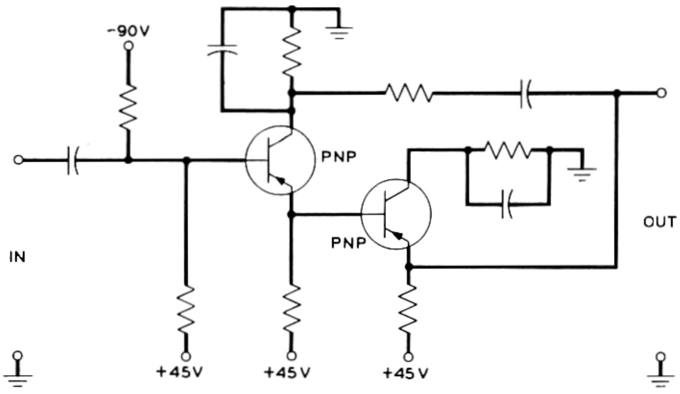

Fig. 12 — Emitter follower.

The above filter sections are then cascaded through isolation sections, which are emitter followers. The negative-impedance converter used in the experimental model is of the type proposed by Larky.[11] Circuit designs of the emitter follower and the converter are shown in Figs. 12 and 13. Note that converters are short-circuit stable at one port and open-circuit stable at the other port. Care must be taken to connect them properly so that the system is stable.

The experimental model is constructed on five 2 × 3 inch cards, with one filter section and an isolation network on each. The complete filter

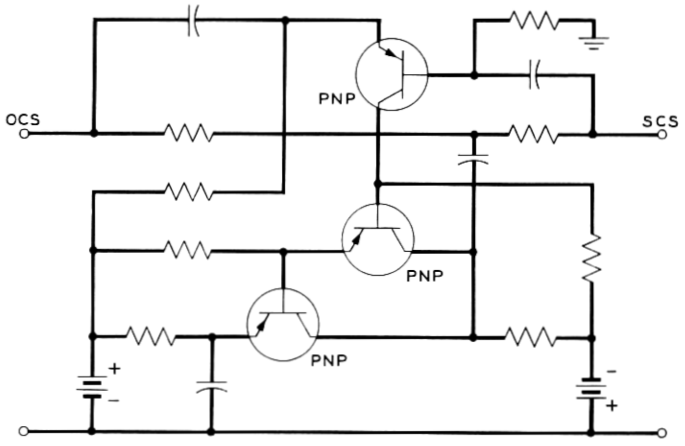

Fig. 13 — Negative impedance converter.

can be placed in a $5\frac{1}{2} \times 4 \times 2\frac{3}{4}$ inch can. Pictures of the filter are shown in Figs. 14 and 15.

The passband transmission characteristic of the experimental model has the five ripples and is flat to within the specified ±0.1 db. The characteristic stays flat to within ±0.15 db over a 10°C temperature variation.

VI. CONCLUSION

It is shown that any driving-point function can be synthesized using passive *RC* networks and one active element. The active element is assumed to be a current-controlled voltage source. A good approximation to such sources is practically possible. The synthesis technique involves synthesis of n-port *RC* networks and n-port *R* networks. Sufficient condition for synthesis of n-port *R* networks without transformers is shown to be the dominant diagonal matrix. Synthesis of n-port *RC* networks is considered in the Appendices. It is shown that this problem can be reduced to the synthesis of $(m + n)$-port *R* network.

The above method of synthesis of driving-point functions leaves much to be desired. The major objection is the balanced structure and the large number of passive elements resulting from the synthesis of $(m + n)$-port *R* network. It should be obvious that the realization of the *R* net-

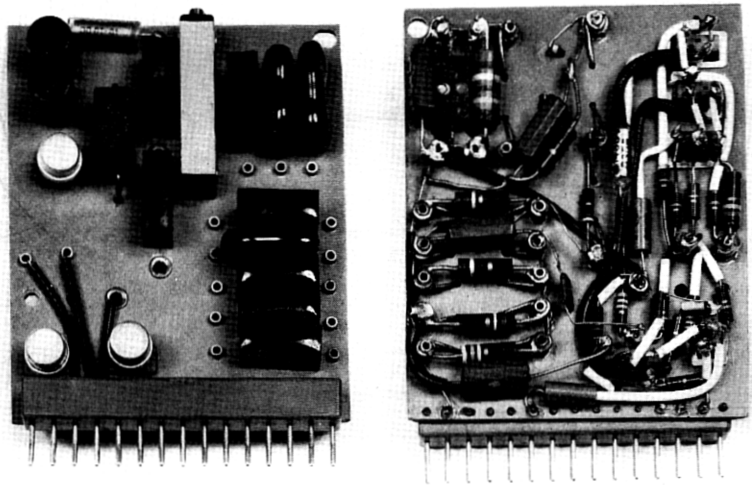

Fig. 14 — Front and rear views of one filter section.

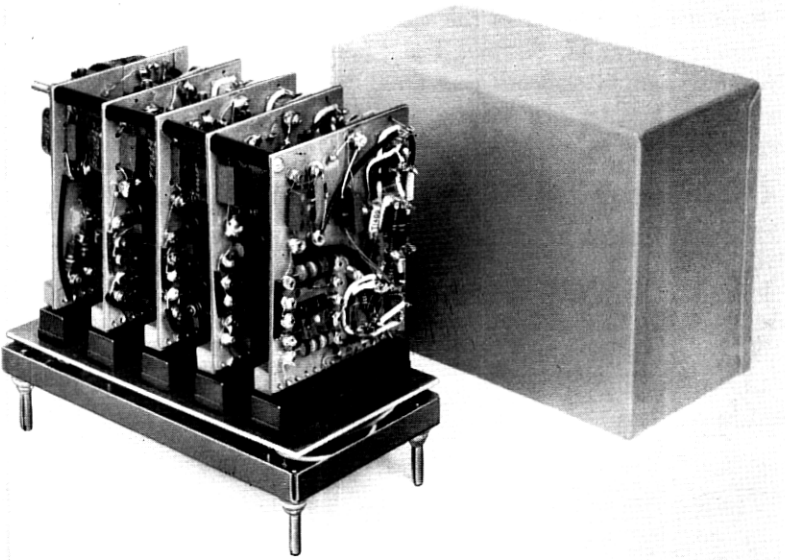

Fig. 15 — Complete bandpass filter.

work with a minimum number of resistors would render this method quite practical.

A more practical synthesis method for driving-point functions is considered next. The cascade method uses the negative-impedance converter as an active element. This method is adapted to the synthesis of voltage transfer functions. Particularly simple structures result when functions of rank 2 are to be realized. A rather elaborate filter is designed using these ideas. An experimental model of such a filter has been built and tested and the results are reported.

Further work on active networks is indicated in several directions. Theoretical investigation may be undertaken to determine the minimum number of active elements required to realize an arbitrary two-port matrix. The use of active elements with *RLC* networks may yield networks with fewer elements. Possibility of using several active elements scattered through the network to yield more stable characteristics may bear some investigation. It is indeed not farfetched to envisage a day when the use of active elements in network design will be viewed with as much equanimity as the use of passive elements is viewed today.

VII. ACKNOWLEDGMENTS

Reduction to practice of the cascade method was carried out by R. P. Abraham and J. M. Sipress. The filter was built and tested by E. P. Greener. Computing work was done by Mrs. H. D. Reinecke.

APPENDIX A

Given $F = N/D$, it will be shown that this may be expressed as

$$F = \frac{N_1 - N_2}{D_1 - D_2}, \tag{48}$$

where N_1/D_1 is an RC admittance and the degrees of N_2 and D_2 are equal to the degrees of N_1 and D_1 respectively.

Choose N_1 and D_1 of the same degree such that N_1/D_1 is an RC admittance. Then

$$\begin{aligned} N_2 &= N_1 - N, \\ D_2 &= D_1 - D. \end{aligned} \tag{49}$$

Degrees of N_2 and D_2 can always be made equal to the degrees of N_1 and D_1, respectively, by a proper choice of the degrees of N_1 and D_1. Thus, the polynomials N_1, D_1, N_2, D_2 are all of the same degree, say n_1. This fact ensures that N_1/D_1, N_2/N_1 and D_2/D_1 have the desirable property (see Appendix B) of having no poles at infinity. Obviously, the degree n_1 should not be less than n, the higher of the degrees of N and D.

It will now be shown that, given a polynomial N, it can always be written as a difference of two polynomials having only negative real roots. This development can in many cases be used to obtain the desired result above. The development is of greater interest in the cascade synthesis method.

Form a function

$$F_1(p) = \frac{N}{\prod_{\nu=1}^{n} (p + \sigma_\nu)}, \tag{50}$$

where σ_ν are real positive numbers and n is either equal to or exceeds by one the degree of N. The residue of the function F_1 in any of the poles is seen to be real. So,

$$F_1(p) = K_\infty + \sum_i \frac{|K_i|}{p + \sigma_i} - \sum_j \frac{|K_j|}{p + \sigma_j}. \tag{51}$$

In (51), the part with positive residues represents a passive *RC* impedance, and so the zeros of this part must be negative real. The part with negative residues is simply an *RC* impedance with a negative sign and so it must also have negative real zeros. Then

$$F_1(p) = \frac{P_1}{P_2} - \frac{P_3}{P_4}, \tag{52a}$$

where all the polynomials P_1, P_2, P_3, P_4 have negative real roots. Further,

$$F_1(p) = \frac{P_1P_4 - P_2P_3}{P_2P_4} = \frac{P_1P_4 - P_2P_3}{\prod_{\nu=1}^{n} (p + \sigma_\nu)} \tag{52b}$$

and

$$N = P_1P_4 - P_2P_3 = N_1 - N_2, \tag{53}$$

where N_1 and N_2 have only negative real roots, as was to be shown.

In a similar manner, the polynomial D can be expressed as $D = D_1 - D_2$, where D_1 and D_2 have only negative real roots. With a proper choice of σ_ν, it may be possible to make N_1/D_1 an *RC* immittance function. No attempt has been made to determine under what conditions the above is possible, since such a general statement does not seem to be of great importance. The difficulty involved in determining the proper σ_ν for a specific problem would govern whether one should apply this method or not.

In (50), if all σ_ν are selected on intervals of the negative real axis where $N(p)$ has the same sign, then the residues of F_1 will have alternating signs at alternate poles. This is easily observed by considering a polynomial N having only complex conjugate roots as shown in Fig. 16.

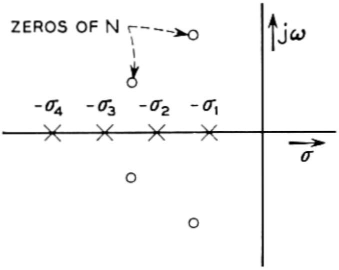

Fig. 16 — Residues with alternating signs in alternate poles.

Along the σ-axis the sign of the function $F_1(p)$ changes on passing a pole but not between the poles. Hence, the residues in these poles have alternating signs at alternate poles. Moreover, if the σ_ν are selected on intervals of the negative σ-axis where $N(p)$ has a positive sign, then the residue of F_1 in the pole closest to the origin, say σ_1, has a positive sign and P_4/P_2 is an RC impedance function.

The results of interest in the cascade method are:

i. From (50) and (51) it is seen that a difference of two RC impedance functions can produce any desired complex zeros.

ii. With a proper choice of σ_ν, it is observed that P_4/P_2 is an RC impedance function.

APPENDIX B

n-Port RC Networks

Assume that it is possible to find a minimum number, m, of capacitors in the desired RC n-port network. The network can then be modified to a $(m + n)$-port terminated in its m-ports in capacitors (Fig. 3). The $(m + n)$-port network is purely resistive and can be investigated to determine if ideal transformers can be avoided. This will not give a complete class of networks without ideal transformers; it will give only a subclass of networks with a minimum number of capacitors and no transformers. In order to obtain the complete class of networks m must be made greater than the minimum number. It will be seen later how the value of m can be chosen to give the desired number of capacitors.

The problem may be stated as follows:

A short-circuit admittance matrix Y of order n representing a n-port RC network is given. It is desired to find a conductance matrix G representing a resistive network such that, when this network is terminated at its m-ports in certain capacitors, the n-port RC network Y is obtained.

Let

$$G = \begin{array}{c} \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right]\begin{array}{l} n \\ m \end{array} \\ \begin{array}{cc} n & \quad m \end{array} \end{array}, \tag{54}$$

where the A_{ij}'s are matrices. When the m-ports are properly terminated in capacitors, the resulting $(n \times n)$ matrix is required to be Y. For convenience, the terminations are assumed to be unit capacitors. Any transformers that appear at these m-ports are eliminated by a simple scaling of these capacitors. Then,

$$Y = A_{11} - A_{12}[A_{22} + Ip]^{-1} \bar{A}_{12}, \tag{55}$$

where

I = unit matrix,
$\bar{A}$ = transpose of A,
$\bar{A}_{12} = A_{21}$, since G is symmetrical.

The short-circuit admittance matrix Y of the RC network may be expressed as

$$Y = Kp + K_\infty - \sum_\nu K_\nu \left(\frac{1}{p + \sigma_v} \right), \tag{56a}$$

where K, K_∞ and K_ν are $(n \times n)$ matrices of residues. For a passive network, these matrices are positive semidefinite. At $p = 0$, $Y(0)$ must also be positive semidefinite for physical realizability. It is desired to identify the expression of (55) with that of (56a). The only way to do this is by making $A_{11} = Kp + K_\infty$. This, of course, does not keep G purely resistive. An alternative approach is to let $Y = Kp + Y_1$, and then identify Y_1 with the expression in (55). Then, the network corresponding to Kp is added in parallel to the network corresponding to Y_1. As will be seen below, the networks obtained are balanced structures and no transformers are needed when the structures are connected in parallel. Even though Y in (56a) can always be realized in this manner, it is not at all desirable, since the realization of Kp requires a large number of additional capacitors. This whole difficulty can be avoided by considering the open-circuit impedance matrix Z and carrying out a development analogous to the one below. No pole at infinity is present in Z and the above difficulty does not arise. The other possible way of avoiding the additional capacitors is to make $K \equiv 0$. It is seen in Appendix A that it is indeed possible to do so when Y is obtained for the active network problem under consideration. Furthermore, since sufficient conditions for realizability of resistance networks without transformers are obtained in terms of the short-circuit admittance matrix G, it is desirable to make the development here in terms of Y. Assuming that K is made equal to zero, Y may be expressed as

$$Y = K_\infty - \sum_\nu K_\nu \left(\frac{1}{p + \sigma_\nu} \right), \tag{56b}$$

where K_∞, K_ν and $Y(0)$ are positive semidefinite.

It is now necessary to put (56b) in such a form that it is readily identified with (55). Let

$$K_\nu = M_\nu \bar{M}_\nu, \tag{57}$$

where M_ν has n rows and m_ν columns. It is then possible to write

$$K_\nu \left(\frac{1}{p + \sigma_\nu} \right) = M_\nu \left[\frac{1}{p + \sigma_\nu} \right]_{m_\nu} \bar{M}_\nu , \tag{58}$$

where

$$\left[\frac{1}{p + \sigma_\nu} \right]_{m_\nu}$$

is a diagonal matrix of order m_ν and elements

$$\left(\frac{1}{p + \sigma_\nu} \right).$$

Let

$$\left[\frac{1}{p + \sigma_\nu} \right]_{m_\nu} = D_{m_\nu} . \tag{59}$$

Then

$$\sum_\nu K_\nu \left(\frac{1}{p + \sigma_\nu} \right) = M_\nu D_{m_\nu} \bar{M}_\nu \tag{60}$$

and

$$\sum_\nu M_\nu D_{m_\nu} M_\nu$$

$$= [M_1 , M_2 , \cdots] \begin{bmatrix} D_{m_1} & 0 & 0 & 0 & 0 \\ 0 & D_{m_2} & 0 & 0 & 0 \\ 0 & 0 & D_{m_3} & 0 & 0 \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \\ \\ \\ \end{bmatrix} = MH\bar{M}, \tag{61}$$

where

$$M = [M_1 , M_2 , \cdots] \tag{62}$$

and

$$H = \begin{bmatrix} D_{m_1} & 0 & 0 & 0 & 0 \\ 0 & D_{m_2} & 0 & 0 & 0 \\ 0 & 0 & D_{m_3} & 0 & 0 \\ 0 & 0 & & & \end{bmatrix}. \tag{63}$$

The columns in M and the order of H are both given by

$$m = \sum_{\nu} m_{\nu} . \tag{64}$$

It is seen from (54) that m also determines the number of capacitors. Thus, m can be arbitrarily chosen by use of (57) and (64) to be any value greater than or equal to the minimum value discussed below. It is, however, necessary to determine G in terms of the known quantities in (56b). Equation (56b) can be rewritten using (61):

$$Y = K_{\infty} - MH\bar{M}. \tag{65}$$

Comparing it with (55),

$$\begin{aligned} A_{11} &= K_{\infty} , \\ A_{12} &= M, \\ [A_{22} + Ip]^{-1} &= H. \end{aligned} \tag{66}$$

Since H is a diagonal matrix, its inverse is given simply by inverting each term of the matrix, each term being of the form $[1/(p + \sigma)]$:

$$H^{-1} = [Ip + S], \tag{67}$$

where I is a unit matrix of order m and

$$S = \begin{bmatrix} \sigma_1 I_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 I_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 I_3 & 0 & 0 \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & \end{bmatrix}, \tag{68}$$

where I_j is a unit matrix of order m_j . Thus,

$$\left.\begin{aligned} A_{11} &= K_{\infty} \\ A_{12} &= M \\ A_{22} &= S \end{aligned}\right\}. \tag{69}$$

If it is desirable to obtain a network with a minimum number of capacitors, it is necessary to obtain a minimum m. This is accomplished by having each m_{ν} a minimum number. Consider the quadratic form of a matrix K_{ν} expressed as sum of the squares

$$\bar{X} \, K_{\nu} X = \sum_{i} [X \, M_{\nu i}]^2,$$

where each $M_{\nu i}$ is a column vector. Then

$$\begin{aligned}\bar{X}K_\nu X &= \sum_i \bar{X}\, M_{\nu i}\bar{M}_{\nu i}X \\ &= \bar{X}\, M_\nu \bar{M}_\nu X,\end{aligned} \tag{70}$$

where

$$M_\nu = [M_{\nu 1}\,,\, M_{\nu 2}\,,\, M_{\nu 3}\,,\, \cdots]. \tag{71}$$

If K_ν is of order n and rank δ_ν, then there are δ_ν independent terms in the quadratic form. The matrix M_ν has n rows and δ_ν columns. This δ_ν is the minimum number of columns in M_ν, since the quadratic form in (70) must have at least δ_ν terms. It is thus necessary to use (70) instead of (57) to obtain a minimum

$$m = \sum_\nu \delta_\nu = \sum_\nu m_\nu\,,$$

since δ_ν is the minimum value of m_ν.

It is easy to show that G, whose submatrices are determined by (69), represents a physically realizable resistance network. Consider the quadratic form of matrix G,

$$\begin{aligned} Q &= [\bar{x},\bar{y}]\begin{bmatrix} A_{11} & A_{12} \\ \bar{A}_{12} & A_{22}\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} \\ &= \bar{x}A_{11}x + \bar{x}A_{12}y + \bar{y}\bar{A}_{12}x + \bar{y}A_{22}y \\ &= \bar{x}K_\infty x + \bar{x}My + \bar{y}\bar{M}x + \bar{y}Sy \\ &= \bar{x}K_\infty x - \bar{x}MS^{-1}\bar{M}x + [\bar{x}M + S\bar{y}]S^{-1}[\bar{M}x + Sy] \\ &= \bar{x}[K_\infty - S^{-1}M\bar{M}]x + [\bar{x}M + S\bar{y}]S^{-1}[\bar{M}x + Sy] \\ &= \bar{x}[Y(0)]x + \bar{z}S^{-1}z,\end{aligned}$$

where $z = \bar{M}x + Sy$ is a column matrix. From (55) and (69), $Y(0)$ is a positive semidefinite matrix and S^{-1} is a diagonal matrix with positive elements. Therefore, $Q > 0$ for all x and z. Hence the quadratic form is positive definite and the matrix G is physically realizable.

It has been shown that, given an n-port RC matrix Y, it is possible to find a matrix G for an auxiliary $(n + m)$-port resistance network. This $(m + n)$-port R network must be terminated at its m-ports in unit capacitors to obtain the desired n-port RC network.

APPENDIX C

It will be shown that, if K_ν is a third-order matrix some of whose elements are specified by (7), then y_{11} and y_{22} may be chosen so that K_ν is positive semidefinite of rank 1.*

From (7) it is clear that

$$R_m(y_{13}y_{23} - y_{12}y_{33}) = \frac{P_2}{Q_1}. \tag{72}$$

Since Q_1 has simple zeros, then $y_{13}y_{23} - y_{12}y_{33}$ must have simple poles. Substituting

$$y_{ij} = k_{ij}^{\infty} - \sum_{\nu} \frac{k_{ij}^{(\nu)}}{p + \sigma_\nu} \tag{73}$$

in the left-hand side of (72) and setting the coefficients of $1/(p + \sigma_\nu)^2$ equal to zero gives

$$k_{13}^{(\nu)}k_{23}^{(\nu)} - k_{12}^{(\nu)}k_{33}^{(\nu)} = 0 \qquad \text{for all } \nu. \tag{74}$$

Thus, y_{11} and y_{22} are driving-point admittances with poles σ_ν but residues as yet unspecified. It is well known that there exists an orthogonal transformation which transforms a positive semidefinite matrix of rank 1 into the diagonal form $[k, 0, 0]$ with $k > 0$. Since the eigenvalues of a matrix A are roots of the equation

$$\lambda^3 - \lambda^2 \operatorname{tr} A + \tfrac{1}{2}\lambda[(\operatorname{tr} A)^2 - \operatorname{tr} A^2] - \det A = 0, \tag{75}$$

where tr means the sum of the diagonal elements, and since $\operatorname{tr} A$, $\operatorname{tr} A^2$ and $\det A$ are invariant under orthogonal transformations, then for a matrix of rank 1

$$\operatorname{tr} A = k, \qquad \operatorname{tr} A^2 = k^2 \qquad \text{and} \quad \det A = 0. \tag{76}$$

Then (75) reduces to

$$\lambda^3 - \lambda^2 k = 0$$

with roots $(k, 0, 0)$. Hence $k_{11}^{(\nu)}$ and $k_{22}^{(\nu)}$ must be chosen so that they are positive and (76) is satisfied for K_ν. This can be done provided (74) is satisfied and

$$k_{11}^{(\nu)} = \frac{[k_{13}^{(\nu)}]^2}{k_{33}^{(\nu)}},$$

---

* The proof is due to Mrs. B. A. Morrison.

and

$$k_{22}^{(\nu)} = \frac{[k_{23}^{(\nu)}]^2}{k_{33}^{(\nu)}}.$$

To determine M_ν, observe that, under the orthogonal transformation $X = TY$, where

$$X = \{x_i\}, \qquad Y = \{y_i\}, \qquad T = \| t_{ij} \| \quad \text{and} \quad T^{-1} = \bar{T},$$

the quadratic form for K_ν becomes

$$\bar{X}K_\nu X = \bar{Y}\bar{T}K_\nu TY = ky_1^2 = k\left[\sum_{j=1}^{3} t_{j1}X_j\right]^2 = \left[\pm\sqrt{k}\sum_{j=1}^{3} t_{j1}X_j\right]^2.$$

Then,

$$M_\nu = \pm\{\sqrt{k}\, t_{j1}\}.$$

Since

$$\bar{T}K_\nu T = \underline{K}_\nu,$$

where $\underline{K}_\nu$ is the diagonal matrix $[k, 0, 0]$, then

$$\bar{T}K_\nu = \underline{K}_\nu\bar{T}$$

and t_{j1} is seen to be the solution to the following set of equations:

$$\sum_{j=1}^{3} t_{j1}[k_{jm}^{(\nu)} - k\delta_{jm}] = 0 \qquad \text{for } m = 1, 2, 3.$$

The result obtained is

$$\bar{M}_\nu = \pm\frac{1}{\sqrt{k_{33}^{(\nu)}}}[k_{13}^{(\nu)}, k_{23}^{(\nu)}, k_{33}^{(\nu)}].$$

APPENDIX D

It will be demonstrated that Q_3/Q_1 is always an *RC* driving-point impedance provided $D(p)$ contains only complex zeros. Further, Q_3'/Q_1'

as well as Q_3''/Q_1'' are also *RC* driving-point impedance functions. Polynomials Q, Q_3 and Q_1 are defined in (27) and (28), and $Q_1' = Q_1 + KP_2$ and $Q_3' = Q_3 + KP_4$ [P_2 and P_4 defined in (28)]. Polynomials Q_1'', and Q_3'' are defined in (42), and are given by $Q_1'' = Q_1p + KP_2$, $Q_3'' = Q_3p + KP_4$.

It follows from the definitions above and from (28) that

$$Q_1'P_4 - Q_3'P_2 = D(p), \tag{77}$$

where Q_1'/P_2, Q_3'/P_4 and P_4/P_2 characterize *RC* driving-point impedance functions. Also, the order of Q_1' and Q_3' equals the order of P_2 and P_4, respectively. Consequently, the roots of $D(p)$ are given by the zeros of an expression of the form

$$1 - R\frac{\prod (p + d_{2i}') \prod (p + f_{1j})}{\prod (p + d_{1k}') \prod (p + f_{2l})} = 0, \tag{78}$$

where

$$Q_1' = k_a' \prod (p + d_{1k}'),$$

$$Q_3' = k_b' \prod (p + d_{2i}'),$$

$$P_2 = k_c \prod (p + f_{1j}),$$

$$P_4 = k_d \prod (p + f_{2l}).$$

Consider the root locus of (78) for $R > 0$. For some specific value of R, the expression yields the roots, and only the roots, of $D(p)$. Consider the constraints that this places on the open-loop poles and zeros of the locus, i.e., the roots of $Q_1'P_4$ and $Q_3'P_2$, respectively, along the negative real axis.

The open-loop root closest to the origin must be a zero of the locus, i.e., a root of P_2, since Q_1'/P_2, Q_3'/P_4 and P_4/P_2 are all *RC*. The next, or second, root is an open-loop pole of the locus. The third root must also be an open-loop pole of the locus, since, if it were an open-loop zero, then it would follow that for all positive values of R there must be a zero of (77) on the negative real axis. This is not permissible, since there must exist a positive value of R for which (78) must yield only the roots of $D(p)$ which are complex. Thus, the second and third roots must be open-loop poles of the locus. The only way this can occur is if one is a root of Q_1' and the other is a root of P_4.

A similar line of reasoning is carried out until all the roots are used. Hence, if the degree of P_2 equals that of P_4, the only possible configuration for the open-loop poles and zeros of the locus is that shown in Fig. 17(a); if the degree of P_2 is one greater than that of P_4, the configuration is as shown in Fig. 17(b). Consequently, the roots of Q_1' and Q_3' must alternate along the negative real axis, with the one closest to the origin being a root of Q_1'. Therefore, Q_3'/Q_1' characterizes an RC driving-point impedance function if the roots of $D(p)$ are all complex.

Thus, Q_3/Q_1 is simply the special case of Q_3'/Q_1' for $K = 0$ in the definitions above. Configurations of the roots are similar to those obtained above and Q_3/Q_1 is an RC impedance function.

In the case of Q_3''/Q_1'', the only difference is that now

$$Q_1'' P_4 - Q_3'' P_2 = pD(p). \tag{79}$$

An expression identical to that of (78) gives the roots of $pD(p)$ which are all complex except for the one at origin. The root-locus is again considered and the configuration of the open-loop zeros and poles is determined. The configuration is shown in Fig. 17(c). The roots of Q_3'' and Q_1'' alternate along the negative real axis with the one closest to the origin being a root of Q_1''; Q_3''/Q_1'' characterizes an RC driving-point impedance function.

APPENDIX E

Conditions are obtained under which it is possible to rationalize z_{12} without use of surplus factors for network functions of rank 2. An additional condition to be satisfied is that the resulting two-port RC network be a ladder structure. Consider a transfer function

$$T(p) = K_0 \frac{p^2 + ap + b}{p^2 + cp + d}, \tag{80}$$

and choose the impedance functions as

$$Z(p) = \frac{p^2 + ap + b}{p^2 + cp + d} \tag{81}$$

or

$$Z(p) = \frac{1}{p} \frac{p^2 + ap + b}{p^2 + cp + d}. \tag{82}$$

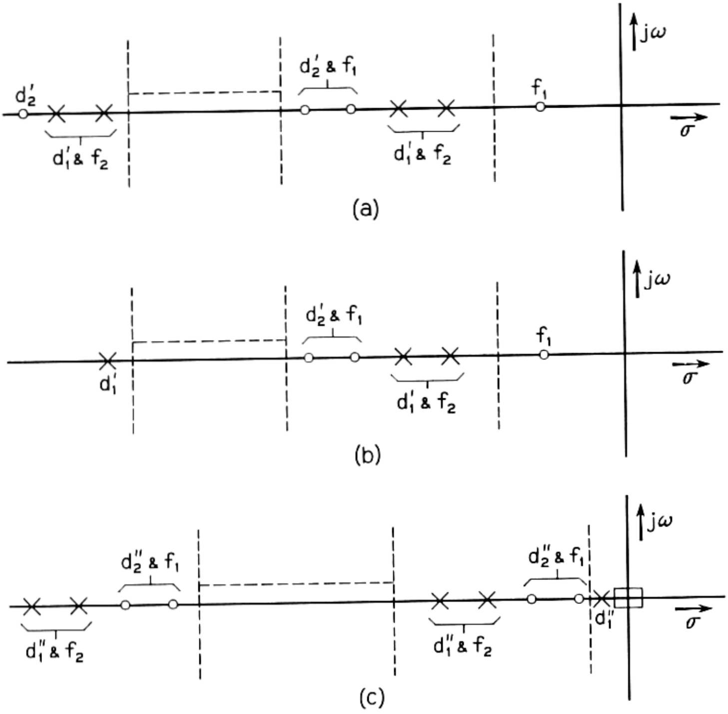

Fig. 17 — Configurations for open-loop poles and zeros of the root locus (a) when P_2 and P_4 are of same degree; (b) when degree of P_2 is one greater than degree of P_4; (c) when polynomials Q_3'' and Q_1'' are used to determine the configuration.

Equation (81) is to be considered when an introduction of constant K is sufficient to rationalize z_{12}, and (82) must be used when K/p is required.

When (81) is used,

$$P_2 = p + \sigma_1,$$

$$P_4 = p + \sigma_2 \qquad\qquad \sigma_2 > \sigma_1 > 0,$$

$$P_1 = p + \frac{\sigma_1\sigma_2 - a\sigma_1 + b}{\sigma_2 - \sigma_1},$$

$$P_3 = \frac{{\sigma_2}^2 - a\sigma_2 + b}{\sigma_2 - \sigma_1}.$$

When the constant K is appropriately introduced,

$$Q_1' = (K + 1)p + \frac{(K + 1)\sigma_1 \sigma_2 - K\sigma_1^2 - c\sigma_1 + d}{\sigma_2 - \sigma_1},$$

$$Q_3' = Kp + \frac{(K + 1)\sigma_2^2 - K\sigma_1 \sigma_2 - c\sigma_2 + d}{\sigma_2 - \sigma_1}.$$

It is required that $P_1 Q_3' - Q_1' P_3 = 0$ have a double root at $p = \eta \leqq 0$, where η is assumed to be nonpositive so as to obtain a ladder network when the two-port is synthesized.

Thus,

$$P_1 Q_3' - Q_1' P_3 = Kp^2 + p\,(x + Ka) + (y + Kb) = 0, \qquad (83)$$

where

$$x = \frac{d - b + \sigma_2(a - c)}{\sigma_2 - \sigma_1}$$

and (84)

$$y = \frac{\sigma_2(b - d) - cb + da}{\sigma_2 - \sigma_1}.$$

For $\eta \leqq 0$,

$$(x + Ka) \geqq 0. \qquad (85)$$

For (83) to have a double root,

$$(x + Ka)^2 = 4K\,(y + Kb),$$

which yields

$$K = \frac{(2a - 4y) \pm \sqrt{(2a - 4y)^2 + 4x^2(4b - a^2)}}{2(4b - a^2)}. \qquad (86)$$

From (86) it is seen that K is positive if $(4b - a^2)$ is positive. It will be assumed that all the zeros and poles in (80) are complex conjugate, so that introduction of K keeps z_{22} an RC impedance, and $(4b - a^2) > 0$. Equation (85) can be satisfied by a proper choice of σ_2, provided that $d > b$.

If $d < b$, (82) is used, and the polynomials Q_1'' and Q_3'' are obtained by introducing K/p, as in (42):

$$Q_1'' = p^2 + p\left(K + \frac{\sigma_1\sigma_2 - c\sigma_1 + d}{\sigma_2 - \sigma_1}\right) + K\sigma_1 ,$$

$$Q_3'' = p\left(K + \frac{\sigma_2^2 - c\sigma_2 + d}{\sigma_2 - \sigma_1}\right) + K\sigma_2$$

and

$$P_1Q_3'' - Q_1''P_3 = p^2(K + y) + p(Ka + x) + Kb = 0, \qquad (87)$$

where

$$x = \frac{da - bc + \sigma_2(b - d)}{\sigma_2 - \sigma_1}, \qquad y = \frac{d - b + \sigma_2 (a - c)}{\sigma_2 - \sigma_1}. \qquad (88)$$

Equation (87) has a double root if

$$(Ka + x)^2 = 4Kb(K + y) \qquad (89)$$

or

$$K = \frac{(2ax - 4by) \pm \sqrt{(2ax - 4by)^2 + 4(4b - a^2)x^2}}{2(4b - a^2)}. \qquad (90)$$

Under the same assumptions as before, $4b - a^2 > 0$, and so $K > 0$. For $\eta \leqq 0$, it is necessary that

$$\frac{Ka + x}{K + y} \geqq 0, \qquad (91)$$

but $K + y > 0$ [see (89)], so (91) implies that $(Ka + x) > 0$. It is seen from (88) that $(Ka + x)$ can be made to be positive if $b > d$.

The impedances z_{11}, z_{12}, z_{22} and Z_L can be determined using the polynomials determined above. The two-port can be realized by using zero-shifting techniques. Only one ideal transformer appears, and it is removed by appropriate impedance scaling. Source conversion is then performed to obtain the desired transfer function to within a constant multiplier. The structures obtained are shown in Fig. 9(a) for $d > b$, and in Fig. 9(b) for $b > d$.

The above development can also be carried out when the zeros are not complex conjugate. In this case, however, one must consider separately the cases when $(4b - a^2) > 0$ and $(4b - a^2) < 0$. Results can also be obtained when some of the coefficients are nonpositive. The case of complex zeros and poles in the left-half plane is considered above, since this is the more important case practically.

REFERENCES

1. Dietzold, R. L., U. S. Patent 2,549,065, April 17, 1951.
2. Bangert, J. T., The Transistor as a Network Element, B.S.T.J., **33,** March 1954, p. 329.
3. Linvill, J. G., *RC* Active Filters, Proc. I.R.E., **42,** March 1954, p. 555.
4. Horowitz, I. M., Synthesis of Active *RC* Transfer Functions, Research Report R-507-56, PIB-437, Microwave Research Inst., Polytechnic Inst. of Brooklyn, November 1956.
5. Yanagisawa, T., *RC* Active Networks Using Current-Inversion-Type Negative-Impedance Converters, Trans. I.R.E., **CT-4,** September 1957, p. 140.
6. Bode, H. W., *Network Analysis and Feedback Amplifier Design,* D. Van Nostrand Co., New York, 1956, p. 67.
7. Rosen, A., A New Network Theorem, J. I. E. E., **62,** 1924, p. 916.
8. Linvill, J. G., Transistor Negative-Impedance Converters, Proc. I.R.E., **41,** June 1953, p. 725.
9. Guillemin, E. A., *Communication Networks,* Vol. 2, John Wiley and Sons, New York, 1951, p. 218.
10. Fialkow, A. and Gerst, I., The Transfer Function of General Two Terminal Pair *RC* Networks, Quart. Appl. Math.. **10,** July 1952, p. 113.
11. Larky, A. I., Negative-Impedance Converters, Trans. I.R.E., **CT-4,** September 1957, p. 124.