Mode Conversion at the Junction of Helix Waveguide and Copper Pipe

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Scattering coefficients for a junction of helix waveguide and copper pipe are calculated. Fields exterior to the helix in the helix waveguide are neglected, but no other approximations are made.

I. INTRODUCTION

The circular electric wave in circular waveguide has an attenuation which theoretically decreases monotonically with increasing frequency. This fact suggests its use as the information carrier in a high-frequency waveguide communication system. However, as the frequency in any circular waveguide is increased, a greater number of the normal modes become propagating modes. Mode conversion at variations from ideal circular cylinder geometry (such as in bends) thus leads to the presence of energy in propagating modes other than the circular electric mode. This eventually leads to rather severe variation of attenuation with frequency for the circular electric mode.

To reduce this variation of attenuation with frequency, mode filter sections, which attenuate unwanted modes much more rapidly than the circular electric mode is attenuated, may be inserted into the line. Short sections of helix waveguide may be used for this purpose.[1] This technique has proven to be effective in smoothing out the attenuation-frequency characteristic of the circular electric mode.

Using mode filters to effect equalization as discussed above suggests the problem of the calculation of mode conversion at the joint between the mode filter and the propagating guide. The problem of calculating the mode conversion at the intersection of the helix waveguide mode filter sections and solid copper wall waveguide has been attacked by others[2] by the use of approximate methods. These methods generally neglect the reflected modes at the junction of the two guides. In this paper, the reflected modes are included, but any currents excited in a shoulder that exists at the junction are neglected. In Section II of this paper the representations of the fields associated with the natural modes of propagation in helix waveguide and copper pipe are given. Useful properties of these representations are also given. In Section III the equations relating mode voltages and currents imposed by boundary conditions at the helix-waveguide-copper-pipe junction are derived. In Section IV these equations are solved.

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GLOSSARY OF SYMBOLS USED

$\mathbf{E}_t$ = transverse electric field,
e_1, e_2 = scale factors associated with the cylindrical coordinates u_1 and u_2,
h = propagation constant of a mode
$\mathbf{H}_t$ = transverse magnetic field,
$\mathbf{i}_1$, $\mathbf{i}_2$ = transverse unit vectors associated with u_1 and u_2,
I = current associated with a mode,
j = $(-1)^{1/2}$,
k^2 = $\omega^2\mu\epsilon$,
l = a scattering coefficient for a helix waveguide and copper pipe junction,
T = a solution of the scalar wave equation,
u_1, u_2 = generalized cylindrical coordinates,
V = voltage associated with a mode,
Y = wave admittance of a mode,
$\mathbf{z}_0$ = unit vector pointed in axial direction of a cylindrical coordinate system,

$$\nabla_t = \frac{\mathbf{i}_1}{e_1}\frac{\partial}{\partial u_1} + \frac{\mathbf{i}_2}{e_2}\frac{\partial}{\partial u_2},$$

$\boldsymbol{\varrho}$ = $\mathbf{i}_1 u_1 + \mathbf{i}_2 u_2$,
χ^2 = a separation constant for the scalar wave equation,
ω = angular temporal frequency,
μ = permeability,
ϵ = permittivity,
δ_{nm} = Kronecker delta.

II. PRELIMINARIES

Before delving into the formulation of the problem at hand, it is necessary to make a few preliminary remarks. In the problem considered herein, it will be necessary to represent the fields associated with the normal modes of propagation in helix waveguide and in copper pipe. The copper pipe is assumed to have perfectly conducting walls and, hence, its normal modes are transverse magnetic and transverse electric modes.

The transverse electric and magnetic fields associated with a transverse magnetic mode in the copper pipe may be represented as

$$\mathbf{E}_{t(n)} = V_{(n)}(z)\nabla_t T_{(n)}(\boldsymbol{\varrho}), \tag{1}$$

$$\mathbf{H}_{t(n)} = I_{(n)}(z)\nabla_t \times [z_0 T_{(n)}(\boldsymbol{\varrho})]. \tag{2}$$

In (1) and (2) the temporal dependence of the fields ($e^{j\omega t}$) is understood, as it is in all equations in this paper. The dependence of the fields upon the distance along the waveguide axis is (z) absorbed in the voltage $V_{(n)}(z)$ and current $I_{(n)}(z)$. The function $T_{(n)}(\boldsymbol{\varrho})$ vanishes for $\boldsymbol{\varrho}$ on the guide wall and is a solution of the scalar wave equation:

$$\nabla_t^2 T_{(n)}(\boldsymbol{\varrho}) = \frac{1}{e_1 e_2}\left[\frac{\partial}{\partial u_1}\left(\frac{e_2}{e_1}\frac{\partial T_{(n)}}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{e_1}{e_2}\frac{\partial T_{(n)}}{\partial u_2}\right)\right] = -\chi_{(n)}^2 T_{(n)}. \tag{3}$$

In this equation, $\chi_{(n)}^2$ is a separation constant related to the propagation constant of the mode being discussed by

$$\chi_{(n)}^2 = k^2 - h_{(n)}^2. \tag{4}$$

The voltage and the current associated with the mode are related via the wave admittance for the mode

$$I_{(n)} = \pm Y_{(n)} V_{(n)} = \pm \frac{\omega\epsilon}{h_{(n)}} V_{(n)}, \tag{5}$$

where the sign depends upon the direction in which the wave is moving.

For transverse electric modes in the copper pipe, the transverse electric and magnetic fields can be expressed in the form

$$\mathbf{E}_{t[n]} = V_{[n]}(z)\nabla_t \times \mathbf{z}_0 T_{[n]}(\boldsymbol{\varrho}), \tag{6}$$

$$\mathbf{H}_{t[n]} = I_{[n]}(z)\nabla_t T_{[n]}(\boldsymbol{\varrho}). \tag{7}$$

The scalar function $T_{[n]}(\boldsymbol{\varrho})$ is a solution of (3) with separation constant $\chi_{[n]}^2$, and the normal derivative of $T_{[n]}(\boldsymbol{\varrho})$ at the waveguide wall vanishes. Relations similar to (4) and (5) can also be written for transverse electric modes:

$$\chi_{[n]}^2 = k^2 - h_{[n]}^2, \tag{8}$$

$$I_{[n]} = \pm Y_{[n]} V_{[n]} = \pm \frac{h_{[n]}}{\omega\mu} V_{[n]}. \tag{9}$$

The transverse vector functions defining the transverse electric fields associated with the transverse electric and transverse magnetic modes in copper pipe form a complete orthogonal set over the internal cross section of the copper pipe. As such, they may be used to represent any transverse electric field in the copper pipe. The same statement can be made of the associated modal transverse magnetic fields and the transverse magnetic field in the copper pipe. The functions $T_{(n)}(\boldsymbol{\rho})$ and $T_{[n]}(\boldsymbol{\rho})$ are assumed to be normalized, so that the following orthonormality conditions hold:

$$\begin{aligned} \chi_{(n)}^2 \iint T_{(n)} T_{(m)}\, ds &= \chi_{[n]}^2 \iint T_{[n]} T_{[m]}\, ds \\ &= \iint (\nabla_t T_{(n)})(\nabla_t T_{(m)})\, ds \\ &= \iint (\nabla_t \times \mathbf{z}_0 T_{(n)})(\nabla_t \times \mathbf{z}_0 T_{(m)})\, ds \\ &= \iint (\nabla_t T_{[n]})(\nabla_t T_{[m]})\, ds \\ &= \iint (\nabla_t \times \mathbf{z}_0 T_{[n]})(\nabla_t \times \mathbf{z}_0 T_{[m]})\, ds \\ &= \delta_{nm}, \end{aligned} \tag{10}$$

where the integration is over the internal cross section of the copper pipe.

In addition to the relations listed in (10), the following relations will be found useful in this paper:

$$\begin{aligned} \iint (\nabla_t T_{(n)})(\nabla_t \times \mathbf{z}_0 T_{[m]})\, ds &= \iint (\nabla_t T_{[n]})(\nabla_t \times T_{[m]})\, ds \\ &= \iint (\nabla_t T_{[m]})(\nabla_t \times \mathbf{z}_0 T_{(n)})\, ds = 0, \end{aligned} \tag{11}$$

which hold for all m and n.

For helix waveguide, two scalar functions are necessary for the specification of the transverse fields since all modes but the circular electric modes are hybrid.

The transverse electric and magnetic fields in helix waveguide may be written in the form

$$\mathbf{E}_{tn} = V_n(\nabla_t T_n + d_n \nabla_t \times \mathbf{z}_0 T'_n), \tag{12}$$

$$\mathbf{H}_{tn} = I_n\left(-\nabla_t \times \mathbf{z}_0 T_n + d_n \frac{h_n^2}{k^2} \nabla_t T'_n\right), \tag{13}$$

which are similar to Unger's representation.[2]

The functions T_n and T_n' are both solutions of (3) with separation constant χ_n^2; they are related to each other by means of the boundary and continuity requirements on the electromagnetic field in the helix waveguide. The constant d_n in (12) and (13) is determined by the fact that the electric field component tangential to the helix wire must vanish at the helix. For a helix of zero pitch,

$$d_n = \left.\frac{\dfrac{\partial T_n}{\epsilon_2 \partial u_2}}{\dfrac{\partial T_n'}{\epsilon_1 \partial u_1}}\right|_{\text{at helix.}} \tag{14}$$

Expressions similar to (4) and (5) can also be written for the helix waveguide modes:

$$\chi_n^2 = k^2 - h_n^2, \tag{15}$$

$$I_n = \pm Y_n V_n = \pm \frac{\omega \epsilon_0}{h_n} V_n . \tag{16}$$

Equation (16) is written for the region interior to the helix, which is assumed to be empty (i.e., $\epsilon = \epsilon_0$ and $\mu = \mu_0$).

Finally, for any two solutions of (3), which are designated here as T_1 and T_2,

$$\begin{aligned}\iint \nabla_t T_1 \cdot \nabla_t T_2 \, ds &= \iint (\nabla_t \times \mathbf{z}_0 T_1)(\nabla_t \times \mathbf{z}_0 T_2 \, ds \\ &= \chi_2^2 \iint T_1 T_2 \, ds + \oint T_1 \frac{\partial T_2}{e_1 \partial u_1} e_2 \, du_2 \\ &= \chi_1^2 \iint T_1 T_2 \, ds + \oint T_2 \frac{\partial T_1}{e_1 \partial u_1} e_2 \, du_2 ,\end{aligned} \tag{17}$$

and hence

$$\iint T_1 T_2 \, ds = \oint \frac{\left(T_1 \dfrac{\partial T_2}{e_1 \partial u_1} - T_2 \dfrac{\partial T_1}{e_1 \partial u_1}\right) e_2 \, du_2}{\chi_1^2 - \chi_2^2} \tag{18}$$

and

$$\iint (\nabla_t \times \mathbf{z}_0 T_1)(\nabla_t T_2)\, ds = \oint T_2 \frac{\partial T_1}{\partial u_2}\, du_2 = -\oint T_1 \frac{\partial T_2}{\partial u_2}\, du_2. \quad (19)$$

In (17), (18) and (19) the line integrals are to be evaluated along the waveguide wall, or along the boundary enclosing the domain of integration of the integrals over the area.

III. THE BOUNDARY PROBLEM — HELIX WAVEGUIDE MODE INCIDENT

The problem under consideration is the calculation of the scattering properties of a junction between helix waveguide and copper pipe. The region interior to the helix in the helix waveguide and the interior of the copper pipe are assumed to be empty. Furthermore, the helix diameter and inside diameter of the copper pipe are assumed to be equal.

To formulate the problem under consideration analytically, assume that a single helix waveguide mode is incident upon the junction from the helix waveguide. This will give rise to reflected helix waveguide modes and transmitted transverse electric and transverse magnetic modes in the copper pipe. The requirement of continuity of the transverse electric field across the plane of the junction indicates that the total transverse electric field due to the incident and reflected helix waveguide modes must equal the total transverse electric field due to the transmitted transverse electric and transverse magnetic modes at the plane of the junction. This equality is represented analytically by

$$\begin{aligned} V_i(\nabla_t T_i + d_i \nabla_t \times \mathbf{z}_0 T_i') &+ \sum_n V_n(\nabla_t T_n + d_n \nabla_t \times \mathbf{z}_0 T_n') \\ &= \sum_{(n)} V_{(n)} \nabla_t T_{(n)} + \sum_{[n]} V_{[n]} \nabla_t \times \mathbf{z}_0 T_{[n]} . \end{aligned} \quad (20)$$

Equation (20) is true in the area interior to the helix of the helix waveguide. In (20) the first term on the left is due to the incident mode, the summation on the left is due to the reflected helix waveguide modes, the first summation on the right is due to the transmitted transverse magnetic modes and the second summation is due to the transmitted transverse electric modes.

In a similar fashion, continuity of the transverse magnetic field across the plane of the junction is expressed as

$$\begin{aligned} I_i \left(-\nabla_t \times \mathbf{z}_0 T_i + d_i \frac{h_i^2}{k^2} \nabla_t T_i' \right) & \\ + \sum_n I_n \left(-\nabla_t \times \mathbf{z}_0 T_n + d_n \frac{h_n^2}{k^2} \nabla_t T_n' \right) & \\ = -\sum_{(n)} I_{(n)} \nabla_t \times \mathbf{z}_0 T_{(n)} + \sum_{[n]} I_{[n]} \nabla_t T_{[n]} . & \end{aligned} \quad (21)$$

Equations (20) and (21), in addition to the specification of the boundary conditions on the helix waveguide field in the region exterior to the helix, would be sufficient to determine the transmitted and reflected mode voltages and currents. In this paper it is assumed that the fields exterior to the helix in the helix waveguide are unimportant. This makes (20) and (21) sufficient for the determination of the transmitted and reflected mode voltages and currents.

Equations (20) and (21) can be converted to linear algebraic equations in the unknown voltages and currents by using some of the relationships presented in the preliminaries. For example, if (20) is multiplied scalarly on both sides by $\nabla_t T_{(p)}$ and the result is integrated over the internal cross section of the copper pipe, there results

$$\frac{V_i \chi_1^2 \oint T_i \frac{\partial T_{(p)}}{e_1 \partial u_1} e_2 \, du_2}{\chi_i^2 - \chi_{(p)}^2} + \sum_n \frac{V_n \chi_n^2 \oint T_n \frac{\partial T_{(p)}}{e_1 \partial u_1} e_2 \, du_2}{\chi_n^2 - \chi_{(p)}^2} = V_{(p)} , \tag{22}$$

where (10), (11), (17), (18) and (19) have been used. The integrals in (22) are along the waveguide wall, which is assumed to be a surface of the form $u_1 =$ a constant.

In a similar fashion, the following equations can be derived:

$$\frac{I_i \chi_i^2 \oint T_i \frac{\partial T_{(p)}}{e_1 \partial u_1} e_2 \, du_2}{\chi_i^2 - \chi_{(p)}^2} + \sum_n \frac{I_n \chi_n^2 \oint T_n \frac{\partial T_{(p)}}{e_1 \partial u_1} e_2 \, du_2}{\chi_n^2 - \chi_{(p)}^2} = I_{(p)} , \tag{23}$$

$$\frac{V_i \chi_i^2 \oint \frac{\partial T_i}{\partial u_2} T_{[p]} \, du_2}{\chi_{[p]}^2 - \chi_i^2} + \sum_n \frac{V_n \chi_n^2}{\chi_{[p]}^2 - \chi_n^2} \oint \frac{\partial T_n}{\partial u_2} T_{[p]} \, du_2 = V_{[p]} , \tag{24}$$

$$\frac{I_i \chi_i^2 \oint \frac{\partial T_i}{\partial u_2} T_{[p]} \, du_2}{\chi_{[p]}^2 - \chi_i^2} + \sum_n \frac{I_n \chi_n^2 \oint \frac{\partial T_n}{\partial u_2} T_{[p]} \, du_2}{\chi_{[p]}^2 - \chi_n^2} = \frac{k^2 I_{[p]}}{h_{[p]}^2} . \tag{25}$$

The voltages and currents of the transmitted transverse electric and transverse magnetic modes can be eliminated from (22) through (25) by use of (5) and (9). Multiplying (22) on both sides by $Y_{(p)}$ and subtracting (23) from the result yields, with application of (16),

$$\frac{V_i \chi_i^2 \oint T_i}{h_i (h_{(p)} + h_i)} \frac{\partial T_{(p)}}{e_1 \partial u_1} e_2 \, du^2 - \sum_n \frac{V_n \chi_n^2 \oint T_n \frac{\partial T_{(p)}}{e_1 \partial u_1} e_2 \, du_2}{h_n (h_{(p)} - h_n)} = 0. \tag{26}$$

The incident helix waveguide mode is treated as a forward-moving wave in deriving (26); the reflected helix waveguide modes are then backward-moving waves and the transmitted copper pipe modes are forward-moving waves.

Consideration of the nature of the T functions can now be used to simplify the set of equations represented by (26). In the coordinate systems of practical interest here, e.g., a circular cylindrical geometry, the T functions for all modes separate, so that

$$T_n = T_n^{(1)}(u_1)T_n^{(2)}(u_2), \tag{27}$$

$$T_{(p)} = T_{(p)}^{(1)}(u_1)T_{(p)}^{(2)}(u_2). \tag{28}$$

The integrals in (26) will vanish unless

$$T_n^{(2)}(u_2) = T_{(p)}^{(2)}(u_2), \tag{29}$$

so that only modes for which (29) is satisfied will appear in (26). Furthermore, the integrals in (26) assume the form

$$\oint T_n \frac{\partial T_{(p)}}{e_1 \partial u_1} e_2 \, du_2 = T_n^{(1)} \frac{\partial T_{(p)}^{(1)}}{\partial u_1} K, \tag{30}$$

where K is a constant independent of n. Using this fact reduces (26) to the form

$$\frac{V_i T_i^{(1)} \chi_i^2}{h_i(h_{(p)} + h_i)} - \sum_n \frac{V_n T_n^{(1)} \chi_n^2}{h_n(h_{(p)} - h_n)} = 0. \tag{31}$$

Applying reasoning similar to that used in deriving (31) to (24) and (25) yields the following set of equations:

$$\frac{V_i T_i^{(1)} \chi_i^2}{h_i(h_i + h_{(p)})} - \sum_n \frac{V_n T_n^{(1)} \chi_n^2}{h_n(h_{(p)} - h_n)} = 0. \tag{32}$$

Equations (31) and (32) must be solved for the unknown reflected mode voltages V_n.

IV. SOLUTION OF THE EQUATIONS

The simultaneous set of equations indicated by (31) and (32) may be solved by successively eliminating the V_n's. For example, multiply the rth equation of the set (31) by

$$\frac{h_{(r)} - h_a}{h_{(p)} - h_a} \tag{33}$$

and subtract the result from the pth equation. The result obtained, after some manipulation, is

$$\sum_n \frac{V_n T_n{}^{(1)} \chi_n{}^2 (h_a - h_n)}{h_n(h_{(r)} - h_n)(h_{(p)} - h_n)} = \frac{V_i T_i{}^{(1)} \chi_i{}^2 (h_i + h_a)}{h_i(h_{(p)} + h_i)(h_{(r)} + h_i)}. \quad (34)$$

Similarly, multiplication of the rth equation of the set (31) by

$$\frac{(h_{(r)} - h_a)}{(h_{[p]} - h_a)}$$

and subtraction of the result from the pth equation of the set indicated by (32) ultimately leads to

$$\sum_n \frac{V_n T_n{}^{(1)} \chi_n{}^2 (h_a - h_n)}{h_n(h_{(r)} - h_n)(h_{[p]} - h_n)} = \frac{V_i T_i{}^{(1)} \chi_n{}^2 (h_i + h_a)}{h_i(h_{[p]} + h_i)(h_{(r)} + h_i)}. \quad (35)$$

The set of equations indicated by (34) and (35) consists of one less equation and one less unknown (V_a) than the set indicated by (31) and (32). The process of elimination of successive V_n's can proceed in this fashion until only one of the V_n's is left, and the system of equations indicated by (31) and (32) is solved. The final result is

$$V_n = \frac{h_n V_i T_i^{(1)} \chi_i{}^2}{h_i T_n^{(1)} \chi_n{}^2} \prod_{\substack{m \\ m \neq n}} \left(\frac{h_m + h_i}{h_m - h_n} \right) \prod_{(m)} \left(\frac{h_{(m)} - h_n}{h_{(m)} + h_i} \right) \prod_{[m]} \left(\frac{h_{[m]} - h_n}{h_{[m]} + h_i} \right). \quad (36)$$

Inspection of (36) reveals the fact that no helix modes which have a propagation constant equal to the propagation constant of a copper pipe mode will be reflected from a helix-waveguide-copper-pipe junction. This result is especially important in the case of the circular electric modes in helix waveguide, since these modes have the same propagation constant in helix waveguide and copper pipe. This is quite reasonable, since the boundary conditions on the circular electric modes are essentially the same in helix waveguide and copper pipe.

Having formally obtained the reflected mode voltages, the transmitted mode voltages can be calculated by inserting the expressions for the V_n's as given by (36) into (22) and (24). Thus, inserting (36) into (22) yields the expression for the transmitted transverse magnetic mode voltages. The result after some manipulation is

$$\begin{aligned} V_{(p)} = V_i \chi_i{}^2 \oint T_i \frac{\partial T_{(p)}}{e_1 \partial u_1} e_2 \, du_2 \Bigg[\frac{1}{(h_{(p)}{}^2 - h_i^2)} + \sum_n \frac{1}{(h_{(p)}{}^2 - h_n{}^2)} \\ \cdot \prod_{\substack{m \\ m \neq n}} \left(\frac{h_m + h_i}{h_m - h_n} \right) \prod_{(m)} \left(\frac{h_{(m)} - h_n}{h_{(m)} + h_i} \right) \prod_{[m]} \left(\frac{h_{[m]} - h_n}{h_{[m]} + h_i} \right) \Bigg]. \end{aligned} \quad (37)$$

In the Appendix it is shown that this can be simplified to

$$V_{(p)} = \frac{V_i \chi_i^2 \oint T_i \frac{\partial T_{(p)}}{e_1 \partial u_1} e_2 \, du_2}{h_i(h_{(p)}^2 - h_i^2)} \prod_m \left(\frac{h_m + h_i}{h_m + h_{(p)}} \right) \cdot \prod_{\substack{(m) \\ (m) \neq (p)}} \left(\frac{h_{(m)} + h_{(p)}}{h_{(m)} + h_i} \right) \prod_{[m]} \left(\frac{h_{[m]} + h_{(p)}}{h_{[m]} + h_i} \right). \tag{38}$$

Similar manipulations yield the following expression for the transmitted transverse electric mode voltages:

$$V_{[p]} = \frac{V_i \chi_i^2 \oint \frac{\partial T_i}{\partial u_2} T_{[p]} \, du_2}{h_i(h_i^2 - h_{[p]}^2)} \prod_m \left(\frac{h_m + h_i}{h_m + h_{[p]}} \right) \cdot \prod_{(m)} \left(\frac{h_{(m)} + h_{[p]}}{h_{(m)} + h_i} \right) \prod_{\substack{[m] \\ [m] \neq [p]}} \left(\frac{h_{[m]} + h_{[p]}}{h_{[m]} + h_i} \right). \tag{39}$$

Equations (36), (38) and (39) are the formal solution to the problem of scattering at a junction of helix waveguide and copper pipe for a helix waveguide mode incident.

The most important results for practical application are the expressions for the transmitted voltages as given by (38) and (39). The quantities on the right in (38) and (39) are the voltage transmission coefficients of the junction multiplied by the incident mode voltages. These coefficients can be converted to scattering coefficients for travelling waves, as calculated by Unger,[2] by multiplying each by the square root of the appropriate wave admittance ratios. Thus, the scattering coefficients are

$$l_{(p)i} = \frac{\chi_i^2}{(h_i h_{(p)})^{1/2}} \frac{\oint T_i \frac{\partial T_{(p)}}{e_1 \partial u_1} e_2 \, du_2}{(h_{(p)}^2 - h_i^2)} \prod_{\substack{m \\ m \neq n}} \left(\frac{h_m + h_i}{h_m + h_{(p)}} \right) \cdot \prod_{\substack{(m) \\ (m) \neq (p)}} \left(\frac{h_{(m)} + h_{(p)}}{h_{(m)} + h_i} \right) \prod_{[m]} \left(\frac{h_{[m]} + h_{[p]}}{h_{[m]} + h_i} \right) \tag{40}$$

and

$$l_{[p]i} = \left(\frac{h_{[p]}}{h_i} \right)^{1/2} \frac{\chi_i^2}{k} \frac{\oint \frac{\partial T_i}{\partial u_2} T_{[p]} \, du_2}{(h_i^2 - h_{[p]}^2)} \prod_m \left(\frac{h_m + h_i}{h_m + h_{[p]}} \right) \cdot \prod_{(m)} \left(\frac{h_{(m)} + h_{[p]}}{h_{(m)} + h_i} \right) \prod_{\substack{[m] \\ [m] \neq [p]}} \left(\frac{h_{[m]} + h_{[p]}}{h_{[m]} + h_i} \right). \tag{41}$$

Furthermore, as pointed out by Unger,[2] proper normalization of the T functions for the helix waveguide insures that these coefficients will be the same whether they indicate conversion from an incident helix waveguide mode to a transmitted copper mode or conversion from an incident copper pipe mode to a transmitted helix waveguide mode.

V. CONCLUSIONS

Circular electric modes are transmitted through a junction of helix waveguide with a zero-pitch helix and copper pipe without any mode conversion or reflection at the junction. Neglecting the fields exterior to the helix in helix waveguide permits solution of the infinite set of equations which arises from requiring continuity of the transverse fields at the junction between helix waveguide and copper pipe.

APPENDIX

Reduction of the Formula for the Transmitted Voltages

The summation appearing in (37) is

$$S = \sum_n \frac{1}{(h_{(p)}{}^2 - h_n{}^2)} \prod_{\substack{m \\ m \neq n}} \left(\frac{h_m + h_i}{h_m - h_n}\right) \prod_{(m)} \left(\frac{h_{(m)} - h_n}{h_{(m)} + h_i}\right) \cdot \prod_{[m]} \left(\frac{h_{[m]} - h_n}{h_{[m]} + h_i}\right). \tag{42}$$

Since all the h's appearing in (42) are for forward-moving waves, they will all lie in the fourth quadrant of the complex h plane ($e^{-ihz} = e^{-i(-i\alpha+\beta)z}$). Thus, S can be seen to be $2\pi i$ times the sum of the fourth quadrant residues of the function

$$f(h) = \frac{1}{2\pi i(h_{(p)}{}^2 - h^2)} \prod_m \left(\frac{h_m + h_i}{h_m - h}\right) \prod_{(m)} \left(\frac{h_{(m)} - h}{h_{(m)} + h_i}\right) \cdot \prod_{[m]} \left(\frac{h_{[m]} - h}{h_{[m]} + h_i}\right) \frac{1}{(h + h_i)}. \tag{43}$$

Thus, S is simply a contour integral in the complex h plane:

$$S = \frac{1}{2\pi i} \oint_{c_1} \frac{1}{(h_{(p)}{}^2 - h^2)} \prod_m \left(\frac{h_m + h_i}{h_m - h}\right) \prod_{(m)} \left(\frac{h_{(m)} - h}{h_{(m)} + h_i}\right) \cdot \prod_{[m]} \left(\frac{h_{[m]} - h}{h_{[m]} + h_i}\right) \frac{dh}{(h + h_i)}, \tag{44}$$

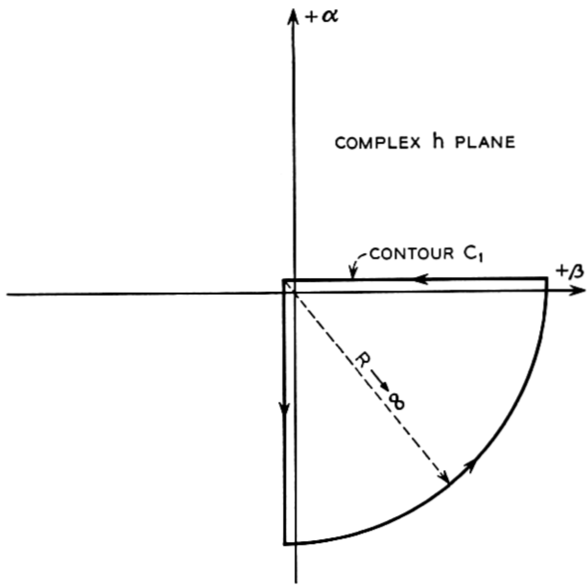

FIG. 1 — Contour for (44).

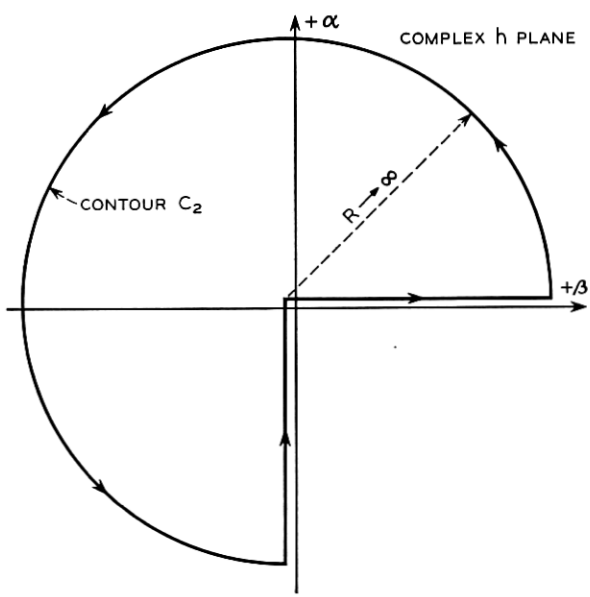

Fig. 2 — Contour for (45).

where the contour is that shown in Fig. 1. Since the contribution to the integral along the circumference of the circle is zero, it can be seen that the value of the integral in (44) is unchanged if the contour is chosen to be that of the other three quarters of the circle enclosing the h plane. Thus, the integral in (44) may be written in the form

$$S = -\frac{1}{2\pi i}\oint_{c_2}\frac{1}{(h_{(p)}{}^2 - h^2)}\prod_m\left(\frac{h_m + h_i}{h_m - h}\right)\prod_{(m)}\left(\frac{h_{(m)} - h}{h_{(m)} + h_i}\right) \cdot \prod_{[m]}\left(\frac{h_{[m]} - h}{h_{[m]} + h_i}\right)\frac{dh}{(h + h_i)}, \tag{45}$$

where the contour is now that shown in Fig. 2.

The integrand in (45) has just two poles in the first, second and third quadrants of the complex h plane, namely, the poles at

$$h = -h_{(p)} \qquad \text{and} \qquad h = -h_i\,,$$

so that S is just $2\pi i$ times the sum of the residues of the integrand at $h = -h_{(p)}$ and $h = -h_i$. Thus,

$$S = -\frac{1}{(h_{(p)}{}^2 - h_i{}^2)} \cdot\left[1 - \prod_m\left(\frac{h_m + h_i}{h_m + h_{(p)}}\right)\prod_{\substack{(m)\\(m)\neq(p)}}\left(\frac{h_{(m)} + h_{(p)}}{h_{(m)} + h_i}\right)\prod_{[m]}\left(\frac{h_{[m]} + h_{[p]}}{h_{[m]} + h_i}\right)\right]. \tag{46}$$

Consideration of (46) in (37) yields (38).

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