Transition Probabilities for Telephone Traffic*

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A stochastic model for the occupancy $N(t)$ of a telephone trunk group is specified by the conditions that arriving calls form a renewal process, that holding times have a negative exponential distribution, and that lost calls are cleared. The transition probabilities of $N(t)$ are determined, and their limits are studied. These transition probabilities have practical value in making theoretical estimates of sampling error in traffic measurements, and in the study of overflow traffic.

I. INTRODUCTION

We shall study a stochastic process $\{N(t), t \geqq 0\}$, which is a mathematical model for the occupancy of N service facilities, with no provisions for delays. For example, $N(t)$ can be interpreted as the number of (fully accessible) telephone channels (trunks) out of a group of N such in use at time t, with lost calls cleared. Also, we can think of $N(t)$ as the number of items on order at time t in an idealized inventory situation in which at most N items can be on order at one time (see Arrow, Karlin and Scarf[1]). Throughout the paper we use terminology appropriate to an application to telephone trunking. The process $N(t)$ is determined by the following assumptions:

i. Holding times of trunks are independent, each with the same negative exponential distribution function, of mean γ^{-1}, γ being the "hang-up rate."

ii. Times between successive attempts to place a call (interarrival times) are independent; each has the distribution function $A(\cdot)$, where $A(\cdot)$ is arbitrary except for the condition $A(0) = 0$. This assumption covers Poisson arrivals as a special case. The mean of $A(\cdot)$, when it exists, is denoted by μ_1.

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iii. There are N trunks, N being finite.

iv. Calls that find all N trunks busy are lost, and are cleared from the system without effect on the flow of arrivals (no retrials). These or similar assumptions appear in Palm,[2] and in Pollaczek;[3,4,5,6] certain properties of $N(t)$ itself have been studied by Takács,[7,8,9] Cohen[10] and Beneš.[11]

II. SUMMARY

The random process of interest is $N(t)$, which is interpreted as the number of trunks in use, or the number of calls in progress, at time t; $N(t)$ is a random step function fluctuating in unit steps from 0 to N. For the most part, we restrict attention to that version of $N(t)$, written $N(t - 0)$, that is continuous from the left.

The present paper is chiefly theoretical in character. It provides (a) formulas for the Laplace transforms of the transition probabilities of the stochastic process $N(t - 0)$, and (b) a statistical description of the calls that *overflow* a trunk group of the kind described in Section I. The formulas will be exemplified and used in a second paper,[12] where specific applications to switch counting and traffic averaging are described.

We begin Section III with a general account telling what transition probabilities are and why they are useful and interesting in traffic theory. The primary result, Theorem 1, can then be stated; it completely characterizes the transition probabilities

$$\Pr\{N(t-0) = n \mid N(0+) = m\}$$

as functions of t by determining their Laplace transforms, under the restriction that $A(\cdot)$ has a probability density. Section III ends with a computation of some important transition probabilities for Poisson arrivals; practical consequences of these results will be developed in the second paper.[12]

We prove Theorem 1 in the Appendix A. If $y(t)$ is the time elapsed since the last call arrival prior to t, the process $\{N(t - 0), y(t)\}$ is Markov, and we calculate its distributions from the usual Kolmogorov equations. The stationary distribution of this Markov process is determined in Appendix B.

In Appendix C the process $N(t - 0)$ is studied directly in terms of renewal theory and regenerative processes, using results of Smith.[13] No assumptions of absolute continuity are made. This procedure leads to an extension of Theorem 1, and other results outlined in the next paragraphs. (Details are omitted.)

Let R_n be this event: a call arrives and finds n trunks in use. Each occurrence of R_n, where $0 \leqq n \leqq N$, is a regeneration point of $N(t - 0)$, in the sense that the history of $N(t - 0)$ prior to the given occurrence of R_n is statistically irrelevant to the development of $N(t - 0)$ after the occurrence. Let $x_{m,n}$ be the time elapsing from an occurrence of R_m to the next occurrence of R_n. We prove $x_{m,n} < \infty$ with probability one, and, if

$$\mu_1 = \int_0^\infty x \, dA(x) < \infty ,$$

then $E\{x_{m,n}\} < \infty$.

The underlying probability functions that we calculate in Appendix C are, for $0 \leqq n \leqq N$:

$$\begin{aligned} Q_n(t) &= \sum_{k=1}^{\infty} \Pr\{k\text{th call arrives before } t \text{ and finds } n \text{ trunks busy}\} \\ &= E\{\text{number of occurrences of } R_n \text{ in } [0,t)\}. \end{aligned} \tag{1}$$

From this interpretation it is apparent that the $Q_n(\cdot)$ are unbounded monotone functions; one may expect them to be ultimately linear. The transition probabilities of $N(t - 0)$ can be represented in terms of the functions $Q_n(\cdot)$ and the transition probabilities of the simple death process with death rate γ per head of population, if the $Q_n(\cdot)$ are evaluated for appropriate initial conditions. This is done in Appendix C. With this representation we investigate the existence of

$$\lim_{t \to \infty} \Pr\{N(t - 0) = n\}.$$

From Theorem 4 and the solutions for the Laplace-Stieltjes transforms of the $Q_n(\cdot)$, this limit, when it exists, can be evaluated explicitly, using the relation

$$E\{x_{n,n}\} = \frac{\mu_1}{p_n},$$

where p_n is the equilibrium probability that an arriving cell will find n trunks in use. (For p_n see Refs. 7 and 11.)

III. TRANSITION PROBABILITIES OF $N(t)$

The *transition probabilities* of a stochastic process x_t tell how likely it is that the random function $x_{(\cdot)}$ take on a value z at a time t, if it is known that it took on the value y at time s. Such a transition probability is written

$$\Pr\{x_t = z \mid x_s = y\}, \tag{2}$$

the vertical bar being read and interpreted as "given that" or "if."

In other words, (2) expresses the relevance of the information that the event $\{x_s = y\}$ has occurred to the likelihood that the event $\{x_t = z\}$ will occur. In still other words, (2) expresses the *dependence* of the event $\{x_t = z\}$ on $\{x_s = y\}$.

The chief practical use of transition probabilities for models of telephone traffic is in computation of covariance functions; these, in turn, are used to compute theoretical estimates of sampling error in actual traffic measurements, such as time averages and switch counts. To see how this happens in a particular case, we consider the use of the continuous time average

$$M(T) = \frac{1}{T}\int_0^T N(t)\,dt$$

as an estimate of the carried load. The variance of M is

$$E\{M^2\} - E^2\{M\} = T^{-2}\int_0^T\int_0^T [E\{N(t)N(s)\} - E\{N(t)\}E\{N(s)\}]ds\,dt. \tag{3}$$

The integrand is just the covariance $R(t,s)$ between $N(t)$ and $N(s)$; if $N(t)$ is stationary in the wide sense, so that $R(t,s) = R(t - s)$, then (3) reduces (by partial integrations) to

$$\text{Var}\,\{M\} = 2T^{-2}\int_0^T (T - t)R(t)\,dt. \tag{4}$$

The covariance $R(t)$ can be written in terms of the transition probabilities of $N(\cdot)$ as

$$R(t) = \sum_{m=0}^{N}\sum_{n=0}^{N} mnp_m \Pr\{N(t) = n \mid N(0) = m\} - \left(\sum_{m=0}^{N} mp_m\right)^2, \tag{5}$$

where $\{p_m\}$ is the stationary distribution of $N(\cdot)$. Formulas (4) and (5) then indicate how the transition probabilities can be used to find the variance of M.

Our basic result concerning transition probabilities is most easily explained and understood after some of the notions used in stating it are discussed. The first few are merely abbreviations; we let

$$A^*(s) = \int_0^\infty e^{-st}dA(t) = a_0(s),$$

$$a_n(s) = A^*(s + n\gamma),$$

$$X_0 = 1,$$

$$X_n = \frac{1 - a_n(s)}{a_n(s)}X_{n-1} = \prod_{j=1}^{n}\frac{1 - a_j(s)}{a_j(s)}.$$

These Laplace transforms enter because we shall be characterizing the Laplace transforms of the transition probabilities in terms of the hang-up rate γ and the transform $A^*(s)$ of the interarrival probability density.

In the summary we have denoted by R_n the event: a call arrives and finds n trunks in use. We let $q_n(t,0)$ be the "density" of R_n at time t, that is, the rate at which R_n is occurring at t, and we let

$$b_n(t) = \sum_{j=n}^{N} \binom{j}{n} q_j(t,0) \tag{6}$$

be the associated binomial moment. From (1), it can be seen that $dQ_n/dt = q_n(t,0)$, when the former exists. The $b_n(\cdot)$ and the $q_n(\cdot,0)$ are also related by the inversion formula

$$q_n(t,0) = \sum_{j=0}^{N-n} (-1)^j \binom{n+j}{n} b_j(t).$$

More generally, we use $q_n(t,u)$ as a density function in the variable u with the heuristic meaning

$$q_n(t,u)du = \Pr\{N(t-0) = n \quad \text{and} \quad u < y(t) \leqq u + du\}.$$

We can now state

Theorem 1: The transition probabilities of $N(t-0)$ may be determined from the generating function formula

$$\begin{aligned} E\{x^{N(t-0)}\} = &\int_0^t \sum_n q_n(t-y,0)[P_y(x)]^{n+1-\delta_{Nn}}[1-A(y)]\,dy \\ &+ \int_t^\infty \sum_n q_n(0,y-t)[P_t(x)]^n \frac{1-A(y)}{1-A(y-t)}\,dy, \end{aligned} \tag{7}$$

where

$$P_u(x) = 1 + (x-1)e^{-\gamma u},$$

and the Laplace transforms of the binomial moments $b_n(\cdot)$ are given by

$$b_n^*(s) = (X_n)^{-1}\left\{b_0^* - \sum_{j=1}^{n}\left[\binom{N}{j-1} b_N^* - \frac{k_n^*}{a_n(s)}\right] X_{j-1}\right\},$$

$$b_0^*(s) = \frac{k_0^*}{1 - A^*(s)},$$

$$b_N^*(s) = \frac{\dfrac{k_0^*}{1-a_0(s)} + \displaystyle\sum_{j=1}^{N} \frac{X_{j-1}k_j^*(s)}{a_j(s)}}{\displaystyle\sum_{n=0}^{N} \binom{N}{n} X_n},$$

where

$$k_n^* = \text{Laplace transform of} \int_0^\infty \sum_{j=n}^N \binom{j}{n} q_j(0,u)\, e^{-n\gamma t} \frac{a(t+u)\, du}{1 - A(u)}.$$

The k_n^* introduce dependence on the boundary conditions at $t = 0$ expressed by the functions $q_n(0,u)$. The Kronecker symbol δ_{Nn} in (7) indicates that a call is lost if it finds all trunks busy.

To show how Theorem 1 can be used we shall compute the Laplace transforms of

$$\Pr\{N(t) = N \mid N(0) = m\}, \qquad m = 0, 1, \cdots, N,$$

in the important special case of Poisson arrivals at rate a, for which a great simplification of the formulas occurs. In this case,

$$A(t) = \begin{cases} 1 - e^{-at}, & t \geqq 0, \\ 0 \quad , & t < 0. \end{cases}$$

Also, we set γ = hang-up rate = 1, which amounts to measuring time in units of mean holding time; then

$$a_n(s) = \frac{a}{a + s + n}. \tag{8}$$

Our choice of the transition probability to the "all trunks busy" condition $\{N(t) = N\}$ as an example is not arbitrary; it turns out that, in many cases, including Poisson arrivals, the mean of $N(t)$ and the covariance depend only on the transition probability to the "boundary" condition $\{N(t) = N\}$. A similar situation occurs in the theory of queues with one server: the mean delay can be written as an integral of the probability of being on the "boundary," i.e., the chance that the server is idle.[14]

Since arrivals are Poisson, the $y(\cdot)$ process is in fact superfluous, and we may assume $N(0) = m, y(0) = 0$, so that

$$\begin{aligned} k_n^*(s) &= \int_0^\infty e^{-st-nt-at} a \binom{m}{n} dt = \binom{m}{n} \frac{a}{a + s + n}, & n \leqq m, \\ &= 0, & n > m. \end{aligned} \tag{9}$$

In formula (7) (Theorem 1) set $x = 1 + w$, and take Laplace transforms with respect to t; the coefficient of w^N is

$$\int_0^\infty e^{-st} \Pr\{N(t) = N \mid N(0) = m\}\, dt$$
$$= \int_C^\infty e^{-st-Nt-at}\, dt[q_N^* + q_{N-1}^* + \delta_{Nm}],$$

where $q_n^*(s)$ is the transform of $q_n(t,0)$. Now, from (6), (9) and (24) we find

$$q_N^* + q_{N-1}^* = \frac{q_N^* - \delta_{Nm} a_N(s)}{a_N(s)};$$

hence

$$\int_0^\infty e^{-st} \Pr \{N(t) = N \mid N(0) = m\} \, dt = \frac{q_N^*}{a}. \tag{10}$$

This result can also be obtained heuristically as follows:

$$\begin{aligned} a &= \text{total density of arrivals at } t \\ &= q_N(t,0) + a[1 - \Pr \{N(t - 0) = N\}]; \end{aligned}$$

taking Laplace transforms, we get (10).

From Theorem 1 and (10), we find

$$\frac{q_N^*}{a} =$$

$$a^{-1} \frac{\binom{m}{0} + \binom{m}{1} \frac{1 - a_0(s)}{a_0(s)} + \cdots + \binom{m}{m} \frac{[1 - a_0(s)] \cdots [1 - a_{m-1}(s)]}{a_0(s) \cdots a_{m-1}(s)}}{\frac{1 - a_0(s)}{a_0(s)} \sum_{n=0}^{N} \binom{N}{n} X_n}.$$

But, for our example, (8) implies

$$\frac{1 - a_n(s)}{a_n(s)} = \frac{s + n}{a};$$

hence, defining (after J. Riordan in the Appendix to Wilkinson[15]) the "sigma" functions $\sigma_k(m)$ by

$$\sigma_0(m) = \frac{a^m}{m!}, \qquad \sigma_k(m) = \sum_{j=0}^{m} \binom{k + j - 1}{j} \frac{a^{m-j}}{(m - j)!},\dagger$$

with m (but not k) an integer, we can show that $a^{-1} q_N^*$ reduces to

$$\int_0^\infty e^{-st} \Pr \{N(t) = N \mid N(0) = m\} \, dt = \frac{a^{N-m} n! \sigma_s(m)}{N! s \sigma_{s+1}(N)}. \tag{11}$$

This and similar results for Poisson arrivals have been found by S. O. Rice in unpublished work.

† The "sigma" functions are related to the Poisson-Charlier polynomials $p_n(x) = a^{n/2}(n!)^{\frac{1}{2}} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j! a^{-j} \binom{x}{j}$ by $\sigma_k(m) = (-a^{\frac{1}{2}})^m (m!)^{-\frac{1}{2}} p_m(-k)$. See Szegö.[16]

Since the event $\{N(t) = N\}$ (the "all trunks busy" condition) is of primary interest, the transition probability

$$p_{NN}(t) = \Pr\{N(t) = N \mid N(0) = N\}$$

has been used (e.g. by Kosten[17]) as a "recovery" or "relaxation" function that is characteristic of the dynamic behavior of the system, especially of its approach to equilibrium from the "all trunks busy" condition. Such a function has been computed from (11) and plotted as Fig. 1, for a (heavy) load of 10 erlangs offered to 5 trunks, giving a loss probability of 0.563.

IV. OVERFLOW TRAFFIC

In the design and engineering of trunking plans in telephony, it is common practice to offer the calls lost by one trunk group to a second or overflow group. It has been discovered that the right choices of group size and the pooling of overflow traffic can lead to efficient trunking arrangements, called *graded multiples*. For this reason, some theoretical work, as well as much empirical study, has been devoted to the statistical behavior of overflowing calls. The principal references are to Brockmeyer,[18] Cohen,[10] Kosten,[19] Palm,[2] Takács,[7,8,9] and Wilkinson.[15]

In accordance with current usage in mathematical literature, let us refer to a sequence of mutually independent, identically distributed, positive random variables as a *renewal process*. The interarrival times that we have assumed in the model describing the trunk group then form a renewal process. It has been shown by Palm[2] that, if calls arriving in

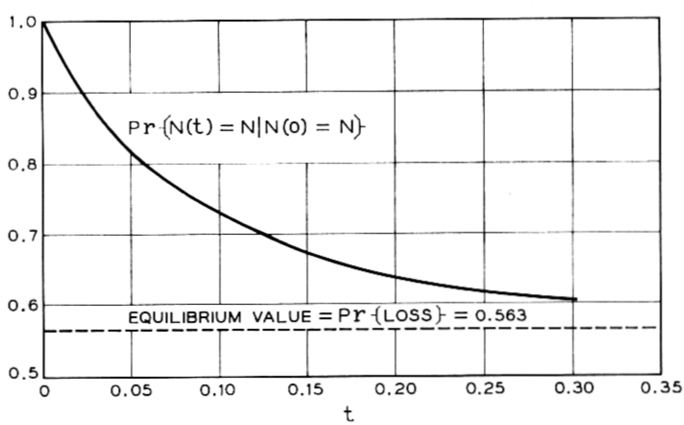

Fig. 1 — "Recovery function" $\Pr\{N(t) = N \mid N(0) = N\}$ for $N = 5$ trunks and $a = 10$ erlangs (heavy traffic).

a renewal process are served by a finite group of trunks, with exponential holding times and lost calls cleared, then the overflowing calls can also be described by a renewal process. That is, the time intervals between successive overflowing calls are mutually independent and identically distributed. Palm also showed how the distribution function of these interoverflow times can be calculated from the interarrival distribution, the hang-up rate and the group size.

We can deduce Palm's results in a simple way from our basic theorem and give a general formula for the Laplace-Stieltjes transform of the interoverflow distribution. Let $O_N(t)$ be the average number of overflows occurring in the closed interval $[0,t]$, assuming that an overflow occurred at time 0. Thus $O_N(t)$ is the particular form of $Q_N(t)$ that arises when $u_1 = 0$ and $N(0-) = N$. We use $G(u)$ to denote the distribution function of the interoverflow times. Since these times are independent, it can be seen that

$$O_N(t) = U(t) + \int_0^t O_N(t-u)\,dG(u), \qquad t \geqq 0, \tag{12}$$

where $U(t)$ is 1 for $t \geqq 0$, and 0 otherwise. If $O_N^*(s)$ and $G^*(s)$ are the respective Laplace-Stieltjes transforms of O_N and G, then (12) implies

$$G^*(s) = \frac{O_N^*(s) - 1}{O_N^*(s)},$$

which determines $G(u)$ uniquely if O_N^* is known.

Since, as noted, O_N^* is the particular case of Q_N^* arising when $u_1 = 0$ and $N(0-) = N$, a formula for it can be found from (32). In the particular case being considered

$$K_n^* = \binom{N}{n},$$

and so O_N^* is given by

$$\frac{1}{1 - A^*(s)} \cdot \frac{1 + \binom{N}{1}\dfrac{1 - a_0(s)}{a_1(s)} + \cdots + \binom{N}{N}\dfrac{[1 - a_0(s)] \cdots [1 - a_{N-1}(s)]}{a_1(s) \cdots a_N(s)}}{\displaystyle\sum_{n=0}^{N} \binom{N}{n} \frac{[1 - a_1(s)] \cdots [1 - a_n(s)]}{a_1(s) \cdots a_n(s)}}. \tag{13}$$

If $\mu_1 = \int_0^\infty x\,dA(x) < \infty$, the mean time between overflows is

$$\frac{\mu_1}{p_N},$$

where p_N is the equilibrium probability of loss (studied in Refs. 7 and 11).

For $N = 1$, (13) gives

$$O_1^* = \frac{1 - A^*(s) + A^*(\gamma + s)}{1 - A^*(s)}, \tag{14}$$

$$G^* = \frac{A^*(\gamma + s)}{1 - A^*(s) + A^*(\gamma + s)}. \tag{15}$$

Since $A^*(\gamma + s)$ is the Laplace-Stieltjes transform of

$$L_1(t) = \int_0^t e^{-\gamma u}\, dA(u),$$

(15) can be inverted to give

$$G(t) = \sum_{n=1}^{\infty} \varphi_n(t) = \{\mathfrak{F}(A)\}(t), \tag{16}$$

where

$$\varphi_1 = L_1,$$
$$\varphi_{n+1} = \varphi_n \bigstar (A - L_1)$$

and "★" denotes Stieltjes convolution.

Formula (13) agrees with the recurrence relation given by Palm[2] for the overflow distribution from N trunks. The "one-trunk" case of (14) through (16) is important theoretically because all other cases can be obtained from it by iteration. Formula (16) defines a mapping $G = \mathfrak{F}(A)$ and the interoverflow distribution for N trunks can be written as $\mathfrak{F}^N(A)$, the Nth iterate.

For one trunk, the first two moments of the interoverflow time u are

$$E\{u\} = \frac{\mu_1}{a_1},$$

$$E\{u^2\} = \frac{\mu^2}{a_1} + \frac{2\mu_1^2}{a_1^2}\left[1 - \frac{1}{\mu_1}\int_0^\infty te^{-\gamma t}\, dA(t)\right],$$

where $\mu_i = \int u^i\, dA$. In particular, the ratio of second to first moment is

$$\frac{E\{u^2\}}{E\{u\}} = \frac{\mu_2}{\mu_1} + \frac{2\mu_1}{a_1}\left[1 - \frac{1}{\mu_1}\int_0^\infty te^{-\gamma t}\, dA(t)\right],$$

so that the mapping $\mathfrak{F}$ always *increases* this ratio, by an amount proportional to $E\{u\}$.

APPENDIX A

Approach Using a Markov Process

Let $N(t)$ be the number of trunks in use at time t. To study the distribution of $N(t)$ we introduce the two-dimensional process $\{N(t),y(t)\}$, where $y(t)$ is the length of the time interval from t back to the last arrival epoch prior to t. We assume that $A(\cdot)$ is absolutely continuous, with a continuous density $a(\cdot)$.

The reason for using the two-dimensional variate is that, unless arrivals are Poisson, the $N(t)$ process by itself is not Markov. To avail ourselves of the functional equations satisfied by the distributions of Markov processes, we include $y(t)$ in the "state of the system." This inclusion does result in a Markov process. The device of "Markovization" by the inclusion of variables has been suggested and developed by Cox,[20] and also has been used by Takács.[7,8,9]

It is natural physically to think of the random functions $N(t)$ and $y(t)$ as being continuous from the right. However, we shall assume only that $y(t)$ is always defined to be equal to $y(t+0)$, and shall study the two processes $\{N(t+0),y(t+0)\}$ and $\{N(t-0),y(t+0)\}$ jointly.

That $N(t-0)$ and $N(t+0)$ are not the same process is clear: $N(t-0) = N$ and $y(t) = 0$ imply $N(t+0) = N$; but, if $N(t+0) = N$, $y(t) = 0$, then $N(t-0) = N$ or $N-1$ according as the call that just arrived is lost or accommodated. The analysis of $N(t-0)$ and $N(t+0)$ shall be carried out in terms of two sets of probability density functions, $p_n(t,y)$ and $q_n(t,y)$, where

$$p_n(t,y)dy = \Pr\{N(t+0) = n \quad \text{and} \quad y < y(t) \leqq y + dy\},$$

$$q_n(t,y)dy = \Pr\{N(t-0) = n \quad \text{and} \quad y < y(t) \leqq y + dy\}.$$

Lemma: $p_n(t,y) = q_n(t,y)$ for almost all y.

Proof: Let P be a basic probability measure determined by our assumptions (i) through (iv) of Section I; P is defined for sets of elements ω in a space Ω. We assume further that $N(t,\omega)$ is separable, so that

$$S_\epsilon = \bigcap_{0<u<\epsilon} \{\omega : N(t-u) = N(t+0) = N(t+u)\}$$

is a measurable set.

Now if $y(t) > \delta > \epsilon$, then $y(t - \epsilon) = y(t) - \epsilon$, almost surely, and

$$\Pr \{S_\epsilon \mid N(t + 0), y(t) > \delta\} \geqq e^{-2\gamma N\epsilon} \frac{1 - A(\delta + \epsilon)}{1 - A(\delta)}$$

independently of $N(t + 0)$, almost everywhere, so that

$$\Pr \{S_\epsilon \mid y(t) > \delta\} \geqq e^{-2\gamma N\epsilon} \frac{1 - A(\delta + \epsilon)}{1 - A(\delta)}.$$

The sets S_ϵ are monotone nondecreasing, so $S_0 = \lim S_\epsilon$ as $\epsilon \to 0$ is measurable, and

$$\Pr \{S_0 \mid y(t) > \delta\} = 1, \qquad \text{almost everywhere } [P], \tag{17}$$

and S_0 is the ω-set on which $N(t)$ is constant in some interval $(t - u, t+u)$. The lemma follows from (17).

It remains to establish the relationship between $p_n(t,y)$ and $q_n(t,y)$ when $y = 0$. From our previous remarks about $N(t - 0)$ and $N(t + 0)$ it can be seen that

$$\begin{aligned} p_N(t,0) &= q_N(t,0) + q_{N-1}(t,0), \\ p_n(t,0) &= q_{n-1}(t,0), \qquad 1 \leqq n \leqq N - 1, \\ p_0(t,0) &= 0. \end{aligned}$$

To formulate the Kolmogorov equations for $p_n(t,\cdot)$ and $q_n(t,\cdot)$, we need the function $\lambda(\cdot)$ defined by

$$\lambda(y) = \frac{a(y)}{1 - A(y)}, \qquad A(y) < 1.$$

This is the probability density that an interarrival time will end in the next dy, given that it has lasted a time y to date. The functions $q_n(t, \cdot)$, where $0 \leqq n \leqq N$, (with $q_{N+1} \equiv 0$), satisfy the difference-differential system

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \gamma n + \lambda(y)\right] q_n = \gamma(n + 1) q_{n+1}, \tag{18}$$

and the behavior of the densities $q_n(t, \cdot)$ for $y = 0$ is determined by the additional condition

$$q_n(t,0) = \int_0^\infty q_n(t,y)\lambda(y)\, dy. \tag{19}$$

We introduce the generating function

$$\psi(x,t,y) = \sum_{n=0}^{N} x^n q_n(t,y),$$

and from (18) obtain

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \gamma(x - 1)\frac{\partial}{\partial x} + \lambda(y)\right]\psi = 0, \tag{20}$$

whose general solution is

$$\psi(x,t,y) = K\{t - y, e^{-\gamma y}(x - 1)\}[1 - A(y)].$$

Before continuing, we note that the functions $p_n(t,y)$ also satisfy the system (18), but that the analog of (19) is

$$p_{n+1-\delta_{N,n}}(t,0) = \int_0^\infty p_n(t,y)\lambda(y)\,dy, \tag{21}$$

where the Kronecker δ symbol is used to indicate that an arriving call is lost if it finds all N trunks busy. The generating function $\varphi(x,t,y)$ of the $p_n(t,y)$ is also a solution of (20).

The function $\psi(x,t,\,\cdot)$ is y-continuous for $y > 0$, so, from the lemma proved previously, we conclude that $\psi(x,t,y) = \varphi(x,t,y)$ almost everywhere in y, and that

$$\lim_{y\to 0} \psi(x,t,y) = \varphi(x,t,0).$$

Because of the "lost calls cleared" assumption, we must have

$$\begin{aligned}\psi(x,t,0+) &= x\psi(x,t,0) - x^N(x - 1)q_N(t,0)\\ &= \varphi(x,t,0),\end{aligned}$$

so that ψ is discontinuous in y at $y = 0$.

Let $P_y = P_y(x)$ abbreviate $1 + (x - 1)e^{-\gamma y}$. It can be verified that the function $K(\cdot,\cdot)$ in the solution of (20) is given by

$$\begin{aligned}K(u, z) &= (1 + z)\psi(1 + z,u,0) - z(1 + z)^N q_N(u,0), &\quad t \geqq y,\\ &= \frac{\psi(1 + ze^{\gamma u},0,-u)}{1 - A(-u)}, & t < y,\end{aligned}$$

for the solution $\psi(x,t,y)$. From this we find that, for $t \geqq y$,

$$\psi(x,t,y) =$$

$$P_y\psi(P_y,t - y,0)[1 - A(y)] - (P_y - 1)P_y^N q_N(t - y,0)[1 - A(y)],$$

while, for $t < y$,

$$\psi(x,t,y) = \psi(P_t,0,y - t)\,\frac{1 - A(y)}{1 - A(y - t)}.$$

The solution for $\varphi(x,t,y)$ is analogous; in view of this and of the close relationship between φ and ψ, only ψ shall be treated from now on.

The function $\psi(x,0,y)$ represents initial conditions, and is considered as given. To find $\psi(x,t,0)$, we use the integral condition (19), and conclude that

$$\begin{aligned}\psi(x,t,0) = &\int_0^t \psi(P_y,t - y,0)P_y a(y)\,dy \\ &- \int_0^t (P_y - 1)P_y^N q_N(t - y, 0)a(y)\,dy \\ &+ \int_t^\infty \psi(P_t,0,y - t)\,\frac{a(y)\,dy}{1 - A(y - t)}.\end{aligned} \tag{22}$$

To solve the functional-integral equation (22), we set $x = 1 + w$, and equate coefficients of like powers of w. This yields

$$\begin{aligned}b_n(t) = &\int_0^t \Big[b_n(t - y) + b_{n-1}(t - y) \\ &- \binom{N}{n-1} b_N(t - y)\Big] e^{-n\gamma y}a(y)\,dy + k_n(t), \qquad n \geqq 0,\end{aligned} \tag{23}$$

where

$$b_n(t) = \sum_{j=n}^{N} \binom{j}{n} q_j(t,0),$$

$$k_n(t) = \int_0^\infty b_n(u)e^{-n\gamma t}\,\frac{a(t + u)\,du}{1 - A(u)}.$$

Note that

$$b_0(t) = \sum_{n=0}^{N} q_n(t,0) = \psi(1,t,0).$$

Let the Laplace transforms of $b_n(\cdot)$, $k_n(\cdot)$ be $b_n^*(s)$, $k_n^*(s)$, respectively. We obtain a simple recurrence for the b_n^* by applying the Laplace transformation to (23). The recurrence is

$$b_n^* = a_n(s)\left\{b_n^* + b_{n-1}^* - \binom{N}{n-1} b_N^*\right\} + k_n^*, \qquad n \geqq 0, \tag{24}$$

where

$$A^*(s) = \int_0^\infty e^{-su}\, dA(u),$$

$$a_n(s) = A^*(s + n\gamma).\dagger$$

To find b_0^*, let x approach 1 in (22); then $P_y(x)$ goes to 1, and we reach the following renewal equation for $b_0(t)$:

$$b_0(t) = \int_0^t b_0(t - y)a(y)\, dy + \int_0^\infty \frac{\psi(1,0,u)}{1 - A(u)}\, a(t + u)\, du. \quad (25)$$

It can be verified that the last term on the right of (25) is just $k_0(t)$; upon solving (25) by transforms, we find that $b_0^* = k_0^*/[1 - A^*]$.

It can be seen that $b_0(t)$ is the density of arrivals at the time t; thus $b_0(t)$ is a familiar function of renewal theory, for which the reader is referred to Smith[21] and the bibliography therein.

The solution of the recurrence (24) is

$$b_n^* = (X_n)^{-1}\left\{b_0^* - \sum_{j=1}^{n}\left[\binom{N}{j-1} b_N^* - \frac{k_n^*}{a_n(s)}\right] X_{j-1}\right\}, \quad (26)$$

where

$$X_0 = 1,$$

$$X_n = \frac{1 - a_n(s)}{a_n(s)}\, X_{n-1}.$$

In particular, the Laplace transform of the density (at t) of arrivals finding all trunks busy is given by

$$b_N^* = q_N^* = \int_0^\infty e^{-st} q_N(t,0)\, dt = \frac{\dfrac{k_0^*}{1 - a_0(s)} + \displaystyle\sum_{j=1}^{N} \frac{X_{j-1}k_j^*(s)}{a_j(s)}}{\displaystyle\sum_{n=0}^{N} \binom{N}{n} X_n}. \quad (27)$$

The generating function of distr $\{N(t - 0)\}$ is

$$\begin{aligned} E\{x^{N(t-0)}\} &= \int_0^\infty \psi(x, t, y)\, dy \\ &= \int_0^t \sum_n q_n(t - y,0)[P_y(x)]^{n+1-\delta_{Nn}}[1 - A(y)]\, dy \quad (28) \\ &+ \int_t^\infty \sum_n q_n(0,y - t)[P_t(x)]^n \frac{1 - A(y)}{1 - A(y - t)}\, dy. \end{aligned}$$

† The functions $a_n(s)$ are to be distinguished from the constants a_n of Ref. 11, which use the same model and notation. The two quantities are related by $a_n = A^*(n\gamma) = a_n(0)$.

APPENDIX B

The Stationary Distribution of $\{N(t),y(t)\}$

We now consider which initial distributions $q_n(0,u)$ for $\{N(0+), y(0+)\}$ are stationary, i.e., are invariant under the transition probabilities of the Markov process $\{N(t-0),y(t)\}$, studied in Appendix A. Intuitively, since we show in Theorem 3 of Appendix C that a limiting distribution exists as $t \to \infty$, we expect this limit to give the stationary distribution. This is the content of

Theorem 2: If $A(u)$ has a continuous derivative and $\mu_1 < \infty$, the x,u function

$$\sum_n p_n P_u^{\,n+1-\delta_{Nn}}(x) \frac{1 - A(u)}{\mu_1}, \qquad u \geqq 0, \quad (29)$$

generates the unique stationary distribution of $\{N(0+),y(0)\}$; (29) is a generating function in x and a probability density in u. The number p_n is the equilibrium probability that an arriving customer find n trunks busy.

To show that (29) generates a stationary distribution, it is sufficient to prove that the choice of (29) for the initial condition makes each $q_n(t,0) = p_n/\mu_1$ for all t. This is equivalent to

$$q_n^*(s) = \frac{p_n}{s\mu_1},$$

or to

$$b_n^*(s) = \frac{b_n}{s\mu_1},$$

with

$$b_n = \sum_n^N \binom{j}{n} p_j .$$

In order to use the recurrence (24) and the formula (27) for q_N^*, we must first calculate the quantities k_n^* imposed by (29). Now $k_n(t)$ is the nth binomial moment associated with the generating function

$$\int_t^\infty \psi(P_t,0,y-t) \frac{dA(y)}{1 - A(y-t)}$$

for $\psi(x,0,u)$ given by (29). Thus, $k_n(t)$ is associated with

$$\mu_1^{-1} \sum_n \int_t^\infty p_n P_{y-t}^{\,1+n-\delta_{Nn}}[P_t(x)]\, dA(y).$$

This is equal to

$$\mu_1^{-1} \sum_n \int_0^\infty p_n P_{u+t}{}^{1+n-\delta_{Nn}}(x)\, dA(u),$$

and so, for $n > 0$,

$$k_n(t) = \mu_1^{-1} \left\{ b_n + b_{n-1} - \binom{N}{n-1} p_N \right\} \int_t^\infty e^{-n\gamma u}\, dA(u).$$

The Laplace transform of this is

$$k_n^* = \left\{ b_n + b_{n-1} - \binom{N}{n-1} p_N \right\} \frac{a_n - a_n(s)}{s\mu_1}, \qquad n > 0.$$

For $n = 0$,

$$k_0(t) = \frac{1 - A(t)}{\mu_1},$$

$$k_0^* = \frac{1 - A^*(s)}{s\mu_1} = \frac{1 - a_0(s)}{s\mu_1}.$$

We now note that, for these k_n^*, the condition $b_N^* = p_N/s\mu_1$ implies $b_n^* = b_n/s\mu_1$ for all lower n. This can be proved by induction from (24). We now substitute these k_n^* in (27) for q_N^* ($= b_N^*$). If we divide out a factor $[1 - a_0(s)]$ in the numerator, the first term is $1/s\mu_1$; the general term is

$$\left[b_n + b_{n-1} - \binom{N}{n-1} p_N \right] \frac{[1 - a_1(s)] \cdots [1 - a_{n-1}(s)][a_n - a_n(s)]}{(s\mu_1)a_1(s) \cdots a_n(s)}.$$

Using the recurrence of Ref. 11,

$$b_n = a_n \left[b_n + b_{n-1} - \binom{N}{n-1} p_N \right], \qquad n > 0,$$

we find after much algebra that $q_N^* = p_N/s\mu_1$, which proves the theorem. The stationary value p_N/μ_1 for the density $q_N(t,0)$ has the following physical interpretation: $1/\mu_1$ is the equilibrium density of arrivals, and p_N is the chance that such an arrival find all trunks busy.

The uniqueness of the stationary distribution follows from that of the limiting distribution as $t \to \infty$. For two distinct stationary distributions of necessity give rise to distinct limits, contradicting Theorem 4 of Appendix C.

The analog of Theorem 2 for the more general formulation of Appendix C is proved by the same form of argument that established Theorem 2, with the difference that Laplace-Stieltjes transforms are involved, and that special mention must be made of the "periodic" case, in which $A(\cdot)$ is concentrated on a lattice.

APPENDIX C

Approach Via Renewal Theory and Regenerative Processes

This last appendix is a quick sketch of results for general distributions $A(\cdot)$; no proofs are given.

Smith[13] has defined a *regenerative* stochastic process $x(t)$ as one for which there is an event R such that, if R occurs at t, then knowledge of $x(s)$ for $s < t$ loses all prognostic value, and the future development of $x(\tau)$ for $\tau > t$ depends only on the fact that R occurred at t. The points at which R occurs are called *regeneration* points of the process.

Let R_n denote the event: an arriving customer finds n trunks busy. Since the interarrival times form a renewal process, each point in time at which R_n occurs is a regeneration point of $N(t - 0)$, for all $0 \leqq n \leqq N$. In fact, we have already[11] made use of this property of the arrival process in constructing the imbedded Markov chain.

We are therefore in a position to use Smith's results[13] directly. The regenerative property of R_n implies that the time intervals between successive occurrences of R_n form a renewal process, i.e., a sequence of independent, identically distributed variates. To apply the results of Ref. 13 we must investigate whether these variates are proper, i.e., finite almost surely, and whether they have finite expectations. We content ourselves with the following result:

Theorem 3: Let $x_{m,n}$ be the time elapsing from an occurrence of R_m until the next occurrence of R_n. Then

$$x_{m,n} < \infty \text{ with probability 1,}$$

and, if the mean interarrival time $\mu_1 = \int x\, dA < \infty$, then

$$E\{x_{m,n}\} < \infty.$$

We use the following notations:

$$u_i = \text{the } i\text{th interarrival time, } i = 1, 2, 3, \cdots,$$

$$r_i(m) = \text{the time interval between the } (i-1)\text{th and the } i\text{th occurrences of } R_m,$$

$$U_k = \sum_1^k u_i = \text{the epoch of the } k\text{th arrival,}$$

$$X_k(m) = \sum_1^k r_i(m) = \text{the epoch of the } k\text{th occurrence of the event } R_m,$$

$$T_t = \text{the epoch of the last arrival prior to } t \text{ and after } 0.$$

The u_i all have the common distribution $A(u)$, except for u_1, which has G. Also, the $r_i(m)$ have a common distribution, except for $r_1(m)$.

During the interval (T_t, t), the process $N(x - 0)$ forms a pure death process whose transition probabilities $P_{m,n}(\cdot)$ are known. Let $U(x)$ be the unit step function at 0 and δ_{mN} the Kronecker delta. The probability that $N(t - 0) = n$ can be represented as

$$\Pr\{N(t-0) = n\} = E\{p_{N(0+),n}(t)U(u_1 - t)\}$$

$$+ \sum_{n-1 \leqq m \leqq N} \int_0^t p_{m+1-\delta_{mN},n}(t-u)\, d_u \Pr\{T_t \leqq u \text{ and } N(T_t - 0) = m\},$$

where the measure implicit in the E operation is the joint distribution of $N(0+)$ and u_1. With the notations just introduced in mind, it can be verified that

$$\Pr\{T_t < u \quad \text{and} \quad N(T_t - 0) = m\} = \sum_{k=1}^{\infty} \Pr\{T_t = X_k(m) \leqq u\}$$

$$= \int_0^u [1 - A(t-v)]\, d_v \sum_k \Pr\{U_k \leqq v \quad \text{and} \quad N(U_k - 0) = m\}$$

$$= \int_0^u [1 - A(t-v)]\, d_v \sum_k \Pr\{X_k(m) \leqq v\},$$

the series being absolutely convergent. By introducing

$$Q_n(t) = \sum_k \Pr\{U_k \leqq t \quad \text{and} \quad N(U_k - 0) = n\},$$

we can write

$$\Pr\{N(t-0) = n\} = E\{p_{N(0+),n}(t)U(u_1 - t)\}$$

$$+ \sum_{n-1 \leqq m \leqq N} \int_0^t p_{m+1-\delta_{mN},n}(t-v)[1 - A(t-u)]\, dQ_m(v).$$

This representation has been used by Takács[7] to study lim Pr $\{N(t - 0) = m\}$ as $t \to \infty$ by methods similar to those used in the proof of Theorem 4.

We can now describe directly some conditions under which Pr $\{N(t -$

$0) = n\}$ goes to a limiting distribution as $t \to 0$, independent of initial conditions. The result is due essentially to Takács.[7]

Theorem 4: If $A(u)$ is not periodic, if $u_1 < \infty$ almost surely, if $\mu_1 = E\{u_i\} < \infty, i > 1$, then

$$\lim_{t\to\infty} \Pr\{N(t-0) = n\} = \sum_{n-1\leqq m\leqq N} \int_0^\infty p_{n+1-\delta_{mN},n}(u) \frac{[1 - A(u)]}{E\{y_{m,m}\}} du.$$

This result follows at once from the previous results of this section and Theorem 2 of Smith,[13] upon noting that $p_{n,k}(u)$ is a linear combination of monotone decreasing functions. From Theorem 3 of Smith[21] there also follows

Theorem 5: If $A(u)$ has period p, if $u_1 < \infty$ almost surely, if $\mu_1 = E\{u_i\} < \infty, i > 1$, and $0 \leqq y < p$, then

$$\lim_{k\to\infty} \Pr\{N(kp + y - 0) = n\} = \sum_{n-1\leqq m\leqq N} \sum_{k\geqq 0} p_{m+1-\delta_{mN},n}(kp + y) \frac{[1 - A(kp + y)]}{E\{x_{m,m}\}}.$$

We now derive and solve equations for the quantities

$$Q_m(u) = \sum_k \Pr\{U_k \leqq u \quad \text{and} \quad N(U_k - 0) = m\},$$

which occur in the representation for the probability $\Pr\{N(t-0) = n\}$. Using the generating variable x and the abbreviation

$$P_y(x) = 1 + (x-1)e^{-\gamma y},$$

we find that

$$\sum_{n=0}^N x^n \Pr\{U_{k+1} \leqq t \quad \text{and} \quad N(U_{k+1} - 0) = n\} =$$

$$\sum_m \int_0^t \int_0^{t-u} P_y^{m+1-\delta_{mN}}(x)\, dA(y)\, d_u \Pr\{U_k \leqq u \quad \text{and} \quad N(U_k - 0) = m\}.$$

The series formed by adding all these equations up on the index k are absolutely convergent; hence no additional generating functions are needed, and we reach the equation:

$$\begin{aligned} \sum_{n=0}^N x^n \sum_{k=1}^\infty \Pr\{U_k \leqq t \quad &\text{and} \quad N(U_k - 0) = n\} = \\ \sum_{n=0}^N x^n \Pr\{u_1 \leqq t \quad &\text{and} \quad N(u_1 - 0) = n\} \\ + \sum_m \int_0^t \int_0^{t-u} P_y^{m+1-\delta_{mN}}(x)\, dA(y)\, d_u \sum_k \Pr\{U_k \leqq u \quad &\text{and} \quad N(U_k - 0) = m\}. \end{aligned} \tag{30}$$

This is an integral-functional equation for the function

$$\Psi(x,t) = \sum_n x^n \sum_k \Pr\{U_k \leqq t \quad \text{and} \quad N(U_k - 0) = n\},$$

which is closely related to the function $\psi(x,t,0)$ treated in Appendix A. In fact, when Ψ is absolutely continuous in t, then ψ is its derivative, and (22) is similar to (30) in the special case where the density ψ exists.

Equation (30) may be solved by the same method as (22), except that Laplace-Stieltjes transforms replace the ordinary Laplace integrals used for (22). We introduce the following notations:

$$Q_n(t) \quad \text{for} \quad \sum_k \Pr\{U_k \leqq t \quad \text{and} \quad N(U_k - 0) = n\},$$

$$B_n(t) \quad \text{for} \quad \sum_{j=n}^{N} \binom{j}{n} Q_j(t),$$

$$K_n(t) \quad \text{for} \quad \sum_{j=n}^{N} \binom{j}{n} \Pr\{u_1 \leqq t \quad \text{and} \quad N(u_1 - 0) = j\}.$$

When each of Q_n, B_n and K_n is absolutely continuous, the corresponding (lower case) quantities $q_n(t,0)$, $b_n(t)$ and $k_n(t)$ are the respective derivatives. Let the respective Laplace-Stieltjes transforms of Q_n, B_n and K_n be Q_n^*, B_n^* and K_n^*. Then (30) leads to the recurrence

$$B_n^* = a_n(s)\left\{B_n^* + B_{n-1}^* - \binom{N}{n-1} B_N^*\right\} + K_n^*. \tag{31}$$

The rest of the solution is in complete analogy with the solution for the b_n^*, q_n^* in Appendix A. We find

$$B_N(t) = Q_N(t),$$

$$B_0(t) = \sum_0^N Q_n(t) = \Psi(1,t).$$

The function $\Psi(1,t)$ satisfies the renewal equation

$$\Psi(1,t) = \int_0^t \Psi(1,t - y)\, dA(y) + G(t),$$

where $G = \text{distr}\{u_1\}$. The Laplace-Stieltjes transform of $\Psi(1,t)$ is

$$B_0^* = \int_0^\infty e^{-st}\, d\Psi(1,t) = \frac{\int_0^\infty e^{-st}\, dG(t)}{1 - A^*(s)} = \frac{K_0^*}{1 - A^*(s)}.$$

The solution of the recurrence (31) is

$$B_n^* =$$

$$\prod_0^n \frac{a_n(s)}{1 - a_n(s)} \left\{B_0^* - \sum_{j=1}^n \left[\binom{N}{j-1} B_N^* - \frac{K_j^*}{a_j(s)}\right] \prod_0^{j-1} \frac{1 - a_i(s)}{a_i(s)}\right\},$$

where the first term of the products is taken to be one. The Q_n^* are given in terms of the B_n^* by the equation

$$Q_n^* = \sum_{j=0}^{N-n} (-1)^j \binom{n+j}{n} B_{n+j}^*.$$

In particular, the analog of (27) is

$$B_N^* = Q_N^* = \int_0^\infty e^{-st}\, dt \sum_k \Pr\{U_k \leqq t \quad \text{and} \quad N(U_k - 0) = N\}$$

$$= [1 - A^*(s)]^{-1} \tag{32}$$

$$\cdot \frac{K_0^* + K_1^* \dfrac{[1 - a_0(s)]}{a_1(s)} + \cdots + K_N^* \dfrac{[1 - a_0(s)] \cdots [1 - a_{N-1}(s)]}{a_1(s) \cdots a_N(s)}}{1 + \dbinom{N}{1} \dfrac{1 - a_1(s)}{a_1(s)} + \cdots + \dbinom{N}{N} \dfrac{[1 - a_1(s) \cdots [1 - a_N(s)]}{a_1(s) \cdots a_N(s)}}.$$

From the representation of $\Pr\{N(t - 0) = n\}$ it can be seen that the generating function of $N(t - 0)$ is

$$E\{x^{N(t-0)}\} = E\{P_t^{N(0+)}(x) U(u_1 - t)\}$$

$$+ \sum_n \int_0^t P_{t-u}^{1+n-\delta_{nN}}(x)[1 - A(t - u)]\, dQ_n(u),$$

with $P_t(x) = 1 + (x - 1)e^{-\gamma t}$, and U the unit step at zero. It follows that the Laplace transform (with respect to t) of the generating function of $N(t - 0)$ is

$$\int_0^\infty e^{-st} E\{x^{N(t-0)}\}\, dt = \int_0^\infty e^{-st} E\{P_t^{N(0+)}(x) U(u_1 - t)\}\, dt$$

$$+ \sum_n Q_n^*(s) \int_0^\infty e^{-sy} P_y^{n+1-\delta_{nN}}(x)[1 - A(y)]\, dy.$$

When $\lim E\{x^{N(t-0)}\}$ exists as $t \to \infty$, we can use Abel's theorem for the Laplace transform to evaluate the limits lim Pr $\{N(t - 0) = n\}$ explicitly. As $s \to 0$,

$$\lim sQ_n^*(s) = \lim \frac{sG_n^*(s)}{1 - F_{n,n}^*(s)}$$

$$= \frac{1}{E\{x_{n,n}\}}.$$

But from (32) we find

$$\lim sQ_n(s) = \frac{p_n}{\mu_1}$$

$$= \frac{p_n}{\int_0^\infty x\,dA(x)},$$

where p_n is the equilibrium probability that an arriving customer finds n trunks busy. (These probabilities have been studied in Takács[7] and Beneš,[11] *inter alia.*) Hence

$$E\{x_{n,n}\} = \text{mean recurrence time of } R_n$$

$$= \frac{\int_0^\infty x\,dA(x)}{p_n} = \frac{\mu_1}{p_n},$$

and, from Theorem 3,

$$\lim_{t\to\infty} \Pr\{N(t - 0) = n\} = \sum_{n-1 \leqq m \leqq N} p_m \int_0^\infty p_{m+1-\delta_{m,N},n}(u)\, \frac{[1 - A(u)]}{m_1}\, du$$

$$\lim_{t\to\infty} E\{x^{N(t-0)}\} = \sum_n p_n \int_0^\infty P_y{}^{n+1-\delta_{N,n}}(x)\, \frac{[1 - A(y)]}{m_1}\, dy,$$

with

$$P_y(x) = 1 + (x - 1)e^{-\gamma y}.$$

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