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# Proving Theorems by Pattern Recognition — II

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*Theoretical questions concerning the possibilities of proving theorems by machines are considered here from the viewpoint that emphasizes the underlying logic. A proof procedure for the predicate calculus is given that contains a few minor peculiar features. A fairly extensive discussion of the decision problem is given, including a partial solution of the* $(x)(Ey)(z)$ *satisfiability case, an alternative procedure for the* $(x)(y)(Ez)$ *case, and a rather detailed treatment of Skolem's case. In connection with the* $(x)(Ey)(z)$ *case, an amusing combinatorial problem is suggested in Section 4.1. Some simple mathematical examples are considered in Section VI.*

*Editor's Note.* This is in form the second and concluding part of this paper' Part I having appeared in another journal.[1] However, an expansion of the author's original plan for Part II has made it a complete paper in its own right.

## I. A SURVEY OF THE DECISION PROBLEM

### 1.1 *The Decision Problem and the Reduction Problem*

With regard to any formula of the predicate calculus, we are interested in knowing whether it is a theorem (the problem of provability), or equivalently, whether its negation has any model at all (the problem of satisfiability). Originally this decision problem was directed to the search for one finite procedure which is applicable to all formulae of the predicate calculus. Since it is known that there can be no such omnipotent procedure, the main problem is to devise procedures effective for classes of formulae which satisfy suitable conditions.

The complementary problem of reduction is to give effective procedures which reduce broader classes to narrower ones while preserving provability or satisfiability. In this way, a decision procedure for a smaller class can be made to apply to a larger one. Thus far, most work on the reduction problem has been directed to the special case of finding procedures which reduce all formulae of the predicate calculus to members of some special class (e.g., those in the Skolem normal form). Each such class is called a reduction class relative to satisfiability or provability according to whether satisfiability or provability is preserved by the transformations (Ref. 2, p. 32). It follows automatically that the corresponding decision problem for each reduction class is unsolvable.

The reduction classes and the procedures employed to obtain them are, being concerned with undecidable cases, only of indirect use for the problem of discovering positive results on the decision problem. More directly relevant are reduction procedures which are applicable when the reduced class is not a reduction class and may in particular be a decidable class. Some very preliminary results on this more general aspect of the reduction problem will be described in Section V.

For both the decision problem and the reduction problem, there is, beyond the "yes or no" as to satisfiability, a further question of determining all models and devising transformation procedures which preserve all models. Such questions have been studied to a certain extent (Ref. 3, p. 23), but will be disregarded in what follows.

It is customary to characterize reduction classes and decidable classes in terms of formulae in the prenex normal form, i.e., with all quantifiers at the beginning. Sometimes, with regard to satisfiability (or provability), conjunctions (or disjunctions) of formulae in the prenex normal form are considered. We shall call this the extended prenex form.

In Section V, a procedure will be given for reducing any formula to a finite set of generally simpler formulae in the extended prenex form such that the original formula is provable if and only if all formulae in the reduced set are. In this and the next few sections, we shall only be concerned with formulae in the extended prenex form. Furthermore, we shall give in Section V a proof-decision procedure for the quantifier-free logic, obtained from the propositional calculus by adding equality, function symbols and individual constants. Any theorem in it is called a quantifier-free tautology, as an extension of the notion of a propositional tautology. We shall make use of the fact that we can always decide whether a given formula is a quantifier-free tautology.

## 1.2 *A Brief Formulation of the Predicate Calculus*

### 1.2.1 *Primitive Symbols*

1.2.1.1 Variables $x$, $y$, $z$, etc. (an infinite set).
1.2.1.2 Individual constants (a finite or infinite set).
1.2.1.3 Propositional (Boolean) operations: $\sim$, $\vee$, &, $\supset$, $\equiv$.
1.2.1.4 Predicate letters (a finite or infinite set).
1.2.1.5 Function letters (a finite or infinite set).
1.2.1.6 Equality: $=$ (a special predicate symbol).
1.2.1.7 Quantification symbols: ( ), ($E$ ).
1.2.1.8 Parentheses.

### 1.2.2 *Inductive Definition of Terms and Formulae*

1.2.2.1 A variable or an individual constant is a term.

1.2.2.2 A function symbol followed by a suitable number of terms is a term.

1.2.2.3 A predicate followed by a suitable number of terms is a formula (and an atomic formula); in particular, if $\alpha$, $\beta$ are terms $=(\alpha,\beta)$ or $\alpha = \beta$ is a formula (and an atomic formula).

1.2.2.4 If $\varphi$, $\psi$ are formulae and $\alpha$ is a variable, then $(\alpha)\varphi$, $(E\alpha)\varphi$, $\sim\varphi$, $\varphi \vee \psi$, $\varphi \,\&\, \psi$, $\varphi \supset \psi$, $\varphi \equiv \psi$ are formulae.

### 1.2.3 *Inductive Definition of Theorems*

1.2.3.1 A quantifier-free tautology is a theorem.

1.2.3.2 If a disjunction $D$ of $n$ alternatives is a theorem, $\varphi\alpha$ is one of the alternatives and $\beta$ is a variable, then:

(a) If $\alpha$ is a term, then the result of replacing $\varphi\alpha$ by $(E\beta)\varphi\beta$ in $D$ is a theorem;

(b) if $\alpha$ is a variable free in $\varphi\alpha$ but not free in the other alternatives and $\beta$ is $\alpha$ or does not occur in $\varphi\alpha$, then the result of replacing $\varphi\alpha$ by $(\beta)\varphi\beta$ in $D$ is a theorem.

1.2.3.3 If $\varphi \vee \cdots \vee \varphi$ is theorem, so is also $\varphi$.

The above formulation is complete only with respect to formulae in the extended prenex form.

## 1.3 *The Fundamental Theorem of Logic*

The main purpose of the next few sections is to study the decision problem on the theoretical foundation of the fundamental theorem of logic, an approach initiated by Skolem[4] and Herbrand,[5] and recently revived by Church,[6,7] and by Klaua[8] and Dreben.[9,10]

Suppose $Mxyz$ is a quantifier-free matrix:

1.3.1 $$(x)(Ey)(z)Mxyz,$$

1.3.2 $$(Ex)(y)(Ez) \sim Mxyz.$$

Let now $D_n$ be $M_1 \vee \cdots \vee M_n$ and $M_i$ be $M1ii'$, $i'$ being an abbreviation for $i + 1$. The fundamental theorem, when applied to 1.3.1, states:

1.3.3 The following three conditions are equivalent:
(a) 1.3.1 is a theorem of the predicate calculus; (b) for some $n$, $D_n$ is a quantifier-free tautology; (c) 1.3.2 is not satisfiable.

If $D_n$ is a quantifier-free tautology, then, by 1.2.3.1, both it and the result of substituting distinct variables for distinct numbers in it are theorems. For example, suppose the result is:

1.3.4 $$Maab \vee Mabc \vee Macd.$$

We have: by 1.2.3.2(b),

$$Maab \vee Mabc \vee (z)Macz;$$

by 1.2.3.2(a),

$$Maab \vee Mabc \vee (Ey)(z)Mayz.$$

Similarly,

$$Maab \vee (Ey)(z)Mayz \vee (Ey)(z)Mayz,$$
$$(Ey)(z)Mayz \vee (Ey)(z)Mayz \vee (Ey)(z)Mayz,$$

by 1.2.3.3,

$$(Ey)(z)Mayz;$$

by 1.2.3.2(b),

$$(x)(Ey)(z)Mxyz.$$

Hence, condition (b) implies conditions (a) and (c) in 1.3.3.

On the other hand, if no $D_n$ is a quantifier-free tautology, then there is, for each $D_n$, some interpretation of the function and predicate symbols on the set $\{1, \cdots, n'\}$ which satisfies $\sim D_n$. By a well-known argument, there is then an interpretation on the domain of all positive integers which satisfies $\sim D_1$, $\sim D_2$, etc. simultaneously. This, however, means that under the interpretation each finite segment of the infinite conjunction

1.3.5 $$\sim M112 \;\&\; \sim M123 \;\&\; \sim M134 \;\&\; \cdots$$

is true. But then there is an integer $x$, viz. 1, such that for every integer $y$, there is an integer $z$, viz. $y'$, such that $\sim Mxyz$. In other words, 1.3.2, the negation of 1.3.1, is true under the interpretation. Hence, the negation of condition (b) implies the negations of conditions (a) and (c).

If we take 1.3.5 as a model of 1.3.2, it seems natural to regard $y$ as an independent variable, $z$ as a dependent variable and $x$ as an initial variable (the limiting case of a dependent variable, a function of zero arguments). The general principle of constructing $M_n$ from 1.3.1 may be summarized by saying that each initial variable gets a constant number, the independent variables taking on all possible positive integers as values and the dependent variables always taking on numbers not used before.

In the general case, we must consider a disjunction (for provability) or conjunction (for satisfiability) of formulae with arbitrary strings of quantifiers. Then we can again construct the related quantifier-free formulae in the same way, with the numbers in each clause proceeding independently.

Thus, if we wish to study the satisfiability problem, we consider any formula of the form:

1.3.6 $$\varphi_1 \;\&\; \cdots \;\&\; \varphi_n \qquad (n \geqq 1),$$

where each $\varphi_i$ is of the form, with $d_1 \geqq 0$, $e_c \geqq 0$, $c \geqq 1$, $e_1, d_2, e_2, \cdots, d_c \geqq 1$:

1.3.7 $$(Ey_1^1) \cdots (Ey_{d_1}^1)(x_1^1) \cdots (x_{e_1}^1) \cdots (Ey_1^c) \cdots$$
$$(Ey_{d_c}^c)(x_1^c) \cdots (x_{e_c}^c) M y_1^1 \cdots x_{e_c}^c.$$

One familiar way of obtaining $M_1$, $M_2$, etc. for the formula 1.3.7 begins by replacing the dependent variables (those with the letter $y$) each with a function (sometimes called a "Skolem function") of all the preceding independent variables (those with the letter $x$), and then dropping all the quantifiers. Let the result be $M^*$. In particular, the initial (dependent) variables are replaced by distinct constants which may be viewed as trivial functions. Suppose $e_1 + \cdots + e_c = p$, $d_1 + \cdots + d_c = q$ in 1.3.7.

The Skolem functions are any functions $g_1, \cdots, g_q$ which, taken together, satisfy the following conditions:

1.3.8 (a) For each $g_i$, $g_i(u_1, \cdots, u_m) \neq u_j$, $j = 1, \cdots, m$, $i = 1, \cdots, q$.

(b) For each $g_i$, $g_i(u_1, \cdots, u_m) = g_i(v_1, \cdots, v_m)$ only when $u_1 = v_1, \cdots, u_m = v_m$.

(c) For any $g_i, g_j, i \neq j$, $g_i(u_1, \cdots, u_m) \neq g_j(v_1, \cdots, v_n)$, for all $u_1, \cdots, u_m, v_1, \cdots, v_n$.

Then we can take the smallest domain which contains the constants for the initial (dependent) variables (or an arbitrary constant when there is no such initial variable) and is closed with respect to the Skolem functions. Once such an (enumerable) domain is available, we can somehow enumerate all the $p$-tuples of members of the domain. Then, for each $i$, $M_i$ is simply the result obtained from $M^*$ when the independent variables are replaced respectively by members of the $i$th $p$-tuple.

The satisfiability problem of 1.3.7 is then reduced to that of the infinite conjunction:

1.3.9 $$M_1 \;\&\; M_2 \;\&\; \cdots$$

Similarly, the satisfiability problem of 1.3.6 can be handled by reducing each $\varphi_i$ separately and then taking the conjunction of the $n$ infinite conjunctions of the form 1.3.9.

It is customary to use the positive integers as the domain, fix some enumeration of the $p$-tuples, and specify the Skolem functions in a natural manner. One familiar enumeration of the $p$-tuples is the following:

1.3.10 $(a_1, \cdots, a_p)$ precedes $(b_1, \cdots, b_p)$. if either

(a) they are permutations of each other but $(a_1, \cdots, a_p)$ precedes $(b_1, \cdots, b_p)$ in the lexicographic order; or

(b) $\max(a_1, \cdots, a_p) = \max(b_1, \cdots, b_p)$, $\Sigma a_i = \Sigma b_i$, but $(a_1, \cdots, a_p)$, rearranged according to nondecreasing magnitude, precedes $(b_1, \cdots, b_p)$, similarly rearranged, in the lexicographic order; or

(c) $\max(a_1, \cdots, a_p) = \max(b_1, \cdots, b_p)$, but $\Sigma a_i < \Sigma b_i$; or

(d) $\max(a_1, \cdots, a_p) < \max(b_1, \cdots, b_p)$.

The Skolem functions are usually chosen by going through the infinite conjunction 1.3.9 from left to right and using each time the smallest unused integer for the next functional expression not yet evaluated. Thus, e.g., $y_1^1, \cdots, y_{d_1}^1$ in 1.3.7 get the constant values $1, \cdots, d_1$, and $M_1$ is:

$$M1 \cdots d_1^1 1 \cdots 1 d_1' \cdots (d_1 + d_2) \cdots (q - d_c + 1) \cdots q1 \cdots 1.$$

Each time a functional expression gets a value, the value is substituted in all later occurrences of the same expression.

In this way we arrive at a form of the fundamental theorem of logic as a generalization of 1.3.3.

It is natural to observe that the infinite conjunction 1.3.9 can be divided into sections (Ref. 4, p. 138):

1.3.11 The first section is the set of those $M_i$'s in which the $p$-tuples replacing the independent variables are made up of integers in the set $\{1, \cdots, d_1\}$, or the set $\{1\}$ if $d_1 = 0$; the $(n + 1)$th section is the set of those $M_i$'s not belonging to the $n$th section in which the $p$-tuples are made up of integers which occur in the union of the first $n$ sections.

This notion has been used by Skolem in explaining some decision procedures (see Section II below).

### 1.4 *Special Cases of the Decision Problem*

The principal known decidable classes are, with regard to satisfiability the following:

I. *The monadic case.* The class of all formulae which contain only monadic predicate letters and no function symbols.

II. *The EA satisfiability case (the AE provability case).* The class of all formulae in the prenex form with prefixes of the form $(Ey_1) \cdots (Ey_m)(x_1) \cdots (x_n)$, $m$, $n \geqq 0$, and no function symbols [or the form $(y_1) \cdots (y_m)(Ex_1) \cdots (Ex_n)$ for provability].

III. *The conjunctive satisfiability case.* Every formula in the prenex form with a matrix which is a conjunction of atomic formulae and their negations. (Equivalently, the disjunctive provability case.)

IV. *The Skolem case.* Every formula in the prenex form with no function symbols such that it has a prefix ending with $(Ey_1) \cdots (Ey_n)$, $n > 0$, and every atomic formula occurring in the matrix contains either one of the variables $y_1, \cdots, y_n$, or all the independent variables. [For provability, $(y_1) \cdots (y_n)$ at the end.]

V. *The $EA_2E$ satisfiability case (the $AE_2A$ provability case).* Every formula containing no function symbols in the prenex form with a prefix $(Ey_1^1) \cdots (Ey_m^1)(x_1)(x_2)(Ey_1^2) \cdots (Ey_n^2)$.

VI. *The Ackermann case.* For satisfiability, every formula which contains no function symbols, no equality sign, only a single dyadic predicate ($G$ say), and has the form $(x)(Ey)Gxy$ & $(x_1) \cdots (x_m)Mx_1 \cdots x_m$, $m \leqq 4$, $M$ quantifier-free.

In addition to these, two other cases may be mentioned:

VII. *The $A_1E_1A_1$ satisfiability case.* Every formula with the prefix $(x_1)(Ey)(x_2)$ and with no function symbols.

VIII. *The Surányi normal form case.* For satisfiability, every formula which has no equality sign, no function symbols, only dyadic predicate symbols, and has the form $(x_1)(x_2)(x_3)Mx_1x_2x_3$ & $(x_1)(x_2)(Ey_3)Nx_1x_2y_3$, $M$, $N$ quantifier-free.

It may be noted that in all the cases, with the single exception of III, no function symbols are permitted. Indeed, very little is known about the decision problem of formulae containing function symbols (compare Ref. 3, pp. 98–107). Unless otherwise stated, we shall always assume that no function symbols occur.

In what follows, cases I and VI will not be considered. So far as the monadic case without equality (a subcase of I) is concerned, it is possible to obtain a decision procedure from one for case II. Some of the problems suggested by the Ackermann case are also encountered by the $A_1E_1A_1$ case, while other implications of this case seem to call for a closer examination of certain arithmetic predicates.

Formulae under case VIII form a reduction class in the sense that there is an effective procedure by which every formula, possibly containing = and function symbols, can be reduced to one in the class with satisfiability preserved (Ref. 2, p. 60). It follows that there exists no decision procedure for this case. It is, however, desirable to find some "semidecision procedure" for the class which is a decision procedure for some subclass of it that is not specified explicitly in advance. It is thought that such semidecision procedures are a useful way of extending the range of formulae decidable by a predetermined finite set of procedures. A brief discussion is included in Section IV to point to the sort of thing which can be done along this line. It should be of interest to design semidecision procedures for case VIII, as well as for other reduction classes.

The case VII is perhaps the best known unsettled case; it has been mentioned in various connections (see, e.g., Ref. 11, p. 576 and Ref. 12, p. 420). In Section IV a procedure will be given which may be a decision procedure for the whole case but has only been shown to terminate for certain special cases. A proof of finiteness of the procedure is wanting. It is thought that, incomplete as the solution is, it is quite suggestive for further works on the decision problem. Some rather amusing combinatorial problems are also related to the considerations on this case.

An alternative decision procedure for the much-studied case V will be given in Section III in the equivalent form $A_2E$ (for satisfiability).

The Skolem case will be examined in considerable detail in Section II, using ideas proposed by Skolem[4] (p. 138) and Church[6] (p. 264). Remarks relevant to machine realizations of the procedure will also be included.

The Skolem case includes the following special cases:

IVa. The $A_1E$ satisfiability case. Because every atomic formula has to include some variable and there is only one independent variable.

IVb. For satisfiability, every formula whose prefix ends with $(Ey_1) \cdots (Ey_n)$, and in which every atomic formula contains at least one of the variables $y_1, \cdots, y_n$.

IVc. For satisfiability, every formula whose prefix is

$$(Ey_1^1) \cdots (Ey_m^1)(x_1) \cdots (x_n)(Ey_1^2) \cdots (Ey_k^2)$$

and in which every atomic formula contains either all of $x_1, \cdots, x_n$ or at least one of $y_1^2, \cdots, y_k^2$.

IVd. For satisfiability, every formula in the Skolem normal form, i.e., with prefix $(x_1) \cdots (x_m)(Ey_1) \cdots (Ey_n)$, such that every atomic formula contains at least $m$ distinct variables.

For the extensive literature on the decision problem, the reader is referred to the bibliographies in Refs. 2 and 3. The writer has not been able to study carefully much of the relevant literature, and is not certain that the procedures described in Sections II and III may not turn out to be inferior to existing ones. Recently, the writer noticed that ideas along the line of the solution of the $E_1A$ provability case given in Section 3 of Part I[1] are contained in Skolem's writings (e.g., Ref. 4, p. 135).

Of the two remaining cases, II and III, some brief comments will suffice.

### 1.5 *Two Simple Cases*

The $EA$ satisfiability case II has agreeable decision procedures not dependent on the fundamental theorem of logic (see Ref. 13, p. 13). It is also easy to devise a decision procedure on the basis of the fundamental theorem. Consider

1.5.1 $$(Ey_1) \cdots (Ey_m)(x_1) \cdots (x_n)My_1 \cdots x_n .$$

This is in fact equivalent to:

1.5.2 $$M_1 \ \& \cdots \& \ M_k, \qquad k = m^n, \qquad \text{or 1 when } m = 0.$$

In fact, this is a limiting case of the fundamental theorem because no Skolem functions are needed, so that the $m$ constants for the initial variables are all we need for fabricating a model. In other words, either the negation of 1.5.2 is a quantifier-free tautology, and the negation of 1.5.1 is a theorem; or 1.5.2 has a model, and 1.5.1 has a model too. The presence of the equal sign is permitted, but the presence of function symbols in 1.5.1 would invalidate the procedure.

The conjunctive satisfiability case III was originally solved by Herbrand (Ref. 5, pp. 44–45). Suppose the matrix is:

1.5.3 $$A_1 \,\&\, \cdots \,\&\, A_m \,\&\, \sim B_1 \,\&\, \cdots \,\&\, \sim B_n ,$$

or, in a different notation:

1.5.4 $$A_1 , \cdots , A_m \nrightarrow B_1 , \cdots , B_n .$$

Assume first that neither equality nor function symbols occur. If no predicate letter occurs both on the left side and on the right side, then we can simply choose to make all predicates occurring on the left side true of all numbers and those on the right false for all numbers, and then the infinite conjunction corresponding to the given formula is true under the interpretation.

Whenever there is one clause on the left and one on the right which contain the same predicate letter, e.g., $A_i$ is $Gabc$ and $B_j$ is $Guvw$, we compare them and ask whether it is possible to assign the same integers to their arguments in some $M_s$ and $M_t$ respectively. If the answer is yes, the original formula can have no model, because the infinite conjunction must be always false. If the answer is no for every such pair, then the original formula has a model.

To compare $A_i$ and $B_j$, we examine the three pairs of corresponding variables. If both variables in some pair are distinct dependent variables, then the two clauses $A_i$ and $B_j$ can never get the same numbers. When this is the case for none of the pairs, we can decide the question by asking whether there are positive integers $s$, $t$ such that $a(s) = u(t)$, $b(s) = v(t)$ and $c(s) = w(t)$, where, for each variable $\alpha$ in the original formula, $\alpha(n)$ is a function giving the number which replaces $\alpha$ in $M_n$. It is possible to give a scheme to generate such function for each given formula. When there are solutions for some pair of clauses, the original formula is not satisfiable.

If the formula 1.5.4 contains function symbols but not $=$, then the comparison of $A_i$ and $B_j$ has to take functions into considerations sometimes. We may have to ask whether $f(a(s)) = g(u(t))$, instead of $a(s) = u(t)$, has a solution. In such cases, there is a solution only when $f$ and $g$ are the same function, because otherwise we can always give different values to $f(a(s))$ and $g(u(t))$ to avoid the incompatibility of $M_s$ and $M_t$.

When the equals sign also occurs, we have to list all the equations among $A_1 , \cdots , A_m$, if there is any, and complete the list by using transitivity. If there are none, we need only to proceed as before, except that we can also reject satisfiability on the ground of, e.g., having an equation $u = v$ among $B_1 , \cdots , B_n$, and $u(p) = v(p)$ has a solution in

$p$. In the general case, we must compare $A_i$ and $B_j$, which have the same predicate letter, in a more complicated manner. One way to do this is to give an effective survey of all the equalities obtainable in $M_1, \cdots, M_t$, for every $t$. And then the question of comparing $Gabc$ and $Guvw$ is reduced to the following: whether there are $p$, $q$, $t$ such that, with the help of the equalities obtainable from $M_1, \cdots, M_t$, we have $a(p) = u(q)$, $b(p) = v(q)$, $c(p) = w(q)$. Since these considerations are only subsidiary for the main purpose of the paper, details for this and other steps sketched above will not be supplied.

## II. THE SKOLEM CASE

### 2.1 *Outline of a General Method*

The subcase IVb, where every atomic formula contains at least one of the last string of dependent variables, is particularly simple. Thus, in every $M_k$, each such variable always gets replaced by some new number so that no atomic formula in $M_k$ can have occurred in any of $M_1, \cdots, M_{k-1}$. Hence, a formula of such a form is satisfiable if and only if $\sim M_1$ is not a quantifier-free tautology.

In the general Skolem case, we make use of the definition of sections given above in 1.3.11. Let $(a_1^k, \cdots, a_p^k)$ be the $p$-tuple which replaces the dependent variables in $M$ to get $M_k$.

Given any member $M_i$ of the $n$th section, the only related instances in the $n$th section are those $M_k$ for which $(a_1^k, \cdots, a_p^k)$ is a permutation of $(a_1^i, \cdots, a_p^i)$, and the only related instances in the $(n + 1)$th section are those $M_j$ for which $(a_1^j, \cdots, a_p^j)$ include only numbers occurring in $M_i$ and at least one number not in the set $\{a_1^i, \cdots, a_p^i\}$.

Hence, it is possible to get a decision procedure by determining whether there exists any set of possibilities which includes models for the instances of the first section, as well as models for all related instances $M_k$ and $M_j$ for every model for $M_i$ in the set.

When the formula is in the Skolem normal form or the form of IVc, somewhat more is true:

2.1.1 If $M_j$ belongs to the $(n + 1)$th section, then it can have common atomic formulae with only at most one $M_i$ in the $n$th section.

This is so because each atomic formula in $M_j$ either contains a new number not occurring in any member of the $n$th section, or otherwise contains all of $\{a_1^j, \cdots, a_p^j\}$ with at least one number (say $a_t^j$) which appeared for the first time in one specific member (say $M_i$) of the $n$th section. In the first case the atomic formula in $M_j$ does not occur in any member of the $n$th section. In the second case, $M_j$ can contain no common atomic formula with any member of the $n$th section except possibly $M_i$, since $a_t^j$ does not occur in any of the other members of the $n$th section.

Detailed considerations will be confined to the treatment of a simple special case.

## 2.2 *An Explicit Procedure for a Special Case*

We consider a very simple special case in which the matrix contains no equals sign (and of course no function symbols), and a single dyadic predicate $G$:

2.2.1 $$(x)(y)(Ez)Mxyz.$$

As an illustration, we use the negation of Example (2) of Part I:[1]

2.2.2 $$(x)(y)(Ez)[(Gxy \ \& \ Gyx \ \& \ {\sim}Gxz \ \& \ {\sim}Gzy \ \& \ {\sim}Gzz) \lor (Gxz \ \& \ Gzy \ \& \ Gzz \ \& \ {\sim}Gxy \ \& \ {\sim}Gyx)].$$

In an alternative notation, the matrix is:

2.2.3 $$Gxy, Gyx \nrightarrow Gxz, Gzy, Gzz;$$
$$Gxz, Gzy, Gzz \nrightarrow Gxy, Gyx.$$

We construct a truth table of all the possibilities which can satisfy the above matrix:

2.2.4

| *Gxy* | *Gyx* | *Gxz* | *Gzx* | *Gyz* | *Gzy* | *Gzz* |
|---|---|---|---|---|---|---|
| t | t | f | | | f | f |
| f | f | t | | | t | t |

The blanks may take either t or f as values. Hence, there are eight rows in all.

For the prefix $(x)(y)(Ez)$, the numbers to substitute for $(x,y,z)$ in $M_1$, $M_2$, $M_3$, $M_4$, etc., are (1,1,2), (1,2,3), (2,1,4), (2,2,5), etc. In order to decide whether a formula of the form 2.2.1 has a model, we ask whether it is possible to make $M112$, $M123$, $M214$, etc., simultaneously true, or, in other words, whether we can find for each $M_i$ one row from the above table according to which $M_i$ is true, such that these infinitely many rows are all compatible in the sense that the same atomic formula always gets the same truth value (t or f).

Among the number triples we can distinguish two classes, those in which $x$ and $y$ get the same numbers, such as (1,1,2), and those in which they get different numbers, such as (2,1,4). The conditions under which a model is possible are roughly: (i) to satisfy $Maab$, a row in the truth table has to behave in a way that $x$ and $y$ are interchangeable; (ii) for each row satisfying *Mabc*, there must be a related row satisfying *Mbac*; (iii) for the two types of row, two corresponding patterns of continuation must be possible, e.g.,

$M112 \to \{M123,\ M225\}$ $\qquad$ $M123 \to \{M136,\ M238,\ M33(10)\}$

These conditions can be formalized more exactly and applied, in particular, to show that 2.2.2 has a model, and therefore its negation is not a theorem. For this purpose, we assume a formula of the form 2.2.1 for which a truth table $T$ like 2.2.4 is constructed. When, for example, *Gxy* in a row $R$ of $T$ gets the same value as *Gzz* in a row $S$ of $T$, we shall use the brief notation $R_{xy} = S_{zz}$ .

2.2.5 A row $S$ in the table $T$ is a uniform row if $S_{xy} = S_{yx}$ , $S_{xz} = S_{yz}$ , $S_{zx} = S_{zy}$ .

Clearly, for a row to satisfy $M112$, it is necessary that it be uniform. If there is no uniform row, then there is no model for the original formula.

2.2.6 A row $S$ in the table $T$ is an heir of a row $R$ in $T$ if $S$ is a uniform row and $R_{zz} = S_{xy}$ .

2.2.7 A row in $T$ is trivial if it has no heir.

Since a row having no heir cannot be continued, we may cross out all trivial rows and be concerned only with nontrivial rows. This is not theoretically necessary because further requirements would cross out trivial rows anyhow, but it makes for efficiency.

2.2.8 A row $R$ in the table $T$ is an ordinary row if there is a row $S$ such that $R_{xy} = S_{yx}$ , $R_{yx} = S_{xy}$ , $R_{xz} = S_{yz}$ , $R_{zx} = S_{zy}$ , $R_{yz} = S_{xz}$ , $R_{zy} = S_{zx}$ . $R$ and $S$ are said to be mates of each other.

This is the condition under which $R$ and $S$ can satisfy ($M123$, $M214$) or ($M214$, $M123$) respectively.

In the table 2.2.4 for the formula 2.2.2, it is easily verified that only the two following rows are uniform rows or ordinary rows:

| | *Gxy* | *Gyx* | *Gxz* | *Gzx* | *Gyz* | *Gzy* | *Gzz* |
|---|---|---|---|---|---|---|---|
| $\alpha$ | t | t | f | f | f | f | f |
| $\beta$ | f | f | t | t | t | t | t |

In fact, $\alpha$ and $\beta$ are the only uniform rows, as well as the only ordinary rows. Each of $\alpha$ and $\beta$ is only a mate of itself.

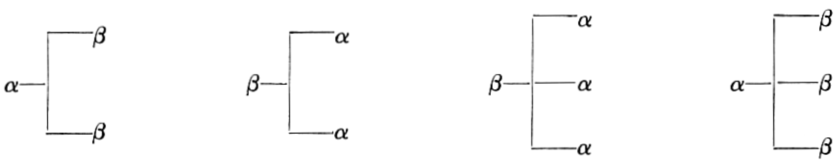
2.2.9 A uniform row $R$ is permanent if (i) it has an heir which is permanent, and (ii) there is a permanent ordinary row $S$ such that $R_{yz} = S_{xy}$, $R_{zy} = S_{yx}$. $S$ is said to be a subordinate of $R$.

2.2.10 An ordinary row $R$ is permanent if (i) it has an heir which is a permanent (uniform) row, (ii) it has a mate that is a permanent ordinary row, and (iii) there are two permanent ordinary rows $P$ and $S$ such that $R_{xz} = P_{xy}$, $R_{zx} = P_{yx}$, $R_{yz} = S_{xy}$, $R_{zy} = S_{yx}$. $P$ and $S$ are said to be a pair of subordinates of $R$.

The two definitions 2.2.9 and 2.2.10 embody a simultaneous recursion. Condition (ii) in 2.2.9 is necessary, if, e.g., $R$ is to satisfy $M112$ and $S$ is to satisfy $M123$. Condition (iii) in 2.2.10 is necessary if, e.g., $R$ is to satisfy $M123$, $P$ is to satisfy $M136$ and $S$ is to satisfy $M238$.

2.2.11 The formula 2.2.1 has a model if and only if its truth table $T$ contains a permanent uniform row.

This assertion will be justified in 2.3. We observe first that both $\alpha$ and $\beta$ are permanent uniform rows for the example 2.2.2. In fact, we have various models for the formula, which are determined, in outline, by the following patterns of continuation:




More exactly, choose, e.g., $\alpha$ as a model of $M112$. As a continuation of this, $\beta$ satisfies $M123$ and $M225$; since $\beta$ is its own mate in the sense of 2.2.8, $\beta$ also satisfies $M214$. Similarly, since $\alpha$ is its own mate, as a continuation of $\beta$ satisfying $M123$, $\alpha$ satisfies $M136$, $M317$, $M238$, $M329$, and $M33(10)$. In this particular case, the model $\beta$ of $M214$ can be continued in the same way. Moreover, the model $\beta$ of $M225$ can be continued by the row $\alpha$, and, e.g., the model $\alpha$ of $M136$ can be continued by the row $\beta$, and so on.

In the general case, a symmetry argument is needed to show that if a model of, e.g., $M123$ can be continued, then a model of $M214$ can also be continued. For example, if $(R, S)$ satisfy $(M123, M214)$ respectively, and $(A, B, C, D)$ satisfy respectively the continuation $(M136, M317, M238, M329)$ of $M123$, then it is easy to see that $(B, A, D, C)$ satisfy the corresponding extension of $M214$. This means that condition (ii) of 2.2.10 can be weakened to require a mate that is an ordinary row with a permanent heir.

The decision procedure implicit in the above definitions may be described explicitly thus:

2.2.12 The decision procedure:

1. Construct a truth table $T$.
2. Find all uniform rows.
3. Cross out all trivial rows.

Let $U_0$ be the set of remaining uniform rows, $V_0$ be the set of remaining ordinary rows. Each time, assume $U_n$ and $V_n$ are given and continue the following four steps:

4. Eliminate every uniform row from $U_n$ which has no subordinate row in $V_n$, thus obtaining $U_{n+1}$ from $U_n$ and $V_n$.

5. Eliminate from $V_n$ every ordinary row which has no mate or no pair of subordinate rows in $V_n$, thus obtaining $V_{n+1}$ from $V_n$.

6. Eliminate every uniform row from $U_{n+1}$ which has no heir in $U_{n+1}$, thus obtaining $U_{n+2}$ from $U_{n+1}$.

7. Eliminate every ordinary row from $V_{n+1}$ which has no heir in $U_{n+2}$, thus obtaining $V_{n+2}$ from $V_{n+1}$ and $U_{n+2}$.

8. The steps 4 through 7 are repeated until one of two things happens: either at some stage we obtain an empty $U_i$ and an empty $V_i$, then we stop and conclude that the original formula 2.2.1 has no model; or else, after a whole round of the steps 4 and 7, we find $U_{n+2}$ and $V_{n+2}$ remain the same as $U_n$ and $V_n$, then we stop and conclude that the original formula 2.2.1 has a model.

In practice, it is more efficient to perform, if possible, each of the steps 4 through 8 repeatedly, before going to the next step.

The procedure is clearly finite, since $U_0$ and $V_0$ are finite, and each round of steps 4 through 8 must reduce the size of $U_n$ or $V_n$ if the procedure has not come to a stop yet. Moreover, the final sets $U_i$ and $V_i$ must be both empty or both nonempty.

## 2.3 *Justification of the Procedure*

As a Skolem case, the formula 2.2.1 must not contain $Gxx$ and $Gyy$. It is, however, not obvious that we are justified in not including two columns $Gxx$ and $Gyy$ in the truth tables such as 2.2.4. For a model constructed on the basis of such reduced tables, it is not evident that, for some positive integer $a$, $Gaa$ might not be compelled to take on the value t at one place, and the value f at another. However, we can prove the following:

2.3.1 In every model obtained on the basis of a truth table not including columns for $Gxx$ and $Gyy$, for every number $a$, $Gaa$ is never compelled to take on two different values.

Take, for example, $G22$. If $Gzz$ occurs in the original formula, $G22$ is compelled to take a fixed value in a model with a row $R$ for $M112$. In the same model, if $S$ is the row for $M225$, then $R_{zz} = S_{xy} = S_{yx}$. Hence, it is harmless that $S_{xx}$ and $S_{yy}$ are compelled to take the same value as both $R_{zz}$ and $S_{xy}$ (or $S_{yx}$). In all other cases, the values for $G22$ can always be given the value of $R_{zz}$ because there is no other place where $G22$ is independently compelled to take a certain truth value.

For the same reason, if neither an atomic formula nor any one obtainable from it by permuting the variables occurs, we may leave out the columns for them. For example, if $Gzz$ does not occur, we can leave it out. If neither $Gxy$ nor $Gyx$ occurs, we can leave both of them out.

On the other hand, if, e.g., $Gxy$ and $Gzy$ occur but $Gyx$ does not, we still must include a column for $Gyx$. Otherwise, since we do not record the value of $Gyx$, it may happen that $R$ satisfies $M112$, with $R_{zy} =$ t, and $S$ satisfies $M214$ with $S_{xy} =$ f. Then no row $P$ can satisfy $M123$, because $P_{yx}$ is compelled to take both the value t and the value f, and this is not recorded without a column for $Gyx$.

To prove 2.2.11, we remark first that there are three types of instances illustrated by $M112$, $M123$, $M214$. For the first kind, an $M_i$ of the form $Maab$, the only $M_j$, $j > i$, which have common atomic formulae with $M_i$ are $Mbbc$, $Mabd$, $Mbae$, because these are the only ways in which both the independent variables $x$ and $y$ can be replaced by numbers occurring in $M_i$, and having only one of the two arguments from $M_i$ yields no common atomic formula. Similarly, if $M_i$ is $Mabc$, $a < b$, there are only five $M_j$, $j > i$ which have common atomic formula with $M_i$. By the symmetry argument preceding 2.2.12, the mate $Mbae$ is also taken care of.

Hence, if there is any permanent uniform row, we can find a model for all instances $M_1$, $M_2$, etc., such that each has some common atomic formula with an earlier one, or, in other words, all those occurring on an infinite tree beginning at $M_1$. This does not exhaust all the instances. For example, $M_{14}$ and $M_{15}$ [i.e., $M34(15)$ and $M43(16)$] are not included. Since, however, they contain no common atomic formulae with the instances already interpreted, we can take two permanent ordinary rows which are mates and get a model for another sequence of instances. In this way, it is seen that, if there is a permanent uniform row in the table $T$, then one can so interpret the predicate $G$ in the domain of the positive integers that the whole sequence $M_1$, $M_2$, etc., are simultaneously satisfied.

The converse is quite obvious. If there is no permanent uniform row, then no interpretation of $M112$ can be continued indefinitely, and there is an $i$. such that $M_1$ & $\cdots$ & $M_i$ is true under no interpretation.

### 2.4 *Questions of Efficiency*

When doing an example by hand, there are shortcuts we find natural to use. These may be viewed as more refined methods which can be mechanized by additional efforts. We give some informal illustration of the type of quick method we tend to use.

Consider the negation of Example (3) given in Part I:[1]

2.4.1 $$(x)(y)(Ez)\{[Gxy \;\&\; (\sim Gyz \vee \sim Gzz)] \vee [(Gxy \;\&\; Hxy) \;\&\; (\sim Hxz \vee \sim Hzz)]\}.$$

In the alternative notation, the matrix of the above formula is:

2.4.2 $$Gxy \nleftrightarrow Gyz;\; Gxy \nleftrightarrow Gzz;\; Gxy, Hxy \nleftrightarrow Hxz;\; Gxy, Hxy \nleftrightarrow Hzz.$$

The truth table for this is:

2.4.3

| | $Gxy$ | $Hxy$ | $Gyx$ | $Hyx$ | $Hxz$ | $Hzx$ | $Gyz$ | $Gzy$ | $Gzz$ | $Hzz$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | t | | | | | | f | | | |
| $\beta$ | t | | | | | | | | f | |
| $\gamma$ | t | t | | | f | | | | | |
| $\delta$ | t | t | | | | | | | | f |

Although the formula contains two predicates instead of just one, it is easy to see that the procedure described above can be extended to cover the case in a very straight-forward manner.

Since there are many blanks in the table, it is essential for efficiency that we do not expand the table by filling in the blanks (there would be $2^{24}$ rows), until we are compelled to do so. In other words, we try to carry out the decision procedure by treating each row containing blanks as a single row and make expansion only when we are not able to eliminate them as single rows.

We observe that for every row, in particular, every uniform row, $Gxy$ gets the value t. It follows that row $\beta$, or more exactly, all the $2^7$ rows obtainable from $\beta$ are trivial by 2.2.7, since an heir of $\beta$ must have $Gxy$ take the value of $Gzz$ in $\beta$, which is f. Hence we may delete row $\beta$ altogether.

In order that row $\alpha$, or any specification $R$ of $\alpha$, be permanent (uniform or ordinary), it is necessary, by 2.2.9 and 2.2.10, that there is a subordinate row $S$, such that $Gxy$ gets the same value in $S$ as $Gyz$ in $R$, or $R_{Gyz} = S_{Gxy}$. But this is impossible because $R_{Gyz}$ is f in every row obtainable from $\alpha$, but $S_{Gxy}$ is $t$ in every row. Hence, we can delete row $\alpha$ altogether, and be concerned only with the rows $\gamma$ and $\delta$.

Since $Hxy$ gets t in all the remaining rows and $Hzz$ gets the value f in $\delta$, every row obtainable from $\delta$ has no heir, and the whole row $\delta$ can be deleted.

However, no permanent ordinary row can be obtained from $\gamma$ alone because, by 2.2.10, for any such row $R$ there must be a subordinate row $P$ such that $R_{Hxz} = P_{Hxy}$, but in row $\gamma$, $Hxz$ is always f and $Hxy$ is always t. Hence, there can also be no permanent uniform row, and, by 2.2.11, the formula 2.4.1 has no model. Therefore, Example (3) in Part I,[1] the negation of 2.4.1, is a theorem.

Another method of deciding 2.4.1 is the following. We begin with $M_1$, which is a disjunction of conjunctions, and choose $M_i$, $M_j$, etc., which contain common atomic formula with $M_1$, in the hope that $M_1$ & $M_i$ & $M_j$ & $\cdots$ as multiplied out into a disjunction of conjunctions will include in each conjunction some atomic formula and its negation. The process may have to be continued.

As we observed before, only $M_2$, $M_3$, $M_4$ can have common atomic formulae with $M_1$. Of these three, on account of the special structure of 2.4.1, $M_3$ has no common part with $M_1$. Hence, we need to consider, to begin with, only $M_1$, $M_2$, $M_4$:

| | (i) | (ii) | (iii) | (iv) |
|---|---|---|---|---|
| $M112$ | $G11 \nleftrightarrow G12$; | $G11 \nleftrightarrow G22$; | $G11, H11 \nleftrightarrow H12$; | $G11, H11 \nleftrightarrow H22$ |
| $M123$ | $G12 \nleftrightarrow G23$; | $G12 \nleftrightarrow G33$; | $G12, H12 \nleftrightarrow H13$; | $G12, H12 \nleftrightarrow H33$ |
| $M225$ | $G22 \nleftrightarrow G25$; | $G22 \nleftrightarrow G55$; | $G22, H22 \nleftrightarrow H25$; | $G22, H22 \nleftrightarrow H55$ |

By the row for $M123$, (i) of $M112$ can be deleted because (i) contains $\sim G12$ (i.e., after $\nleftrightarrow$), while each clause in the row for $M123$ contains $G12$. It can be seen then that every row in column (i) can be deleted in the same way. Similarly, (ii) of the row for $M112$ can be deleted because it contains $\sim G22$, while each clause in the row for $M225$ contains $G22$; therefore, the whole column (ii) can be deleted eventually, and we need only consider the columns (iii) and (iv). But then (iii) of the row for $M112$ can also be deleted because it contains $\sim H12$, and all the remaining columns of the row for $M123$ contain $H12$. Finally, we have only column (iv) left. Now, however $\sim H22$ occurs in the row for $M112$ and $H22$ occurs in the row for $M225$. Hence, the conjunction of the three rows of column (iv) is a contradiction, and 2.4.1 has no model.

### 2.5. *The Inclusion of Equality*

The decision procedure in 2.2 can be extended to deal with cases where the equal sign occurs in the given formula:

2.5.1 $$(x)(y)(Ez)Mxyz, \qquad \text{with} = \text{occurring.}$$

Additional considerations are needed to take care of the special properties of $=$. First we bring $Mxyz$ into a disjunction of conjunctions of atomic formulae and their negations, in the usual manner. Then we modify the resulting matrix to take care of the properties of $=$. (a) Each conjunction that contains an inequality of the form $v \neq v$, $v$ being $x$ or $y$ or $z$, is deleted. (b) In each conjunction, a clause of the form $v = v$ is deleted. (c) Within each conjunction, if $u = v$ is a clause with distinct variables $u$ and $v$, we add also, as new clauses (if not occurring already), $v = u$ and the result of replacing any number of occurrences of $u$ by $v$ (or $v$ by $u$) in each clause of the conjunction; this is repeated for every equality until no new clause is generated. (d) Repeat the steps (a) and (b) on the result obtained by step (c); in addition, any conjunction which contains both an atomic formula and its negation is deleted.

We now construct the truth table on the basis of the new matrix (in a disjunctive normal form). Uniform rows, ordinary rows and permanence can be defined in a similar manner as before, except that a uniform row has to satisfy the additional condition that $x = y$ and $y = x$ both get the truth value $t$ (not only that they just get a same value). In this way, we can obtain a decision procedure for all formulae of the form 2.5.1.

It is believed that the same type of consideration can be used to extend all the cases considered in this paper to include also the equal sign. In the next two sections, equality will be left out and attention will be confined to formulae not containing the equals sign (nor, of course, function symbols).

## III. THE $A_2E$ SATISFIABILITY CASE

We give an alternative treatment of this case which, it is conjectured, is in general more efficient than the method of Schütte[11] as reformulated by Klaua.[8] The method will be explained with the special case when only one dependent variable and only one dyadic predicate $G$ occur:

3.1 $$(x)(y)(Ez)Mxyz.$$

The main difference between this case and the case solved in 2.2 above is that $Gxx$ and $Gyy$ are permitted to occur in $Mxyz$. As a result, for example, $M123$ may contain common atomic formula with any $Mabc$ in which $a$ or $b$ is one of 1, 2, 3.

As an example, we choose arbitrarily the following:

3.2 $(x)(y)(Ez)[\sim Gxx \;\&\; (Gxy \supset \sim Gyx) \;\&\; Gxz \;\&\; (Gzy \supset Gxy)]$.

The matrix may be rewritten as:

3.3 $$Gxz \nrightarrow Gxx,Gxy,Gzy;\; Gxz,Gxy \nrightarrow Gxx,Gyx;$$
$$Gxz \nrightarrow Gxx,Gyz,Gzy.$$

The truth table is:

3.4

| $Gxx$ | $Gxy$ | $Gyx$ | $Gyy$ | $Gxz$ | $Gyz$ | $Gzx$ | $Gzy$ | $Gzz$ |
|---|---|---|---|---|---|---|---|---|
| f | f | | | t | | | f | |
| f | | f | | t | | | f | |
| f | t | f | | t | | | | |

The problem is, as before, to decide whether there is a model that satisfies $M_1$, $M_2$, etc., simultaneously. The conditions are rather similar to those in 2.2 except that for any two rows $R$ and $S$ which, say, satisfy $Mabc$ and $Mdef$ in a model, there must be two rows which satisfy $Mcfg$ and $Mfch$ in the model. There is also a related requirement for a row satisfying $M_1$, because the number 1 is never used to replace a dependent variable. The various conditions may be stated:

3.5 A row $R$ is uniform if $R_{xx} = R_{xy} = R_{yx} = R_{yy}$, $R_{xz} = R_{yz}$, $R_{zx} = R_{zy}$.

3.6 A row $S$ is an heir of a row $R$ if $S$ is uniform and $R_{zz} = S_{xx}$.

3.7 Two rows $R$ and $S$ form a parallel pair if $R_{xx} = S_{yy}$, $R_{xy} = S_{yx}$, $R_{yx} = S_{xy}$, $R_{yy} = S_{xx}$, $R_{xz} = S_{yz}$, $R_{yz} = S_{xz}$, $R_{zx} = S_{zy}$, $R_{zy} = S_{zx}$.

Two rows of a parallel pair are said to be mates of each other.

If $R$ and $S$ are to satisfy $Mabc$ and $Mbae$, it is necessary that they form a parallel pair. In general, for a row satisfying $Mabc$, there must also be two parallel pairs of related rows satisfying $Macd$, $Mcae$, $Mbcf$, $Mcbg$. When $a = b$, the two parallel pairs become one. This, plus the requirement that every row in a model must have an heir may be summarized in the following condition.

3.8 A row $R$ is normal if the following conditions are all satisfied:

3.8.1 It has a normal row as a mate;

3.8.2 It has an heir which is a normal row;

3.8.3 There are two normal rows $P$ and $S$ such that $R_{xx} = P_{xx}$, $R_{xz} = P_{xy}$, $R_{zx} = P_{yx}$, $R_{zz} = P_{yy}$, and $R_{yy} = S_{xx}$, $R_{yz} = S_{xy}$, $R_{zy} = S_{yx}$, $R_{zz} = S_{yy}$. Such rows $P$ and $S$ are said to be subordinates of $R$.

A uniform row is its own mate, although a self-mated row is not always a uniform row. For a uniform row, 3.8.1 is a redundant condition, and $P$ and $S$ coincide in 3.8.3. The definition 3.8 of normality is clearly recursive.

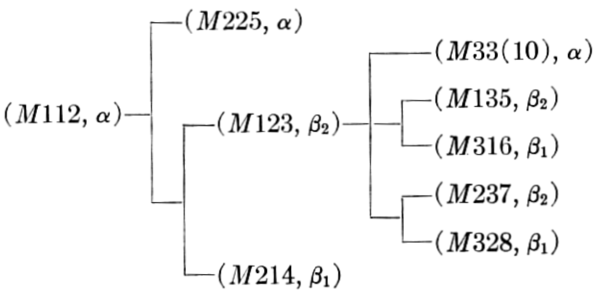
In the table 3.4, we observe that, because $Gxx$ always takes the value f, $Gzz$ can only take the value f in order that the row has an heir. Moreover, since $Gxx$ always gets the value f and $Gxz$ always gets the value t, in order that a row has a mate, $Gyy$ must always take the value f and $Gyz$ always t. Hence, we need consider only the following eight rows which result from filling the remaining gaps:

3.9

| | $Gxx$ | $Gxy$ | $Gyx$ | $Gyy$ | $Gxz$ | $Gyz$ | $Gzx$ | $Gzy$ | $Gzz$ |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | f | f | f | f | t | t | f | f | f |
| $\beta_1$ | f | f | t | f | t | t | f | f | f |
| $\beta_2$ | f | t | f | f | t | t | f | f | f |
| a | f | f | t | f | t | t | t | f | f |
| b | f | t | f | f | t | t | f | t | f |
| c | f | f | f | f | t | t | t | f | f |
| d | f | t | f | f | t | t | t | f | f |
| e | f | t | f | f | t | t | t | t | f |

Row e has no mate, beacuse of the columns 5 to 8. Rows c and d have no mate, because b, the only row satisfying the condition on $Gzx$ and $Gzy$, does not satisfy the condition on $Gxy$ and $Gyx$. Neither row a nor row b has subordinates as required by 3.8.3. Hence, we have only the remaining rows $\alpha$, $\beta_1$, $\beta_2$ to consider.

$\alpha$ is the only uniform row, $(\beta_1, \beta_2)$ form a parallel pair, and $\beta_2$ is both $P$ and $S$ in 3.8.3 for all the three rows $\alpha$, $\beta_1$, $\beta_2$. Hence, we have, for example:



In particular, ($M214$,$\beta_1$) can be continued in the same way as ($M123$, $\beta_2$). Indeed, continuation in every branch can be made similarly. In other words, $\alpha$, $\beta_1$, $\beta_2$ are all normal by 3.8. This, however, does not yet secure a model for the formula 3.2. There are, for example, those instances in which (1,5), (5,1), (3,4), (4,3), etc., replace $(x,y)$ of $Mxyz$; they also have common atomic formulae with the instances shown in the above graph.

3.10 A formula 3.1 has a model if and only if (a) it has a nonempty table of normal rows, (b) this table has a nonempty subtable $T'$ such that:

3.10.1 For every pair $(R,S)$ in $T'$, there is a parallel pair $(P,Q)$ in $T'$ such that $P_{xx} = R_{zz}$, $Q_{xx} = S_{zz}$.

3.10.2 There is a uniform row $R$ in $T'$ such that for every row $S$ in $T'$, there is a parallel pair $(P,Q)$ in $T'$, for which $P_{xx} = R_{xx}$, $Q_{xx} = S_{zz}$.

These are the additional requirements mentioned after 3.4. In the example under consideration, the table consisting of all the three normal rows $\alpha$, $\beta_1$, $\beta_2$ satisfies the requirements on $T'$. Hence, 3.2 does have models. One model for the predicate $G$ is the relation $<$ among positive integers. That is, however, not the only model, because the model of $G$ does not have to be transitive. For example, $G15$ and $G51$ can be (t,f) or (f,t) or (f,f).

It can be verified that the conditions in 3.10 are indeed necessary and sufficient.

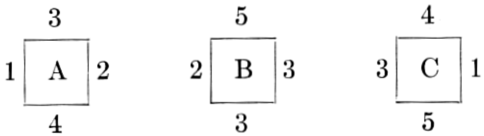
## IV. THE $A_1E_1A_1$ SATISFIABILITY CASE

### 4.1 *A Generalized Game of Dominoes*

The study of the decision problem of the present case has suggested a related abstract mathematical problem which can easily be stated in everyday language. The problem appears to be of interest even to those who are not concerned with questions in mathematical logic.

Assume we are given a finite set of square plates of the same size with edges colored, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are not permitted to rotate or reflect a plate. The question is to find an effective procedure by which we can decide, for each given finite set of plates, whether we can cover up the whole plane (or, equivalently, an infinite quadrant thereof) with copies of the plates subject to the restriction that adjoining edges must have the same color.

For example, suppose a set consists of the three plates:




Then we can easily find an infinite solution by the following argument. The following configuration satisfies the constraint on the edges:

$$\begin{array}{ccc} A & B & C \\ C & A & B \\ B & C & A \end{array}$$

Now the colors on the periphery of the above block are seen to be the following:

$$\begin{array}{ccccc} & 3 & 5 & 4 & \\ 1 & & & & 1 \\ 3 & & & & 3 \\ 2 & & & & 2 \\ & 3 & 5 & 4 & \end{array}$$

In other words, the bottom edge repeats the top edge, and the right edge repeats the left edge. Hence, if we repeat the $3 \times 3$ block in every direction, we obtain a solution of the given set of three plates. In general, we define a "cyclic rectangle."

4.1.1 Given any finite set of plates, a cyclic rectangle of the plates is a rectangle consisting of copies of some or all plates of the set such that: (a) adjoining edges always have the same color; (b) the bottom edge of the rectangle repeats the top edge; (c) the right edge repeats the left edge.

Clearly, a sufficient condition for a set of plates to have a solution is that there exists a cyclic rectangle of the plates.

What appears to be a reasonable conjecture, which has resisted proof or disproof so far, is:

4.1.2 *The fundamental conjecture*: A finite set of plates is solvable (has at least one solution) if and only if there exists a cyclic rectangle of the plates; or, in other words, a finite set of plates is solvable if and only if it has at least one periodic solution.

It is easy to prove the following:

4.1.3 If 4.1.2 is true, we can decide effectively whether any given finite set of plates is solvable.

Thus, we proceed to build all possible rectangles from copies of the plates of different sizes, using smaller ones first. If 4.1.2 is true, the process will always terminate in one of two ways: either at some stage we arrive at a cyclic rectangle and, therefore, the original set is solvable; or else we arrive at a size such that there is no rectangle of that size in which adjoining edges always have the same colors. The latter alternative is in fact a necessary and sufficient condition under which the original set is not solvable. However, if 4.1.2 is not true, it would be possible that a set has a solution, but we can never see this fact by the latter criterion at any finite stage: there would always be the possibility that for the next size there exist no rectangles with same-colored adjoining edges.

There is a naturally uneasy feeling about the effectiveness of such a procedure. The argument is essentially the familiar one that if a set and its complement are both recursively enumerable, then the set is recursive. It shows that the procedure always terminates (provided 4.1.2 is true) but gives no indication in advance as to how long it might take in each case.

If 4.1.2 is proved, it seems likely that it would be proved in a stronger form by exhibiting some simple recursive function $f$ with the following property. For any set of plates with $m$ distinct colors and $n$ distinct plates, if the set is solvable, there is a cyclic square of the size $k \times k$, where $k = f(m,n)$. If that happens, or even if we have not exhibited such a function $f$ but 4.1.2 can be proved by fairly elementary arguments, we would have some estimate in advance of how long the procedure takes in each case.

As it is, we can make the testing procedure quite systematic even though we do not know whether 4.1.2 is true. The procedure would be a decision procedure and presumably quite an efficient one, if 4.1.2 is true. If 4.1.2 should turn out to be false, then the procedure would only be a semidecision procedure. In fact, it is possible to show that the procedure does work in several classes of cases, e.g., when a set has unique solution apart from translations, or whenever either horizontally or vertically no color can be followed by different colors. But we shall not delay over such partial results.

If 4.1.2 should be false, then there would be two possibilities: either the set of all solvable finite sets of plates is not recursive, or it is recursive but requires a more complex decision procedure.

The problem can clearly be generalized to higher dimensions: for example, to cubes with colored surfaces instead of squares with colored edges.

We return now to the $A_1E_1A_1$ satisfiability case.

### 4.2 *Preliminary Definitions and an Example*

The general form of the case is:

4.2.1 $$(x)(Ey)(z)Mxyz,$$

where $M$ is a quantifier-free matrix containing neither function symbols nor the equality sign. From the fundamental theorem, it follows that 4.2.1 is satisfiable (solvable) if and only if each finite subset of the infinite set of matrices $Mii'j$ $(i,j = 1, 2, \cdots)$ is solvable (not contradictory). Since the second number is always the successor of the first, we shall write $Mij$ for $Mii'j$.

We illustrate the general case by considering the special case where $Mxyz$ contains only a single dyadic predicate $G$. The negation of Example (4) given in the introduction of Part I[1] will be the concrete example:

4.2.2 $$(x)(Ey)(z)[\sim Gxx \;\&\; Gxy \;\&\; (Gyz \supset Gxz)].$$

In the alternative notation, the matrix is

4.2.3 $$Gxy, Gxz \nrightarrow Gxx;\; Gxy \nrightarrow Gyz, Gxx.$$

The truth table is:

4.2.4

| $Gxx$ | $Gxy$ | $Gyx$ | $Gyy$ | $Gxz$ | $Gzx$ | $Gyz$ | $Gzy$ | $Gzz$ |
|---|---|---|---|---|---|---|---|---|
| f | t | | | t | | t | | |
| f | t | | | t | | f | | |
| f | t | | | f | | f | | |

Since there are five blank columns, there are altogether $3 \times 2^5$ or 96 rows. The problem now is to decide whether we can choose one row for each matrix $Mij(i,j = 1, 2, \cdots)$ such that, taken together, all the matrices come out true. This really involves both the problem of finding the pieces and the problem of putting them together. Thus, if $j$ is distinct from $i$ and $i'$, any row can satisfy $Mij$ alone, if we substitute $i$, $i'$, $j$ for $x$, $y$, $z$ in the truth table; but a row can satisfy $Mij$ when $j$ is $i$ or $i'$ only in case certain related columns get the same truth values. This is the problem of finding the pieces. When there are such pieces, there is the harder problem of putting them together. For example, if there are rows satisfying $M11$ and $M12$ separately, there may yet be no pair of rows which satisfy $M11$ and $M12$ simultaneously because the common atomic formulae in both matrices must get identical values.

Since the putting-together part is quite complex, it seems natural to combine small pieces into blocks first. For this purpose, we consider row pairs and row quadruples (i.e., pairs of pairs).

D4.1 Two rows $P,Q$ in the truth table $T$ form a basic row pair $(P,Q)$ if, for some $i$, they can simultaneously satisfy $Mii'$ and $Mii$ respectively. More explicitly, the conditions are:

i. $P_{yy} = P_{yz} = P_{zy} = P_{zz}$, $P_{xy} = P_{xz}$, $P_{yx} = P_{zx}$;

ii. $Q_{xx} = Q_{xz} = Q_{zx} = Q_{zz}$, $Q_{xy} = Q_{zy}$, $Q_{yx} = Q_{yz}$;

iii. $P_{xx} = Q_{xx}$, $P_{xy} = Q_{xy}$, $P_{yx} = Q_{yx}$, $P_{yy} = Q_{yy}$.

In the table 4.2.4, it is easy to verify that there are only two basic row pairs $(\alpha,\beta)$ and $(\gamma,\delta)$:

4.2.5

| | $Gxx$ | $Gxy$ | $Gyx$ | $Gyy$ | $Gxz$ | $Gzx$ | $Gyz$ | $Gzy$ | $Gzz$ |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | f | t | f | f | t | f | f | f | f |
| $\beta$ | f | t | f | f | f | f | f | t | f |
| $\gamma$ | f | t | f | t | t | f | t | t | t |
| $\delta$ | f | t | f | t | f | f | f | t | f |

Obviously basic row pairs are necessary for building a model of 4.2.1. In fact, given any formula 4.2.1, if its truth table $T$ contains no basic row pairs, then it has no model and, indeed, the conjunction of $M11$ and $M12$ is a contradiction.

We shall consider pairs of row pairs, called row quadruples, which are useful in chaining row pairs together.

D4.2. Given any two row quadruples $(A,B;\ C,D)$ and $(P,Q;\ R,S)$, if $C = P$, $D = Q$, then the former is a predecessor of the latter and the latter is a successor of the former.

D4.3. Four rows $P$, $Q$, $R$, $S$ form a basic row quadruple $(P,Q;\ R,S)$ if, for some $i$, they satisfy simultaneously $Mii'$, $Mii$, $Mi'i''$, $Mi'i'$, respectively, or, more explicitly, if:

i. $(P,Q)$ and $(R,S)$ are basic row pairs;

ii. $P_{yy} = R_{xx}$;

iii. $(P,Q;\ R,S)$ has a successor which is a basic row quadruple.

In the table 4.2.4, there is only one basic row quadruple, viz., $(\alpha,\beta;\ \alpha,\beta)$. The quadruple $(\alpha,\beta;\ \gamma,\delta)$ satisfies i and ii, but not iii. It is easy to see that, given any formula 4.2.1, if its truth table $T$ contains no basic row quadruples, then it has no solution and, indeed, the conjunction of $M12$, $M11$, $M23$, $M22$, $M34$, $M33$ is a contradiction.

Clearly, if a row $R$ satisfies $Mij'$ in a model, then there must be one row $S$ which satisfies $Mji$, one basic row quadruple $(A,B;\ C,D)$ which satisfies $Mii'$, $Mii$, $Mi'i''$, $Mi'i'$, and one basic quadruple which satisfies $Mjj'$, $Mjj$, $Mj'j''$, $Mj'j'$. In particular, when $j$ is $i$, we get the basic row pairs which occur in some basic quadruple.

D4.4 Two rows $R,S$ form an ordinary row pair $(R,S)$ if

i. $R_{xx} = S_{zz}$, $R_{xz} = S_{zy}$, $R_{zx} = S_{yz}$, $R_{zz} = S_{yy}$;

ii. There is a basic quadruple $(A,B;\ C,D)$ such that $A_{xx} = R_{xx}$, $A_{xy} = R_{xy}$, $A_{yx} = R_{yx}$, $A_{yy} = R_{yy}$;

iii. There is a basic quadruple $(P,Q;\ K,L)$ such that $P_{xx} = S_{xx}$, $P_{xy} = S_{xy}$, $P_{yx} = S_{yx}$, $P_{yy} = S_{yy}$.

In the table 4.2.4, since the only basic quadruple is $(\alpha,\beta;\ \alpha,\beta)$, it is relatively simple to find all the rows which do occur in ordinary row pairs. Since every row which is to satisfy some $Mij$ in any solution must occur as one row in some ordinary row pair, we tabulate all such rows together and, from now on, confine our attention to them. It happens in this example that all these rows have in common five columns:

| $Gxx$ | $Gxy$ | $Gyx$ | $Gyy$ | $Gzz$ |
|---|---|---|---|---|
| f | t | f | f | f |

Therefore, we only have to list the remaining columns:

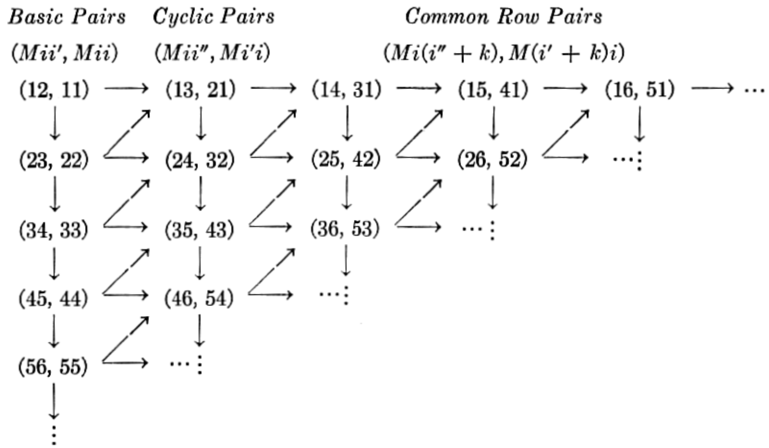
4.2.6

| | $Gxz$ | $Gzx$ | $Gyz$ | $Gzy$ | *ordinary pairs* |
|---|---|---|---|---|---|
| $\alpha$ | t | f | f | f | $(\alpha,\beta)$ |
| $\beta$ | f | f | f | t | $(\beta,\alpha)$ |
| $\delta_1$ | t | t | t | t | $(\delta_1, \delta_1)$ |
| $\delta_2$ | f | f | f | f | $(\delta_2, \delta_2)$ |
| $\delta_3$ | t | f | f | t | $(\delta_3, \delta_3)$ |
| $\delta_4$ | t | f | t | f | $(\delta_4, \delta_5)$ |
| $\delta_5$ | f | t | f | t | $(\delta_5, \delta_4)$ |
| $\delta_6$ | t | f | t | t | $(\delta_6, \delta_7)$ |
| $\delta_7$ | t | t | f | t | $(\delta_7, \delta_6)$ |

In fact, if only the four columns have to be considered, there are 12 rows in the original table 4.2.4, and the two rows $(R,S)$ in each ordinary row pair satisfy the condition: $R_{xz} = S_{zy}$, $R_{zx} = S_{yz}$. Hence, it is easy to get the above table. Briefly, the relevant information for the example is the nine ordinary pairs given above and the basic quadruple $(\alpha,\beta;\ \alpha,\beta)$.

Thus far we have been concerned only with rather elementary properties of the rows in the truth table. The more involved part is to design a scheme of extending recursively the construction of models. In order to explain how this is done, we introduce a chart.

4.2.7 Chart for $(x)(Ey)(z)Mxyz$:




In the chart, the ordinary row pairs satisfying $(Mij', Mji)$ are divided into three classes: basic when $i = j$, cyclic when $i' = j$, common otherwise. The general plan of the procedure is as follows. The existence of basic row quadruples assures that we can find a model for all the matrices $M12$, $M11$, $M23$, $M22$, etc. in the first column. Similarly, we can define cyclic quadruples to give an effective condition for the existence of a model for all matrices appearing in the second column of the chart, and so on. But in order that these models can be combined to give a model for all the matrices and therewith for a given formula 4.2.1, each column must be related to the column on its left in a suitable manner. This situation with two infinite dimensions seems to be the chief cause of the complexity of the $A_1E_1A_1$ case.

In the chart of 4.2.7, each row pair $(R,S)$ that is not basic is subordinate to a quadruple $(A,B; C,D)$ made up of the two row pairs $(A,B)$, $(C,D)$ on its left with arrows leading to it. The quadruple is said to be superior to the pair $(R,S)$.

D4.5 An ordinary row pair $(R,S)$ is a subordinate of a quadruple $(A,B; C,D)$ if

i. $R_{xx} = A_{xx}$, $R_{xy} = A_{xy}$, $R_{yx} = A_{yx}$, $R_{yy} = A_{yy}$, $R_{yz} = C_{xz}$, $R_{zy} = C_{zx}$, $R_{zz} = C_{zz}$;

ii. $S_{xx} = D_{xx}$, $S_{xy} = D_{xy}$, $S_{yx} = D_{yx}$, $S_{xz} = A_{zx}$, $S_{zx} = A_{xz}$.

A quadruple $(R,S; P,Q)$ is subordinate to a row sextuple $(A,B; C,D; K,L)$ if $(R,S)$ is subordinate to $(A,B; C,D)$, and $(P,Q)$ to $(C,D; K,L)$.

D4.6 Two rows $R,S$ form a cyclic row pair $(R,S)$ if

i. $(R,S)$ is an ordinary row pair;

ii. $R_{xy} = S_{zx}$, $R_{yx} = S_{xz}$, $R_{yy} = S_{xx}$, $R_{yz} = S_{xy}$, $R_{zy} = S_{yx}$.

Obviously, given 4.1, if its table contains no two rows forming a cyclic pair, then the conjunction, briefly $C_6$, of $M12$, $M11$, $M23$, $M22$, $M13$, $M21$ is a contradiction.

In the table 4.2.6, there are, among the nine ordinary row pairs, only one that is cyclic, $(\delta_4, \delta_5)$. Since there are only one basic quadruple, each has only one superior. This is of course not always the case, it is only due to special features of the example 4.2.2.

In order to find out whether there is any succession of cyclic pairs which will satisfy all rows of the column for cyclic pairs in the chart, we study cyclic quadruples.

D4.7 Four rows $P,Q,R,S$ form a cyclic quadruple $(P,Q; R,S)$ if

i. $(P,Q)$ and $(R,S)$ are cyclic row pairs;

ii. $Q_{xx} = R_{xx}$, $Q_{xy} = R_{xy}$, $Q_{yx} = R_{yx}$, $Q_{yy} = R_{yy}$;

iii. There is a basic sextuple $(A,B; C,D; K,L)$; which is respectively superior to $(P,Q; R,S)$;

iv. $(P,Q; R,S)$ has a successor which is also a cyclic quadruple.

Obviously, given a formula 4.2.1, if its table contains no rows that form a cyclic quadruple, then the conjunction of $C_6$, $M34$, $M33$, $M24$, $M32$ is a contradiction.

The existence of a cyclic quadruple certainly assures that we can satisfy all the rows of the second column of the chart simultaneously. It assures a bit more: the two pairs $(P,Q)$, $(R,S)$ of a cyclic quadruple are always compatible with any three pairs $(A,B)$, $(C,D)$, $(K,L)$ which form two basic quadruples, respectively superior to them. This is, however, insufficient to secure that all the rows in the first two columns of the chart can be simultaneously satisfied, because it is possible that no cyclic quadruple beginning with $(R,S)$ is subordinate to any quadruple beginning with $(K,L)$. In other words, the blocks might not fit together.

As it happens, this problem does not arise with the example 4.2.2. Since there is only one cyclic pair $(\delta_4, \delta_5)$, there can be at most one cyclic quadruple, viz, $(\delta_4, \delta_5; \delta_4, \delta_5)$. It can be verified by D4.7 that this is indeed a cyclic row quadruple. Since there is only one basic quadruple $(\alpha,\beta; \alpha,\beta)$, we see immediately that by using $(\alpha,\beta)$ for $(Mii', Mii)$ $(i = 1, 2, \cdots)$ and $(\delta_4, \delta_5)$ for $(Mii'', Mi'i)(i = 1, 2, \cdots)$, all these matrices (of the first two columns of the chart) are simultaneously satisfied. Moreover, this is the only possible model for the two initial infinite columns of matrices.

We shall first define common row quadruples, settle 4.2.2, and then come back to the more general question.

D4.8 Two ordinary row pairs $(R,S)$, $(P,Q)$ form a common quadruple $(R,S;\ P,Q)$ of order $k$ [i.e., in the $(2 + k)$th column of the chart] if

i. When $k = 1$, there is a cyclic row sextuple which is superior to $(R,S;\ P,Q)$; or when $k = n + 1$, for some positive integer $n$, there is a common row sextuple of order $n$ which is superior to $(R,S;\ P,Q)$.

ii. $(R,S;\ P,Q)$ has a successor which is a common quadruple of order $k$.

By this definition, we can successively find the common row quadruples of orders 1, 2, etc. In the actual procedure, we examine each time to determine whether we have already enough information to decide the original formula. Only when this is not the case do we find the common quadruples of the next order.

In the case of 4.2.2, since $(\delta_4, \delta_5;\ \delta_4, \delta_5)$ is the only cyclic quadruple, it is easy to verify, by 4.2.6 and D4.5 that $(\delta_4, \delta_5;\ \delta_4, \delta_5)$ is the only common quadruple of order 1. Thus, by D4.5, if $(R,S)$ is subordinate to the cyclic quadruple $(\delta_4, \delta_5;\ \delta_4, \delta_5)$, $R_{yz} = (\delta_4)_{xz} = t$, $R_{zy} = (\delta_4)_{zx} = f$, and $S_{xz} = (\delta_4)_{zx} = f$, $S_{zx} = (\delta_4)_{xz} = t$. By 4.2.6, $(R,S)$ must be $(\delta_4, \delta_5)$.

From this, it follows that, for every $n$, there is exactly one common quadruple of order $n$, viz. $(\delta_4, \delta_5;\ \delta_4, \delta_5)$. This is an immediate consequence of D4.8 and the above transition from the cyclic column to the first common column in the chart. Hence, we have obtained a model for 4.2.2. It is easy to verify that the model for $G$ is just the usual ordering relation $<$ among positive integers.

This completes the solution of the example 4.2.2, which, however, is not a sufficient illustration of the general case. We have to discuss a procedure by considering more complex situations.

### 4.3 *The Procedure*

One possible procedure is to add one infinite column at a time. Thus, it is possible to represent all possible solutions of each column by a graph, and to represent the solutions satisfying all the initial $n'$ columns by a finite set of graphs if it is possible so to represent all solutions satisfying the initial $n$ columns. Since the common columns enjoy a measure of uniformity, simultaneous solutions for all the columns would be assured if suitable repetitions occur. An exact explanation of such a procedure would be quite lengthy. In any case, a successful choice of patterns of repetition has not been found to assure that for every solvable table, such repetition always occurs.

Instead of elaborating the above procedure, we transform the problem to something similar to the abstract question of 4.1. Thus, given any formula of the form 4.2.1, we can, as in 4.2, construct its truth table and find all the common row pairs in the table. Among the common row pairs, some are also cyclic row pairs and some are also basic row pairs.

If now we take the common row pairs $a$, $b$, $c$, $d$, etc., as elementary units which are to fill up the infinite quadrant as shown in the chart given under 4.2.7, then the following scheme appears to be feasible. Suppose the points in the infinite quadrant are to be filled by $a_{ij}$, $i,j = 1, \delta, \cdots$, then we may consider instead all the $2 \times 2$ matrices:

$$\begin{pmatrix} a_{ij} & a_{ij'} \\ a_{i'j} & a_{i'j'} \end{pmatrix}, \qquad \text{for all } i,j = 1,2,\cdots$$

In other words, given the common row pairs, we can form all possible $2 \times 2$ matrices of them which satisfy the relations of subordination. These $2 \times 2$ matrices are then the basic pieces from which we are to obtain an infinite solution subject to the conditions: (a) consecutive rows or columns from two matrices are the same; (b) only basic and cyclic row pairs are permitted in the first two columns.

It can be verified that the problem of finding a model for the original formula is equivalent to that of finding a way to fill up the infinite quadrant by such derived $2 \times 2$ blocks of row pairs.

The abstract problem is: given any finite set of $2 \times 2$ matrices of the form

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

to decide whether it is possible to fill up the infinite quadrant with copies of these pieces. This is not quite the same as the problem of colored plates described in 4.1, because here what is done amounts to coloring the corners, or imposing connections between neighboring sides within a same square.

Any set of such $2 \times 2$ matrices can also be construed as a set of colored plates. Conversely, given any set of colored plates, we can also find in a systematic manner a corresponding set of such matrices such that the solvability problems for them are equivalent. For example, we may replace a colored plate by a block of nine $2 \times 2$ matrices so that the restriction on neighboring sides no longer operates.

It is possible to use a procedure similar to the one described roughly in 4.1. Some change is needed to take care of the additional conditions on the first two columns. Thus, a sufficient condition is to get a cyclic rectangle $m \times n$ on which we can attach a frill of two columns on the left to obtain a rectangle $m \times (2 + n)$ such that: (a) the tops of the first two columns are the same as the bottoms; (*b*) the additional requirements of being basic or cyclic are satisfied by the frills.

### 4.4 *Further Problems*

The discussions so far seem to have barely scratched the surface of a group of rather difficult problems, among which the basic one is probably that of measuring the complexity of formulae in the predicate calculus.

One may measure the complexity of a formula in many different ways. The "simplest" model of a formula may be taken as a semantic measure. The quantifier prefix or graph of a formula may be taken as a syntactic measure. In addition, for formulae with a same prefix, we may also classify the possible matrices by the truth tables. Our knowledge on using these criteria to give detailed classifications seems very limited. One example of the ignorance is the following open problem (Ref. 2, p. 177): whether there is any class of formulae which is neither decidable, nor a reduction class. It appears reasonable to conjecture that there must be such classes, although the first examples which one will get are likely to be artificial ones.

Some of the reduction classes are, formally speaking, surprisingly simple. For example, from the Surányi normal form given above as case VIII, it follows that, for satisfiability, one reduction class is:

4.4.1 Formulae with prefix $(x)(y)(Ez)(w)Mxyzw$, where $M$ contains neither function symbols, nor $=$, nor predicate letters which are not dyadic.

Since each matrix $M$ is effectively determined by a truth table on the atomic formulae in $M$, the class may be viewed as a union of a simple sequence of finite classes $C_1$, $C_2$, etc., where $C_n$ is the subclass of formulae each containing exactly $n$ predicates (or, equivalently, the first $n$ predicates in some enumeration). There is a sense in which the decision problem for each finite set of formulae is solvable, and yet usually we as a matter of fact only solve the problem as a corollary to a solution for some infinite class.

To obtain a semidecision procedure for the class VIII or 4.4.1, we need more complicated arrangements of triples or quadruples of positive integers than the case $A_1E_1A_1$. Take, for example, the class in case VIII. We have to consider not only the triples $(a,b,c)$ with $b = a'$, but all the triples for the first half of the formula, and among them those for the $A_2E_1$ case are used simultaneously for the second half of the formula.

An example is:

4.4.2 $(x)(y)(z)(\sim Gxy \vee \sim Gyz \vee Gxz) \ \& \ (x)(y)(Eu)(\sim Gxx \ \& \ Gyu).$

If we use the Skolem function $g$ of the $A_2E_1$ case, we can rewrite the above as

4.4.3 $$(\sim Gxy \vee \sim Gyz \vee Gxz) \ \& \ (\sim Gxx \ \& \ Gygxy).$$

In general, we are concerned with deciding the satisfiability of formulae of the form

4.4.4 $$Mxyz \ \& \ Nxygxy.$$

As $(x,y,z)$ runs through all triples of positive integers, we get an infinite sequence from 4.4.4, and a semidecision procedure is to decide, for certain cases, whether such an infinite sequence can be simultaneously satisfied.

For example, we may throw together all permutations of a given triple, and confine ourselves to the triples $(a,b,c)$ with $a \leqq b \leqq c$, assigning each of them a lattice point:

$$f(x,y,z) = (x - 1, z - x, z - y),$$
$$f^{-1}(x,y,z) = (x + 1, x + y, x + y + z).$$

The correlation uses all lattice points $(x,y,z)$ of nonnegative integers. For instance, (1,3,5) gets the point (0,2,2).

We might try to create different types of cubes each with eight vertices from $(i,j,k)$ to $(i',j',k')$ and piece them together. But it is not easy to see how to find a procedure analogous to that described in 4.1 which would at the same time take into consideration the second half of the formula.

## V. A PROOF PROCEDURE FOR THE PREDICATE CALCULUS

### 5.1 *The Quantifier-Free Logic F*

Given the definition of formulae in 1.2, we can define sequents, antecedents, consequents, as in Ref. 13, p. 5. The sequents in $F$ are those containing no quantifiers and the rules for $F$ are exactly the same as those for $P_e$ (Ref. 13, p. 8), except for containing not only variables but also functional expressions as terms.

*Example 1.* $1 \neq x', x = x + 1 \rightarrow 1 \neq x + 1$

By the rules $P2$a and $P2$b (Ref. 13, p. 5), this is a theorem if the following is:

$$1 = x + 1, x' = x + 1 \rightarrow 1 = x'.$$

This is a theorem by $P7$ and $P8$ (Ref. 13, p. 8).

*Example 2.* $x + y' = (x + y)', y \neq x + y, y' = v' \supset y = v, v = x + y \rightarrow y' \neq x + y$

By $P2$a, $P2$b, and $P5$b, this is a theorem if the following two sequents are:

i. $x + y' = (x + y)', y = v, v = x + y, y' = x + y \rightarrow y = x + y;$

ii. $x + y' = (x + y)', v = x + y, y' = x + y' \rightarrow y' = v', y = x + y.$

i. is a theorem by $P7$ and $P8$ since we can replace $y$ and $x + y$ by $v$.

ii. is also a theorem because we can replace $v'$ by $(x + y)'$ and then $y'$ by $x + y'$ in the first clause of the consequent and the result is a theorem by $P1$.

These rules in fact yield a decision procedure for all quantifier-free sequents. In order to see this, we use a more efficient method to speed up applications of P7 and P8.

Given an atomic sequent which contains equality but is not yet a theorem by $P1$ or $P7$. List every pair $(a,b)$ if $a = b$ occurs in the antecedent. Extend repeatedly the set of pairs by symmetry and transitivity. Join each pair by the equals sign and add all of them to the antecedent. Now compare each clause in the antecedent with each clause in the consequent to see whether there is a pair of clauses which can be obtained from each other by substituting equals for equals; moreover, examine each equality in the consequent to see whether it can turn into $\alpha = \alpha$ by substituting equals for equals. If either case occurs, the sequent is a theorem. If neither is the case, then we can find an interpretation of the functions and predicates so that the antecedents are all true but the consequents are all false.

### 5.2 *The Rules for Quantifiers.*

In the present formulation of the predicate calculus, one emphasis is on separating out reversible rules of proof which serve to supply decision procedures as well, because they have the property that not only the premises imply the conclusion but also conversely.

The rules governing quantifiers were given in Part I.[1]*

* "S4. When the input problem contains quantifiers, the following preliminary simplifications are made: (i) All free variables are replaced by numbers, distinct numbers for distinct variables. (ii) Vacuous quantifiers, i.e., quantifiers whose variables do not occur in their scopes, are deleted. (iii) Different quantifiers are to get distinct variables; for example, if $(x)$ occurs twice, one of its occurrences is replaced by $(z)$, $z$ being a new variable. This last step of modification is specially useful when occurrences of a same quantifier are eliminated more than once at different stages.

The justification of the reduction to subproblems (Part I, T2.1) is obvious because all truth-functional rules are reversible and $(x)(Gx \,\&\, Hx)$ is a theorem if and only if $(x)Gx$ and $(x)Hx$ both are.

Usually T2.2 (Part I) is true, but restrictions are necessary, as the following example would show:

$$(x)(Ey)[(z)Gyz \,\&\, Hxy].$$

Although $x$ does not occur in the scope of $(z)$, there is no way to bring $(z)$ out of the scope of $(x)$ because the variable $y$ ties up the two clauses in the formula. There are several possible alternatives: one may make exact the restrictions needed, or record the scope of each quantifier in the usual manner, or use the easy simplification that when a quantifier governs a formula with two halves joined by a logical connective but the variable of the quantifier occurs only in one of the two halves, the scope is just that half.

The test of connectedness of variables and functors (Part I, T2.3) is meant as a device to simplify the interconnections between quantifiers. In particular, the test gives a method for ascertaining that certain apparently complex sequents fall under the *AE* provability case. In order, however, actually to bring such a set of sequents into the *AE* form, we need in general transformations similar to those used in reducing a sequent to the miniscope form. Since the process can be tedious, one may prefer an alternative method of not carrying out the transformation but merely determining a bound $k$ such that either the original sequent is a theorem or has a counter-model with no more than $k$ objects. If this alternative is chosen, a method for calculating the bound $k$ has to be devised.

In any case, when we have a finite set of atomic sequents and a set of governing relations among the variables and functors, we should further simplify the matrix, i.e., the set of atomic sequents by the familiar methods of dropping repetitions and immediate consequences.

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"S5. After the above preliminary simplifications, each problem is reduced to as many subproblems as possible in the following manner: (i) Eliminate in the usual manner every truth-functional connective which is not governed by any quantifiers. (ii) Drop every initial positive quantifier (i.e., universal in the consequent or existential in the antecedent that is not in the scope of any other quantifier) and treat its variable as free, i.e., replace all its occurrences by those of a new number. (i) and (ii) are repeated for as long as possible. As a final result of this step, each problem is reduced to a finite set of subproblems such that the problem is a theorem if and only if all the subproblems are.

"T2.1 The original problem is a theorem if and only if all its subproblems (in the above sense) are.

"T2.2 We can separate out $Q$ and its scope from those quantifiers whose variables do not occur in the scope of $Q$.

"T2.3 If two symbols, each a functor or a variable, are not connected in the final matrix, we can always so transform the original sequent as to separate the two quantifiers which give way to them."

If there are two subsets of the set of atomic sequents which contain neither common variables nor common functors, then they can be separated.

Moreover, each atomic formula that contains neither variables nor functors can be eliminated by the familiar method of replacing $F(p)$ by $F(t)$ & $F(f)$. In other words, it can simply be dropped on the ground of the following consideration. E.g., take

$$Guv, G11 \rightarrow Gvk.$$

This is equivalent to the conjunction of:

$$Guv, \mathrm{t} \rightarrow Gvk;$$
$$Guv, \mathrm{f} \rightarrow Gvk.$$

But the second sequent is always true and can be dropped; the t in the first sequent can be dropped, so that we have

$$Guv \rightarrow Gvk.$$

After all the above steps, we arrive at a finite set of finite sets of atomic sequents which, taken together, are equivalent to the original problem. We may consider each finite set of atomic sequents separately and proceed according to the governing relations between their variables and functors.

We can view the set as a formula in the prenex form with a matrix in a conjunctive normal form. Or, if we prefer, we may replace $\rightarrow$ by $\nrightarrow$ and construe the variables as universal quantifiers, the functors as existential quantifiers. Then we get a negation of the formula in prenex form with a matrix in the disjunctive normal form.

In either case, the remaining problem is to be handled by considerations such as those explained in Sections II through IV.

There is an easily mechanizable procedure by which we can, in theory, not only prove all provable formulae, but also refute all formulae which have finite countermodels. All we have to do is test, besides the sequence $M_1$, $M_2$, $M_3$, etc., whether a formula is satisfiable in a domain with one object, or two objects, or etc. For example, given

$$(x)(y)(Ez)Mxyz, \tag{1}$$

if some of $M112$, $M123$, $\cdots$ is contradictory, then the negation of (1) is a theorem; if relative to some finite domain, (1) can be satisfied, then the negation of (1) is not a theorem. For example, (1) is satisfiable in a domain with one object if and only if $M111$ is satisfiable; with two objects, if and only if

$$(x)(y)(Mxy1 \vee Mxy2)$$

or

$$(x)[(Mx11 \vee Mx12) \;\&\; (Mx21 \vee Mx22)]$$

or

$$[(M111 \vee M112) \;\&\; (M121 \vee M122)] \\ \&\; [(M211 \vee M212) \;\&\; (M221 \vee M222)]$$

is satisfiable.

## VI. REMARKS ON MATHEMATICAL DISCIPLINES

Besides the contrast between proving and calculating, there is a contrast between symbol manipulation and number manipulation. There are problems such as proving trigonometric identities, factorization, differentiation and integration, which all appear to be mechanizable. In numerical calculations, it appears likely that the process of choosing one or another method of calculation can also be mechanized in many cases.

There is the problem of applying the methods considered so far to deal with concrete examples.

One example referred to in Part I[1] (p. 231) is Hintikka's derivation of a contradiction from his own formal system.[14] Here, intuitive understanding is required to select from the set of all axioms suitable members which are sufficient to produce contradictions. Experience, however, shows that, even after a reasonable selection is made, to actually give an exact derivation of a contradiction remains quite a dreary affair. In such a case, the sort of procedure discussed in this paper can be useful.

In fact, Hintikka uses five axioms to derive a contradiction. Write briefly:

$$Hayz \quad \text{for} \quad z \neq a \;\&\; z \neq y \;\&\; z \in y \;\&\; y \in z.$$

The conjunction of the axioms is:

$$\begin{aligned} &(Ex)(Ey)(x \neq y) \;\& \\ &\quad (Ea)(Eb)(Ec)(Ed)(y)\,\{[y \neq a \supset (y \in a \equiv (Ez)Hayz)] \;\& \\ &\qquad\qquad [y \neq b \supset (y \in b \equiv \sim(Ez)Hbyz)] \;\& \qquad (2) \\ &\quad [y \neq c \supset (y \in c \equiv (y = a \vee y = b))] \;\& \\ &\qquad\qquad [y \neq d \supset (y \in d \equiv y = c)]\}. \end{aligned}$$

The assertion is that (2) leads to a contradiction. In other words, (2) has no model, and its negation is a theorem of the predicate calculus. To decide whether this assertion is true, we only have to test (2) by essentially the method of Section III because (2) can be transformed into a formula with $EA_2E$ prefix. Such a method yields also a proof or a refutation of the assertion that (2) gives a contradiction.

In a different direction, we may consider some simple examples in the arithmetic of positive integers.

First, we consider the example, $x' \neq x$. We wish, in other words, to prove, with the help of induction, that this is a consequence of the axioms:

$$x' \neq 1,$$
$$x' \neq y' \rightarrow x \neq y.$$

As a general principle, we try to use induction. Since there is only one variable, we reduce the problem to:

$$(x)x' \neq 1,\ (x)(y)(x' = y' \supset x = y) \rightarrow 1' \neq 1, \tag{3}$$

$$(x)x' \neq 1,\ (x)(y)(x' = y' \supset x = y),\ x' \neq x \rightarrow x'' \neq x'. \tag{4}$$

These can be dealt with by the program described in Part I, except that, to avoid confusion, we use now $a$, $b$, $c$, etc., instead of numerals to replace the positive variables. We have:

$$1' = 1,\ u = v \rightarrow x' = 1,$$
$$1' = 1 \rightarrow x' = 1,\ u' = v',$$
$$u = v,\ a'' = a' \rightarrow x' = 1,\ a' = a,$$
$$a'' = a' \rightarrow x' = 1,\ a' = a,\ u' = v'.$$

These sequents are all true by substitution: 1 for $x$ in the first two; $a'$ for $u$ and $a$ for $v$ in the last two.

As a somewhat more complex example, we take the commutativity of addition. In order to prove $x + y = y + x$, we may use induction either on $x$ or on $y$. We arbitrarily take the earliest variable:

$$1 + y = y + 1, \tag{5}$$

$$x + y = y + x \rightarrow x' + y = y + x'. \tag{6}$$

To prove $1 + y = y + 1$, we make induction on $y$:

$$1 + 1 = 1 + 1,$$
$$1 + a = a + 1 \rightarrow 1 + a' = a' + 1.$$

The first is a theorem by the property of equality. To prove the second, we use another general principle, viz., when a defined symbol occurs, we make use of the definition. In this particular case, we make use of the recursive definition of addition, and try to prove

$$u + 1 = u', u + v' = (u + v)', 1 + a = a + 1 \rightarrow 1 + a' = a' + 1.$$

In order to derive the consequent from the antecedent, we start from $1 + a'$ and $a' + 1$, use the equalities in the antecedent to transform them, and attempt to find a chain to join them. Thus, we may try to make all possible applications of the three equalities in the antecedent:

$$(1 + a) + 1 \text{---} (a + 1) + 1$$
$$1 + a' \text{---} (1 + a)' \text{---} (a + 1)' \text{---} (a')' \text{---} a' + 1$$
$$1 + (a + 1) \text{---} 1 + (1 + a)$$

$$a' + 1 \text{---} (a + 1) + 1 \text{---} (1 + a) + 1$$
$$(a')' \text{---} (a + 1)' \text{---} (1 + a)' \text{---} 1 + a'$$

In general, we may begin two trees simultaneously from both sides of the equality, do not write down any term which has already occurred in the same tree, and stop when a common term appears on both trees. When we get to the more complicated situations, we have to investigate two additional things. First, it would take too long to search through trees, so that it is desirable to organize available informations in forms which are more quickly accessible. Second, we may exhaust two trees and still fail to get a common term. Then we need to prove some lemma which would join up the two trees.

For example, the above graphs give us a proof of (5). To prove the other induction hypothesis, viz. (6), we may try to do the same with:

$$u + 1 = u', u + v' = (u + v)', a + b = b + a \rightarrow a' + b = b + a',$$

$$a' + b \text{---} (a + 1) + b$$

$$b + a' \text{---} (b + a)' \text{---} (a + b)' \text{---} a + b' \text{---} a + (b + 1)$$
$$b + (a + 1) \quad (b + a) + 1 \text{---} (a + b) + 1$$

In this way, we have exhausted the applicable cases of the equalities in the antecedent. Since we have proved the first induction hypothesis (5), we can add it to the antecedent. Then we get some further extensions:

$$\begin{aligned} &(a + 1) + b \text{——} (1 + a) + b, \\ &b + (a + 1) \text{——} b + (1 + a), \\ &a + (b + 1) \text{——} a + (1 + b). \end{aligned}$$

At this stage, we would ask whether any other given theorem can be used to join up the two trees for $a' + b$ and $b + a'$, or, if not, what a reasonable lemma would be. If the associative law has been proved, we may observe that the missing link is supplied by:

$$(a + 1) + b = a + (1 + b). \tag{7}$$

Otherwise we should try to make a "reasonable" selection of some suitable lemma and prove it. If, for example, we have chosen (7), we would try to establish it by induction on $a$ or on $b$.

It is possible that the quantifier-free theory of positive integers, including arbitrary simple recursive definitions, can be handled mechanically with relative ease, and yield fairly interesting results. The restriction to quantifier-free methods means that we are concerned only with quantifier-free theorems to be proved without using quantifiers in, e.g., applying the principle of mathematical induction. It is clear from works in the literature that this restricted domain of number theory is rather rich in content. It goes beyond logic in an essential way because of the availability of (quantifier-free) mathematical induction.

With regard to the general questions of using machines to assist mathematical research, there is a fundamental contrast between problem and method. While it seems natural to choose first the objective (e.g., number theory or geometry) and then look for methods, it is likely that a more effective approach is to let the methods lead the way. For example, since the known interesting decidable classes of formulae of the predicate calculus either do not contain function symbols or do not contain quantifiers, we are led to the simple examples above: quantifier-free number theory or function-free set theory.

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