The Covariance Function of a Simple Trunk Group, with Applications to Traffic Measurement*

By V. E. BENEŠ

(Manuscript received July 6, 1960)

Erlang's classical model for telephone traffic in a loss system is considered: N trunks, calls arriving in a Poisson process and negative exponential holding times; calls which cannot be served at once are dismissed without retrials. Let $N(t)$ be the number of trunks in use at t. An explicit formula for the covariance $R(\cdot)$ of $N(\cdot)$ in terms of the characteristic values of the transition matrix of the Markov process $N(\cdot)$ is obtained. Also, $R(\cdot)$ is expressed purely in terms of constants and the "recovery" function, i.e. the transition probability $Pr\{N(t) = N \mid N(0) = N\}$; $R(\cdot)$ is accurately approximated by $R(0)e^{r_1 t}$, with r_1 the largest negative characteristic value, itself well approximated (underestimated) by $-E\{N(\cdot)\}/R(0)$. Exact and approximate formulas for sampling error in traffic measurement are deduced from these results.

I. INTRODUCTION

A theoretical study of sampling fluctuations in telephone traffic measurements is useful both in designing procedures for measuring traffic loads and in interpreting field observations. Hayward[1] and Palm[2] have given an approximate formula for the sampling error incurred when observations of the numbers of calls in existence are made at fixed intervals of time. Their formula has the disadvantage that it is derived for a probabilistic model (of the traffic) in which there is an infinite number of available trunks. Thus there is no limit to the number of calls which can be in progress at one time, and no congestion. Two important parameters, the number N of trunks in the group, and the probability p_N of loss, are left out of account. For this reason the practical

* This work was completed in part while the author was visiting lecturer at Dartmouth College, Fall-Winter, 1959–60.

application of this model is usually restricted to large groups of trunks which are lightly loaded.

In this paper we derive and study the *covariance function* of the simplest stochastic model of a finite group of N trunks. The sampling error in traffic measurements can be calculated exactly from the covariance. We find formulas for the magnitudes of fluctuations of observed traffic for both periodic and continuous observation. The exact formulas lead to simple approximations similar to Hayward's, which take account of the number of trunks. Our results are summarized and discussed in Section II.

We shall use A. K. Erlang's classical probabilistic model for a group of trunks, described as follows:

i. Holding times of trunks are mutually independent, each with a negative exponential distribution. Time is measured in units of mean holding time.

ii. Epochs at which calls arrive form a Poisson process of intensity $a > 0$, independently of the holding times. The offered load is then a erlangs.

iii. There are $N < \infty$ trunks; calls which find all N of these trunks busy are "lost," and are cleared from the system.

These assumptions determine a Markov stochastic process $N(t)$, $-\infty < t < \infty$, the number of trunks in use at time t. $N(\cdot)$ is a random step-function fluctuating in unit steps between 0 and N. As is well known, $N(\cdot)$ has stationary probabilities $\{p_n, n = 0,1,\cdots, N\}$ given by the (first) Erlang distribution

$$p_n = \frac{\dfrac{a^n}{n!}}{\displaystyle\sum_{k=0}^{N} \frac{a^k}{k!}} \tag{1}$$

$$= \text{equilibrium probability that } n \text{ trunks are busy.}$$

With this choice of absolute probabilities, $N(\cdot)$ is a strictly stationary process, whose mean and variance are respectively

$$m_1 = a(1 - p_N),$$

$$\sigma^2 = m_1 - ap_N(N - m_1).$$

The probability p_N of loss is shown in Fig. 1, the fractional occupancy $N^{-1}m_1$ in Fig. 2, and the variance σ^2 in Fig. 3.

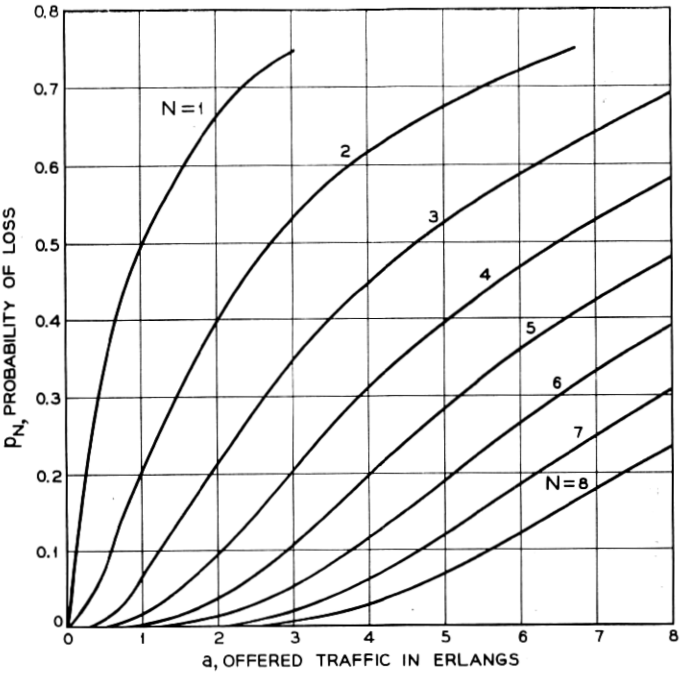

Fig. 1 — Probability p_N of loss.

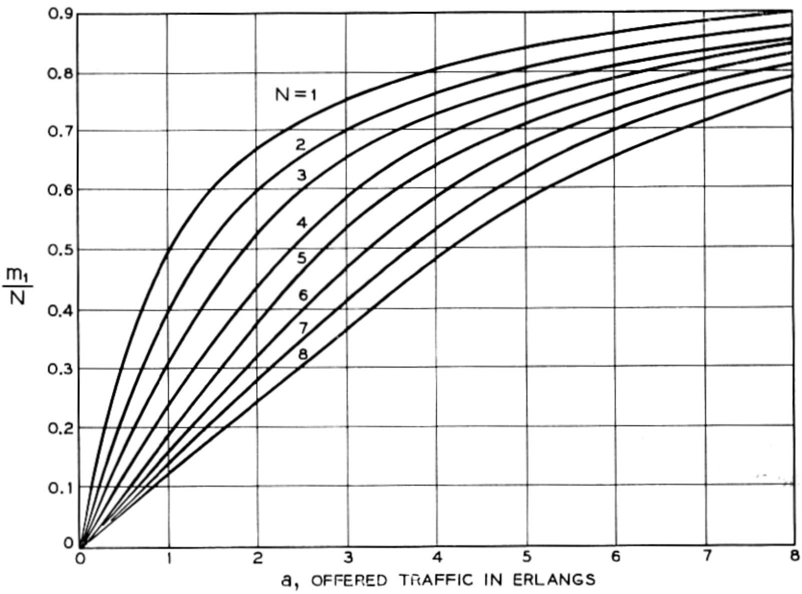

Fig. 2 — Fractional occupancy m_1/N.

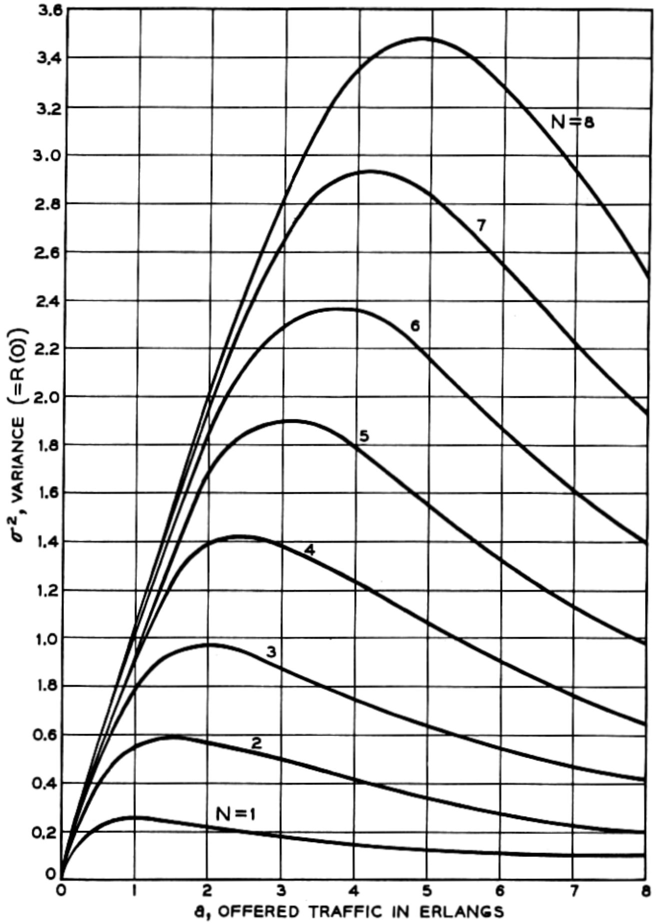

Fig. 3 — Equilibrium variance.

II. DISCUSSION, SUMMARY AND CONCLUSIONS

The covariance $R(t,s)$ between samples $N(t),N(s)$ of the stochastic process $N(\cdot)$ is the average of the product of $N(t)$ and $N(s)$, minus the product of the averages:

$$R(t,s) = E\{N(t)N(s)\} - E\{N(t)\}E\{N(s)\}.$$

Since $N(\cdot)$ is a stationary real process, we have $R(t,s) = R(\,|\,t - s\,|\,)$. The function $R(\cdot)$ is called the *covariance* function of the process $N(\cdot)$.

It can be written as

$$R(t) = \lim_{u\to\infty} E\{N(t+u)N(u)\} - E\{N(t+u)\}E\{N(u)\}$$
$$= \sum_{m=0}^{N} mp_m \sum_{n=0}^{N} n \Pr\{N(t) = n \mid N(0) = m\} - \left(\sum_{m=0}^{N} mp_m\right)^2, \tag{2}$$

where $\{p_m\}$ are the stationary (or equilibrium) probabilities given by (1), and

$$\Pr\{N(t) = n \mid N(0) = m\}$$

denotes the transition probability that n trunks are busy at time t if m were busy at time 0. The function $R(\cdot)$ expresses the average dependence or correlation between samples of $N(\cdot)$ taken at times t apart.

The principal practical use of the covariance function $R(\cdot)$ in the theory of telephone traffic is in computing theoretical estimates of sampling error incurred in traffic load measurements. Two methods of measuring traffic, the *switch-count* and the *time-average*, are considered in this paper. In the switch-count, n observations $\{x_1, \cdots, x_n, x_j = N(j\tau), j = 1, \cdots, n\}$ of the random process are made at intervals τ apart; the average

$$\frac{1}{n}\sum_{j=1}^{n} N(j\tau) = \frac{1}{n}\sum_{j=1}^{n} x_j = n^{-1}S_n$$

is then used as an estimate of the carried load $m_1 = a - ap_N$. This method is important economically because it is cheaper to scan trunk groups periodically than to observe them continuously. The number τ is the *scan interval*, and the number $S_n = x_1 + \cdots + x_n$ is called the (total) *number of paths in service*, in n observations. Table I lists actual mean holding times, scan intervals used and resulting values of τ for

Table I — Holding Times, Scan Intervals and Values of τ

Type of Call	Typical Holding Time (seconds)	Scan Interval (seconds)		Ratio τ of Scan Interval to Holding Time	
		U.S.A.	Europe	U.S.A.	Europe
Local Call	100–200	100	36	1 to $\frac{1}{2}$	$\frac{1}{3}$ to $\frac{1}{6}$
Long Distance Call	200–600	100	36	$\frac{1}{2}$ to $\frac{1}{6}$	$\frac{1}{6}$ to $\frac{1}{20}$
Originating Register Holding Time	10–15	10 or 100	36	1 to $\frac{2}{3}$ or 10 to 7	4 to 2
No. 5 Marker Holding Time	0.25–1.0	10	—	—	—

various types of call. The variance of $n^{-1}S_n$ is expressible in terms of the covariance $R(\cdot)$ as

$$\operatorname{Var}\{n^{-1}S_n\} = n^{-2} \sum_{j=-n}^{n} (n - |j|)R(j\tau). \tag{3a}$$

In the *time-average*, the continuously recorded sample average

$$M(T) = T^{-1} \int_0^T N(t)\, dt,$$

is used to estimate the carried load. The variance of this estimate is

$$\operatorname{Var}\{M(T)\} = 2T^{-2} \int_0^T (T - t)R(t)\, dt. \tag{3b}$$

Thus the mean square error of both these methods of traffic measurement can be calculated theoretically if the covariance $R(\cdot)$ is known.

In formula (2) the covariance function is expressed in terms of the stationary probabilities $\{p_n\}$ given by the Erlang distribution, and the transition probabilities

$$p_{mn}(t) = \Pr\{N(t) = n \mid N(0) = m\}.$$

In the theory of telephone traffic, the particular transition probability

$$p_{NN}(t) = \Pr\{N(t) = N \mid N(0) = N\}$$

has been singled out (in Refs. 3 and 4, for example) as a suitable "recovery" or "relaxation" function that is characteristic of the dynamic behavior of the Markov process $N(\cdot)$ in point of the undesirable "all trunks busy" condition.

We shall show that a much more cogent reason than this can be adduced to support the importance of the recovery function to traffic theory: the covariance function $R(\cdot)$ can be expressed entirely in terms of the recovery function and the offered load a. In other words, a single one of the $(N + 1)^2$ transition probabilities appearing in formula (2) suffices for determining the covariance function, and this one is the recovery function $p_{NN}(\cdot)$. This fact is a theoretical justification of the intuitive view that the recovery function is important, for now the variances of $n^{-1}S_n$ and of $M(T)$ are expressible using only the recovery function.

We next give a summary of the contents of the remaining sections; this is followed by an account of specific results and conclusions.

An exact formula for the covariance $R(\cdot)$ is stated and discussed in Section III, and derived in Section VII. The formula readily yields a rigorous upper bound which appears to give a close approximation to $R(\cdot)$ itself. In Section IV the recovery function $p_{NN}(\cdot)$ is given, and it is shown how the covariance may be expressed in terms of the recovery function by a convolution integral. The variance of $n^{-1}S_n$ is studied in Section V; an exact formula, and an approximating upper bound [based on the upper bound for $R(\cdot)$], are both obtained. The variance of the time-average $M(T)$ is considered in Section VI; again, an exact formula and an approximating upper bound are found.

The covariance function $R(\cdot)$ is bounded from above and closely approximated by a single exponential function

$$R(t) \leqq \sigma^2 e^{r_1 t}, \qquad \sigma^2 = R(0), \qquad r_1 < 0.$$

Here

$$\begin{aligned} \sigma^2 &= R(0) \\ &= \text{equilibrium variance of } N(\cdot) \\ &= (\text{load carried}) - (\text{load lost})(\text{average number of idle trunks}), \end{aligned}$$

and the reciprocal time constant r_1 in the exponent is the dominant* characteristic value of the "rate" or "transition" matrix of the differential equations satisfied by the transition probabilities. Alternately, r_1 is the root of least magnitude of a Poisson-Charlier polynomial. The root r_1 is shown as a function of offered traffic a for $N = 1, \cdots, 8$ in Fig. 4, and is tabulated in Table II.

A lower bound for r_1, *depending only on the mean and variance of* $N(\cdot)$, is derived in Section VIII by making use of the fact that the matrix of the differential equations for the transition probabilities is symmetrizable. For low values of offered traffic per trunk, i.e., $a/N < 1$, this bound can be used to approximate r_1. In any case, the bound is a convenient starting place for the use of Newton's method. The bound is the ratio $-m_1/\sigma^2$, which satisfies the inequality

$$-\frac{m_1}{\sigma^2} \leqq r_1 < -1,$$

with

$$\begin{aligned} m_1 &= \text{equilibrium mean of } N(\cdot) \\ &= \text{load carried} = a(1 - p_N), \\ \sigma^2 &= \text{equilibrium variance of } N(\cdot) \\ &= (\text{load carried}) - (\text{load lost})(\text{average number of idle trunks}). \end{aligned}$$

The approximation $r_1 \cong -m_1/\sigma^2$ is illustrated in Fig. 5.

* I.e., that of least magnitude (among the nonzero characteristic values).

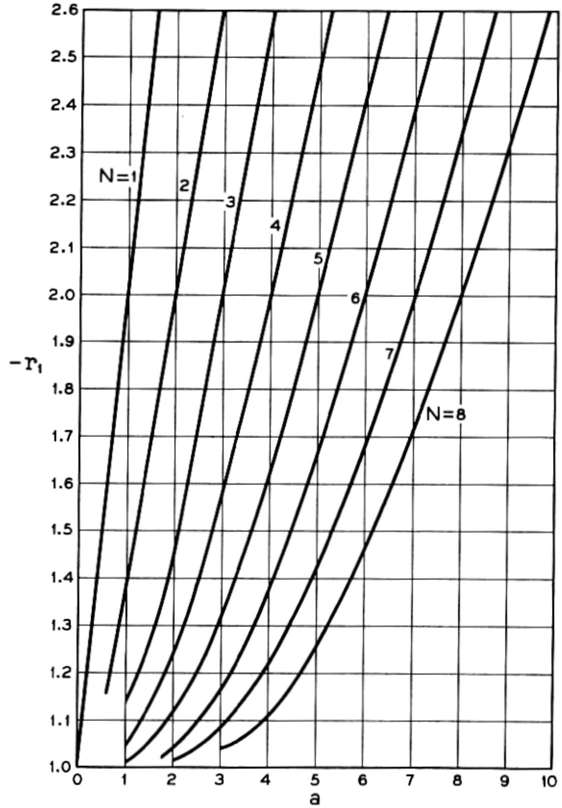

Fig. 4 — Negative of the root r_1 of smallest magnitude as a function of load a for $N = 1, \cdots, 8$.

TABLE II — NEGATIVE OF DOMINANT CHARACTERISTIC VALUE r_1

a	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$
1	1.043967	1.011448	1.002421	1.000421	1.000062
2	1.249464	1.112166	1.045044	1.015806	1.004800
3	1.582363	1.326321	1.172257	1.084025	1.037229
4	2.000000	1.629624	1.383389	1.222707	1.121762
5	2.477548	2.000000	1.663799	1.427870	1.265214
6	3.000000	2.422137	2.000000	1.689991	1.463798
7	3.557618	2.885474	2.381627	2.000000	1.710891
8	4.143703	3.382497	2.800900	2.350437	2.000000
9	4.753426	3.907677	3.251918	2.735363	2.325514
10	5.383178	4.456828	3.730121	3.150052	2.682770

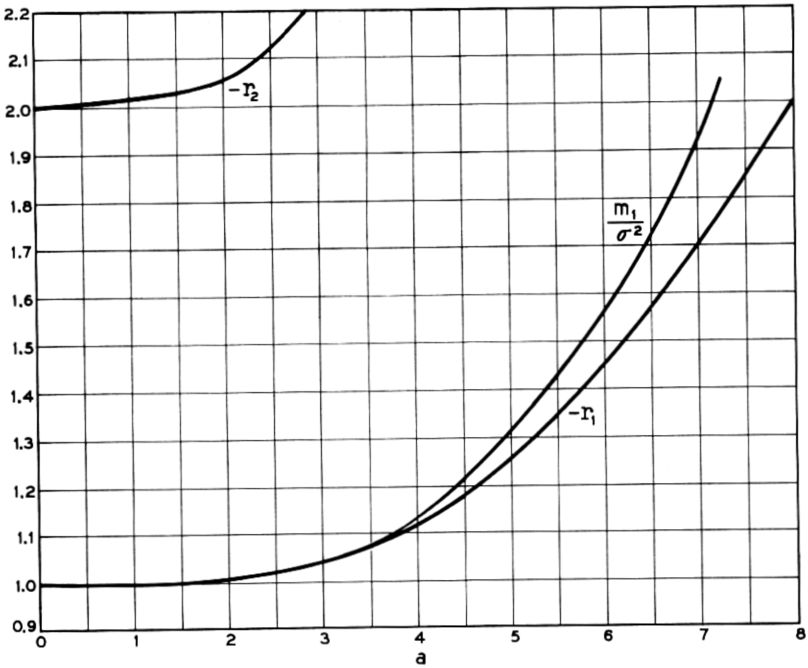

Fig. 5 — Illustration of the approximation $r_1 \cong -m_1/\sigma^2$.

By the "infinite trunk" model we shall henceforth mean the stochastic model for telephone traffic determined by all the same assumptions that we made in the Introduction, except that $N = \infty$; i.e., an unlimited number of trunks is postulated. Riordan[5] and Beneš[6,7] have considered this model; Hayward[1] based his sampling error formula on it.

It is widely believed that the "infinite trunk" model is applicable to large groups of lightly loaded trunks. Such a belief is gratuitous until comparisons with a model having a finite number of trunks are made. Studying the covariance function of the simple finite trunk group enables us to make some of the needed comparisons; e.g., the variances of $n^{-1}S_n$ and $M(T)$ in the two models are of particular interest. Knowledge of the covariance $R(\cdot)$, however, is also relevant to the other three cases to which engineers are loath to apply the "infinite trunk" model, viz.:

i. large groups of heavily loaded trunks,
ii. small groups of lightly loaded trunks,
iii. small groups of heavily loaded trunks.

The variance of $n^{-1}S_n$ is bounded from above and approximated by the formula

$$\mathrm{Var}\{n^{-1}S_n\} \leqq n^{-1}\sigma^2 \left\{\mathrm{ctnh}\,\lambda - \frac{1 - e^{-2n\lambda}}{2n}\,\mathrm{csch}^2\,\lambda\right\}, \tag{4}$$

where n is the number of observations, and

$$\lambda = -\frac{\tau r_1}{2} = -\tfrac{1}{2}\,(\text{scan interval})\,(\text{dominant characteristic value}).$$

The *exact* formula for the variance of $n^{-1}S_n$ in the "infinite trunk" model is

$$n^{-1}a\left\{\mathrm{ctnh}\,\tfrac{1}{2}\tau - \frac{1 - e^{-n\tau}}{2n}\,\mathrm{csch}^2\,\tfrac{1}{2}\tau\right\}. \tag{5}$$

The upper bound (4) for the finite group is compared with the exact formula (5) for the "infinite trunk" model in Fig. 6, which shows each formula as a function of the scan interval τ for various n, for $a = 20$ erlangs offered to 20 trunks. The curves suggest that the upper bound for $\mathrm{Var}\{n^{-1}S_n\}$ for $N < \infty$ is consistently less than the corresponding variance in the "infinite trunk" model. As might be expected, increasing the scan interval τ improves accuracy for the same number of observations. This is because the covariance function is positive, and monotone in $|t|$.

The variance of $M(T)$ is bounded from above and approximated by

$$\mathrm{Var}\{M(T)\} \leqq 2\sigma^2\,\frac{e^{r_1T} - 1 - r_1T}{T^2r_1^2}, \tag{6}$$

where T is the length of the time-interval of continuous observation, and σ^2 and r_1 are, as before, the variance of $N(\cdot)$ and the dominant characteristic value, respectively. The exact formula for the variance of $M(T)$ in the "infinite trunk" model is

$$2a\,\frac{e^{-T} - 1 + T}{T^2}. \tag{7}$$

Since $r_1 < -1$, and σ^2 is always less than a if $N < \infty$, the "infinite trunk" model overestimates the variance of $M(T)$ if applied to a finite group. This conclusion is illustrated in Fig. 7, which shows the formulas (6) and (7) for a load of 20 erlangs offered to 20 trunks. For an observation time of 10 mean holding times the "infinite trunk" formula (7) applied here would overestimate the variance by about 500 per cent. This is about as extreme a case as would occur in practice. Fig. 7 also

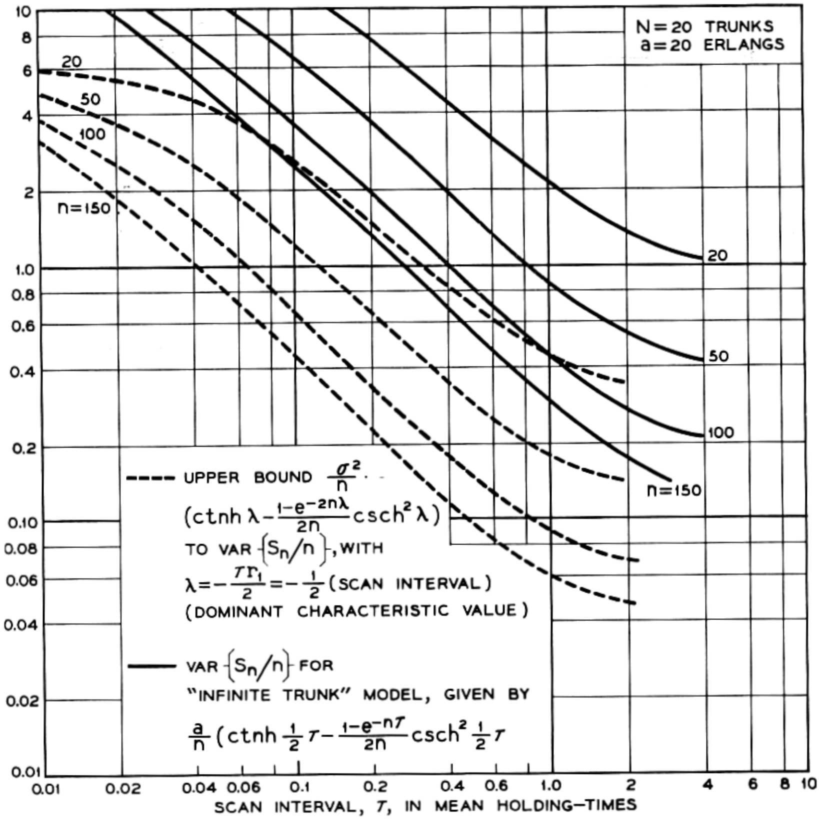

Fig. 6 — Comparison of variance of S_n/n for finite and infinite trunk models.

depicts a "mixed" formula obtained by replacing a by σ^2 in the "infinite trunk" formula (6); for 10 mean holding times the "mixed" formula only overestimates the variance by about 100 per cent. Thus most of the discrepancy is due to the difference between σ^2 and a.

Our conclusions are set down in the following list:

1. The average dynamic behavior of the process $N(\cdot)$, as described by the covariance function $R(\cdot)$, can be adequately determined from the dominant characteristic value r_1 and the variance σ^2 of $N(\cdot)$.

2. The same parameters, r_1 and σ^2, suffice to give simple approximating upper bounds for the sampling error incurred in both periodic and continuous observation of $N(\cdot)$. These bounds depend on the size N of the trunk group.

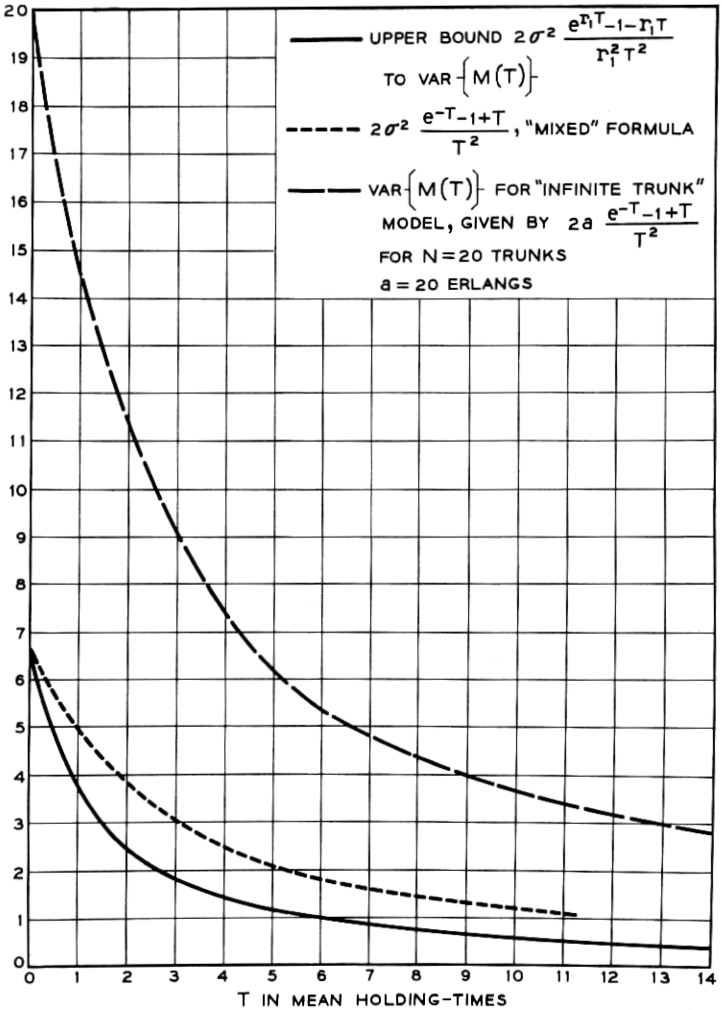

Fig. 7 — Comparison of variance of $M(T)$ for finite and infinite trunk models.

3. In terms of r_1 and σ^2 it is possible to check the applicability, for theoretical estimates of sampling error, of the "infinite trunk" model which assumes $N = \infty$.

4. The "infinite trunk" model, applied to finite trunk groups, consistently and often grossly overestimates the sampling error. The overestimation occurs largely because σ^2 is always less, and for heavy traffic

is much less, than a, the (Poisson) variance of $N(\cdot)$ in the "infinite trunk" model.

5. In terms of r_1 and σ^2 it is possible to design sampling procedures for traffic measurement that depend explicitly on the number N of trunks in the group. By these methods, a given accuracy can be obtained with less observation, and thus at lower cost, than the "infinite trunk" model would require.

6. Hence for finite groups of trunks traffic sampling procedures which are based on the "infinite trunk" model tend to be wasteful, particularly for heavy traffic. The parameters r_1 and σ^2 provide a systematic way of tailoring the measurement procedure to the number of trunks in the group.

III. THE COVARIANCE FUNCTION

To state the formula for $R(\cdot)$ we need the "sigma" functions* defined (see Riordan[9]) as

$$\sigma_0(m) = \frac{a^m}{m!},$$

$$\sigma_k(m) = \sum_{j=0}^{m} \binom{k+j-1}{j} \frac{a^{m-j}}{(m-j)!},$$

with m (but not k) a nonnegative integer. These functions are connected with the Poisson-Charlier polynomials

$$p_n(x) = a^{n/2}(n!)^{\frac{1}{2}} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j! a^{-j} \binom{x}{j}$$

by the relation

$$\sigma_k(m) = (-a^{\frac{1}{2}})^m (m!)^{-\frac{1}{2}} p_m(-k).$$

(See Ref. 10, p. 33.)

For fixed N and a, let $r_1, r_2, \cdots, r_N$ be (in order of increasing magnitude) the N zeros in the variable s of the polynomial $\sigma_{s+1}(N)$. In Section VII the covariance is shown to be given by (the exact formula)

$$R(t) = -a^2 p_N \sum_{j=1}^{N} \frac{e^{r_j t}}{r_j(1+r_j)^2} \prod_{i \neq j} [1 - (r_j - r_i)^{-1}] \qquad (8)$$

where p_N is the probability of loss. It has been shown† that the zeros r_j are all real, negative, and distinct; all are less than -1, and consecu-

---

* The σ notation is copied from unpublished work of H. Nyquist. The functions themselves were introduced into traffic theory by Palm.[8]

† The earliest reference appear to be Haantjes[11] in 1938. See also Ledermann and Reuter.[12]

tive pairs are separated by at least unity. Fig. 8 shows these roots for $N = 1, 2, 3$ as functions of a.

Now r_j is always negative, and the terms of the product satisfy

$$1 - \frac{1}{r_j - r_i} > 0; \tag{9}$$

hence the sum in (8) has all terms negative, so that

$$R(t) \geqq 0, \qquad \text{all } t.$$

The correlation between successive samples is thus always positive. It is obvious from (8) that

$$-a^2 p_N \sum_{j=1}^{N} \frac{1}{r_j(1 + r_j)^2} \prod_{i \neq j} \frac{r_j - 1 - r_i}{r_j - r_i} = \sigma^2 = R(0). \tag{10}$$

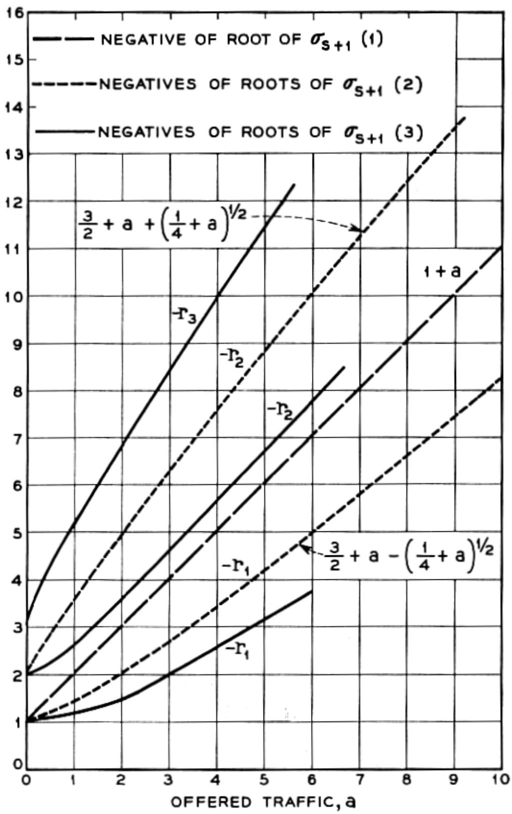

Fig. 8 — Roots of the first three σ-functions.

Since r_1 is the root closest to zero, the value of (8) is only increased if the r_j in the exponents of (8) are replaced by r_1. Using (9) and (10), we conclude that

$$0 \leqq \sigma^2 e^{r_1 t} - R(t) = \xi(t), \tag{11}$$

where

$$\xi(t) = a^2 p_N \sum_{j=2}^{N} \frac{e^{r_j t} - e^{r_1 t}}{r_j(1 + r_j)^2} \prod_{i \neq j} \frac{r_j - 1 - r_i}{r_j - r_i},$$

and

$$\xi(t) \leqq a^2 p_N e^{-t} \sum_{j=2}^{N} (j + 1)^{-2} \leqq a^2 p_N \frac{4\pi^2 - 30}{24} e^{-t}$$

$$\leqq (0.3933) a^2 p_N e^{-t}.$$

The approximation $R(t) \cong \sigma^2 e^{r_1 t}$ is illustrated in Figs. 9 and 10. It appears to be fairly accurate, especially for light loads.

The upper bound $\sigma^2 e^{r_1 t}$ for $R(t)$ should be compared with the rigorous formula (see Riordan[5] and Beneš[7])

$$R(t) = ae^{-t},$$

which holds for the "infinite trunk" model. In this model the equilibrium distribution of occupancy is Poisson, so that

$$R(0) = \sigma^2 = \mathrm{Var}\{N(t)\} = E\{N(t)\} = a,$$

and the "time constant" of the exponential is unity, since time is measured in units of mean holding time.

The difference between the "infinite trunk" model and the "finite trunk" model in point of the covariance can be understood by considering the effect of congestion, which is present in the latter. Congestion affects the upper bound formula most directly through the value of the variance σ^2. It is obvious intuitively, and borne out in Fig. 3, that as a increases σ^2 must eventually decrease to zero. This behavior is not mimicked by the "infinite trunk" model, for which $\sigma^2 = a$.

The finitude of N, i.e., congestion, affects the bound $\sigma^2 e^{\{r_1 t\}}$ in two ways: (a) the "time constant" is not unity but the smaller number $-(r_1)^{-1}$, so that the rate at which dependence between samples of $N(\cdot)$ decreases (as a function of the interval between samples) is larger than in the "infinite trunk" model; this "time constant" decreases as the traffic a increases, because, as illustrated by Fig. 4, r_1 is a monotone

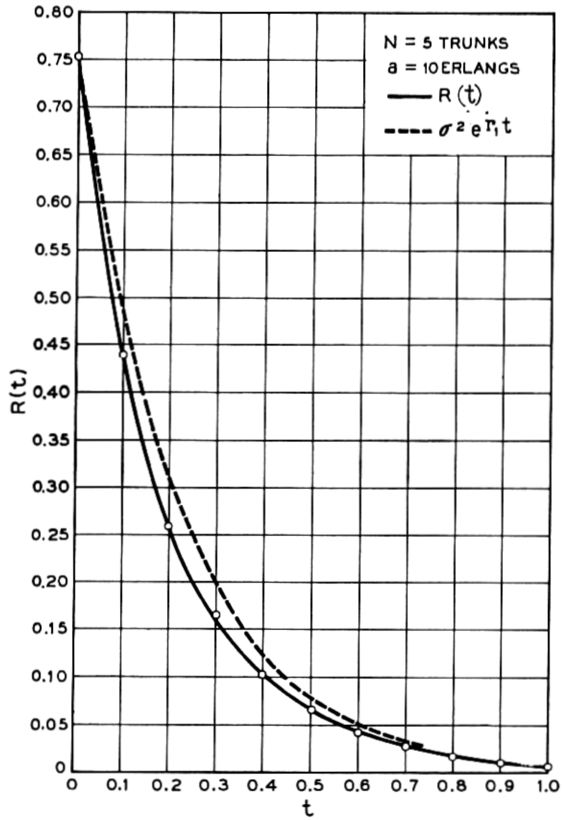

Fig. 9 — The covariance $R(t)$ for $N = 5$ trunks, $a = 10$ erlangs, with the approximate formula $R(t) \sim \sigma^2 e^{r_1 t}$.

decreasing function of a; (b) the value of $R(0)$ $(=\sigma^2)$ is not a but the generally much smaller number

$$\sigma^2 = a(1 - p_N)\left[1 - ap_N\left(\frac{N}{a - ap_N} - 1\right)\right],$$
$$= a[1 - p_N(1 + N - a + ap_N)].$$

The last form shows that $\sigma^2 < a$ for all a and N. In fact, it is obvious intuitively that

$$\sigma^2 = m_1 - ap_N(N - m_1) < m_1 < a.$$

A simple approximation for the dominant root r_1 can sometimes be

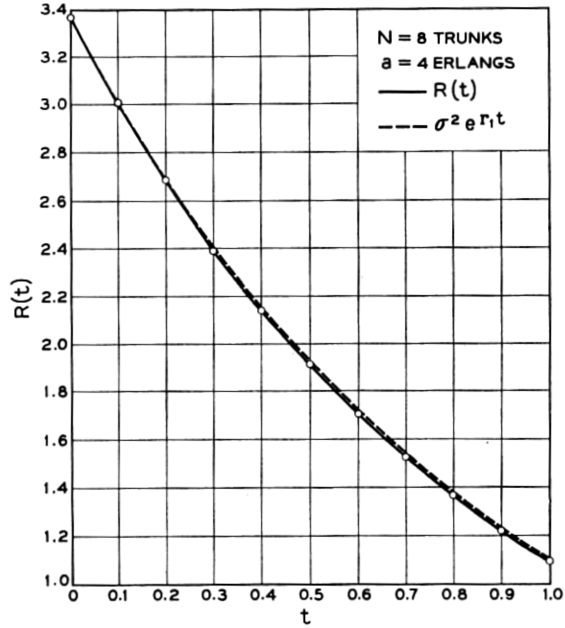

Fig. 10 — The covariance $R(t)$ for $N = 8$ trunks, $a = 4$ erlangs, with the approximate formula $R(t) \sim \sigma^2 e^{r_1 t}$.

used to make the approximation $R(t) \cong \sigma^2 e^{\{r_1 t\}}$ more useful. It is shown in Section VIII that

$$-\frac{m_1}{\sigma^2} = -\frac{\text{carried load}}{\text{load variance}} \leqq r_1 ;$$

i.e., $-m_1/\sigma^2$ is a rigorous lower bound to r_1. Fig. 5 suggests this bound gives a fairly good approximation to r_1 if $a/N < 1$. Hence a simple approximate formula for $R(\cdot)$, valid for $a/N < 1$, is given by

$$R(t) \cong \sigma^2 e^{\{-m_1 t/\sigma^2\}}$$

$$\cong (\text{load variance}) \exp\left\{-\frac{\text{carried load}}{\text{load variance}}\, t\right\}. \qquad (12)$$

We know that $R(t) \leqq \sigma^2 e^{\{r_1 t\}}$ and that $-m_1/\sigma^2 < r_1$; hence replacing r_1 by $-m_1/\sigma^2$ tends to correct the error in the upper bound formula. The formula (12) is illustrated in Fig. 11.

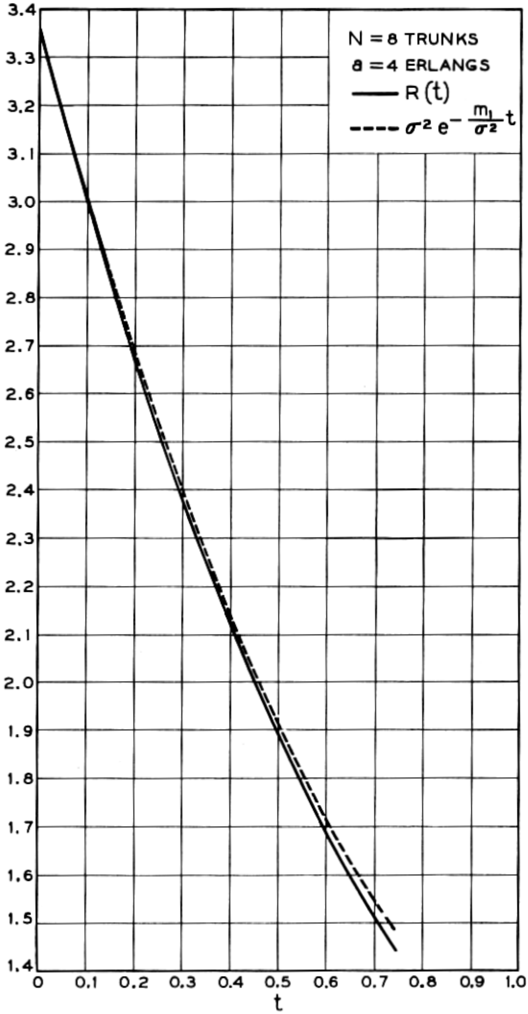

Fig. 11 — Comparison of $R(t)$ with $\sigma^2 e^{-(m_1/\sigma^2)t}$ for $N = 8$ trunks, $a = 4$ erlangs.

IV. THE COVARIANCE IN TERMS OF THE RECOVERY FUNCTION

It has been shown[3] that the Laplace transform of $p_{NN}(\cdot)$ is

$$\frac{\sigma_s(N)}{s\sigma_{s+1}(N)}.$$

By expansion in partial fractions we find that

$$p_{NN}(t) = p_N - \sum_{j=1}^{N} \frac{e^{r_j t}}{r_j} \prod_{i \neq j} \left(1 - \frac{1}{r_j - r_i}\right). \tag{13}$$

The sum assumes only negative values, and so $p_{NN}(\cdot)$ decreases monotonically to the loss probability p_N. The recovery function is illustrated in Fig. 12.

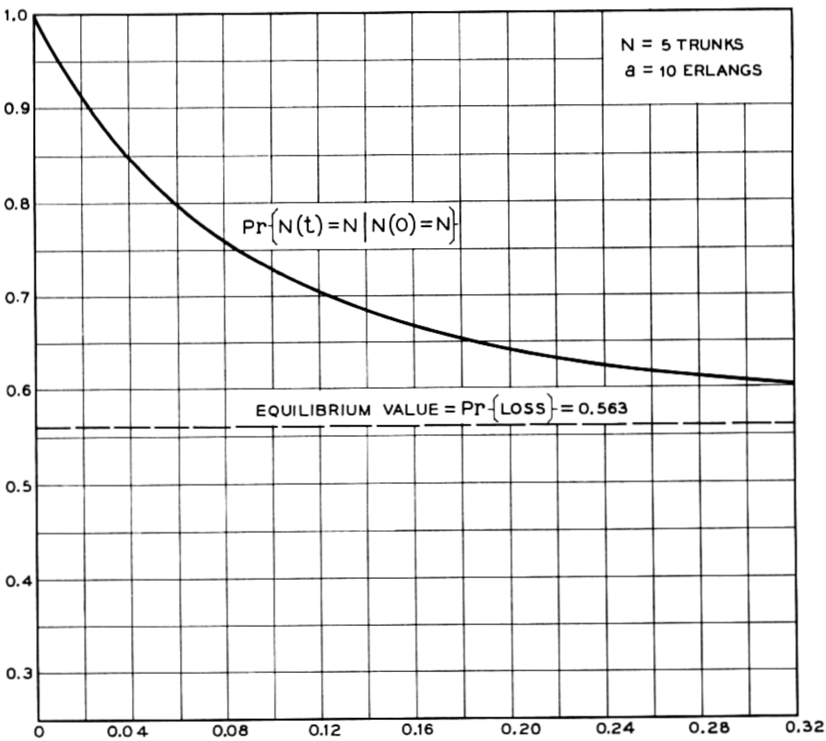

Fig. 12 — Recovery function for $N = 5$ trunks, $a = 10$ erlangs.

We now observe that for each $j = 1, \cdots, N$,

$$\int_0^t (t - u)\, e^{-(t-u)+r_j u}\, du = \frac{e^{r_j t} - e^{-t}}{(r_j + 1)^2} - \frac{t e^{-t}}{r_j + 1}. \tag{14}$$

By comparison of formulas (8) and (13), and use of (14), one finds that

$$R(t) = a^2 p_N \int_0^t (t - u) e^{-(t-u)} [p_{NN}(u) - p_N]\, du + \sigma^2 e^{-t} + C t e^{-t}, \tag{15}$$

where

$$C = -a^2 p_N \sum_{j=1}^{N} \frac{1}{r_j(r_j + 1)} \prod_{i \neq j} \left(1 - \frac{1}{r_j - r_i}\right).$$

This formula expresses $R(\cdot)$ in terms of $p_{NN}(\cdot)$ by a simple convolution. To evaluate C explicitly we note that

$$C = -a^2 p_N \left[\frac{\sigma_{s+1}(N - 1)}{(1 + s)\sigma_{s+1}(N)} - \frac{a_{-1}}{1 + s} \right]_{s=0},$$

where a_{-1} is the first coefficient in the power series expansion of the left-hand term in the bracket. One finds

$$a_{-1} = \frac{\sigma_0(N - 1)}{\sigma_0(N)} = \frac{N}{a},$$

$$\begin{aligned} C &= a^2 p_N \left(\frac{N}{a} - 1 + p_N \right) \\ &= a p_N (N - m_1) \\ &= \text{(load lost) (average number of idle trunks)}. \end{aligned}$$

V. THE VARIANCE OF THE NUMBER OF PATHS IN SERVICE

We assume that n observations $\{x_j, j = 1, \cdots, n\}$ of $N(\cdot)$ are made during an interval of equilibrium, so that

$$\operatorname{Cov}\{x_i, x_j\} = R(|i - j|\, \tau),$$

where τ is the scan interval. Then with

$$\begin{aligned} S_n &= x_1 + x_2 + \cdots + x_n \\ &= \text{number of paths found in service}, \end{aligned}$$

we find that

$$\begin{aligned}\operatorname{Var}\{S_n\} &= E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j\right\} - E^2\left\{\sum_{i=1}^{n} x_i\right\} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{Cov}\{x_i, x_j\} \qquad (16) \\ &= \sum_{j=-n}^{n} (n - |j|)R(j\tau).\end{aligned}$$

To give an exact formula for $\operatorname{Var}\{S_n\}$ we note that

$$\sum_{m=-\infty}^{\infty} e^{-2|m|u} = \operatorname{ctnh} u,$$

$$\sum_{m=-\infty}^{\infty} |m| e^{-2|m|u} = -\frac{d}{du}\sum_{m=1}^{\infty} e^{-2mu} = \tfrac{1}{2}\operatorname{csch}^2 u$$

and

$$\begin{aligned}\sum_{m=n}^{\infty} (m-n)e^{-2mu} &= \tfrac{1}{2}e^{-2nu}\sum_{m=-\infty}^{\infty} |m| e^{-2|m|u} \\ &= \tfrac{1}{4}e^{-2nu}\operatorname{csch}^2 u.\end{aligned}$$

Then also

$$\begin{aligned}\sum_{j=-n}^{n} (n - |j|)e^{-2|j|u} &= n\sum_{m=-\infty}^{\infty} e^{-2|m|u} - \sum_{m=-\infty}^{\infty} |m| e^{-2|m|u} \\ &\quad + 2\sum_{m=n}^{\infty} (m-n)e^{-2mu} \qquad (17) \\ &= n\operatorname{ctnh} u - \frac{(1-e^{-2nu})}{2}\operatorname{csch}^2 u.^*\end{aligned}$$

Since the covariance $R(\cdot)$ is a symmetric function given by (8), it can be seen that

$$\begin{aligned}\operatorname{Var}\{n^{-1}S_n\} &= \\ &-n^{-1}a^2 p_N \sum_{j=1}^{N} \frac{\left[\operatorname{ctnh}\left(-\frac{\tau r_j}{2}\right) - \frac{1-e^{n\tau r_j}}{2n}\operatorname{csch}^2\left(-\frac{\tau r_j}{2}\right)\right]}{r_j(1+r_j)^2} \qquad (18) \\ &\cdot \prod_{i\neq j}\left(1 - \frac{1}{r_j - r_i}\right).\end{aligned}$$

* Use of this identity was suggested by unpublished work of J. W. Tukey to which the author had access.

This formula is exact, given the assumptions. It is easily shown from formula (17) that the exact formula for the variance of $n^{-1}S_n$ in the "infinite trunk" model is

$$n^{-1}a\left\{\operatorname{ctnh}\tfrac{1}{2}\tau - \frac{1 - e^{-n\tau}}{2n}\operatorname{csch}^2\tfrac{1}{2}\tau\right\},$$

illustrated in Fig. 6.[13]

Returning to the case of finitely many trunks, we can obtain approximating upper bounds to formula (18) for $\operatorname{Var}\{n^{-1}S_n\}$ by using the results of Section III on the covariance function. It can be seen from the arguments leading to (17) that replacing the roots r_j by r_1 in the hyperbolic functions in (18) increases the values of the expressions in square brackets; this replacement is equivalent to using the upper bound

$$\sigma^2 e^{r_1 t}$$

for $R(t)$ in formula (16). Hence

$$\operatorname{Var}\{n^{-1}S_n\} \leqq n^{-1}\sigma^2\left\{\operatorname{ctnh}\lambda - \frac{1 - e^{-2n\lambda}}{2n}\operatorname{csch}^2\lambda\right\}, \tag{19}$$

where

$$\lambda = -\frac{\tau r_1}{1} = -\tfrac{1}{2}(\text{scan interval})(\text{dominant characteristic value}).$$

Since $\sigma^2 e^{r_1 t}$ is close to $R(t)$, we may expect that the overestimate (19) gives a good approximation to the actual variance. This approximation is conveniently plotted as a function of λ for various n in Fig. 13.

VI. THE VARIANCE OF TIME AVERAGES

It follows from formulas (3b) and (8) that

$$\operatorname{Var}\left\{\int_0^T N(t)\,dt\right\} = c_0 + c_1 T + o(e^{-T}) \tag{20}$$

as $T \to \infty$, where

$$c_0 = 2a^2 p_N \sum_{j=1}^{N} \frac{\prod_{i\neq j}\left(1 - \frac{1}{r_j - r_i}\right)}{r_j^{\,3}(1 + r_j)^2}$$

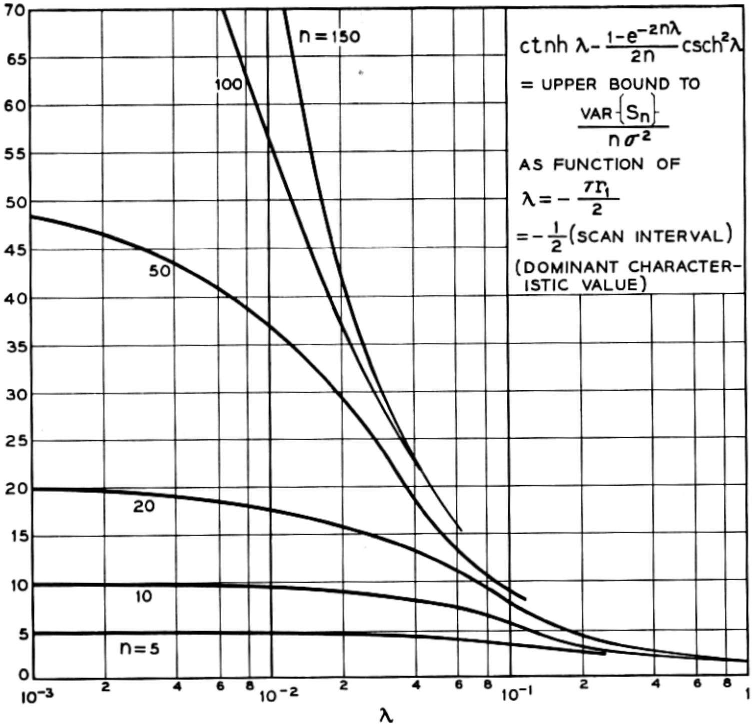

Fig. 13 — Upper bound to $\text{Var}\{S_n\}/n\sigma^2$.

is a negative constant, and

$$c_1 = \int_0^\infty R(u)\,du = 2a^2 p_N \sum_{j=1}^{N} \frac{\prod_{i \neq j}\left(1 - \frac{1}{r_j - r_i}\right)}{r_j^2(1 + r_j)^2}.$$

Note that c_0 and c_1 differ only in the power of r_j that occurs in the denominators. The third term of (20) is positive; is given by

$$o(e^{-T}) = -2a^2 p_N \sum_{j=1}^{N} \frac{e^{r_j T}}{r_j^3(1 + r_j)^2} \prod_{i \neq j}\left(1 - \frac{1}{r_j - r_i}\right);$$

equals $-c_0$ at $T = 0$; and is of smaller order than e^{-T} because $r_1 < -1$. To evaluate c_1 explicitly, we note that

$$c_1 = -2a^2 p_N\left[\frac{\sigma_{s+1}(N-1)}{s(1+s)^2\sigma_{s+1}(N)} - \frac{1 - p_N}{s} - \frac{a_{-1}}{1+s} - \frac{a_{-2}}{(1+s)^2}\right]_{s=0}, \quad (21)$$

where a_{-2}, a_{-1} are respectively the first and second coefficients in the power series expansion of the leftmost term in the bracket of (21); these are given by

$$a_{-2} = \frac{\sigma_0(N-1)}{\sigma_0(N)} = \frac{N}{a},$$

$$a_{-1} = \frac{d}{dx}\frac{\sigma_{x+1}(N-1)}{x\sigma_{x+1}(N)}\bigg]_{x=-1}$$

$$= -\frac{N}{a} + \frac{1-p_N}{ap_N}.$$

To find c_1 we must compute

$$\lim_{s\to 0}\left[\frac{\sigma_{s+1}(N-1)}{s(1+s)^2\sigma_{s+1}(N)} - \frac{1-p_N}{s}\right] = -\frac{d}{dx}\frac{\sigma_{x+1}(N-1)}{(1+x)^2\sigma_{x+1}(N)}\bigg]_{x=0}.$$

This equals

$$2(1-p_N) - \frac{d}{dx}\frac{\sigma_{x+1}(N-1)}{\sigma_{x+1}(N)}\bigg]_{x=0},$$

or

$$(1-p_N)\left[2 - \frac{d}{dx}\log\sigma_{x+1}(N-1) + \frac{d}{dx}\log\sigma_{x+1}(N)\right]_{x=0}.$$

Now the generating function of the σ-functions is

$$\Phi(s,z) = \sum_{n=0}^{\infty} z^n\sigma_s(n) = (1-z)^{-s}e^{az}$$

so that

$$\frac{\partial}{\partial s}\Phi(s,z) = \Phi(s,z)\sum_{n=1}^{\infty}\frac{z^n}{n},$$

$$\frac{d}{ds}\sigma_s(n) = \frac{\sigma_s(0)}{n} + \frac{\sigma_s(1)}{n-1} + \cdots + \frac{\sigma_s(n-1)}{1},$$

$$\xi_n = \frac{d}{dx}\log\sigma_{x+1}(n)\bigg]_{x=0} = \sum_{j=0}^{n-1}\frac{\sigma_1(j)}{(n-j)\sigma_1(n)}.$$

It follows that

$$c_1 = 2a^2p_N[a_{-2} + a_{-1} + (1-p_N)(2-\xi_{N-1}+\xi_N)]$$

$$= 2\sigma^2 + 2a^2p_N(1-p_N)(1-\xi_{N-1}+\xi_N) + 2aNp_N.$$

The constant c_0 can be evaluated in a similar fashion.

From the bounds (11) for $R(\cdot)$ we conclude that

$$0 \leqq 2\sigma^2 \left(\frac{e^{r_1 T} - 1 - r_1 T}{T^2 r_1^2} \right) - \mathrm{Var}\{M(T)\}$$

$$\leqq (0.3933) a^2 p_N \left(\frac{e^{-T} - 1 + T}{T^2} \right),$$

and since $R(t) \cong \sigma^2 e^{r_1 t}$, we may expect that the *overestimate*

$$2\sigma^2 \frac{e^{r_1 T} - 1 - r_1 T}{T^2 r_1^2} \tag{22}$$

is a good approximation to the variance of $M(T)$. This approximation has the same form as the exact formula (7) for the "infinite trunk" model, because in both cases a single exponential is used for $R(\cdot)$ in formula (8). The overestimate (22) is depicted graphically in Fig. 14. It was convenient to plot the ratio

$$\frac{\mathrm{Var}\{M(T)\}}{\sigma^2}$$

as a function of the single parameter

$$\mu = r_1 T = -(\text{dominant characteristic value})\ (\text{observation time}).$$

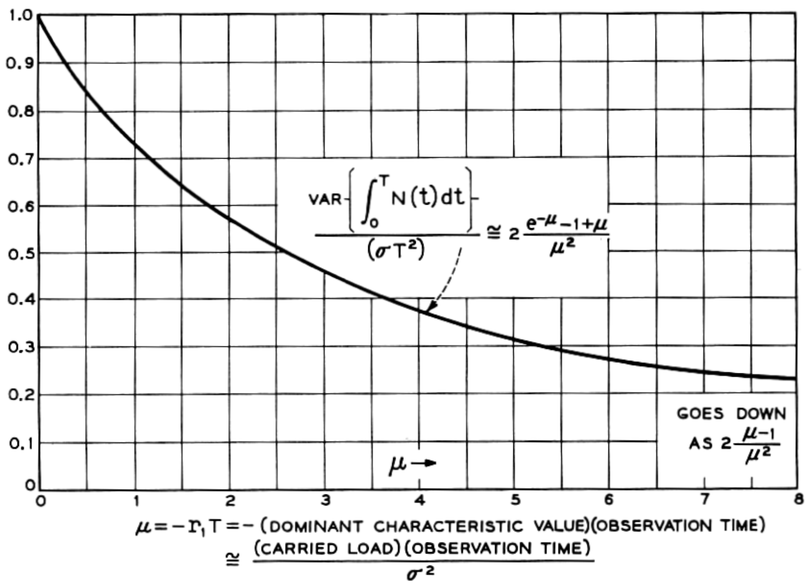

Fig. 14 — Approximation to $\mathrm{Var} \int_0^T \{N(t)dt\}/(\sigma T)^2$.

A simpler form of (22), valid for $a/N < 1$, results when we replace r_1 by its lower bound

$$-\frac{m_1}{\sigma^2} = -\frac{\text{carried load}}{\text{load variance}} \leqq r_1 .$$

This replacement decreases the value obtained, i.e., moves the approximation in the direction of $\mathrm{Var}\{M(T)\}$.

VII. DERIVATION OF THE COVARIANCE

The transition probabilities

$$p_{mn}(t) = \Pr\{N(t) = n \mid N(0) = m\}$$

of $N(\cdot)$ satisfy the Kolmogorov equations

$$\begin{aligned} p_{mn}(0) &= \delta_{mn} , \\ \frac{d}{dt} p_{mN} &= a p_{m(N-1)} - N p_{mN} , \\ \frac{d}{dt} p_{mn} &= (n+1) p_{m(n+1)} + a p_{m(n-1)} - (a+n) p_{mn} , \qquad 0 < n < N, \\ \frac{d}{dt} p_{m0} &= p_{m1} - a p_{m0} . \end{aligned} \tag{23}$$

Multiplying the nth equation by n, and summing on the index n, we find

$$\frac{d}{dt} E\{N(t) \mid N(0) = m\} = -E\{N(t) \mid N(0) = m\} + a[1 - p_{mN}(t)],$$

whence

$$E\{N(t) \mid N(0) = m\} = m e^{-t} + a \int_0^t e^{-(t-u)} [1 - p_{mN}(u)] \, du.$$

By formula (2), the covariance is then

$$\begin{aligned} R(t) &= \sum_{m=0}^{N} m p_m E\{N(t) \mid N(0) = m\} - m_1^2 \\ &= m_2 e^{-t} + a m_1 (1 - e^{-t}) - m_1^2 - a \int_0^t e^{-(t-u)} \sum_{m=0}^{N} m p_m p_{mN}(u) \, du, \end{aligned}$$

where

$$m_i = \sum_{n=0}^{N} n^i p_n$$

for $i = 1, 2$ and $\{p_n\}$ are the stationary probabilities given by (1). In particular,

$$m_1 = a(1 - p_N), \tag{24}$$

$$\sigma = (m_2 - m_1^2)^{\frac{1}{2}} = [m_1 - ap_N(N - m_1)]^{\frac{1}{2}}. \tag{25}$$

The Laplace transform of

$$\Pr\{N(\cdot) = N \mid N(0) = m\}$$

has been determined[3] to be

$$\frac{a^{N-m} m! \sigma_s(m)}{N! s \sigma_{s+1}(N)}.$$

Therefore that of $R(\cdot)$ is

$$\begin{aligned} R^*(s) = \int_0^\infty e^{-st} R(t)\, dt &= \frac{m_2}{1+s} + \frac{am_1}{s(1+s)} - \frac{m_1^2}{s} \\ &\quad - \frac{a}{s(1+s)\sigma_{s+1}(N)} \sum_{m=1}^{N} m p_m \frac{a^{N-m} m! \sigma_s(m)}{N!}. \end{aligned} \tag{26}$$

By (1), the last term of (26) is

$$-\frac{ap_N}{s(1+s)\sigma_{s+1}(N)} \sum_{m=1}^{N} m\sigma_s(m).$$

It has been shown[9] that the "sigma" functions satisfy the recurrences

$$\sigma_s(m) = \sigma_{s+1}(m) - \sigma_{s+1}(m-1), \tag{27}$$

$$m\sigma_s(m) = a\sigma_s(m-1) + s\sigma_{s+1}(m-1), \tag{28}$$

so that

$$\begin{aligned} \sum_{m=1}^{N} m\sigma_s(m) &= a\sum_{k=0}^{N-1} \sigma_s(k) + s\sum_{k=0}^{N-1} \sigma_{s+1}(k) \\ &= a\sigma_{s+1}(N-1) + s\sigma_{s+2}(N-1), \end{aligned}$$

and

$$\frac{\sigma_{s+2}(N-1)}{\sigma_{s+1}(N)} = \frac{N}{s+1} \frac{a\sigma_{s+1}(N-1)}{(s+1)\sigma_{s+1}(N)}.$$

The foregoing identities yield the following simplified formula for $R^*(s)$:

$$R^*(s) = \frac{m_2}{1+s} + \frac{am_1}{s(1+s)} - \frac{aNp_N}{(1+s)^2} - \frac{a^2 p_N}{1+s}\left[\frac{\sigma_{s+1}(N-1)}{s(1+s)\sigma_{s+1}(N)}\right] - \frac{m_1^2}{s}. \tag{29}$$

From (27) we find that the partial fraction expansion

$$\frac{\sigma_{s+1}(N-1)}{\sigma_{s+1}(N)} = \sum_{j=1}^{N} - \frac{\sigma_{r_j}(N)N!}{(s-r_j)\prod_{i\neq j}(r_j - r_i)}$$
$$= \sum_{j=1}^{N} (s-r_j)^{-1} \prod_{i \neq j} \frac{r_j - 1 - r_i}{r_j - r_i},$$

is valid, where $\{r_j\}$ are the zeros of $\sigma_{s+1}(N)$.

By a similar argument, since $p_N = \sigma_0(N)/\sigma_1(N)$,

$$\frac{\sigma_{s+1}(N-1)}{s(1+s)\sigma_{s+1}(N)} = \frac{1-p_N}{s} - \frac{N}{a(1+s)} + \sum_{j=1}^{N}(s-r_j)^{-1}\frac{1}{r_j}\frac{1}{1+r_j}\prod_{i\neq j}\frac{r_j-1-r_i}{r_j-r_i}.$$

Hence formula (29) can be inverted to give, for $t \geqq 0$,

$$\begin{aligned} R(t) &= m_2 e^{-t} + am_1[1 - e^{-t}] - aNp_N te^{-t} - m_1^2 \\ &\quad - a^2 p_N \int_0^t e^{-(t-u)}\left[1 - p_N - \frac{N}{a}e^{-u} + \sum_{j=1}^{N}\frac{e^{r_j u}}{r_j(1+r_j)} \cdot \prod_{i\neq j}\frac{r_j-1-r_i}{r_j-r_i}\right]du, \\ &= \sigma^2 e^{-t} + a^2 p_N K e^{-t} - a^2 p_N \sum_{j=1}^{N}\frac{e^{r_j t}}{r_j(1+r_j)^2}\prod_{i\neq j}\frac{r_j-1-r_i}{r_j-r_i}, \end{aligned} \tag{30}$$

where

$$\sigma^2 = m_2 - m_1^2 = \text{equilibrium variance},$$

and

$$K = \sum_{j=1}^{N}\frac{1}{r_j}\frac{1}{(1+r_j)^2}\prod_{i\neq j}\frac{r_j-1-r_i}{r_j-r_i}.$$

To evaluate K explicitly we observe that

$$K = -\left[\frac{\sigma_{s+1}(N-1)}{(1+s)^2\sigma_{s+1}(N)} - \frac{a_{-1}}{1+s} - \frac{a_{-2}}{(1+s)^2}\right]_{s=0}, \qquad (31)$$

where a_{-2}, a_{-1} are respectively the first and second coefficients in the power series expansion of the leftmost term in the bracket of (31). Thus

$$K = a_{-1} + a_{-2} - 1 + p_N .$$

Now

$$\frac{\sigma_{s+1}(N-1)}{(1+s)^2\sigma_{s+1}(N)} = (1+s)^{-2}\frac{\sigma_0(N-1)}{\sigma_0(N)} + (1+s)^{-1} \cdot\left[\frac{d}{dx}\frac{\sigma_{x+1}(N-1)}{\sigma_{x+1}(N)}\right]_{x=-1} + \sum_{j=1}^{N}\frac{(s-r_j)^{-1}}{(1+r_j)^2}\prod_{i\neq j}\frac{r_j-1-r_i}{r_j-r_i}.$$

From the recurrence (28) for the σ-functions we find that

$$s\,\frac{\sigma_{s+1}(N-1)}{\sigma_s(N)} - N + a\,\frac{\sigma_s(N-1)}{\sigma_s(N)} = 0;$$

differentiating with respect to s and setting $s = 0$, we obtain

$$a_{-1} = \frac{d}{ds}\frac{\sigma_{s+1}(N-1)}{\sigma_{s+1}(N)}\bigg|_{s=-1} = \frac{1}{a}\left(-\frac{\sigma_1(N-1)}{\sigma_0(N)}\right) = -\frac{1-p_N}{ap_N}.$$

Clearly,

$$a_{-2} = \frac{\sigma_0(N-1)}{\sigma_0(N)} = \frac{N}{a},$$

and so

$$K = -\frac{1-p_N}{ap_N} + \frac{N}{a} - 1 + p_N,$$

$$a^2p_NK = -\sigma^2.$$

Thus the formula (30) for the covariance function $R(\cdot)$ simplifies to

$$R(t) = -a^2p_N\sum_{j=1}^{N}\frac{e^{r_jt}}{r_j(1+r_j)^2}\prod_{i\neq j}\frac{r_j-1-r_i}{r_j-r_i}. \qquad (32)$$

VIII. APPROXIMATION TO THE DOMINANT CHARACTERISTIC VALUE

The differential equations (23) can be written in the form

$$\frac{d}{dt}P(t) = QP(t),$$

where $P(t)$ is the matrix of transition probabilities $\{p_{mn}(t)\}$ and Q is the matrix of the "transition rates":

$$Q = \begin{pmatrix} -a & 1 & 0 & 0 \cdots 0 & & 0 \\ a & (-a-1) & 2 & 0 & & 0 \\ 0 & a & (-a-2) & 3 & & \\ \vdots & & & & & \vdots \\ & & & & 0 & \\ & & & & N-1 & 0 \\ 0 & 0 & \cdots & a & (-a-N+1) & N \\ 0 & 0 & \cdots & 0 & a & -N \end{pmatrix}.$$

The characteristic values of Q are $0, r_1, r_2, \cdots, r_N$. We define

$$\mu_n = \frac{1}{p_n} = n!a^{-n} \sum_{j=0}^{N} \frac{a^j}{j!}, \qquad n = 0,1,\cdots,N,$$

and we introduce an inner product for the space $L_2(\mu)$ of $(N + 1)$-tuples of complex numbers by the definition

$$(x,y) = \sum_{n=0}^{N} x_n \bar{y}_n \mu_n .$$

The matrix Q represents a symmetric operator on $L_2(\mu)$, i.e.,

$$(Qx,y) = (x,Qy), \qquad x,y \in L_2(\mu).$$

It is easily seen that

$$\sum_{n=0}^{N} \frac{1}{\mu_n} = \sum_{n=0}^{N} p_n = 1, \tag{33}$$

$$Qp = 0, \qquad \text{for} \quad p = (p_0, p_1, \cdots, p_N), \tag{34}$$

$$Q_{ij}\mu_i = Q_{ji}\mu_j, \qquad i,j = 0,1,\cdots,N. \tag{35}$$

The last identity implies that

$$(Qx,y) = -\frac{1}{2} \sum_{i,j} \overline{(y_i\mu_i - y_j\mu_j)} \frac{Q_{ji}}{\mu_i} (\mu_i x_i - \mu_j x_j),$$

$$(Qx,x) \leqq 0,$$

so (as we already know) all characteristic values of Q are nonpositive, being of the form (Qx,x) for some $x \in L_2(\mu)$.

From the extremal properties of the characteristic values of symmetric operators (e.g., Zaanen,[14] p. 383, Theorem 3) we conclude that

$$r_1 = \max(Qx,x),$$

the maximum being over all $x \in L_2(\mu)$ which are not identically zero, and satisfy $(x,x) = 1$, $(x,p) = 0$, p being the vector of stationary probabilities, as in (34).

We can now estimate r_1 from below by choosing an appropriate vector x. We choose

$$x_n = \frac{n - m_1}{\sigma \mu_n}, \qquad n = 0,1,\cdots, N,$$

where m_1 and σ are the mean and standard deviation of $N(\cdot)$ in equilibrium, given by formulas (24) and (25) respectively. Clearly, $(x,x) = 1$ and $(x,p) = 0$, and

$$\begin{aligned}(Qx,x) &= -a \sum_{n=0}^{N-1} p_n \left(\frac{n - m_1}{\sigma} - \frac{n + 1 - m_1}{\sigma}\right)^2 \\ &= -\frac{a(1 - p_N)}{\sigma^2} \\ &= -\frac{m_1}{\sigma^2} \leqq r_1 .\end{aligned}$$

(See Kramer.[15])

This approximation is illustrated in Fig. 5.

IX. ACKNOWLEDGMENTS

The author thanks S. O. Rice and J. W. Tukey for access to unpublished manuscripts and helpful discussions, and A. Descloux for giving the manuscript a careful reading. The suggestions of R. I. Wilkinson and J. Riordan are also appreciated.

REFERENCES

1. Hayward, W. S., Jr., The Reliability of Telephone Traffic Load Measurements by Switch Counts, B.S.T.J., **31,** 1952, p. 357.
2. Palm, C., Tekniska Medelanden fran Kungl. Telegrafstyrelsen, 1941, nr. 7–9.
3. Beneš, V. E., Transition Probabilities for Telephone Traffic, B.S.T.J., **39,** 1960, p. 1297.
4. Kosten, L., Over de invloed van herhaalde oproepen in de theorie der blokkeringskansen, De Ingenieur, **47,** 1947, p. 123.
5. Riordan, J., Telephone Traffic Time Averages, B.S.T.J., **30,** 1951, p. 1129.
6. Beneš, V. E., A Sufficient Set of Statistics for a Simple Telephone Exchange Model, B.S.T.J., **36,** 1957, p. 939.

7. Beneš, V. E., Fluctuations of Telephone Traffic, B.S.T.J., **36,** 1957, p. 965.
8. Palm, C., Calcul exact de la perte dans les groupes de circuits échelonnés, Ericsson Tech., **4,** 1936, p. 41.
9. Riordan, J., appendix to Wilkinson, R. I., Theories for Toll Traffic Engineering in the U.S.A., B.S.T.J., **35,** 1956, p. 507.
10. Szegö, G., *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, New York, 1938.
11. Haantjes, J., Wiskundige Opgaven, **17,** 1938.
12. Ledermann, W. and Reuter, G. E., Spectral Theory for the Differential Equations of Simple Birth and Death Processes, Phil. Trans. Roy. Soc. (London), **236A,** 1954, p. 321.
13. Olsson, K. M., Calculation of Dispersion in Telephone Traffic Recording Values for Pure Chance Traffic, Tele (Eng. Ed.), **2,** 1959, p. 71.
14. Zaanen, A. C., *Linear Analysis*, Interscience, New York, 1953.
15. Kramer, H. P., Symmetrizable Markov Matrices, Ann. Math. Stat., **30,** 1959, p. 149.