Synthesis of *N*-Port Active *RC* Networks

By I. W. SANDBERG

(Manuscript received August 16, 1960)

The following basic theorem concerning active RC networks is proved:
Theorem: An arbitrary N × N matrix of real rational functions in the complex-frequency variable (a) can be realized as the short-circuit admittance matrix of a transformerless active RC N-port network containing N real-coefficient controlled sources, and (b) cannot, in general, be realized as the short-circuit admittance matrix of an active RC network containing less than N controlled sources.

I. INTRODUCTION

It is often desirable to avoid the use of magnetic elements in synthesis procedures, since resistors and capacitors are more nearly ideal elements and are usually cheaper, lighter and smaller. This is especially true in control systems in which, typically, exacting performance is required at very low frequencies. The rapid development of the transistor has provided the network synthesist with an efficient low-cost active element and has stimulated considerable interest in active *RC* network theory during the past decade.

Several techniques have been proposed for the active *RC* realization of transfer and driving-point functions.[1-18] It has been established that any real rational fraction can be realized as the transfer or driving-point function of a transformerless active *RC* network containing one active element. In particular, Linvill's technique[3] has been the basis for much of the later work.

Recently, Sipress[18] has shown that any two of the four short-circuit admittance parameters of a two-port network can be chosen arbitrarily and realized with a structure requiring only one active element. It follows that all four parameters can be realized with three active elements.

The problem of determining the minimum number of controlled sources required to realize all N^2 parameters of an arbitrary N-port immittance matrix is of considerable theoretical importance and has been of interest to network theorists for several years. The solution to this problem is stated in the abstract; its proof is the subject of this paper.

In Section II we derive some fundamental properties of N-port networks containing less than N controlled sources. The results are formulated in terms of inequalities involving the ranks of certain matrices. It follows from this study that at least N controlled sources are required for the realization of an arbitrary $N \times N$ immittance matrix. In Section III we make use of our previous results to establish an approach to the realization problem. This approach leads to a constructive proof that N controlled sources are in fact sufficient. A numerical example illustrating the essential points in the synthesis technique is presented in the Appendix.

II. N-PORT NETWORKS CONTAINING CONTROLLED SOURCES

A controlled source is ordinarily understood to be an ideal two-port network-representation of a single branch-branch constraint. The four types of elementary controlled sources are shown in Fig. 1. Note that the two "hybrid sources" [Fig. 1(a) and (b)] form a complete set, since they can be appropriately connected in cascade to realize each of the other two.

For our purposes it is convenient to generalize the definition of a controlled source to refer to any voltage or current source whose value is a weighted sum of certain prescribed voltages and currents. Specifically, if the value of a controlled voltage or current source is denoted by a_p,

$$a_p = \sum_{i=1}^{j+k} c_{pi} b_i , \tag{1}$$

where $b_1, b_2, \cdots, b_j$ are controlling currents and $b_{j+1}, b_{j+2}, \cdots, b_{j+k}$ are controlling voltages. It is assumed that the a_p, c_{pi} and b_i are Laplace-

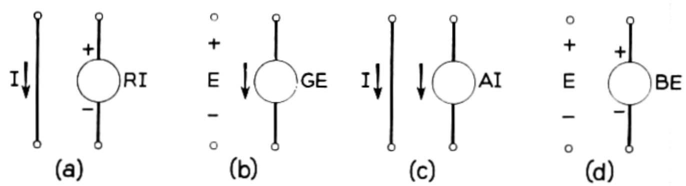

Fig. 1 — The four elementary controlled sources.

transformed quantities and that the c_{pi} are real rational functions of the complex frequency variable s.

2.1 *The Short-Circuit Admittance Matrix of an N-Port Network Containing Controlled Sources*

Consider the evaluation of the short-circuit admittance matrix of an N-port network containing a controlled source subnetwork as shown in Fig. 2. Denote by **E** and **I** respectively the column matrices of voltages and currents at the N accessible ports:

$$\mathbf{E} = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_N \end{bmatrix}, \qquad \mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}. \tag{2}$$

Let **A** be the column matrix of all l controlled current sources and m controlled voltage sources, and let **B** be the column matrix of all j cur-

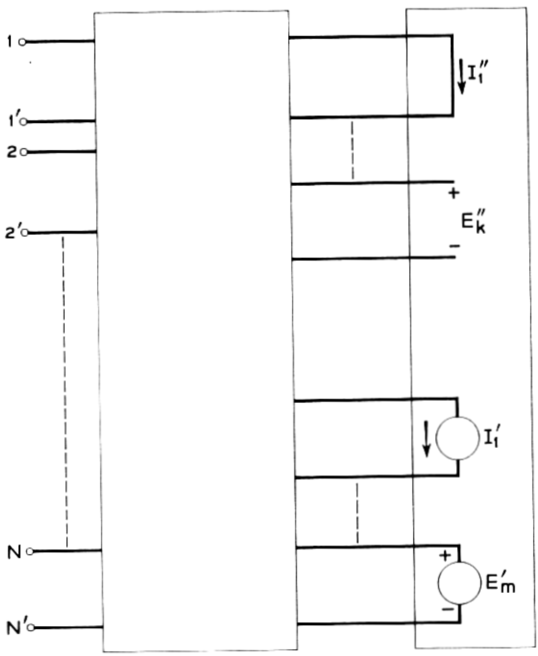

Fig. 2 — N-port network containing a controlled-source subnetwork.

rents and k voltages influencing the controlled sources:

$$\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_l \\ a_{l+1} \\ \vdots \\ a_{l+m} \end{bmatrix} = \begin{bmatrix} I_1' \\ I_2' \\ \vdots \\ I_l' \\ E_1' \\ \vdots \\ E_m' \end{bmatrix}, \tag{3}$$

$$\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_j \\ b_{j+1} \\ \vdots \\ b_{j+k} \end{bmatrix} = \begin{bmatrix} I_1'' \\ I_2'' \\ \vdots \\ I_j'' \\ E_1'' \\ \vdots \\ E_k'' \end{bmatrix}. \tag{4}$$

The relationship between **A** and **B** is assumed to be given by

$$\mathbf{A} = \mathbf{CB}, \tag{5}$$

where **C** is a $(l + m) \times (j + k)$ matrix of real rational functions in the complex frequency variable.

With **E** and **A** treated as independent variables, we apply the superposition theorem to obtain

$$\mathbf{I} = \mathbf{Y}_0\mathbf{E} + \mathbf{DA}, \tag{6}$$

where $\mathbf{Y}_0$ and **D** are defined by the equation. In particular, $\mathbf{Y}_0$ is the $N \times N$ short-circuit admittance matrix of the N-port network with the value of all controlled sources set equal to zero.

Similarly, we can express **B** as

$$\mathbf{B} = \mathbf{FE} + \mathbf{GA}, \tag{7}$$

where the matrices **F** and **G** are defined by the equation. From (5) and (7),

$$\mathbf{A} = [\mathbf{U} - \mathbf{CG}]^{-1}\mathbf{CFE}, \tag{8}$$

where $\mathbf{U}$ is the identity matrix of order $l + m$. Using (6),

$$[\mathbf{Y} - \mathbf{Y}_0] = \mathbf{D}[\mathbf{U} - \mathbf{CG}]^{-1}\mathbf{CF}, \tag{9}$$

where $\mathbf{Y}$ and $\mathbf{Y}_0$ are the short-circuit admittance matrices of the N-port network with all controlled sources respectively operative and set equal to zero. In certain degenerate cases, $\mathbf{Y}_0$ and/or the right-hand side of (9) will not exist. In such instances the network can be treated as a limiting case of a structure for which this difficulty does not occur.

2.2 *The Rank of* $[\mathbf{Y} - \mathbf{Y}_0]$

Consider the maximum rank of the $N \times N$ matrix $[\mathbf{Y} - \mathbf{Y}_0]$. Since the rank of a matrix product cannot exceed the rank of any of its constituent factors,[19]

$$\text{rank } [\mathbf{Y} - \mathbf{Y}_0] \leqq \text{rank } [\mathbf{C}] = R_c \,. \tag{10}$$

The elements of $[\mathbf{Y} - \mathbf{Y}_0]$ are real rational functions in the complex frequency variable. Assuming that this matrix has finite poles at $s = s_1, s_2, \cdots, s_m$ of multiplicity $n_1, n_2, \cdots, n_m$ respectively, it can be expressed as

$$[\mathbf{Y} - \mathbf{Y}_0] = \sum_{k=0}^{p} \mathbf{A}_k s^k + \sum_{l=1}^{m} \sum_{k=1}^{n_l} \mathbf{B}_{-k}^{(l)} \frac{1}{(s - s_l)^k}, \tag{11}$$

where the $\mathbf{A}_k$ and $\mathbf{B}_{-k}^{(l)}$ are coefficient matrices and in particular, the $\mathbf{B}_{-1}^{(l)}$ are residue matrices.

From (11), the matrix of coefficients of the first term in the Laurent expansion at the pole $s = s_l$ is

$$\mathbf{B}_{-n_l}^{(l)} = (s - s_l)^{n_l}[\mathbf{Y} - \mathbf{Y}_0] \,|_{s=s_l} \,. \tag{12}$$

In view of (10), we have

$$\text{rank } [\mathbf{B}_{-n_l}^{(l)}] \leqq R_c \,. \tag{13}$$

Similarly, the leading coefficient of the matric polynomial in (11) is given by

$$\mathbf{A}_p = \lim_{s \to \infty} \frac{1}{s^p} [\mathbf{Y} - \mathbf{Y}_0], \tag{14}$$

and hence

$$\text{rank } [\mathbf{A}_p] \leqq R_c \,. \tag{15}$$

Consequently, when $R_c < N$, all k-rowed minors of the matrices $\mathbf{B}_{-n_l}^{(l)}$ and $\mathbf{A}_p$ vanish, where $k = R_c + 1, R_c + 2, \cdots, N$.

Inequalities (13) and (15) shed considerable light on the fundamental properties of an N-port network containing a controlled source subnetwork. In fact, at poles of $\mathbf{Y}$ which are not poles of $\mathbf{Y}_0$, these conditions yield explicit restrictions on the $\mathbf{Y}$ matrix. For example, let $\mathbf{Y}$ be the admittance matrix of an active RC network and take $\mathbf{Y}_0$ to be the corresponding passive RC matrix obtained from $\mathbf{Y}$ by setting all controlled source coefficients equal to zero. It is well known that $\mathbf{Y}_0$ must be regular everywhere in the complex plane except at infinity and at points on the negative-real axis where only simple poles may occur. Hence $\mathbf{Y}_0$ cannot influence the coefficient matrices $\mathbf{B}_{-n_l}^{(l)}$ at any multiple-order pole or at any pole not on the negative-real axis. In particular, the rank of the residue matrix at any simple complex pole cannot exceed R_c, the rank of the matrix $\mathbf{C}$.

The rank of $\mathbf{C}$, of course, cannot exceed the number of its rows or columns, whichever is smaller. That is,

$$R_c \leqq \min [j + k, l + m]. \tag{16}$$

This means that R_c cannot exceed the number of controlled sources or the total number of controlling voltages and currents, whichever is smaller. Consequently, if any of the prescribed $\mathbf{B}_{-n_l}^{(l)}$ are to have full rank at a pole of $\mathbf{Y}$ which is not a pole of $\mathbf{Y}_0$, the controlled source subnetwork must include at least N controlled sources and at least N distinct control ports.†

A similar development, of course, can be carried out in terms of the open-circuit impedance matrices $\mathbf{Z}$ and $\mathbf{Z}_0$. Note that these results are valid for controlled source coefficients c_{pi} which may be any set of real rational functions in the complex-frequency variable. Note also that a driving-point immittance can be regarded as a controlled source, since such immittances impose a constraint which is merely a special case of (1).

III. N-PORT ACTIVE RC REALIZATION

We begin the study of the N-port realization problem by considering an active RC network containing one controlled source. Specifically, consider an $(N + 2)$-port passive RC network characterized by the $(N + 2) \times (N + 2)$ short-circuit admittance matrix $\tilde{\mathbf{Y}}$, and suppose

† The realization of an arbitrary $N \times N$ matrix of constants as the short-circuit admittance matrix of an N-port network containing positive resistors, ideal transformers and controlled sources also requires, in general, at least N controlled sources. This follows from the fact that, in this case, $\mathbf{Y}_0$ is the matrix of a nonnegative quadratic form, and hence it is possible to prescribe constant matrices $\mathbf{Y}$ such that, for the entire class of matrices $\mathbf{Y}_0$, $[\mathbf{Y} - \mathbf{Y}_0]$ is of rank N.

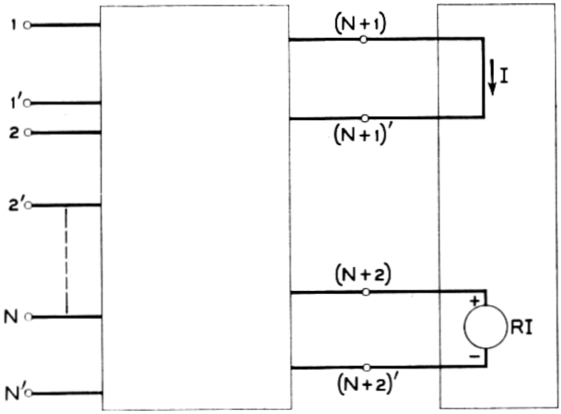

Fig. 3 — Active RC network containing one controlled source—canonical subnetwork.

that a current-controlled voltage source is connected between ports $N + 1$ and $N + 2$ as shown in Fig. 3. Denote by $\mathbf{Y}_0$ and $\mathbf{Y}$ the $N \times N$ short-circuit admittance matrices relating the column vectors

$$\mathbf{E} = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_N \end{bmatrix} \qquad \text{and} \qquad \mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix} \tag{17}$$

when the controlled source coefficient respectively vanishes and is equal to R.

The matrix $\mathbf{Y}$ is given, as a special case of (9), by

$$\mathbf{Y} - \mathbf{Y}_0 = -\frac{R}{1 + R\tilde{y}_{N+1,N+2}} \begin{bmatrix} \tilde{y}_{1,N+2} \\ \vdots \\ \tilde{y}_{N,N+2} \end{bmatrix} [\tilde{y}_{N+1,1} \cdots \tilde{y}_{N+1,N}], \tag{18}$$

where $\mathbf{Y}_0$ is the matrix of elements in the first N rows and columns of $\tilde{\mathbf{Y}}$. It is convenient to express (18) as

$$\mathbf{Y} - \mathbf{Y}_0 = -\frac{R}{q(q + R\tilde{p}_{N+1,N+2})} \begin{bmatrix} \tilde{p}_{1,N+2} \\ \vdots \\ \tilde{p}_{N,N+2} \end{bmatrix} [\tilde{p}_{N+1,1} \cdots \tilde{p}_{N+1,N}], \tag{19}$$

where q and the $\tilde{p}_{jk}$ are polynomials, and

$$\tilde{y}_{jk} = \frac{\tilde{p}_{jk}}{q}. \tag{20}$$

It is evident from (18) or (19) that, as anticipated, $[\mathbf{Y} - \mathbf{Y}_0]$ has unit rank.

3.1 *N-Port Synthesis*

Our objective is to obtain an expression involving $\mathbf{Y}$ similar to (9) with a right-hand side of rank N. We know that a network characterized by such a relationship will require at least N controlled sources.

It is well known that a rank N matrix can be expressed as a sum of N rank 1 matrices.[19] This suggests that the realization of $\mathbf{Y}$ can be accomplished with N networks connected in parallel. We shall specifically consider the parallel connection of N networks of the type shown in Fig. 3.

Assuming that the scalar coefficient on the right-hand side of (19) is the same function of s for each of the N subnetworks, we obtain†

$$\mathbf{Y} - \sum_{i=1}^{N} \mathbf{Y}_{0i} = -\frac{R}{q(q + R\tilde{p}_{N+1,N+2})} \sum_{i=1}^{N} \begin{bmatrix} \tilde{p}_{1,N+2}^{(i)} \\ \vdots \\ \tilde{p}_{N,N+2}^{(i)} \end{bmatrix} [\tilde{p}_{N+1,1}^{(i)} \cdots \tilde{p}_{N+1,N}^{(i)}], \tag{21}$$

where

$$\tilde{\mathbf{Y}}^{(i)} = \frac{1}{q} [\tilde{p}_{jk}^{(i)}] \tag{22}$$

and

$$\tilde{p}_{N+1,N+2}^{(i)} = \tilde{p}_{N+1,N+2}, \qquad i = 1,2,\cdots, N. \tag{23}$$

The sum of matrix products in (21) can be written as a single matrix with the element in the jth row and kth column given by

$$\sum_{i=1}^{N} \tilde{p}_{j,N+2}^{(i)} \tilde{p}_{N+1,k}^{(i)}. \tag{24}$$

This matrix can therefore be written as the product of the following two matrices:

† The networks are assumed to be such that admittance matrices add without the use of ideal transformers. This is justified later by employing balanced structures.

$$\mathbf{P}_1\mathbf{P}_2 = \begin{bmatrix} \tilde{p}_{1,N+2}^{(1)} \cdots \tilde{p}_{1,N+2}^{(N)} \\ \vdots \quad\quad \vdots \\ \tilde{p}_{N,N+2}^{(1)} \cdots \tilde{p}_{N,N+2}^{(N)} \end{bmatrix} \begin{bmatrix} \tilde{p}_{N+1,1}^{(1)} \cdots \tilde{p}_{N+1,N}^{(1)} \\ \vdots \quad\quad \vdots \\ \tilde{p}_{N+1,1}^{(N)} \cdots \tilde{p}_{N+1,N}^{(N)} \end{bmatrix}. \tag{25}$$

From (21) and (25),

$$\mathbf{Y} - \mathbf{Y}_{0T} = -\frac{R}{q(q + R\tilde{p}_{N+1,N+2})}\mathbf{P}_1\mathbf{P}_2, \tag{26}$$

where

$$\mathbf{Y}_{0T} = \sum_{i=1}^{N} \mathbf{Y}_{0i}.$$

Let the prescribed short-circuit admittance matrix $\mathbf{Y}$ be given as

$$\mathbf{Y} = \frac{1}{D}[N_{ij}], \tag{27}$$

where D is the common denominator polynomial of the elements in $\mathbf{Y}$ and $[N_{ij}]$ is a matrix of polynomials. Similarly, write $\mathbf{Y}_{0T}$ as

$$\mathbf{Y}_{0T} = \frac{1}{q}[p_{ij}] = \frac{1}{q}\sum_{k=1}^{N}[p_{ij}^{(k)}]. \tag{28}$$

From (26), (27) and (28),

$$\frac{1}{qD}[Dp_{ij} - qN_{ij}] = \frac{R}{q(q + R\tilde{p}_{N+1,N+2})}\mathbf{P}_1\mathbf{P}_2. \tag{29}$$

In (29) let terms be identified as follows:

$$\tilde{p}_{N+1,N+2} = \frac{1}{R}(D - q), \tag{30}$$

$$\mathbf{P}_1\mathbf{P}_2 = \frac{1}{R}[Dp_{ij} - qN_{ij}]. \tag{31}$$

At this point we have reduced the synthesis of the N-port admittance matrix $\mathbf{Y}$ to the determination of N realizable $(N + 2) \times (N + 2)$ RC network matrices $\tilde{\mathbf{Y}}^{(i)}$ whose elements satisfy (30) and (31).

3.2 *Sufficient Conditions for the Realization of* $\mathbf{Y}$

The matrices $\tilde{\mathbf{Y}}^{(i)}$ can be expressed as

$$\tilde{\mathbf{Y}}^{(i)} = s\mathbf{K}_\infty^{(i)} + \mathbf{K}_0^{(i)} + \sum_{j=1}^{\deg q}\mathbf{K}_j^{(i)}\frac{s}{s + \sigma_j}, \tag{32}$$

where "deg q" means the degree of the polynomial q, and where the σ_j are real and satisfy

$$0 < \sigma_1 < \sigma_2 \cdots < \sigma_{\deg q} .$$

If the coefficient matrices $\mathbf{K}_\infty$, $\mathbf{K}_0$ and $\mathbf{K}_j$ are "dominant-diagonal" matrices,† (32) can be realized as a transformerless balanced $RC(N + 2)$-port network.[20]

Assume that $\mathbf{Y}_{0T}$ has been chosen so that

(a) its coefficient matrices satisfy the dominant-diagonal condition with the inequality sign;†

(b) the matrix $(1/R)[Dp_{ij} - qN_{ij}]$ can be expressed as the product of two polynomial matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ with the property that $(1/q)\mathbf{P}_1$ and $(1/q)\mathbf{P}_2$ are matrices of realizable RC transfer admittances (these admittances are assumed to have poles at infinity only when $\mathbf{Y}_{0T}$ has a pole at infinity); and

(c) the function $\tilde{p}_{N+1,N+2}$ satisfies the realizability and regularity constraints stated in (b).

If (a) is satisfied, we can write $\mathbf{Y}_{0T}$ as the sum of N matrices $\mathbf{Y}_{0i}$, each of which has coefficient matrices that satisfy the dominant-diagonal condition with the inequality sign. Recall that $\mathbf{Y}_{0i}$ is the matrix of elements in the first N rows and columns of the $(N + 2) \times (N + 2)$ matrix $\tilde{\mathbf{Y}}^{(i)}$. To obtain $\tilde{\mathbf{Y}}^{(i)}$, we border $\mathbf{Y}_{0i}$ with two additional rows and columns of elements. All but three of the required numerator polynomials are determined by the entries in the polynomial matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ which satisfy (b). Of the remaining three polynomials, $\tilde{p}_{N+1,N+2}$ is given by (30) and is assumed to satisfy (c), while $\tilde{p}_{N+1,N+1}^{(i)}$ and $\tilde{p}_{N+2,N+2}^{(i)}$ may be chosen freely to assist realizability, since they are unrestricted by (30) or (31).

The realizability of $\tilde{\mathbf{Y}}^{(i)}$ can be ensured by having it exhibit the dominance characteristic, and this can always be done by choosing the scale factors of the polynomials $\tilde{p}_{N+1,N+1}^{(i)}$ and $\tilde{p}_{N+2,N+2}^{(i)}$ as well as the value of R, the controlled source coefficient, to be sufficiently large.

Hence (a), (b) and (c) are sufficient for the realization of $\mathbf{Y}$. To make further progress, we next establish conditions that permit $\mathbf{P} = (1/R)[Dp_{ij} - qN_{ij}]$ to be written as the product of two matrices with polynomial elements of lower degree.

† A dominant-diagonal matrix $\mathbf{M}$ has elements m_{jk} which satisfy

$$m_{jj} \geqq \sum_{k \neq j} | m_{jk} |.$$

3.3 *Factorization of the Matric Polynomial* P

Let L be the degree of the highest degree polynomial in $\mathbf{P} = (1/R)\cdot[Dp_{ij} - qN_{ij}]$, and suppose that the zeros of

$$\det \mathbf{P} = \sum_{k=0}^{NL} a_k s^k$$

include K distinct real zeros at $s = s_i$, $(i = 1, 2, \cdots, N, \cdots, K)$.

Consider the result of determining a nonsingular matrix $\mathbf{Q}$ with real constant elements such that every element in the ith column of $\mathbf{PQ}$ has a zero at $s = s_i$, $(i = 1, 2, \cdots, N)$. If indeed this can be done, $\mathbf{P}$ can be written as

$$\mathbf{P} = (\mathbf{PQ})\mathbf{Q}^{-1} = \mathbf{P}'(\mathbf{DQ}^{-1}), \tag{33}$$

where $\mathbf{D}$ is the diagonal matrix diag $[s - s_1, s - s_2, \cdots, s - s_N]$, and the degree of the highest degree polynomial in $\mathbf{P}'$ is $L - 1$. This is equivalent to removing a linear factor of the matric polynomial $\mathbf{P}$:

$$\begin{aligned} \mathbf{P} = \sum_{j=1}^{L} s^j \mathbf{A}_j &= \left[\sum_{j=1}^{L-1} s^j \mathbf{A}_j'\right] \mathbf{DQ}^{-1} \\ &= \left[\sum_{j=1}^{L-1} s^j \mathbf{A}_j' \mathbf{Q}^{-1}\right] \mathbf{QDQ}^{-1} \\ &= \left[\sum_{j=1}^{L-1} s^j \mathbf{A}_j''\right] (s\mathbf{U} - \mathbf{B}), \end{aligned} \tag{34}$$

where $\mathbf{U}$ is the identity matrix of order N and

$$\mathbf{B} = \mathbf{Q} \text{ diag } [s_1, s_2, \cdots, s_N]\mathbf{Q}^{-1}. \tag{35}$$

If $(N - 1)L < K$, a matrix $\mathbf{Q}$ having the required properties exists and can be constructed as follows. First, note that at any zero of det $\mathbf{P}$, say at $s = s_l$, the column rank of $\mathbf{P}$ is necessarily less than N, and hence there exists a relationship of the form

$$0 = \sum_{j=1}^{N} \alpha_{jl}[\mathbf{P}_j(s_l)], \tag{36}$$

where $[\mathbf{P}_j(s_l)]$ is the jth column vector of $\mathbf{P}$ evaluated at $s = s_l$, and the α_{jl} are not all zero. Note also that at no more than $(N - 1)L$ of the zeros of det $\mathbf{P}$ is it possible to determine alphas, not all zero, which satisfy

$$0 = \sum_{j\neq k}^{N} \alpha_{jl}[\mathbf{P}_j(s_l)], \tag{37}$$

where k is any one of the integers $[1,2,\cdots,N]$. This follows at once from the fact that all nonidentically vanishing determinants, formed from det $\mathbf{P}$ by replacing the kth column of det $\mathbf{P}$ with a column of constants, vanish at most at $(N - 1)L$ points. Therefore, if $(N - 1)L < K$, there must exist at least one equation of the type (36) for a real zero and with $\alpha_{kl} \neq 0$. In other words, there exists a nonsingular matrix of real elements

$$\mathbf{Q}_k = \begin{bmatrix} 1 & & & q_{1k} & & & \\ & \ddots & & & & & \\ & & 1 & \vdots & & & \\ & & & q_{kk} & & & \\ & & & \vdots & 1 & & \\ & & & & & \ddots & \\ & & & q_{Nk} & & & 1 \end{bmatrix} \qquad (38)$$

such that every element in the kth column of $\mathbf{PQ}_k$ has a real zero at $s = s_k$. Note that the elements in all columns except the kth remain unchanged. Hence the matrix $\mathbf{Q}$ can be constructed as a product of N matrices $\mathbf{Q}_j$ chosen so that every element in the ith column of

$$\mathbf{P} \prod_{j=1}^{m} \mathbf{Q}_j , \qquad i = 1,2,\cdots, m.$$

has a real zero at $s = s_i$.

To summarize, if $(N - 1)L < K$, N distinct real zeros of det $\mathbf{P}$ can be removed as a linear factor of the matric polynomial $\mathbf{P}$. The remaining polynomial is of degree $L - 1$ and all coefficient matrices are real.†

To simplify the discussion, we have not considered certain extensions of the factorization technique. It is possible, for example, to carry out a similar development with respect to the rows of $\mathbf{P}$. This permits the removal of a linear factor that premultiplies the remaining matric polynomial.

3.4 *Consideration of Conditions (a), (b) and (c)*

The admittance matrix $\mathbf{Y}_{0T}$ can be made to have dominant-diagonal coefficient matrices by choosing any $N \times N$ realizable RC admittance

† This implies that the matric polynomial $\mathbf{P}$ can be written as

$$\mathbf{P} = \mathbf{C} \prod_{i=1}^{L} (s\mathbf{U} - \mathbf{B}_i)$$

when det $\mathbf{P}$ has NL distinct zeros. When these zeros are all real the coefficient matrices $\mathbf{C}$ and $\mathbf{B}_i$ are also real.

matrix, with elements of suitable degree as determined subsequently, and multiplying each diagonal entry by a sufficiently large positive real constant ρ. Hence condition (a) is easily satisfied. Denote the matrix determined in this way by

$$\mathbf{Y}_{0T} = \frac{1}{q}\begin{bmatrix} \rho p'_{11} & p_{12} & \cdots & p_{1N} \\ \vdots & \rho p'_{22} & & \\ & & \ddots & \\ p_{N1} & & & \rho p'_{NN} \end{bmatrix}. \tag{39}$$

The polynomial det $\mathbf{P}$ can be written as

$$\det \mathbf{P} = \det \frac{1}{R}\,[Dp_{ij} - qN_{ij}] = \left(\frac{\rho}{R}\right)^N \left\{D^N \prod_{i=1}^{N} p'_{ii} + \frac{R(s)}{\rho^N}\right\}, \tag{40}$$

where $R(s)/\rho^N$ is a polynomial with degree not exceeding NL and with all coefficients that approach zero as ρ approaches infinity. We shall assume that the degree of p_{ii}, deg p_{ii}, has been chosen to be independent of the index i. Note that, as ρ approaches infinity, N deg p_{ii} zeros of det $\mathbf{P}$ approach the zeros of

$$\prod_{i=1}^{N} p'_{ii}.$$

The zeros of this product can be chosen to be distinct and different from those of D. Hence, for a sufficiently large value of ρ, (a) is satisfied and det $\mathbf{P}$ has at least N deg p_{ii} distinct real zeros.

Next, consider condition (b). The degree of the highest degree polynomial in $\mathbf{P}$ is given by

$$\begin{aligned} L &= \max\,[\max \deg p_{ij} + \deg D, \max \deg N_{ij} + \deg q] \\ &= \max\,[\deg p_{ii} + \deg D, \max \deg N_{ij} + \deg q]. \end{aligned} \tag{41}$$

Hence,

$$\begin{aligned} L &= \deg p_{ii} + \max\,[\max \deg N_{ij} - \epsilon, \deg D] \\ &= \deg p_{ii} + L_\epsilon\,, \end{aligned} \tag{42}$$

where

$$\begin{aligned} \epsilon &= 0, \qquad \deg p_{ii} = \deg q \\ \epsilon &= 1, \qquad \deg p_{ii} = \deg q + 1. \end{aligned} \tag{43}$$

To remove k linear factors of the matric polynomial $\mathbf{P}$ as described in Section 3.3, it is sufficient, after removal of the $(k - 1)$th factor, that

$$(N - 1)[\deg p_{ii} + L_\epsilon - (k - 1)] < N \deg p_{ii} - N(k - 1). \quad (44)$$

If $k = L_\epsilon$ factors are removed, **P** could be written as the product of two matrices, one of degree L_ϵ and the other of degree $\deg p_{ii}$. Substituting this value of k into (44) gives the required relationship between L_ϵ and $\deg p_{ii}$:

$$NL_\epsilon - 1 < \deg p_{ii}. \quad (45)$$

Hence conditions (a) and (b) are satisfied† with $\deg p_{ii} = NL_\epsilon$. Finally, it is evident that condition (c) can be satisfied simultaneously, since $\tilde{p}_{N+1,N+2}$ can be chosen to have any degree not exceeding $\deg p_{ii}$.

This proves the theorem stated in the abstract.

IV. CONCLUSION

We have proven that N is the sufficient and, in general, minimum number of controlled sources required to realize an arbitrary $N \times N$ matrix of real rational functions as a transformerless active RC N-port network. A canonical structure is a parallel combination of N networks, each containing a single controlled source. The type of controlled source employed is one of the two basic elementary controlled sources. Similar developments can be carried out for other types of controlled sources.

Further work is indicated in several directions. It is desirable to avoid the use of balanced networks. A detailed investigation of matric polynomial factorization may shed some light on this possibility. A major difficulty stems from the fact that relatively little is known about the realization of transformerless passive RC networks. Even so, it is almost certain that more practical canonical structures will be discovered.

It is noteworthy that the analytical machinery employed here provides insight into other fundamental questions. For example, it is easy to show that all N resistors in Oono's passive N-port realization[21] are in fact necessary. Similarly all N-negative and N-positive resistors in Carlin's active N-port realization[22] are necessary.

V. ACKNOWLEDGMENT

The author wishes to acknowledge the helpful and stimulating discussions held with S. Darlington and B. McMillan. The technique of

† This is a pessimistic statement of the required degree of p_{ii}, and results from the particular technique employed to factor **P**. A more detailed study of matric polynomial factorization, as yet incomplete, indicates that the degree of p_{ii} can generally be reduced by a factor of N.

factoring a matric polynomial presented in Section 3.3 is based on a suggestion by S. Darlington.

APPENDIX

Synthesis of a Two-Port Network — A Numerical Example

To illustrate the main points in the synthesis technique presented in Section III, we consider in detail the synthesis of a two-port network. This example demonstrates also that (45) is not a necessary condition.

Let the prescribed 2 × 2 matrix be

$$\begin{aligned}\mathbf{Y} &= \frac{1}{D}[N_{ij}] \\ &= \frac{1}{s^2+s+1}\begin{bmatrix} s^2+s+2 & s^2+s+3 \\ s^2+s+4 & s^2+s+5 \end{bmatrix}.\end{aligned} \tag{46}$$

We choose $\mathbf{Y}_{0T}$ as the following matrix that obviously satisfies the dominance condition with the inequality sign:

$$\begin{aligned}\mathbf{Y}_{0T} &= \frac{1}{q}[p_{ij}] \\ &= \frac{1}{(s+2)(s+4)}\begin{bmatrix} 5(s+1)(s+3) & 0 \\ 0 & 5(s+1)(s+3) \end{bmatrix}.\end{aligned} \tag{47}$$

From (30), (31), (46) and (47),

$$\tilde{p}_{3,4} = -\frac{1}{R}(5s+7), \tag{48}$$

$$\begin{aligned}\mathbf{P}_1\mathbf{P}_2 &= \mathbf{P} = \frac{1}{R} \\ &\cdot\begin{bmatrix} 4s^4+18s^3+24s^2+15s-1 & -s^4-7s^3-17s^2-26s-24 \\ -s^4-7s^3-18s^2-32s-32 & 4s^4+18s^3+21s^2-3s-25 \end{bmatrix}.\end{aligned} \tag{49}$$

Consider the factorization of **P** into two matric polynomials of the second degree. The factors of

$$\begin{aligned}R^N \det \mathbf{P} = 15s^8 &+ 130s^7 + 420s^6 + 555s^5 - 152s^4 \\ &- 1629s^3 - 2474s^2 - 1972s - 743,\end{aligned}$$

determined with a digital computer, are

$$\begin{aligned}&(s+1.0707018)(s-1.6223931)(s+3.0014915)(s+2.6871002) \\ &\cdot(s+1.3191886 \pm j1.2215876)\,(s+0.4456939 \pm j0.9460882).\end{aligned} \tag{50}$$

Denote by s_1, s_2, s_3 and s_4 respectively the zeros of the first four factors in (50).

First, we determine a matrix $\mathbf{Q}_1$ such that both elements in the first column of $\mathbf{PQ}_1$ have a zero at $s = s_1$. At $s = s_1$,

$$\alpha_{11}[\mathbf{P}_1(s_1)] + \alpha_{21}[\mathbf{P}_2(s_1)] = 0. \tag{51}$$

By evaluating the pair of polynomials in either row of (49) at $s = s_1$ we obtain:

$$0.76249\ \alpha_{11} + \alpha_{21} = 0.$$

Hence,

$$\mathbf{Q}_1 = \begin{bmatrix} 1 & 0 \\ -0.76249 & 1 \end{bmatrix}. \tag{52}$$

From (49) and (52),

$$\mathbf{PQ}_1 = \frac{1}{R}\,[a_{ij}], \tag{53}$$

where

$$a_{11} = (4.7625s^3 + 18.2382s^2 + 17.4347s + 16.1575)(s + 1.0707),$$

$$a_{21} = -(4.0499s^3 + 16.3886s^2 + 16.4651s + 12.0835)(s + 1.0707),$$

$$a_{12} = -(s^4 + 7s^3 + 17s^2 + 26s + 24),$$

$$a_{22} = (4s^4 + 18s^3 + 21s^2 - 3s - 25).$$

Next we find a matrix $\mathbf{Q}_2$ such that the first column of $\mathbf{PQ}_1\mathbf{Q}_2$ is identical to that of $\mathbf{PQ}_1$, and both elements in the second column of $\mathbf{PQ}_1\mathbf{Q}_2$ have a zero at $s = s_2$. The evaluation of polynomials as before leads to

$$\mathbf{Q}_2 = \begin{bmatrix} 1 & 0.48643 \\ 0 & 1 \end{bmatrix}. \tag{54}$$

At this point, $\mathbf{P}$ can be expressed as

$$\mathbf{P} = \frac{1}{R}\,[b_{ij}]\ \text{diag}\,[s + 1.0707, s - 1.6223]\mathbf{Q}^{-1}, \tag{55}$$

where $\mathbf{Q}^{-1} = (\mathbf{Q}_1\mathbf{Q}_2)^{-1}$, and

$$b_{11} = 4.7625s^3 + 18.2382s^2 + 17.4347s + 16.1575,$$

$$b_{21} = -(4.0499s^3 + 16.3886s^2 + 16.4651s + 12.0835),$$

$$b_{12} = 1.3166s^3 + 6.4881s^2 + 11.5058s + 9.6064,$$

$$b_{22} = 2.0300s^3 + 11.2124s^2 + 22.6465s + 19.2894.$$

A second linear factor of $\mathbf{P}$ can be removed by repeating this process. Specifically, if the zeros at $s = s_3$ and $s = s_4$ are removed respectively from the first and second columns of $[b_{ij}]$, $\mathbf{P}$ can be expressed as

$$\mathbf{P} = \mathbf{P}_1\mathbf{P}_2$$

with

$$\mathbf{P}_1 = \beta \begin{bmatrix} 4.3560s^2+3.1605s+4.3961 & 1.3166s^2+2.9503s+3.5781 \\ -(4.6766s^2+5.8136s+6.0077) & 2.0300s^2+5.7576s+7.1753 \end{bmatrix},$$

$$\mathbf{P}_2 = \frac{1}{\beta R} \begin{bmatrix} 0.6292s^2+2.5622s+2.0221 & -0.4863s^2-1.9803s-1.5629 \\ 0.9568s^2+1.5419s-2.7648 & 0.8499s^2+0.5007s-4.7910 \end{bmatrix}, \tag{56}$$

where β is an arbitrary nonzero real parameter.

To determine $\tilde{\mathbf{Y}}^{(1)}$ and $\tilde{\mathbf{Y}}^{(2)}$, first write $\mathbf{Y}_{0T}$ as the sum of two matrices, $\mathbf{Y}_{01}$ and $\mathbf{Y}_{02}$, that satisfy the dominance condition with the inequality sign. The following choice is clearly acceptable:

$$\mathbf{Y}_{01} = \mathbf{Y}_{02} = \tfrac{1}{2}\mathbf{Y}_{0T} \cdot$$

Hence, $\tilde{\mathbf{Y}}^{(1)}$ and $\tilde{\mathbf{Y}}^{(2)}$ are given by

$$\tilde{\mathbf{Y}}^{(i)} = \frac{1}{q} \begin{bmatrix} \tilde{p}_{11}^{(i)} & 0 & \tilde{p}_{13}^{(i)} & \tilde{p}_{14}^{(i)} \\ 0 & \tilde{p}_{22}^{(i)} & \tilde{p}_{23}^{(i)} & \tilde{p}_{24}^{(i)} \\ \tilde{p}_{31}^{(i)} & \tilde{p}_{32}^{(i)} & \tilde{p}_{33}^{(i)} & \tilde{p}_{34}^{(i)} \\ \tilde{p}_{41}^{(i)} & \tilde{p}_{42}^{(i)} & \tilde{p}_{43}^{(i)} & \tilde{p}_{44}^{(i)} \end{bmatrix}, \tag{57}$$

where

$$\tilde{p}_{34}^{(1)} = \tilde{p}_{34}^{(2)} = -\frac{1}{R}(5s + 7),$$

$$\tilde{p}_{11}^{(1)} = \tilde{p}_{11}^{(2)} = \tilde{p}_{22}^{(1)} = \tilde{p}_{22}^{(2)} = \tfrac{5}{2}(s + 1)(s + 3).$$

The polynomials $\tilde{p}_{33}^{(1)}$, $\tilde{p}_{33}^{(2)}$, $\tilde{p}_{44}^{(1)}$, and $\tilde{p}_{44}^{(2)}$ are unrestricted by (31) and hence, for simplicity, can be chosen to be $\frac{5}{2}(s + 1)(s + 3)$. The remaining polynomials are obtained from (25) with $\mathbf{P}_1$ and $\mathbf{P}_2$ given explicitly in (56). It is evident that finite, nonzero parameters β and R can be determined so that the matrices $\tilde{\mathbf{Y}}^{(1)}$ and $\tilde{\mathbf{Y}}^{(2)}$ satisfy the dominance condition. The realization of each of these matrices takes the form shown

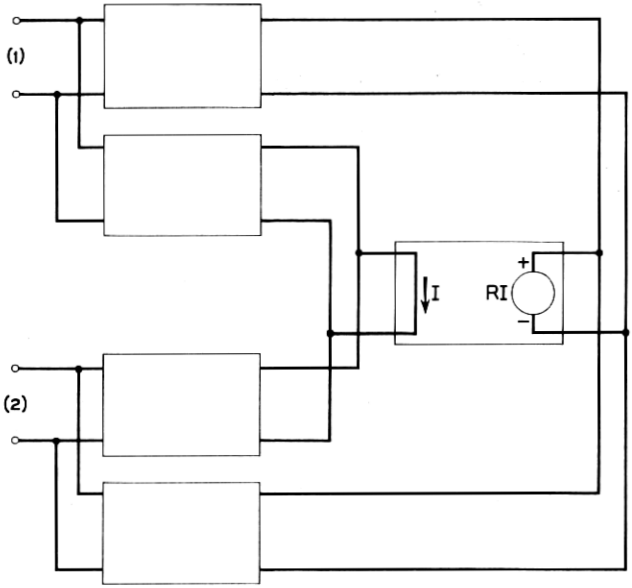

Fig. 4 — Realization of $\tilde{Y}^{(1)}$ or $\tilde{Y}^{(2)}$ for two-port network example.

in Fig. 4, where the rectangles enclose transformerless passive balanced *RC* structures.[20]

REFERENCES

1. Dietzold, R. L., Frequency Discriminative Electric Transducer, U. S. Patent 2,549,965, April 17, 1951.
2. Bangert, J. T., The Transistor as a Network Element, B.S.T.J., **33,** 1954, p. 329.
3. Linvill, J. G., RC Active Filters, Proc. I.R.E., **42,** 1954, p. 555.
4. Armstrong, D. B. and Reza, F. M., Synthesis of Transfer Functions by Active RC Networks, I.R.E. Trans., **CT-1,** 1954, p. 8.
5. Sallen, R. P. and Key, E. L., A Practical Method of Designing RC Active Filters, I.R.E. Trans., **CT-2,** 1955, p. 74.
6. Horowitz, I. M., RC-Transistor Network Synthesis, Proc. Nat. Elect. Conf. October 1956, p. 818.
7. Horowitz, I. M., Synthesis of Active RC Transfer Functions, Research Report R-507-56 PIB-437, Microwave Research Inst., Polytechnic Inst. of Brooklyn, 1956.
8. Sipress, J. M., Active RC Partitioning Synthesis of High-Q Bandpass Filter, thesis, Polytechnic Inst. of Brooklyn, 1957.
9. Bongiorno, J. J., Synthesis of Active RC Single-Tuned Bandpass Filters, I.R.E. Nat. Conv. Rec., March 1958, Pt. 2, p. 30.
10. Sandberg, I. W., Active RC Networks, Research Report R-662-58 PIB-590, Microwave Research Inst., Polytechnic Inst. of Brooklyn, 1958.
11. DeClaris, N., Synthesis of Active Networks — Driving-Point Functions, I.R.E. Nat. Conv. Rec., March 1959, Pt. 2, p. 23.
12. Myers, B. R., Transistor-RC Network Synthesis, I.R.E. Wescon Conv. Rec., August 1959, Pt. 2, p. 65.

13. Kinariwala, B. K., Synthesis of Active RC Networks, B.S.T.J., **38,** 1959, p. 1269.
14. Horowitz, I. M., Optimization of Negative Impedance Converter Synthesis Techniques, I.R.E. Trans., **CT-6,** 1959, p. 296.
15. Blecher, F. H., Application of Synthesis Techniques to Electronic Circuit Design, I.R.E. Nat. Conv. Rec., March 1960, Pt. 2, p. 210.
16. Kuh, E. S., Transfer Function Synthesis of Active RC Networks, I.R.E. Nat. Conv. Rec., March 1960, Pt. 2, p. 134.
17. Sandberg, I. W., Synthesis of Driving-Point Impedances with Active RC Networks, B.S.T.J., **39,** July 1960, p. 947.
18. Sipress, J. M., Synthesis of Active RC Networks, to be published.
19. Halmos, P. R., *Finite-Dimensional Vector Spaces*, D. Van Nostrand Co., New York, 1942, pp. 92, 93.
20. Slepian, P. and Weinberg, L., Synthesis Applications of Paramount and Dominant Matrices, Proc. Nat. Elect. Conf., October 1958, p. 611.
21. Oono, Y., Synthesis of a Finite 2-N Terminal Network by a Group of Networks Each of Which Contains Only One Ohmic Resistance, J. Math. Phys., **29,** 1950, p. 13.
22. Carlin, H. J., General N-Port Synthesis with Negative Resistors, Proc. I.R.E. **48,** 1960, p. 1174.