THE BELL SYSTEM TECHNICAL JOURNAL

VOLUME XL — MARCH 1961 — NUMBER 2

Pulse Transmission by AM, FM and PM in the Presence of Phase Distortion

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(Manuscript received October 13, 1960)

In pulse transmission systems, pulses modulated in various ways to carry information may be transmitted by amplitude, phase or frequency modulation of a carrier, and with each type of modulation various methods of detection are possible. An important consideration in many applications is the performance of various modulation and detection methods in the presence of phase distortion or equivalent envelope delay distortion, which may be appreciable in certain transmission facilities. The principal purpose of this presentation is a theoretical evaluation of transmission impairments resulting from certain representative types of delay distortion. These transmission impairments are reflected in the need for increased signal-to-noise ratio at the detector input to compensate for the effect of delay distortion.

The performance in pulse transmission by various carrier modulation and detection methods can be formulated in terms of a basic function common to all, known as the carrier pulse transmission characteristic, which is related by a Fourier integral to the amplitude and phase characteristics of the channel. Numerical values are given here for the carrier pulse transmission characteristic with linear and quadratic delay distortion, together with the maximum transmission impairments caused by these fairly representative forms of delay distortion with various methods of carrier modulation and signal detection. These include amplitude modulation with envelope and with synchronous detection, two-phase and four-phase modulation with synchronous detection and with differential phase detection and binary frequency modulation.

In determining the effect of delay distortion, a raised cosine amplitude spectrum of the pulses at the detector input has been assumed in all cases, together with the minimum pulse interval permitted with this spectrum and ideal implementation of each modulation and detection method. Furthermore, optimum adjustments from the standpoint of slicing levels and sampling instants at the detector output are assumed for each particular case of delay distortion. These idealizations insure that only the effect of delay distortion is evaluated and considered in comparing modulation methods, and that this effect is minimized by appropriate system adjustments.

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I. INTRODUCTION

Binary pulse transmission by various methods of carrier modulation has been dealt with elsewhere on the premise of ideal amplitude and phase characteristics of the carrier channels.[1] An important consideration in many applications is the performance in the presence of phase distortion or equivalent envelope delay distortion, which may be appreciable in certain transmission facilities. An ideal amplitude spectrum of received pulses can be approached with the aid of appropriate terminal filters with gradual cutoffs, such that the associated phase characteristic is virtually linear. Nevertheless, pronounced phase distortion may be encountered in pulse transmission over channels with sharp cutoffs outside the pulse spectrum band, as in frequency division carrier system channels designed primarily for voice transmission.

The principal purpose of the present analysis is a theoretical evaluation of transmission impairments resulting from certain representative types of delay distortion in pulse transmission by various methods of carrier modulation and signal detection. These transmission impairments are reflected in the need for increased signal-to-noise ratio at the detector input to compensate for the effect of delay distortion.

The performance in pulse transmission by various carrier modulation and detection methods can be related to a basic function known as the carrier pulse transmission characteristic. This basic function gives the shape of a single carrier pulse at the channel output, i.e., the detector input, under ideal conditions or in the presence of the particular kind of transmission distortion under consideration. From this basic function can be determined the envelopes of carrier pulse trains at the detector input, together with the phase of the carrier within the envelope. The shape of demodulated pulse trains with various methods of carrier modulation and detection can, in turn, be determined for various combinations of transmitted pulses, together with the maximum transmission impairment from a specified type of channel imperfection, such as delay distortion dealt with here.

The carrier pulse transmission characteristic is related by a Fourier integral to the amplitude and phase characteristic of the channel. It has been determined elsewhere[2] for pulses with a raised cosine spectrum and cosine variation in transmission delay over the channel band, and for pulses with a gaussian spectrum with linear variation in delay. A cosine variation in delay is approximated in some transmission facilities and has certain advantages from the standpoint of analysis, both as regards numerical evaluation and interpretation in terms of pulse echoes.

A somewhat similar form of delay distortion that affords a satisfactory approximation in many cases is quadratic (or parabolic) delay distortion. Quadratic delay distortion is in theory approached near midband of a flat bandpass channel with sharp cutoffs, such as a carrier system voice channel, and usually affords a satisfactory approximation over the more important part of the transmission band of such channels. Linear delay distortion is approximated when a bandpass channel with gradual cutoffs is established to one side of midband of a flat bandpass channel with sharp cutoffs. These and other types of delay distortion do not lend themselves to convenient analytical evaluation of the Fourier integrals for the pulse transmission characteristic. However, at present, these integrals can be accurately evaluated by numerical integration with the aid of digital computers for any specified pulse spectrum and phase distortion.

Numerical values are given here for the carrier pulse transmission characteristics with linear and quadratic delay distortion, together with the maximum transmission impairments caused by these limiting and fairly representative forms of delay distortion with various methods of carrier modulation and signal detection. These include amplitude modulation with envelope and with synchronous detection, two-phase and four-phase modulation with synchronous detection and with differential phase detection and binary frequency modulation. In determining the effect of delay distortion, a raised cosine amplitude spectrum of the pulses at the detector input has been assumed in all cases, together with the minimum pulse interval permitted with this spectrum, ideal implementation of each modulation and detection method and optimum design from the standpoint of slicing levels and sampling instants at the detector output. These idealizations insure that only the effect of delay distortion is evaluated and considered in comparing modulation methods, a condition that is difficult to realize with experimental rather than analytical comparisons.

As mentioned above, the present analysis involves a basic function common to all modulation methods, which in general would be determined with the aid of digital computers. This approach has certain advantages in comparison of modulation methods and from the standpoint of optimum system design over direct computer simulation of each modulation method. The latter direct approach may be preferable for any specified modulation method and type of transmission impairment and has been used in connection with a binary double-sideband AM system with envelope detection, for cosine and sine variations in transmission delay over the channel band and for combinations of these.[3] The transmission impairments caused by linear and quadratic delay distortion, and combinations thereof, have been determined experimentally for a binary vestigial-sideband amplitude modulation data transmission system employing envelope detection.[4]

The present analysis is concerned with certain "coarse structure" variations in transmission delay that ordinarily predominate over smaller "fine structure" variations, except in transmission facilities where elaborate phase equalization is used. Transmission impairments from small irregular fine structure gain and phase deviations over the channel band can be evaluated by methods discussed elsewhere[2] and are not considered here.

II. CARRIER PULSE TRAINS AND MODULATION METHODS

2.1 *General*

In carrier pulse modulation systems the pulse trains at the transmitting end modulate a carrier in amplitude, phase or frequency. In AM the demodulated signal depends on the envelope of the received carrier pulse train at sampling instants, in PM on the phase of the carrier within the envelope and in FM on the time derivative of the phase at sampling instants. To determine the performance of these various methods in the presence of transmission distortion it is necessary to formulate the received carrier pulse trains.

The received carrier pulse trains at the channel output, i.e., the detector input, can in all cases be formulated in terms of the carrier pulse transmission characteristic, that is, the received carrier pulse in response to a single transmitted pulse. This pulse transmission characteristic is related to the shape of the modulating pulses at the transmitting end, and to the amplitude and phase characteristic of the channel, by a Fourier integral, as discussed and illustrated for special cases in the Appendix. The general formulation of the pulse trains at the detector input and the resultant demodulated pulse trains with various methods of carrier modulation and signal detection is dealt with in the following sections.

2.2 *Carrier Pulse Transmission Characteristics*

It will be assumed that a carrier pulse of rectangular or other suitable envelope is applied at the transmitting end of a bandpass channel. The received pulse with carrier frequency ω_c can then be written in the general form [Ref. 2, Equation (2.09)]

$$P_c(t) = \cos(\omega_c t - \psi_c)R_c(t) + \sin(\omega_c t - \psi_c)Q_c(t) \tag{1}$$

$$= \cos[\omega_c t - \psi_c - \varphi_c(t)]\bar{P}_c(t), \tag{2}$$

where

$$\bar{P}_c(t) = [R_c^2(t) + Q_c^2(t)]^{\frac{1}{2}}, \tag{3}$$

$$\varphi_c(t) = \tan^{-1} [Q_c(t)/R_c(t)], \tag{4}$$

$$R_c(t) = \bar{P}_c(t) \cos \varphi_c(t), \tag{5}$$

$$Q_c(t) = \bar{P}_c(t) \sin \varphi_c(t). \tag{6}$$

In the above relations R_c and Q_c are the envelopes of the in-phase and quadrature components of the received carrier pulse and $\bar{P}_c$ the resultant envelope. The time t is taken with respect to a conveniently chosen origin, for example the midpoint of a pulse interval or the instant at which R_c or $\bar{P}_c$ reaches a maximum value.

With a carrier frequency ω_0 rather than ω_c, relation (1) is modified into

$$P_0(t) = \cos (\omega_0 t - \psi_0) R_0(t) + \sin (\omega_0 t - \psi_0) Q_0(t) \tag{7}$$

and relations (2) through (6) are correspondingly modified by replacing c by the subscript 0.

When the carrier frequency is changed from ω_0 to ω_c the spectrum of a received pulse will change, provided the transmission-frequency characteristic of the channel remains fixed, except in the limiting case of a carrier pulse of infinitesimal duration having a flat spectrum. However, by appropriate modification of the transmission-frequency characteristic the amplitude spectrum of a pulse at the channel output, i.e., the detector input, can be made the same regardless of the carrier frequency. On the latter premise of equal amplitude spectra at carrier frequencies ω_0 and ω_c, the following relations apply [Ref. 2, Equation (2.18)]:

$$\begin{aligned} R_c(t) &= \cos [\varphi_0(t) + \omega_y t - \psi_y] \bar{P}_0(t) \\ &= \cos (\omega_y t - \psi_y) R_0(t) - \sin (\omega_y t - \psi_y) Q_0(t), \end{aligned} \tag{8}$$

$$\begin{aligned} Q_c(t) &= \sin [\varphi_0(t) + \omega_y t - \psi_y] \bar{P}_0(t) \\ &= \cos (\omega_y t - \psi_y) Q_0(t) + \sin (\omega_y t - \psi_y) R_0(t), \end{aligned} \tag{9}$$

where

$$\varphi_0(t) = \tan^{-1} [Q_0(t)/R_0(t)],$$

$$\omega_y = \omega_c - \omega_0,$$

$$\psi_y = \psi_c - \psi_0.$$

Relations (8) and (9) apply when $R_c(t)$ and $Q_c(t)$ are referred to the carrier phase ψ_c in (1) rather than the carrier phase ψ_0 in (7). If a carrier phase ψ_0 is used as reference, $\psi_y = 0$ in (8) and (9), and

$$R_{c,0} = \cos \omega_y t R_0(t) - \sin \omega_y t Q_0(t), \tag{10}$$

$$Q_{c,0} = \cos \omega_y t Q_0(t) + \sin \omega_y t R_0(t). \tag{11}$$

With (8) and (9) in (3), or (10) and (11) in (3):

$$\bar{P}_c(t) = \bar{P}_0(t) = [R_0^2(t) + Q_0^2(t)]^{\frac{1}{2}}. \tag{12}$$

The resultant envelope of a single pulse is thus the same regardless of carrier frequency and phase, on the premise of a fixed pulse spectrum at the channel output as assumed above.

2.3 *Pulse Trains at Detector Input*

Let carrier pulses be transmitted at intervals T, and let t be the time from the midpoint of a selected interval. The following designation will be introduced for convenience

$$\begin{aligned} R_c(t + nT) &= R_c\left[T\left(\frac{t}{T} + n\right)\right] = R_c(x + n), \\ Q_c(t + nT) &= Q_c\left[T\left(\frac{t}{T} + n\right)\right] = Q_c(x + n), \end{aligned} \tag{13}$$

where n is the time expressed in an integral number of pulse intervals of duration T and x the time in a fraction of a pulse interval.

Let $a(-n)$ and $\psi_c(-n)$ be the amplitude and phase of the carrier pulse transmitted in the nth interval prior to the interval 0 under consideration, and $a(n)$, $\psi_c(n)$ the corresponding quantities for the nth subsequent interval. The received pulse train in the interval $-T/2 < t < T/2$ is then

$$\begin{aligned} W_0(x) = &\sum a(n) \cos [\omega_c t - \psi_c(n)] R_c(x - n) \\ &+ \sum a(n) \sin [\omega_c t - \psi_c(n)] Q_c(x - n) \end{aligned} \tag{14}$$

$$= \sum a(n) \cos [\omega_c t - \psi_c(n) - \varphi_c(x - n)] \bar{P}_c(x - n), \tag{15}$$

where the summations are between $n = -\infty$ and $n = \infty$.

During the next interval, T to $2T$, the received wave is obtained by replacing $a(n)$ and $\psi_c(n)$ by $a(n + 1)$ and $\psi_c(n + 1)$ and is thus

$$W_1(x) = \sum a(n + 1) \cos [\omega_0 t - \psi_c(n + 1) - \varphi_c(x - n)] \bar{P}_c(x - n) \tag{16}$$

where t and x refer to midpoint of interval 1.

In pulse modulation systems as considered herein it is assumed that the modulating pulses are rectangular in shape and of duration equal to the pulse interval. For equal phases $\psi_c(n) = \psi_c$ of all the modulating pulses, (14) then becomes

$$W_0 = \cos(\omega_c t - \psi_c) \sum a(n) R_c(x - n) + \sin(\omega_c t - \psi_c) \sum a(n) Q_c(x - n). \tag{17}$$

When $a(n) = a$ is a constant the input is a continuous carrier, so that evaluation of (17) will give

$$W_0 = aA_c \cos(\omega_c t - \psi_c), \tag{18}$$

where A_c is the amplitude of the transmission-frequency characteristic of the channel at $\omega = \omega_c$ and it is assumed in the determination of R_c and Q_c that the phase characteristic is zero at $\omega = \omega_c$. That is, a constant transmission delay is ignored, which is permissible without loss of generality.

When $R_c(t)$ and $Q_c(t)$ are determined from the channel transmission-frequency characteristic by the usual Fourier integral relations, in the form represented by (159) through (163) of the Appendix, the following relations apply for rectangular modulating pulses of duration T equal to the pulse interval:

$$A_c = \sum_{n=-\infty}^{\infty} R_c(x - n), \tag{19}$$

$$0 = \sum_{n=-\infty}^{\infty} Q_c(x - n). \tag{20}$$

2.4 *Amplitude Modulation*

In AM systems $\psi_c(n) = \psi_c =$ constant and (14) becomes

$$W_0(x) = \cos(\omega_c t - \psi_c) \sum a(n) R_c(x - n) + \sin(\omega_c t - \psi_c) \sum a(n) Q_c(x - n). \tag{21}$$

With synchronous detection, also referred to as homodyne and coherent detection, the received wave is applied to a product demodulator together with a demodulating wave $\cos(\omega_0 t - \psi_c)$. After elimination of higher frequency demodulation products by low-pass filtering the demodulated baseband output becomes, when a factor of one-half is omitted for convenience

$$U_0(x) = \sum a(n) R_c(x - n). \tag{22}$$

If ω_s is the bandwidth of the modulating signal, the high-frequency output of the product demodulator will have a lowest frequency $2\omega_c - \omega_s$, which can be separated from the modulating wave by low-pass filtering provided $2\omega_c - \omega_s \geqq \omega_s$, or if $\omega_c \geqq \omega_s$.

At sampling instants $x = 0$, the desired signal is $a(0)R(0)$ and the remaining terms in (22) represent intersymbol interference in systems where $R_c(n) \neq 0$ for $n = \pm 1, \pm 2$, etc.

Owing to elimination of the quadrature components, synchronous detection is simpler from the standpoint of analysis than envelope detection, in which the demodulated signal depends on the envelope of the received wave (21) as given by

$$\bar{W}_0(x) = \{[\sum a(n)R_c(x-n)]^2 + [\sum a(n)Q_c(x-n)]^2\}^{\frac{1}{2}}. \quad (23)$$

The desired signal at sampling instants $x = 0$ is $a(0)[R_c^2(0) + Q_c^2(0)]^{\frac{1}{2}}$ and the remaining terms in (23) represent intersymbol interference.

2.5 *Phase Modulation with Synchronous Detection*

In phase modulation systems the amplitude $a(n) = a =$ constant and the phase $\psi_c(n)$ is varied from one pulse interval to the next. The received wave (15) then becomes

$$W_0(x) = \sum \cos[\omega_c t - \psi_c(n) - \varphi_c(x-n)]\bar{P}_c(x-n). \quad (24)$$

In a multiphase system, the received wave is in general applied to several product demodulators together with a demodulating wave cos $(\omega_c t - \psi)$. In the particular case of two-phase modulation a single demodulator suffices, and the demodulator output after elimination of high-frequency demodulation products by low-pass filtering and omitting a factor of one-half is of the general form

$$U_0(x) = \sum \cos[\psi_c(n) - \psi + \varphi_c(x-n)]\bar{P}_c(x-n). \quad (25)$$

At the sampling instants

$$U_0(0) = \sum \cos[\psi_c(n) - \psi + \varphi_c(-n)]\bar{P}_c(-n), \quad (26)$$

where as before the summation is between $n = -\infty$ and $n = \infty$.

The desired signal is represented by the term for $n = 0$ and is

$$\begin{aligned} U_0(0) &= \cos[\psi_c(0) - \theta]\bar{P}_c(0) \\ &= [\cos\theta\cos\psi_c(0) + \sin\theta\sin\psi_c(0)]\bar{P}_c(0), \end{aligned} \quad (27)$$

where

$$\theta = \psi - \varphi_c(0). \quad (28)$$

When the phase ψ of the demodulating wave is so chosen that $\theta = 0$, and if $\psi_c(0) = 0$ or π as in two-phase modulation, then $U_0(0) = \pm\bar{P}_c(0)$.

In four-phase modulation two product demodulators are required, with the demodulating waves displaced 90° in phase. The output of the second demodulator is then, in place of (26),

$$V_0(0) = \sum \sin [\psi_c(n) - \psi + \varphi_c(x - n)]\bar{P}_c(-n) \qquad (29)$$

and the desired output at sampling instants is, in place of (27),

$$\begin{aligned} V_0(0) &= \sin [\psi_c(0) - \theta]\bar{P}_c(0) \\ &= [\cos \theta \sin \psi_c(0) - \sin \theta \cos \psi_c(0)]\bar{P}_c(0). \end{aligned} \qquad (30)$$

The preferable choice of the phase of the demodulating wave in the above relations may depend on certain considerations in the implementation of modulators and demodulators. In Table I are given the four possible combined outputs as determined by the carrier phase $\psi_c(0)$ for the particular cases $\theta = 0$ and $\theta = \pi/4$. For convenience the outputs for $\theta = \pi/4$ are normalized to unit amplitude, the actual amplitudes being $\pm\frac{1}{2}\sqrt{2}$.

It will be noted that with $\theta = 0$ the output U_0^0 determines whether one carrier is modulated in phase by $\psi_c = 0$ or π, while the output V_0^0 determines whether the quadrature carrier is modulated in phase by $\psi_c = 0$ or π. The two carriers can thus be modulated and demodulated independently, without the need for circuitry to convert the two demodulator output to carrier phase, as would be required with $\theta = \pi/4$. With differential phase detection, to be discussed in the next section, such converters would be required both with $\theta = 0$ and $\theta = \pi/4$. In this case $\theta = \pi/4$ may be preferable for the reason that only two states $(-1,1)$ are possible for each demodulator, rather than three states $(-1,0,1)$ with $\theta = 0$.

Table I — Demodulator Outputs U_0^0 and V_0^0 in Four-Phase Systems as Determined by Carrier Phase $\psi_c(0)$ for Demodulating Waves with Phases $\theta = 0$ and $\pi/4$

$\psi_c(0)$	$\theta = 0$		$\theta = \pi/4$	
	U_0^0	V_0^0	U_0^0	V_0^0
0	1	0	1	−1
$\pi/2$	0	1	1	1
π	−1	0	−1	1
$3\pi/2$	0	−1	−1	−1

2.6 *PM with Differential Phase Detection*

An alternative method of demodulation that will be considered in connection with phase modulation is differential phase detection. With this method $W_0(x)$ as given by (15) is applied to one pair of terminals of a product demodulator, and $W_1(x)$ as given by (16) to the other pair with a suitable phase shift θ. The demodulator output is then, with $a(n)$ constant as in phase modulation,

$$\begin{aligned} U_{01}(x) = {} & \{\sum \cos [\omega_c t - \psi_c(n) - \varphi_c(x-n) + \theta]\bar{P}_c(x-n)\} \\ & \cdot \{\sum \cos [\omega_c t - \psi_c(n+1) - \varphi_c(x-n)]\bar{P}_c(x-n)\}, \end{aligned} \tag{31}$$

where as before the summations are between $n = -\infty$ and $n = \infty$.

After elimination of high-frequency components present in (31) by low-pass filtering and omitting a factor of one-half, the resultant baseband output can be written

$$U_{01}(x) = \sum S_n(x)\bar{P}_c(x-n), \tag{32}$$

$$U_{01}(0) = \sum S_n(0)\bar{P}_c(-n), \tag{33}$$

in which the summations are between $n = -\infty$ and ∞ and

$$\begin{aligned} S_n(x) = \sum_{m=-\infty}^{\infty} \bar{P}_c(x-m) \cos [& \psi_c(-n) - \psi_c(-m+1) \\ & + \varphi_c(x-n) - \varphi_c(x-m) - \theta], \end{aligned} \tag{34}$$

$$\begin{aligned} S_n(0) = \sum_{m=-\infty}^{\infty} \bar{P}_c(-m) \cos [& \psi_c(-n) - \psi_c(-m+1) \\ & + \varphi_c(-n) - \varphi_c(-m) - \theta]. \end{aligned} \tag{35}$$

The desired output is represented by the term in (33) for $n = 0$ and is

$$U_{01}^{0}(0) = S_0(0)\bar{P}_c(0) \tag{36}$$

where, in accordance with (35),

$$\begin{aligned} S_0(0) = \sum_{m=-\infty}^{\infty} \bar{P}_c(-m) \cos [& \psi_c(0) - \psi_c(-m+1) \\ & + \varphi_c(0) - \varphi_c(-m) - \theta]. \end{aligned} \tag{37}$$

In the absence of intersymbol interference $\bar{P}_c(-m) = 0$ for $m \neq 0$ and

$$S_0^{0}(0) = \bar{P}_c(0) \cos [\psi_c(0) - \psi_c(1) - \theta] \tag{38}$$

so that (36) become

$$U_{01}^{0}(0) = \bar{P}_c^{2}(0) \cos [\psi_c(0) - \psi_c(1) - \theta]. \tag{39}$$

It will be recognized that this expression is of the same form as (27) except that $\psi_c(0)$ is replaced by the phase change $\psi_c(0) - \psi_c(1)$ between two successive sampling instants. In the particular case of four-phase transmission the phase θ may be chosen as, say, $\theta = 0$ or $\pi/4$, in which case the outputs of the two product demodulators would be as indicated in Table I for phase modulation with synchronous detection, except that $\psi_c(0)$ is replaced by the phase difference $\psi_c(0) - \psi_c(1)$.

With a signal of bandwidth ω_s the high-frequency part of (31) will have a lowest frequency $2(\omega_c - \omega_s)$, since it is the product of two high-frequency components each of lowest frequency $\omega_c - \omega_s$. The baseband signal represented by (32) will have a maximum frequency $2\omega_s$, so that a flat low-pass filter of minimum bandwidth $2\omega_s$ is required to avoid distortion of the baseband signal. In order that the filter also eliminate the high-frequency components in the demodulator output, it is necessary that $2(\omega_c - \omega_s) \geqq 2\omega_s$ or $\omega_c \geqq 2\omega_s$. With synchronous detection it was necessary that $\omega_c \geqq \omega_s$.

2.7 *Binary FM with Frequency Discriminator Detection*

In frequency modulation $a(n)$ in (14) is constant and $\psi_c(n)$ varies with time so that the time derivative of $[\omega_c t - \psi_c(n)]$ represents a variable frequency. Pulse transmission without intersymbol interference over a channel of the same bandwidth as required for double-sideband AM is in this case possible for certain ideal amplitude and phase characteristics of the channels, as shown elsewhere [Ref. 1, Section 5]. The formulation is here modified to include any amplitude and phase characteristic of the channels.

It will be assumed that a space is represented by a frequency $\omega_0 - \bar{\omega}$ and a mark by a frequency $\omega_0 + \bar{\omega}$. Discontinuity in a transition from mark to space can then be avoided for rectangular modulating pulses of duration T provided,

$$\bar{\omega}T = k\pi, \qquad k = 1,2,3. \tag{40}$$

In a system of minimum bandwidth $k = 1$, and in this case intersymbol interference can be avoided with a channel band no wider than required for double-sideband AM.

When a mark is preceded and followed by a space during the nth pulse interval, the envelope of the resultant carrier pulse is obtained with $t_0 = (t + nT)$ in Equation (23) of Ref. 1 and becomes

$$\begin{aligned} \bar{E}_{sm}{}^0(t + nT) &= 2 \cos \bar{\omega}(t + nT) \\ &= (-1)^n 2 \cos \bar{\omega}t, \end{aligned} \tag{41}$$

where the last relation follows from (40) with $k = 1$.

The resultant carrier pulse during interval 0 is of the general form

$$\begin{aligned} P(t) &= \cos(\omega_0 t + \varphi_0)(-1)^n R_0(t - nT) \\ &\quad + \sin(\omega_0 t + \varphi_0)(-1)^n Q_0(t - nT), \end{aligned} \tag{42}$$

where t is the time from the midpoint of interval 0.

When $\psi(-\bar{\omega})$ is the phase distortion* at the frequency $\omega_0 - \bar{\omega}$, Equation (34) of Ref. 1 is modified into

$$\begin{aligned} E(t) &= -\cos(\omega_0 t + \varphi_0)[A(-\bar{\omega}) \cos y - (-1)^n R_0(t - nT)] \\ &\quad + \sin(\omega_0 t + \varphi_0)[A(-\bar{\omega}) \sin y - (-1)^n Q_0(t - nT)], \end{aligned} \tag{43}$$

where

$$y = \bar{\omega} t + \psi(-\bar{\omega}). \tag{44}$$

When a sequence of marks and spaces is transmitted, the resultant wave at the detector input becomes

$$\begin{aligned} W_0(t) &= \cos(\omega_0 t + \varphi_0)[A(-\bar{\omega}) \cos y - \alpha_0(x)] \\ &\quad + \sin(\omega_0 t + \varphi_0)[A(-\bar{\omega}) \sin y - \beta_0(x)], \end{aligned} \tag{45}$$

where

$$\alpha_0(x) = \sum_{n=-\infty}^{\infty} (-1)^n a(n) R_0(x - n), \tag{46}$$

$$\beta_0(x) = \sum_{n=-\infty}^{\infty} (-1)^n a(n) Q_0(x - n), \tag{47}$$

in which the notation is in accordance with (13), $x = t/T$ and $y = \pi x + \psi(-\bar{\omega})$.

The phase of the wave (45) is given by

$$\tan \Psi_0(t) = -\frac{\sin y - \mu\beta_0(x)}{\cos y - \mu\alpha_0(x)}, \tag{48}$$

where

$$\mu = 1/A(-\bar{\omega}) \tag{49}$$

Expression (41) of Ref. 1 for a single pulse is replaced by the following for the demodulated pulse train at $x = t/T$:

$$\begin{aligned} U_0(x) &= \frac{\mu}{2D} [\mu(\alpha_0^2 + \beta_0^2) - \alpha_0 \cos y - \beta_0 \sin y \\ &\quad - \frac{1}{\bar{\omega}} (\alpha_0' \sin y - \beta_0' \cos y) - \frac{\mu}{\bar{\omega}} (\beta_0' \alpha_0 - \alpha_0' \beta_0)], \end{aligned} \tag{50}$$

* As in Ref. 1, the linear component of the phase characteristic is disregarded ·nce it only represents a fixed transmission delay.

where

$$D = 1 + \mu^2(\alpha_0^2 + \beta_0^2) - 2\mu(\alpha_0 \cos y + \beta_0 \sin y), \tag{51}$$

in which

$$\alpha_0 = \alpha_0(x), \quad \beta_0 = \beta_0(x), \quad \alpha_0' = d\alpha_0/dt, \quad \beta_0' = d\beta_0/dt.$$

2.8 *Signal-to-Noise Ratios in Binary FM*

Since binary FM with frequency discriminator detection is a nonlinear modulation method, determination of the optimum signal-to-noise ratio at the detector input for a given error probability presents a very difficult analytical problem, at least when consideration is given to minimum bandwidth requirements together with appropriate shaping of bandpass and postdetection low-pass filters. In Ref. 1 these various factors were taken into account, but the signal-to-noise ratios at sampling instants were evaluated on the approximate basis of a steady state carrier representing a continuing space or mark and a relatively high signal-to-noise ratio. On this basis it turned out that, in the absence of a postdetection low-pass filter, binary FM would have a disadvantage in signal-to-noise ratio of about 4.5 db compared to an optimum bipolar AM or phase reversal system. This would be reduced to about a 1.5-db disadvantage by addition of an optimum low-pass filter. The analysis further indicated that, for a specified postdetection low-pass filter, there would be an optimum division of shaping between the transmitting and receiving bandpass filters that would give a slight advantage in signal-to-noise ratios over an optimum bipolar AM system. In view of the approximations involved, the above analysis does not prove that such an advantage exists. Rather, it is probable that optimum bipolar AM has some advantage in signal-to-noise ratio over optimum binary FM. This is indicated by other analyses that do not assume a high signal-to-noise ratio but introduce other approximations in that they do not consider frequency discriminator detection or the shaping of band-pass filters or the effect of a postdetection low-pass filter.

It is well known that an approximation is involved in assuming high signal-to-noise ratios and thus ignoring the breaking phenomenon in FM. Moreover, even in the absence of intersymbol interference, it is an approximation to assume a steady state carrier over a short sampling interval, regardless of the transmitted code, as shown below.* Referring to Equation (202) of Ref. 1, random noise at the detector input can be written in the form

$$e_i(t) = r_i(t) \cos(\omega_0 t + \varphi_0) + q_i(t) \sin(\omega_0 t + \varphi_0). \tag{52}$$

* This was shown by A. P. Stamboulis.

When this noise is combined with the signal as given by (43), equation (48) for the phase in the presence of interference becomes

$$\tan \psi_{0,i} = -\frac{\sin y - \mu[\beta_0(x) + q_i(x)]}{\cos y - \mu[a_0(x) + r_i(x)]}, \tag{53}$$

where $x = t/T$.

In the absence of intersymbol interference at sampling instants, $a_0(0) = 0$ or 1, $\beta_0(0) = 0$, $y = 0$ and $\mu = 2$. In this case appropriate modification of (50) gives for the demodulated signal plus noise at sampling instants

$$U_0(U) + U_i(0) = \frac{(\alpha_0 + r_i)[2(\alpha_0 + r_i) - 1] + q_i'/\bar{\omega} + 2\alpha_0' q_i/\bar{\omega}}{[2(\alpha_0 + r_i) - 1]^2}, \tag{54}$$

where $r_i = r_i(0)$, $q_i = q_i(0)$ and $q_i' = dq_i(t)/dt$ for $t = 0$.

If $r_i \ll 1$, the last equation is approximated by

$$U_0(0) + U_i(0) \cong \alpha(0) - r_i + q_i'/\bar{\omega} + 2\alpha_0' q_i/\bar{\omega}, \tag{55}$$

where $U_0(0) = \alpha(0) = 0$ for space and 1 for mark, and the interfering voltage after demodulation is

$$U_i(0) \cong -r_i + q_i'/\bar{\omega} + 2\alpha_0' q_i/\bar{\omega} \tag{56}$$

The first two terms represent the conventional approximation for a continuing mark or space and a high signal-to-noise ratio.

In order to neglect the third term in (56) it is necessary that $\alpha_0'(0) = 0$. This is not the case except for a continuing space, a continuing mark or a mark preceded and followed by a continuing space. For other combinations of transmitted pulses there is some contribution from the third term. In the particular case of a raised cosine pulse spectrum, as considered herein, the maximum effect for a random pulse train is less than 0.15 db and can thus be ignored. For narrower pulse spectra the effect may be appreciably greater.

In the analysis that follows, transmission impairments from intersymbol interference owing to phase distortion will be evaluated on the same basis for FM as for the other modulation methods, although the approximations involved may be somewhat greater.

2.9 *Slicing Levels and Noise Margins*

As indicated by the preceding derivations, the demodulated wave is related to the received carrier wave $W_0(x)$ in a manner that depends on the carrier modulation and detection method. In general the demodulated wave at sampling instants may assume a number of different amplitudes. Let $U^{(s)}$ designate the demodulated wave for one particular amplitude or state a_s of the transmitted signal and $U^{(s+1)}$ the demodulated wave at a sampling instant for an adjacent amplitude or state s_{s+1} of the transmitted signal. There will then be a certain sequence of transmitted pulses for which a maximum value $U_{\max}{}^{(s)}$ is obtained, owing to intersymbol interference, and also a certain sequence resulting in a minimum value $U_{\min}{}^{(s+1)}$. If there is equal probability of a_s and a_{s+1} and of positive and negative noise voltages, the optimum level for distinction between $U^{(s)}$ and $U^{(s+1)}$ is

$$L_0{}^{(s)} = \tfrac{1}{2}[U_{\min}{}^{(s+1)} + U_{\max}{}^{(s)}]. \tag{57}$$

In the presence of $U^{(s)}$ the margin for distinction from $U^{(s+1)}$ is

$$M^{(s)} = L_0{}^{(s)} - U^{(s)} \tag{58}$$

and in the presence of $U^{(s+1)}$ the margin for distinction from $U^{(s)}$ is

$$M^{(s+1)} = U^{(s+1)} - L_0{}^{(s)}. \tag{59}$$

The minimum margins are obtained with $U^{(s)} = U_{\max}{}^{(s)}$ and with $U^{(s+1)} = U_{\min}{}^{(s+1)}$ in (58) and (59). The minimum margins thus become

$$M_{\min}{}^{(s+1)} = M_{\min}{}^{(s)} = \tfrac{1}{2}[U_{\min}{}^{(s+1)} - U_{\max}{}^{(s)}]. \tag{60}$$

For sequences of marks and spaces, or other signal patterns, such that the minimum margins for distinction between adjacent signal states are obtained, an error will occur if the noise voltage at the sampling instant exceeds $M_{\min}$ in amplitude and has the appropriate polarity. (Polarity is immaterial except for the two extreme signal states.) For other signal patterns the tolerable amplitude of the noise voltage is greater. The value of $M_{\min}$ relative to the value in the absence of intersymbol interference thus gives the maximum transmission impairment. The average impairment obtained by considering various pulse train patterns and the corresponding values of $M^{(s)}$ and $M^{(s+1)}$ as given by (58) and (59) will be less, as discussed below.

2.10 *Evaluation of Transmission Impairments*

By way of illustration it will be assumed that all values of M between $M_{\min}$ and a maximum value $M_{\max}$ are equally probable, and that the noise has a gaussian amplitude distribution. With a given fixed value of M the probability of an error can be written as

$$p_e = \tfrac{1}{2} \operatorname{erfc}(aM) \tag{61}$$

where erfc = 1 − erf is the error function complement and a is a factor that depends on the ratio of signal power to noise power.

Considering all noise margins between the limits mentioned above, the average error probability becomes

$$\bar{p}_e = \frac{1}{M_{\max} - M_{\min}} \frac{1}{2} \int_{M_{\min}}^{M_{\max}} \operatorname{erfc}(aM)dM$$
$$= \frac{1}{2} \frac{1}{M_{\max} - M_{\min}} \left[M_{\max} \operatorname{erfc} B - M_{\min} \operatorname{erfc} A + \frac{1}{a\sqrt{\pi}} (e^{-A^2} - e^{-B^2}) \right], \tag{63}$$

where

$$A = a\, M_{\min},$$
$$B = a\, M_{\max}.$$

With

$$M_{\max} = k\, M_{\min}, \tag{64}$$

(63) becomes

$$\bar{p}_e = \frac{1}{2} \frac{1}{k-1} \left[k \operatorname{erfc}(kA) - \operatorname{erfc} A + \frac{1}{A\sqrt{\pi}} (e^{-A^2} - e^{-k^2A^2}) \right]. \tag{65}$$

For $k = 1$, the latter expression conforms with (61).

The maximum error probability would be obtained by considering a fixed noise margin equal to $M_{\min}$ and would be

$$\hat{p}_e = \tfrac{1}{2} \operatorname{erfc} A. \tag{66}$$

The error committed in assuming $M_{\min}$ can be determined by writing $\bar{p}_e$ as given by (65) in the form

$$\bar{p}_e = \tfrac{1}{2} \operatorname{erfc}(cA), \tag{67}$$

where $c \geqq 1$ is so chosen that (67) equals (65).

The average noise margin is then

$$\bar{M} = c\, M_{\min} \tag{68}$$

By way of numerical illustration let A be so chosen that $\hat{p}_e$ as given by (66) in one case is 10^{-4} and in another case 10^{-5}. The results given in Table II are then obtained from (65) and (67).

As will be shown in a later section, a factor $k = 3$ may correspond to

TABLE II — RATIO $c = \bar{M}/M_{min}$ FOR EQUAL PROBABILITY OF ALL NOISE MARGINS BETWEEN M_{min} AND $M_{max} = k\,M_{min}$ FOR NOISE WITH A GAUSSIAN AMPLITUDE DISTRIBUTION

	$A = 2.62$; $\hat{p}_e = 10^{-4}$			$A = 3.0$; $\hat{p}_e = 10^{-5}$		
$k =$	1	2	3	1	2	3
$\bar{p}_e$	10^{-4}	10^{-5}	5×10^{-6}	10^{-5}	1.5×10^{-6}	7.5×10^{-7}
c	1	1.15	1.17	1	1.1	1.12
c (in db)	0	1.2	1.4	0	0.8	1.0

a transmission impairment of about 10 db based on the minimum noise margin, whereas the actual impairment would be 1.4 db less for an error probability of 10^{-4}, about 1 db less for an error probability 10^{-5}. For an error probability of 10^{-6} or less the error committed in evaluating transmission impairments on the basis of the minimum noise margin can be disregarded. This also applies for greater error probabilities when the transmission impairment based on the minimum noise margin is small, in which case $k < 2$.

III. SYNCHRONOUS AM AND PM

3.1 *General*

Amplitude modulation can be used in conjunction with envelope detection and synchronous detection. The former method is simplest from the standpoint of implementation, but synchronous detection, also referred to as homodyne and coherent detection, affords an improvement in signal-to-noise ratio. Since synchronous detection is also the simplest method from the standpoint of analysis, it will be considered here, except for a comparison of envelope and synchronous detection for binary double-sideband AM.

Amplitude modulation in general implies several pulse amplitudes, and can be used with double-sideband and with vestigial-sideband transmission. The particular case of bipolar binary AM with synchronous detection is equivalent to two-phase modulation.

With amplitude modulation and synchronous detection it is possible to transmit pulse trains on two carriers at quadrature with each other, and under certain idealized conditions to avoid mutual interference. The special case of bipolar binary AM on each of the two carriers is equivalent to four-phase modulation.

The signal-to-noise ratio as related to error probability is discussed elsewhere (Ref. 1, Section XVIII) for various optimized binary AM or

PM systems on the premise of ideal synchronous detection. Ideal synchronous detection for AM or PM as assumed here can in principle be approached without penalty in signal-to-noise ratio, by various methods of implementation. For example, a demodulating wave for a product demodulator can be derived with the aid of a resonator of sufficiently narrow bandwidth (high Q) tuned to the carrier frequency, or the second or the fourth harmonic thereof, depending on the particular method and on whether two-phase or four-phase modulation is used. A demodulating wave can also be supplied from an oscillator at the receiving end, the phase of which would be controlled by comparison with that of the carrier of the received signal. Such phase-locked oscillator methods have been devised for analog signal transmission by suppressed carrier double-sideband AM[5] and vestigial-sideband AM.[6] With any one of the above methods, noise in the demodulating wave would be virtually absent, as would the effect of phase distortion in the channel. Actually some penalty in signal-to-noise ratio as compared to ideal synchronous detection would be incurred, owing to unavoidable fluctuations in the amplitude and phase of the demodulating wave, resulting from the finite bandwidth of the resonators and mistuning, or from imperfect oscillator control. A common property of these methods is that a rather long time, as measured in pulse intervals, is required to establish a demodulating wave, if the above fluctuations in amplitude and phase are to be held within tolerable limits. This may be a disadvantage in certain applications, which in the case of phase modulation can be overcome by differential phase detection, in exchange for a penalty in signal-to-noise ratio resulting from the presence of both noise and phase distortion in the demodulating wave, as discussed in Section IV.

A general formulation is given here of intersymbol interference and resultant maximum transmission impairment as related to the carrier pulse transmission characteristic, together with illustrative applications to the particular cases of linear and quadratic delay distortion. The formulation is, however, applicable to any given gain and phase deviation over the channel band, provided the carrier pulse transmission characteristic has been determined, which in general would entail Fourier integral evaluation with the aid of computers.

3.2 *Synchronous AM and Two-Phase Modulation*

With synchronous detection (22) applies, or alternately, with $x = 0$ and $U_0(0) = U$,

$$U = a(0)R_c(0) + \sum_{n=1}^{\infty} [a(-n)R_c(n) + a(n)R_c(-n)]. \qquad (69)$$

The following notation will be used:

$$r_c^+ = \sum_{n=1}^{\infty} [R_c^+(n) + R_c^+(-n)], \tag{70}$$

$$r_c^- = \sum_{n=1}^{\infty} [R_c^-(n) + R_c^-(-n)], \tag{71}$$

where R_c^+ designates positive values of R_c and R_c^- absolute values when R_c is negative.

Let there be l different amplitude levels, between a minimum amplitude $a_{\min}$ and a maximum amplitude $a_{\max}$. When a pulse of amplitude $a_s = a_s(0)$ is transmitted, the maximum value of (69) is

$$U_{\max}^{(s)} = a_s R_c(0) + a_{\max} r_c^+ - a_{\min} r_c^-. \tag{72}$$

For the next higher pulse amplitude $a_{s+1} = a_s + (a_{\max} - a_{\min})/(l - 1)$, the minimum value of (69) is

$$U_{\min}^{(s+1)} = a_{s+1} R_c(0) + a_{\min} r_c^+ - a_{\max} r_c^-. \tag{73}$$

The minimum noise margin is, in accordance with (60),

$$M_{\min} = \frac{a_{\max} - a_{\min}}{2} \left[\frac{R_c(0)}{l - 1} - r_c^+ - r_c^- \right]. \tag{74}$$

In the absence of intersymbol interference $r_c^+ = 0$, $r_c^- = 0$ and $R_c(0) = R_c^{(0)}(0)$, so that

$$M^0 = \frac{a_{\max} - a_{\min}}{2} \frac{R_c^0(0)}{l - 1}. \tag{75}$$

The value of $M_{\min}$ as given by (74) is smaller than M^0 in the absence of intersymbol interference by the factor

$$\eta_{\min} = \frac{R_c(0)}{R_c^0(0)} \left[1 - (l - 1) \frac{r_c^+ + r_c^-}{R_c(0)} \right] \tag{76}$$

$$= \frac{R_c(0)}{R_c^0(0)} \left[1 - (l - 1) \frac{\bar{r}_c}{R_c(0)} \right], \tag{77}$$

where

$$\bar{r}_c = \sum_{n=1}^{\infty} [\bar{R}_c(-n) + \bar{R}_c(n)], \tag{78}$$

in which $\bar{R}_c$ designates the absolute value of R_c.

The factor $R_c(0)/R_c^0(0)$ represents the transmission impairment owing to reduction in pulse amplitude at sampling instants. The summation term represents transmission impairments owing to intersymbol interference.

Relation (77) applies regardless of the polarity of the transmitted pulses and for both symmetrical (double sideband) and asymmetrical (vestigial sideband) systems. The special case $l = 2$ and $a_{\min} = -a_{\max}$ represents binary bipolar AM, which can also be regarded as two-phase transmission.

3.3 *Quadrature Carrier AM and Four-Phase Modulation*

With synchronous detection it is possible under certain ideal conditions to transmit signals on two carriers at quadrature without mutual interference. In general, however, the quadrature component in (21) will in this case give rise to interference and (69) is replaced by

$$\begin{aligned} U = a(0)R_c(0) + \sum_{n=1}^{\infty} [a(-n)R_c(n) + a(n)R_c(-n)] \\ + b(0)Q_c(0) + \sum_{n=1}^{\infty} [b(-n)Q_c(n) + b(n)Q_c(-n)], \end{aligned} \tag{79}$$

where $b(n)$ are the pulse amplitudes in the quadrature system.

For equal differences between maximum and minimum amplitudes in the two systems, i.e., $a_{\max} - a_{\min} = b_{\max} - b_{\min}$, (77) is replaced by

$$\eta_{\min} = \frac{R_c(0)}{R_c^0(0)} \left\{ 1 - \frac{l-1}{R_c(0)} [Q_c(0) + \bar{r}_c + \bar{q}_c] \right\}, \tag{80}$$

where $\bar{r}_c$ is defined by (78), and similarly

$$\bar{q}_c = \sum_{n=1}^{\infty} \left[\bar{Q}_c(-n) + \bar{Q}_c(n) \right], \tag{81}$$

where $\bar{Q}_c$ designates the absolute values of Q_c.

In general the phase of the demodulating carrier can be so chosen that $Q_c(0) = 0$, as is demonstrated later.

Expression (80) applies regardless of pulse polarities in the two quadrature systems. The special case of two binary bipolar AM systems, i.e., $l = 2$ and $a_{\min} = b_{\min} = -a_{\max} = -b_{\max}$ can also be regarded as four-phase transmission.

3.4 *Even Symmetry Pulse Spectrum and Delay Distortion*

When the spectrum of a received pulse at the detector input has even symmetry about the carrier frequency, and the phase characteristic has odd symmetry (even symmetry delay distortion), the quadrature components $Q_c(n)$ vanish (see the Appendix). In this case (77) and (80) are identical, so that there is no mutual interference between pulse trains transmitted on two carriers at quadrature. In this special case it is thus possible by quadrature carrier AM to realize a two-fold increase in pulse transmission rate, without increased intersymbol interference. An alternative means to the same end is to use vestigial sideband transmission, as discussed below.

Let T be the pulse interval in double-sideband AM, in which case the pulse interval in vestigial-sideband AM would be $T' = T/2$. Returning to (10) and (11) let $R_0(t)$ be the in-phase component in double-sideband AM, and let $Q_0(t) = 0$ for an amplitude spectrum with even symmetry about ω_0 and a phase characteristic with odd symmetry. Let ω_y be the carrier frequency from midband in vestigial-sideband transmission. By appropriate choice of ω_y it is possible to make $\omega_y T' = \pi/2$, in which case $\cos \omega_y T' = 0$, $\sin \omega_y T' = 1$. The following relations are thus obtained:

At even sampling points, i.e., $m = 0,2,4,6, \cdots$,

$$R_{c,0}(mT') = (-1)^{m/2} R_0(mT') = (-1)^{m/2} R_0(mT/2),$$
$$Q_{c,0}(mT') = 0. \tag{82}$$

At odd sampling points, i.e., $m = 1,3,5,7, \cdots$,

$$\begin{aligned} R_{c,0}(mT') &= 0, \\ Q_{c,0}(mT') &= (-1)^{(m-1)/2} R_0(mT') = (-1)^{(m-1)/2} R_0(mT/2). \end{aligned} \tag{83}$$

In accordance with the above relations, at even sampling points only the in-phase components are present and are the same as in double-sideband AM. At odd sampling points the quadrature components are present, but are eliminated with synchronous detection and need not be considered.

In summary, when the amplitude spectrum at the detector input has even symmetry about the midband frequency, and the phase characteristic has odd symmetry, relation (77) applies for double-sideband AM, quadrature double-sideband AM, vestigial-sideband AM, as well as special cases thereof, such as two-phase and four-phase modulation.

In the next section numerical results are given for the special case of a raised cosine spectrum at the detector input with quadratic delay distortion about the midband frequency.

3.5 *Raised Cosine Spectrum and Quadratic Delay Distortion*

In the following numerical illustration, the spectrum at the detector input will be assumed to have a raised cosine shape, as shown in Fig. 1.

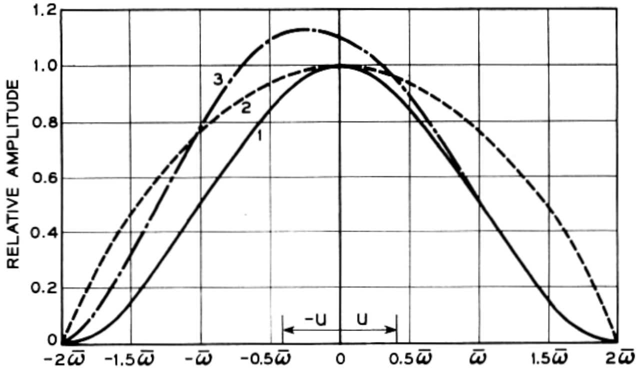

Fig. 1 — Raised cosine pulse spectrum and transmission-frequency characteristic of channel for double- and vestigial-sideband AM. Curve 1: Spectrum of carrier pulse envelope at detector input (channel output); $S(u)/S(0) = \cos^2 (\pi u/4\bar{\omega})$. Curve 2: Transmission-frequency characteristic of channel with rectangular transmitted carrier pulses of duration $T = \pi/\bar{\omega}$ and carrier at midband; $A(u)/A(0) = (\pi u/4\bar{\omega})/\tan (\pi u/4\bar{\omega})$. Curve 3: Transmission frequency characteristic of channel with rectangular transmitted carrier pulses of duration $T/2$ and carrier at $u = \bar{\omega}$; $A(u)A(0) = \cos^2 (\pi u/4\bar{\omega})\{[(u - \bar{\omega})/4\bar{\omega}]/\sin [\pi(u - \bar{\omega})/4\bar{\omega}]\}$.

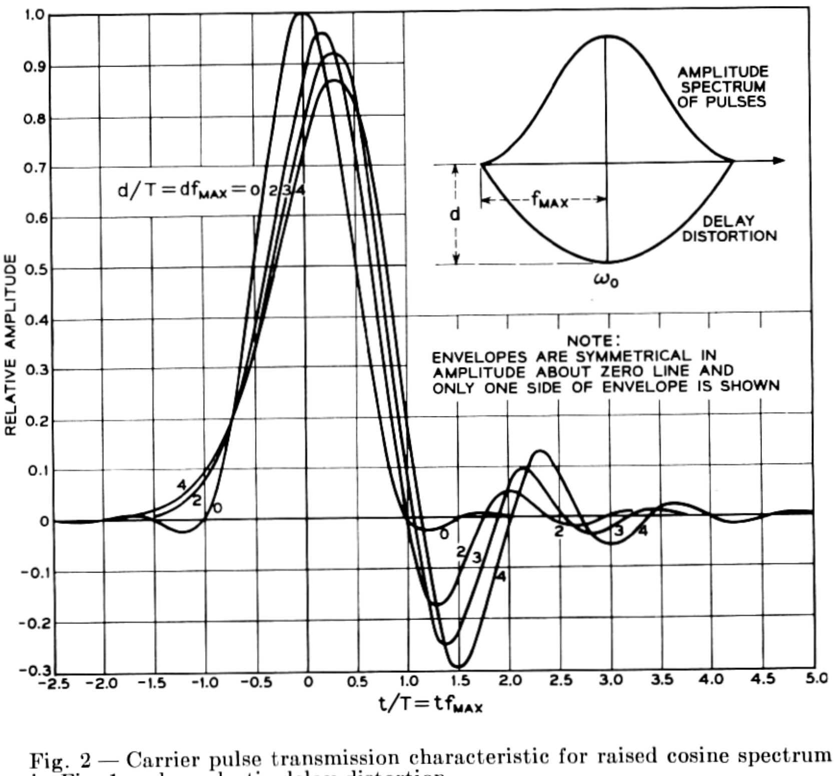

Fig. 2 — Carrier pulse transmission characteristic for raised cosine spectrum as in Fig. 1 and quadratic delay distortion.

The shape of the transmission-frequency characteristic of the channel required to this end depends on the shape of the transmitted pulses. It is shown in Fig. 1 for rectangular modulating pulses, with the carrier at midband and also with the carrier to one side of midband, as in vestigial-sideband transmission. These characteristics, together with the optimum division of shaping between transmitting and receiving filters, are discussed in Section XIV of Ref. 1. Even though the amplitude characteristics of the detector input spectra are the same in double- and vestigial-sideband transmission, it is necessary to use different shaping of transmitting filters, as indicated in Fig. 1, since the rectangular modulating pulses have different carrier frequencies and different durations.

The phase characteristic is assumed to contain a linear component, together with phase distortion component varying as the third power of frequency from midband, which corresponds to delay distortion increasing as the second power of frequency from midband, as indicated in Fig. 2. The function $R_0(t/T) = R_0(x + n)$ for this case has been determined by numerical integration, as discussed in the Appendix. It is given in Table XIX of the Appendix and shown in Fig. 2. The values for $x = 0$, i.e., integral values of t/T are given in Table III.

TABLE III — FUNCTION $R_0(n)$ FOR RAISED COSINE SPECTRUM AND QUADRATIC DELAY DISTORTION

$n = t/T$	d/T				
	0	1	2	3	4
−3	0	−0.0006	0.0025	0	0
−2	0	−0.0013	0.0011	0.0017	0.0028
−1	0	0.0467	0.0756	0.0891	0.0986
0	1	0.9633	0.8795	0.7956	0.7336
1	0	−0.0341	−0.0098	0.0827	0.2045
2	0	0.0196	0.0543	0.0655	0.0142
3	0	0.0044	0.0020	−0.0231	−0.0584
4	0	0.0014	−0.0014	−0.0087	−0.0037
5	0	0.0006	−0.0008	−0.0022	0.0040
$\sum_{-3}^{5} R_0(n)$	1.0	1.0000	1.0004	1.0006	0.9957

With exact evaluation of $R_0(n)$ the summation $\sum R_0(n)$ between $n = -\infty$ and $n = \infty$ should equal 1.

With the values of $R_0(n)$ given in Table III, the values of $\bar{r}_c = \bar{r}_0$ obtained from (78), and of $\eta_{\min}$ as obtained from (77) are given in Table IV.

These factors are shown in Fig. 3 and apply for double- and vestigial-

TABLE IV — FACTOR η_{min} FOR RAISED COSINE SPECTRUM AND QUADRATIC DELAY DISTORTION FOR SYNCHRONOUS AM WITH l AMPLITUDE LEVELS*

d/T	0	1	2	3	4
r_0	0	0.109	0.148	0.273	0.385
$l = 2$	1	0.855	0.732	0.523	0.347
$l = 3$	1	0.746	0.585	0.250	−0.037
$l = 4$	1	0.637	0.437	−0.053	—
$l = 5$	1	0.529	0.300	−0.296	—

* Results also apply with envelope detection (Section 3.9)

sideband AM and for quadrature double-sideband AM, and special cases thereof, such as two-phase and four-phase transmission. Since the quadrature component is absent the factors also apply for double-sideband AM with envelope detection. It should be noted that T in all cases is the pulse interval in double-sideband AM, which is twice the pulse interval

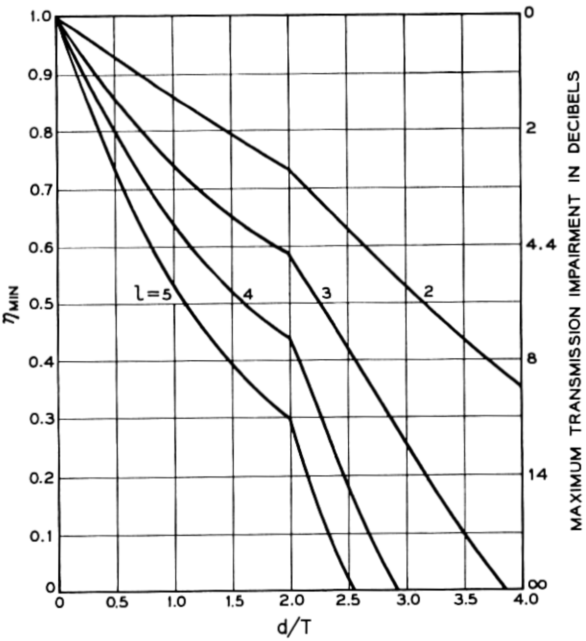

Fig. 3 — Factor η_{min} for raised cosine pulse spectrum and quadratic delay distortion as in Fig. 2, for AM systems employing synchronous detection and l pulse amplitudes. Factor applies for double- and vestigial-sideband AM and quadrature double-sideband AM.

in vestigial-sideband AM or twice the combined pulse interval in quadrature double-sideband AM.

In the above evaluation it was assumed that the pulses were sampled at $t = 0$, which is not at the peak of a pulse except for $d/T = 0$. For $d/T = 4$ the pulse peak is nearly at $t/T = x = 0.25$. Sampling at $x = 0.25$ gives, for $l = 2$, $\eta_{\min} = 0.356$ rather than 0.347.

The factor $\eta_{\min}$ expressed in decibels, as in Fig. 3, indicates the maximum transmission impairment, i.e., the maximum increase in signal-to-noise ratio required at the detector input to compensate for the effect of delay distortion. This maximum impairment would be closely approached for signal-to-noise ratios such that the error probability is sufficiently small, say less than 10^{-7}. However, for error probabilities in the range ordinarily considered the transmission impairment will be less, in accordance with the discussion in Section 2.10. For example, for $d/T = 4$, $R_0(0) = 0.734$ and $\bar{r}_0(0) = 0.385$. The maximum noise margin for $l = 2$ is in this case

$$M_{\max} = R_0(0) + \bar{r}_0(0) \cong 1.12$$

and the minimum margin is

$$M_{\min} = R_0(0) - \bar{r}_0(0) \cong 0.397.$$

For $k = M_{\max}/M_{\min} = 3.2$, the results given in Table II indicate that the transmission impairments would be less than the maximum by about 1 db and 1.4 db for error probabilities of 10^{-5} and 10^{-4} respectively. For $d/T = 4$ and $l = 2$ the maximum impairment indicated in Fig. 3 is $-20 \log_{10} M_{\min} \cong 9.2$ db, whereas impairments of about 8.2 and 7.8 db would be expected for error probabilities of 10^{-5} and 10^{-4}, on the premises underlying the evaluation in Section 2.10.

3.6 *Even Symmetry Spectrum and Odd Symmetry Delay Distortion*

When the pulse spectrum at the detector input has even symmetry about the midband frequency and the phase characteristic has a component of even symmetry, i.e., odd symmetry delay distortion, the in-phase and quadrature components both have even symmetry with respect to t. That is,

$$R_0(-t) = R_0(t); \qquad Q_0(-t) = Q_0(t) \tag{84}$$

With synchronous detection the phase of the demodulating carrier would preferably be so chosen that $Q_0(t)$ would vanish at $t = 0$, since this would give the maximum amplitude of the demodulated pulse at a sampling constant, equal to $[R_0^2(0) + Q_0^2(0)]^{\frac{1}{2}}$. For purposes of analysis it is therefore convenient to modify the phase angle so that the quadrature component vanishes at $t = 0$. The modified quantities are related to R_0 and Q_0 by (see Appendix)

$$R_{00}(t) = \frac{R_0(0)R_0(t) + Q_0(0)Q_0(t)}{[R_0^2(0) + Q_0^2(0)]^{\frac{1}{2}}}, \tag{85}$$

$$Q_{00}(t) = \frac{R_0(0)Q_0(t) - Q_0(0)R_0(t)}{[R_0^2(0) + Q_0^2(0)]^{\frac{1}{2}}}. \tag{86}$$

In the case of double-sideband transmission (77) applies, with

$$\bar{r}_c = \bar{r}_{00} = 2\sum_{n=1}^{\infty} \bar{R}_{00}(n). \tag{87}$$

With quadrature double-sideband transmission (80) applies, with $Q_c(0) = Q_{00}(0) = 0$ and

$$\bar{q}_c = \bar{q}_{00} = 2\sum_{n=1}^{\infty} \bar{Q}_{00}(n), \tag{88}$$

where $\bar{R}_{00}$ and $\bar{Q}_{00}$ designate absolute values.

In the case of vestigial-sideband transmission at pulse intervals $T' = T/2$, the in-phase component referred to a carrier at frequency $\omega_0 + \bar{\omega}$ is obtained from (10) and becomes

$$R_{c,00} = \cos\left(\frac{\bar{\omega}T}{2}\, m\right) R_{00}(m) - \sin\left(\frac{\bar{\omega}T}{2}\, m\right) Q_{00}(m). \tag{89}$$

With $\bar{\omega}T = \pi$, i.e., $\bar{\omega}T' = \pi/2$, (89) gives at even sampling points, $m = 0,2,4,6, \cdots$,

$$R_{c,00}^{(m)} = (-1)^m R_{00}(m). \tag{90}$$

At odd sampling points, $m = 1,3,5,7, \cdots$,

$$R_{c,00}^{(m)} = (-1)^{(m+1)/2} Q_{00}(m). \tag{91}$$

In this case (77) applies, with $R_c(0) = R_{00}(0)$ and

$$\bar{r}_c = 2\sum_{m=2,4,6\cdots}^{\infty} \bar{R}_{00}(m) + 2\sum_{m=1,3,5\cdots}^{\infty} \bar{Q}_{00}(m). \tag{92}$$

3.7 *Raised Cosine Spectrum and Linear Delay Distortion*

For the special case of a raised cosine spectrum and linear delay distortion the functions R_0 and Q_0 have been determined by numerical integration, as discussed further in the Appendix. They are given in Table XX for certain ratios d/T, where d is the difference in delay be-

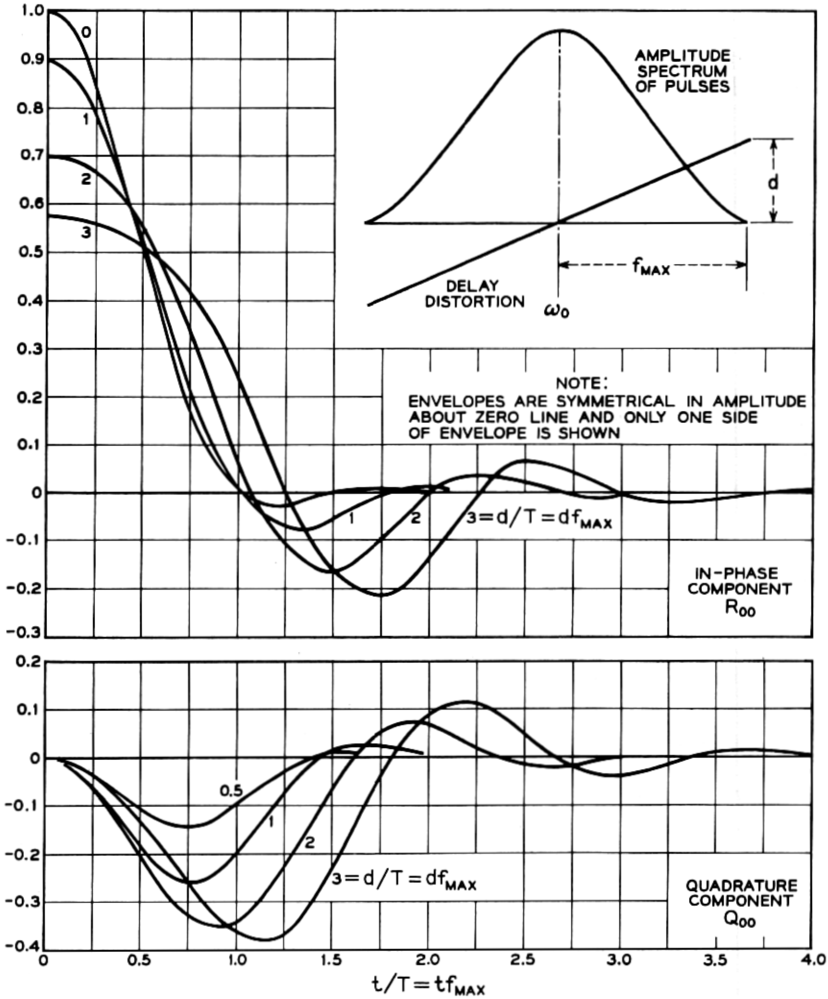

Fig. 4 — Carrier pulse transmission characteristics for raised cosine spectrum as in Fig. 1 and linear delay distortion.

tween the midband frequency and maximum sideband frequency as illustrated in Fig. 4. The modified functions R_{00} and Q_{00} are given in Table XXI and are shown in Fig. 4. For negative values of t/T, R_{00} and Q_{00} are the same as shown in Fig. 4 for positive values.

For double-sideband transmission the factors $\eta_{\min}$ given in Table V are obtained from (77), with $\bar{r}_c$ taken in accordance with (87). These factors are shown in Fig. 5. The case $l = 2$ corresponds to two-phase transmission.

TABLE V — FACTOR $\eta_{\min}$ FOR DOUBLE-SIDEBAND AM AND l = 2, 3, 4 AND 5 PULSE AMPLITUDES

l	d/T				
	0	0.5	1	2	3
2	1	0.959	0.860	0.517	−0.144
3	1	0.947	0.826	0.336	−0.860
4	1	0.935	0.792	0.155	−1.57
5	1	0.923	0.758	−0.026	—

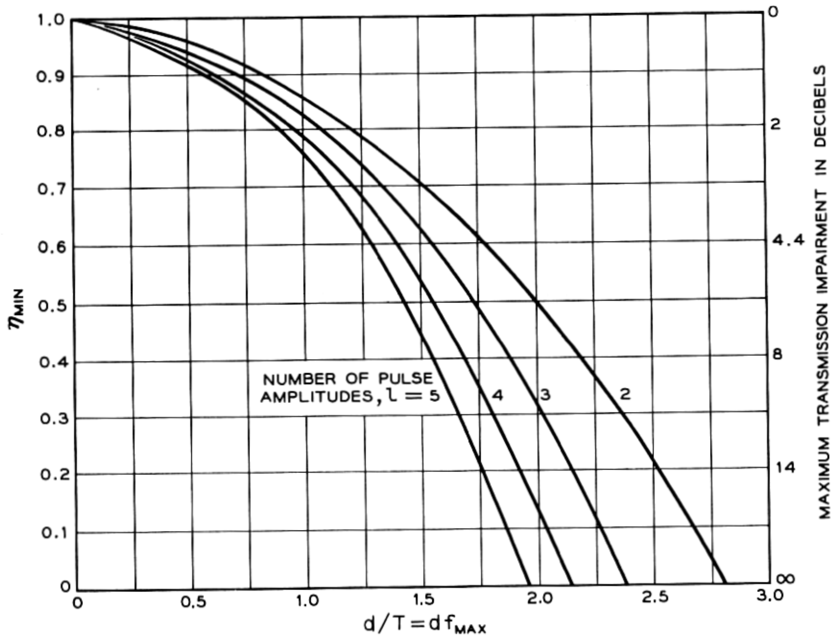

Fig. 5 — Factor $\eta_{\min}$ for raised cosine pulse spectrum and linear delay distortion as in Fig. 4 for double-sideband AM systems employing synchronous detection and l pulse amplitudes.

For quadrature double-sideband AM the factors in Table VI are obtained from (80) with $\bar{r}_c$ and $\bar{q}_c$ taken in accordance with (87) and (88). These factors are shown in Fig. 6. The case l = 2 corresponds to four-phase transmission.

For vestigial-sideband transmission the factor $\eta_{\min}$ is determined from (77), with $\bar{r}_c$ taken in accordance with (92). The factors given in Table VII are thus obtained.

TABLE VI — FACTOR η_{min} FOR QUADRATURE DOUBLE-SIDEBAND AM FOR l = 2 AND 3 PULSE AMPLITUDES

l	d/T			
	0	0.5	1	2
2	1	0.735	0.458	−0.32
3	1	0.499	0.022	−1.34

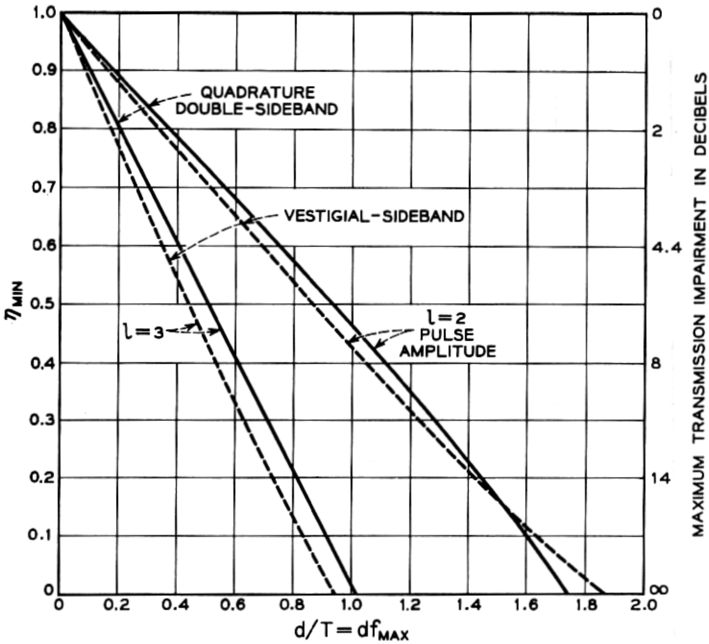

Fig. 6 — Factor η_{min} for raised cosine pulse spectrum and linear delay distortion as in Fig. 4 for quadrature double-sideband AM systems (solid lines) and vestigial-sideband AM systems (dashed lines) employing synchronous detection and l = 2 and 3 pulse amplitudes.

TABLE VII — FACTOR η_{min} FOR VESTIGIAL-SIDEBAND AM FOR l = 2 AND 3 PULSE AMPLITUDES

l	d/T			
	0	0.5	1	2
2	1	0.703	0.42	−0.061
3	1	0.435	−0.05	−0.82

3.8 *Vestigial-Sideband vs. Quadrature Double-Sideband AM*

The factors $\eta_{\min}$ for vestigial sideband AM are compared in Fig. 6 with the corresponding factors for quadrature double-sideband AM. With ideal transmission-frequency characteristics and ideal synchronous detection the two methods are equivalent from the standpoint of channel bandwidth requirements and optimum signal-to-noise ratio for a given error probability. As shown in Section 3.4, this also applies for pulse spectra at the detector input with even symmetry about the midband frequency, in the presence of delay distortion with even symmetry. The equations in Section 3.6 and the curves in Fig. 6 show that the above two methods are not equivalent in the presence of delay distortion with odd symmetry about the midband frequency. With linear delay distortion the factor $\eta_{\min}$ is, however, very nearly the same with both methods. For practical purposes quadrature double-sideband AM and vestigial-sideband AM can be regarded as equivalent with any type of delay distortion that would be expected in actual facilities. This equivalence would apply on the premise of ideal synchronous detection but not necessarily with actual implementation of synchronous detection, for the reason that the penalty in signal-to-noise ratio incurred in deriving a demodulating wave may not be the same with both methods.

3.9 *Envelope Detection vs. Synchronous Detection*

In the preceding analysis ideal synchronous detection was assumed, which permits the use of bipolar pulses. An alternative method that is simpler in implementation is envelope detection, which, however, entails the use of unipolar pulse transmission and for this reason has a certain disadvantage in signal-to-noise ratio as compared to synchronous detection.[1] In addition, transmission impairments by phase distortion may in certain cases be greater with envelope than with synchronous detection, as shown below.

When both the pulse spectrum and delay distortion have even symmetry about the carrier frequency, so that the quadrature component is absent, the effect of delay distortion is the same as with synchronous detection. The results given in Table IV thus apply also for double-sideband AM with envelope detection.

When a quadrature component is present in the carrier pulse transmission characteristic the resultant demodulated wave is in accordance with (23) given by

$$U(0) = (r_0^2 + q_0^2)^{\frac{1}{2}}, \tag{93}$$

where

$$r_0 = r_0(x) = \sum_{n=-\infty}^{\infty} a(n)R_0(x - n), \tag{94}$$

$$q_0 = q_0(x) = \sum_{n=-\infty}^{\infty} a(n)Q_0(x - n). \tag{95}$$

The modified values R_{00} and Q_{00} can be used in place of R_0 and Q_0.

Owing to the presence of both in-phase and quadrature components, it does not appear feasible to derive a simple general expression for $U_{\max}^{(s)}$ and $U_{\min}^{(s+1)}$ similar to (72) and (73). These values can, however, be determined by examining several combinations of transmitted pulses, as illustrated below for binary pulse transmission, a raised cosine pulse spectrum at the detector input and linear delay distortion. Using values of R_{00} and Q_{00} given in Table XXI of the Appendix, the results are as shown in Table VIII. Since both $R_0(t)$ and $Q_0(t)$ in this case have even symmetry about $t = 0$ the maximum effect of delay distortion is encountered for pulse trains with even symmetry about the sampling point, i.e., $a(-n) = a(n)$. Hence, only pulse trains with this property need to be considered.

From Table VIII can be obtained $W_{\max}^{(0)}$ and $W_{\min}^{(1)}$, as indicated by asterisks, together with the optimum slicing level given by (57) and the factor $\eta_{\min} = W_{\min}^{(1)} - W_{\max}^{(0)}$. These are given in Table IX.

Table VIII — Values of $U(0)$ for Raised Cosine Spectrum and Linear Delay Distortion for Various Combinations of Marks = 0 and Spaces = 1

d/T	$a(0)$	$a(-1)$ $a(1)$	$a(-2)$ $a(2)$	$a(-3)$ $a(3)$	r_0	q_0	$U(0)$
0.5	0	1	0	0	0.036	−0.191	0.195
	0	1	1	0	0.046	−0.194	0.199*
	0	1	1	1	0.046	−0.194	0.199
	1	0	0	0	0.952	0.194	0.970*
	1	1	0	0	0.988	0.003	0.988
	1	1	1	0	0.998	0	0.998
1	0	1	0	0	0.1452	−0.3584	0.384*
	0	1	1	0	0.1676	−0.3370	0.374
	0	1	1	1	0.1688	−0.3322	0.374
	1	0	0	0	0.8309	0.3306	0.90*
	1	1	0	0	0.9761	−0.0278	0.98
	1	1	1	0	0.9985	−0.0064	0.99
2	0	1	0	0	0.5192	−0.4878	0.72*
	0	1	1	0	0.5156	0.3724	0.64
	1	0	0	0	0.5786	0.3895	0.70
	1	0	1	0	0.5020	0.5040	0.71
	1	0	0	1	0.5598	0.1380	0.68*

TABLE IX — FACTOR η_{min} WITH BINARY AM AND ENVELOPE DETECTION FOR RAISED COSINE SPECTRUM AND LINEAR DELAY DISTORTION

d/T	0	0.5	1	2
$W_{max}^{(0)}$	0	0.199	0.384	0.72
$W_{min}^{(1)}$	1	0.970	0.900	0.68
L	0.50	0.58	0.63	0.70
η_{min}	1	0.77 (0.959)†	0.60 (0.86)†	−0.04 (0.517)†

† From Table V for binary AM with synchronous detection.

It will be recognized that synchronous detection has a significant advantage over envelope detection as regards transmission impairments caused by pronounced linear delay distortion, for the reason that the effect of the quadrature component is eliminated. In general, delay distortion will have a component of even symmetry and a component of odd symmetry about the carrier frequency, in which case the quadrature component will be smaller. The advantage of synchronous detection as regards transmission impairments caused by delay distortion will then be less than indicated in Table IX. The principal advantage of synchronous detection is that it permits the use of bipolar transmission, which in the case of binary systems as considered above affords about 3 db improvement in the ratio of average signal power to average noise power for a given error probability (Ref. 1, Table VII).

In the case of vestigial-sideband transmission a pronounced quadrature component is present even in the absence of phase distortion. The advantage of synchronous detection over envelope detection is in this case significantly greater than for double-sideband transmission considered above, for the reasons that bipolar transmission can be used and quadrature component is eliminated. In the absence of phase distortion and with a raised cosine pulse spectrum at the detection input, synchronous detection has about a 9 db advantage over envelope detector in the ratio of average signal power to average noise power for a given error probability (6 db owing to elimination of quadrature component and 3 db owing to bipolar transmission).

Evaluation of transmission impairments from phase distortion is more complicated for envelope than for synchronous detection. These impairments have been determined experimentally for a binary vestigial-sideband system with an approximately raised cosine spectrum at the detector input, for linear and quadratic delay distortion and combinations thereof.[4] They are significantly greater than determined herein for synchronous detection. Hence envelope detection entails more phase equalization than synchronous detection, unless a greater disparity in signal-to-noise ratio is accepted than the 9 db applying in the absence of phase distortion.

IV. PM WITH DIFFERENTIAL PHASE DETECTION

4.1 *General*

In phase modulation with differential phase detection, the demodulator output would under ideal conditions depend on changes in carrier phase between two successive pulse intervals of duration T. In its simplest and ideal form, the signal with two-phase modulation would be applied to one pair of terminals of a product demodulator, while the signal delayed by a pulse interval T would be applied to the other pair. With four-phase modulation two product demodulators are required, each with a delay network at one pair of terminals. In addition, a phase shift of 90° must be provided between all frequencies of the demodulating waves of the two demodulators, as indicated in Fig. 7. Such a phase shift over a frequency band can be realized in principle and closely approached with actual networks.[7] The modulator outputs would be applied to low-pass filters of appropriate bandwidth for elimination of high-frequency demodulation products, and the output of these would be sampled at interval T. The phase of the carrier would be indicated by the combined output as discussed in Sections 2.5 and 2.6.

With the above method it is possible with ideal channel characteristics to avoid intersymbol interference at sampling instants, without the need for a wider channel band than required with synchronous detection. However, the two methods are not in all respects equivalent from the

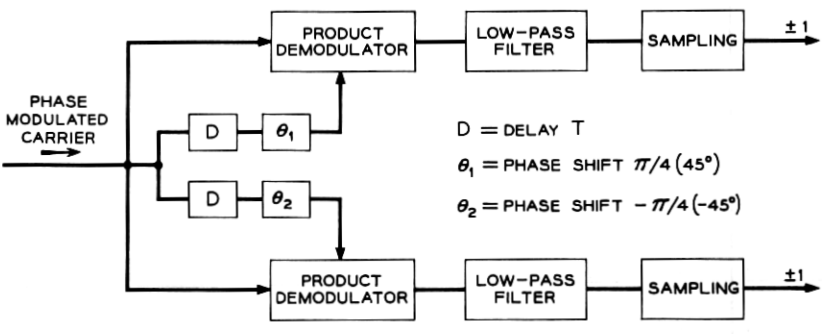

Fig. 7 — Basic demodulator arrangement for four-phase modulation with differential phase detection.

standpoint of bandwidth utilization. As discussed in Sections 2.5 and 2.6, with synchronous detection the carrier frequency must be at least equal to the maximum baseband signal frequency, whereas with differential phase detection it must exceed twice the maximum baseband signal frequency. This requirement does not impose a limitation on bandwidth utilization with differential phase modulation provided the midband frequency of the available channel is at least twice the lowest frequency, or that this condition is realized through frequency translation prior to demodulation.

With differential phase detection the demodulating wave is established without the need for a long delay (measured in pulse intervals) as required with certain other methods mentioned in Section 3.1. Moreover, a substantial fluctuation in carrier phase can be tolerated, since only the difference in phase between adjacent pulses need be considered. These advantages are realized in exchange for a penalty in signal-to-noise ratio as compared to ideal synchronous detection owing to the presence of noise in the demodulating wave. For very small error probabilities, and assuming ideal implementation in all respects, this impairment is about 1 db for two-phase and about 2.3 db for four-phase modulation.[8,9] Comparable penalties in signal-to-noise ratio as compared to ideal synchronous detection may be incurred with the other methods of providing a demodulating wave mentioned in Section 3.1, owing to small unavoidable amplitude and phase fluctuations in the demodulating wave resulting from other causes than noise. However, in the case of differential phase detection, greater transmission impairments would be expected from phase distortion, since the effect of phase distortion, like that of noise, is present in both the signal and the demodulating wave. The transmission impairments resulting from quadratic delay distortion are determined here, and compared with that encountered with ideal synchronous detection.

Other implementations of differential phase detection than assumed herein have been used, but in principle these entail a wider channel band than with ideal synchronous detection. For example, the two demodulator inputs or outputs could be integrated over a pulse integral T with the aid of a narrow-band resonator tuned to the carrier frequency, and then be rapidly quenched before the next signal interval. When the channel bandwidth is limited, the phase of the demodulating carrier will then depend on the phases of the carrier during several pulse intervals. Thus some intersymbol interference from bandwidth limitation is encountered even in the absence of phase distortion, and the effect of phase distortion will be greater than that determined herein. However, excessive transmission impairments from bandwidth limitation and phase distortion can be avoided by appropriate techniques, as when a large number of narrow channels are provided within a common band of much greater bandwidth than that of the individual channels.[10]

4.2 *Basic Expressions*

In differential phase modulation the carrier would be at midband, i.e., $\omega_c = \omega_0$. With $U_{01} = V$, expression (33) for the demodulated signal becomes

$$V = \sum_{n=-\infty}^{\infty} S_n(0)\bar{P}_0(-n), \tag{96}$$

where

$$S_n(0) = \sum_{m=-\infty}^{\infty} \bar{P}_0(-m) \cos [\psi_0(-n) - \psi_0(-m+1) + \varphi_0(-n) - \varphi_0(-m) - \theta]. \tag{97}$$

The above expressions apply for the output of the single demodulator in two-phase systems, in which $\theta = 0$. In four-phase systems the output of one demodulator is obtained with $\theta = \theta_1$ and the output of the other with $\theta = \theta_2 = \theta_1 \pm \pi/2$.

Examination of (97) shows that the term for $m = n + 1$ is independent of the phase difference $\psi_0(-n) - \psi_0(-m+1)$ and is given by

$$\cos [\varphi_0(-n) - \varphi_0(-n-1) - \theta]\bar{P}_0(-n-1).$$

This represents a DC or bias component. The total bias component in (96) is

$$V_0 = \sum_{n=-\infty}^{\infty} \cos [\varphi_0(-n) - \varphi_0(-n-1) - \theta]\bar{P}_0(-n)\bar{P}_0(-n-1). \tag{98}$$

Determination of transmission impairments becomes rather difficult except for the special case in which $\varphi_0(n) = 0$, which will be considered in further detail below.

4.3 *Even Symmetry Spectrum and Delay Distortion*

When the pulse spectrum has even symmetry about the midband frequency, and the phase characteristic has odd symmetry (i.e., even symmetry delay distortion) the quadrature component of $P_0(t)$ vanishes, i.e., $\varphi(-n) = 0$. In this case, $\bar{P}_0 = R_0$ and (96) becomes

$$V = \sum_{n=-\infty}^{\infty} S_n(0)R_0(-n) = S_0R_0(0) + \sum_{n=1}^{\infty} [S_nR_0(-n) + S_{-n}R_0(n)], \tag{99}$$

while (97) and (98) simplify to

$$S_n = S_n(0) = \sum_{m=-\infty}^{\infty} R_0(-m) \cos [\psi_0(-n) - \psi_0(-m+1) - \theta], \tag{100}$$

$$V_0 = \sum_{m=-\infty}^{\infty} R_0(-n)R_0(-n-1) \cos \theta. \tag{101}$$

With $\theta = \pi/4$ in (100) and (101), these expressions can be written, after introduction of a normalizing factor $\sqrt{2}$,

$$S_n = \sum_{m=-\infty}^{\infty} R_0(-m)a_m(n), \tag{102}$$

$$V_0 = \sum_{m=\infty}^{\infty} R_0(-n)R_0(-n-1), \tag{103}$$

$$a_m(n) = \sqrt{2} \cos [\psi_0(-n) - \psi_0(-m+1) - \pi/4] = \cos [\psi_0(-n) - \psi_0(-m+1)] + \sin [\psi_0(-n) - \psi_0(-m+1)]. \tag{104}$$

With $\psi_0(-n) - \psi_0(-m+1) = 0, \pi/2, \pi$ or $3\pi/2$ the following relations apply

$$a_m(n) = \pm 1 \quad \text{for} \quad m \neq n+1, \qquad = 1 \quad \text{for} \quad m = n+1. \tag{105}$$

In view of (105), (102) can alternately be written in the form

$$S_n - [1 - a_{n+1}(n)]R_0(-n-1) = \sum_{m=-\infty}^{\infty} a_m(0)R_0(-m), \tag{106}$$

where in the summation $a_m(n)$ can be chosen as -1 or as 1, also for $m = n + 1$.

4.4 *Two-Phase Modulation*

With synchronous detection, two-phase modulation could be used in conjunction with both double-sideband and vestigial-sideband transmission. With differential phase detection, however, vestigial-sideband transmission is not practicable, since severe transmission impairments would be incurred even in the absence of phase distortion, owing to the presence of the quadrature component. Hence only double-sideband two-phase modulation is considered here.

For $n = 0$, (106) gives

$$S_0 - [1 - a_1(0)R_0(-1)] = \sum_{m=-\infty}^{\infty} a_m(0)R_0(-m). \tag{107}$$

Assume that in (107) a sequence of values of $a_m(0)$ has been chosen, for example $a_{-3}(0) = 1$, $a_{-2}(0) = 1$, $a_{-1}(0) = 1$, $a_0(0) = 1$, $a_1(0) = -1$, $a_2(0) = 1$, etc. For any other value of n than $n = 0$, the sequence of $a_m(n)$ will either be identical with that for $a_m(0)$, or all signs will be reversed. This follows from (104), since $\psi_0(-n)$ will differ from $\psi_0(0)$ by 0 or π. Hence, for $n \neq 0$, the right-hand side of (106) can be replaced by the left-hand side of (107), so that

$$S_n - [1 - a_{n+1}(n)]R_0(-n-1) = \pm [S_0 - [1 - a_1(0)]R_0(-1)] \tag{108}$$

In the absence of intersymbol interference, V as given by (99) would be -1 or 1. In the following, the minimum possible value of V will be determined, on the assumption that $V = 1$ without intersymbol interference; i.e., $a_0(0) = 1$.

Consider first the term $S_0R_0(0)$ in (99). The minimum possible value of S_0 is obtained from (107) by choosing $a_m(0) = -1$ for $R_0(-m) > 0$ and $a_m(0) = 1$ for $R_0(-m) < 0$. The following relation is thus obtained for the minimum possible value of S_0, on the above premise of $a_0(0) = 1$:

$$\begin{aligned} S_{0,\min} = {} & [1 - a_1(0)]\, R_0(-1) + R_0(0) \\ & - \sum_{m=1}^{\infty} [\bar{R}_0(-m) + \bar{R}_0(m)], \end{aligned} \tag{109}$$

where $\bar{R}_0$ designates the absolute values. In the above expression $a_1(0)$ would be taken as $a_1(0) = 1$ if $R_0(-1) < 0$ and as $a_1(0) = -1$ if $R_0(-1) > 0$. The term $[1 - a_1(0)]R_0(-1)$ can therefore be written alternatively as $R_0(-1) + \bar{R}_0(-1)$, in which case (109) becomes

$$S_{0,\min} = R_0(-1) + \bar{R}_0(-1) + U_{\min}, \tag{110}$$

where

$$U_{\min} = R_0(0) - \sum_{m=1}^{\infty} [\bar{R}_0(-m) + \bar{R}_0(m)]. \tag{111}$$

It will be recognized that $U_{\min}$ is the same as the minimum possible value of the demodulated voltage with synchronous detection, in the presence of a mark, as given in a somewhat more general form by (73).

Having thus determined $S_{0,\min}$ it follows from (108) and (110) that the two possible associated values of $S_{n,\min}$ are given by

$$S_{n,\min} = R_0(-n-1) + \bar{R}_0(-n-1) \pm U_{\min}, \tag{112}$$

where the term $[1 - a_{n+1}(n)]R_0(-n-1)$ in (108) has been replaced by the equivalent representation by the first two terms in (112).

To obtain the minimum value of V as given by (99), each term in the series must be made to have the maximum negative value. To this end the negative sign in (112) for $U_{\min}$ is chosen if $R_0(n)$ is positive, and the positive sign if $R_0(n)$ is negative. The minimum possible value of V thus obtained with (110) and (112) in (99) is

$$\begin{aligned} V_{\min} = {} & [R_0(-1) + \bar{R}_0(-1) + U_{\min}]R_0(0) \\ & + \sum_{n=1}^{\infty} [R_0(-n-1) + \bar{R}_0(-n-1)]R_0(-n) \\ & + \sum_{n=1}^{\infty} [R_0(n+1) + \bar{R}_0(n+1)]R_0(n) \\ & - U_{\min} \sum_{n=1}^{\infty} [\bar{R}_0(-n) + \bar{R}_0(n)], \end{aligned} \tag{113}$$

$$\begin{aligned} = {} & U_{\min} \left\{ R_0(0) - \sum_{n=1}^{\infty} [\bar{R}_0(-n) + \bar{R}_0(n)] \right\} \\ & + \sum_{n=-\infty}^{\infty} [R_0(-n-1) + \bar{R}_0(-n-1)]R_0(-n), \end{aligned} \tag{114}$$

where the first term can also be written $U_{\min}^2$.

In accordance with the discussion in Section 4.2, the demodulator output contains a bias or DC component V_0 given by (103). Optimum performance is obtained when the threshold level for distinction between $V = 1$ and $V = -1$ is made equal to V_0. When V_0 is subtracted from both sides of (114) the following expression is obtained for two-phase modulation:

$$V_{\min}^0 = V_{\min} - V_0 = U_{\min}^2 + \Sigma, \tag{115}$$

where

$$\Sigma = \sum_{n=-\infty}^{\infty} \bar{R}_0(-n-1)R_0(-n). \tag{116}$$

When intersymbol interference is absent at sampling instants, $R_0(n) = 0$ for $n \neq 0$, and for $n = 0$ is $R_0^0(0)$. In this case $V_{\min}^0 = U_{\min}^2 = [R_0^0(0)]^2$. The voltage given by (115) is smaller than in the absence of intersymbol interference by the factor

$$\eta_{\min}^0 = \eta_{\min}^2 + \Sigma/[R_0^0(0)]^2, \tag{117}$$

where $\eta_{\min}$ applies for synchronous detection.

4.5 *Four-Phase Modulation*

The basic difference between two-phase and four-phase modulation is that relation (108) does not apply for four-phase modulation. Returning to the discussion following (107), if a sequence $a_m(0)$ is chosen in four-phase transmission, the sequence $a_m(n)$ can be chosen independently. This follows from the (104), which shows that if $a_m(0)$ has a given value, say $a_m(0) = 1$, it is possible to make each $a_m(n)$ equal to $+1$ or -1 by appropriate choice of $\psi(-n)$.

For this reason the minimum value (or maximum negative value) of the right-hand side of (106) is now, for $n \neq 0$:

$$\begin{aligned}[S_n - [1 - a_{n+1}(n)]R_0(-n-1)]_{\min} \\ = -R_0(0) - \sum_{m=1}^{\infty} [\bar{R}_0(-m) + \bar{R}_0(m)].\end{aligned} \tag{118}$$

The right-hand side of (118) is smaller than for two-phase transmission as given by (112) by $-2R_0(0)$. When this modification is introduced, the following expression is obtained for four-phase modulation, in place of (115) for two-phase modulation:

$$\begin{aligned}V_{\min}^0 = U_{\min}^2 - 2R_0(0) \sum_{n=1}^{\infty} [\bar{R}_0(-n) + \bar{R}_0(n)] \\ + \sum_{n=-\infty} \bar{R}_0(-n-1)R_0(-n)\end{aligned} \tag{119}$$

or

$$V_{\min}^0 = U_{\min}^2 - 2R_0(0)[R_0(0) - U_{\min}] + \Sigma, \tag{120}$$

where Σ is given by (116).

The voltage given by (120) is smaller than in the absence of intersymbol interference by the factor

$$\eta_{\min}^0 = [V_{\min}^0/R_0^0(0)]^2. \tag{121}$$

Table X — Minimum Amplitudes of Demodulated Pulse Trains in Two-Phase Modulation with Differential Phase Detection

d/T	0	1	2	3	4
$U_{\min}^2$	1	0.730	0.536	0.273	0.120
Σ	0	0.012	0.058	0.137	0.222
$V_0 = L_0$	0	0.012	0.058	0.137	0.222
$V_{\min}^0 = \eta_{\min}^0$	± 1	± 0.74	± 0.594	± 0.41	± 0.342
$V_{\min}^+ = V_{\min}^0 + V_0$	1	0.75	0.652	0.547	0.564
$V_{\min}^- = -V_{\min}^0 + V_0$	-1	-0.73	-0.536	-0.273	-0.12

4.6 *Raised Cosine Spectrum and Quadratic Delay Distortion*

The function $R_0(n)$ for this case is given in Table III. The values of $U_{\min}$ for synchronous detection are given in Table IV for $l = 2$. In Table X are given the various quantities appearing in expression (115) for the minimum amplitudes of received pulse trains at sampling instants with optimum slicing lead equal to the DC component V_0. The values of $V_{\min}^0 = \eta_{\min}^0$ are shown in Fig. 8.

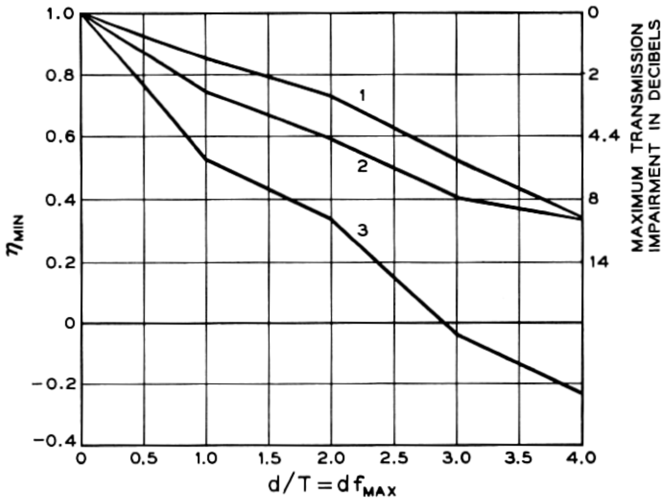

Fig. 8 — Factor $\eta_{\min}$ for raised cosine spectrum and quadratic delay distortion as in Fig. 2 for synchronous detection and differential phase detection. Curve 1: Ideal synchronous detection — applies for two-phase and four-phase modulation with carrier at midband and pulses at intervals T, and for vestigial-sideband transmission with pulses at intervals $T/2$. Curve 2: Ideal differential phase detection — two-phase modulation with pulse interval T. Curve 3: Ideal differential phase detection — four-phase modulation with pulse interval T.

TABLE XI — MINIMUM AMPLITUDES OF DEMODULATED PULSE TRAINS IN FOUR-PHASE MODULATION WITH DIFFERENTIAL PHASE DETECTION

d/T	0	1	2	3*	4*
$V_0 = L_0$	0	0.012	0.058	−0.137	−0.232
$V_{\min}^0 = \eta_{\min}^0$	±1	±0.53	±0.33	±0.04	±0.23
$V_{\min}^+ = V_{\min}^0 + V_0$	+1	+0.54	+0.39	+0.10	0
$V_{\min}^- = -V_{\min}^0 + V_0$	−1	−0.52	−0.27	+0.18	+0.45

* Reversal of sign indicates a reversal in sign of the demodulated pulses.

With the accuracy used herein it turns out that Σ and V_0 are numerically equal but are not identical.

It will be noted that, when delay distortion is pronounced, the bias component V_0 is appreciable, and that a significant penalty can be incurred if the threshold or slicing level is taken as 0 rather than V_0. For example, with $d/T = 4$ and 0 threshold level the minimum amplitude of a demodulated pulse for a carrier phase $\psi = 0$ would be 0.564, and the minimum negative amplitude for a carrier phase $\psi = \pi$ would be −0.12. With the optimum threshold level the minimum amplitudes are ±0.342. Hence the tolerable peak noise amplitudes would be greater by a factor 0.342/0.12 = 2.85.

With four-phase modulation the values given in Table XI are obtained from (120). The values of $V_{\min}^0 = \eta_{\min}^0$ are shown in Fig. 8.

In the above illustrative examples it was assumed that pulses were transmitted at the minimum interval T permitted if intersymbol interference is to be avoided in the absence of delay distortion. The effect of delay distortion may or may not be reduced by increasing the pulse interval, that is, in exchange for a slower transmission rate. By way of

TABLE XII — FUNCTION $R_0(n)$ FOR RAISED COSINE SPECTRUM AND QUADRATIC DELAY DISTORTION WITH 50 PER CENT INCREASE IN PULSE INTERVAL

n^*	d/T				
	0	1	2	3	4
−2	0	−0.0006	0.0025	0	0
−1	0	0.0053	0.0081	0.0145	0.0202
0	1	0.9633	0.8795	0.7956	0.7336
1	0	−0.0341	−0.1161	−0.2200	−0.2909
2	0	0.0044	0.0020	−0.0231	−0.0584
3	0	−0.0009	0.0011	0.0044	−0.0030

$n^* = \frac{2}{3}$ of the values n given in Table XIX.

TABLE XIII — MINIMUM AMPLITUDES OF DEMODULATED PULSES WITH 50 PER CENT INCREASE IN PULSE INTERVALS

d/T	0	1	2	3	4
$U_{\min}$	1	0.92	0.75	0.50	0.36
Σ	0	−0.028	−0.10	−0.17	−0.08
$V_{\min}^0$ (2-phase)	1	0.80	0.46	0.08	0.045
$V_{\min}^0$ (4-phase)	1	0.72	0.23	−0.40	−0.50

illustration it will be assumed that the pulse interval is increased by a factor 1.5, in which case the values of R_0 are as given in Table XII.

With this modification, the various quantities are as given in Table XIII.

In Fig. 9 values of $U_{\min}$ and $V_{\min}^0$ are compared with those for the minimum interval between pulses. It will be noted that there is no significant difference in the case of two-phase or four-phase modulation with synchronous detection. With differential synchronous detection some advantage is realized for small delay distortion in exchange for a disadvantage with pronounced delay distortion.

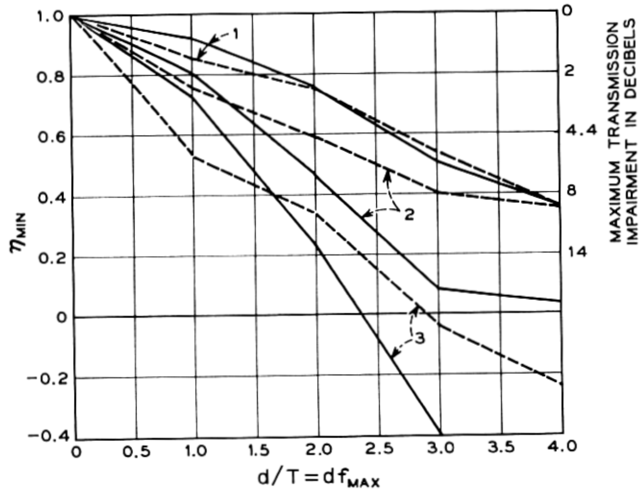

Fig. 9 — Effect of pulse interval on factor $\eta_{\min}$ for raised cosine spectrum with quadratic delay distortion (dashed curves: pulse interval T, as in Fig. 8; solid curves: pulse interval $1.5T$). Curves 1: Ideal synchronous detection — applies for two-phase and four-phase modulation with carrier at midband and pulses at intervals $1.5T$ and for vestigial-sideband transmission with pulses at intervals $0.75T$. Curves 2: Ideal differential phase detection — two-phase modulation with pulse intervals $1.5T$. Curves 3: Ideal differential phase detection — four-phase modulation with pulse intervals $1.5T$.

V. BINARY FREQUENCY MODULATION (FSK)

5.1 *General*

As shown elsewhere,[1] with optimum design, binary FM or frequency shift keying requires the same minimum bandwidth as double-sideband AM. In the absence of transmission distortion from gain and phase deviations, the optimum signal-to-noise ratio required at the detector input for a given error probability is slightly greater than for two-phase transmission with ideal synchronous detection, but would be expected to be about the same as for two-phase transmission with ideal differential phase detection. Binary FM may be preferable to the latter method from the standpoint of implementation and has an advantage over the simpler method of binary AM with envelope detection from the standpoint of signal-to-noise ratio and performance during sudden transmission level variations.

The performance of binary FM is determined here for channels with linear and quadratic delay distortion and compared with that of the other methods mentioned above. In this analysis ideal frequency discriminator detection is assumed, in which the demodulated signal is proportional to the time derivative of the phase of the received wave. This condition may be closely approached with actual detectors when the channel bandwidth is small in relation to the midband frequency. However, when this is not the case, ideal FM detection is only approximated with conventional frequency discriminators or zero crossing detectors.

5.2 *Basic Expression*

Expression (48) for the demodulated pulse train applies for any amplitude and phase characteristic of the channel. In the case of a continuing space, $a(n) = 0$ in (46) and (47) and $U_0(t) = 0$. With a continuing mark, $a(n) = 1$ in the above expressions and

$$\alpha_0(x) = \sum_{n=-\infty}^{\infty} (-1)^n R_0(x - n), \tag{122}$$

$$\beta_0(x) = \sum_{n=-\infty}^{\infty} (-1)^n Q_0(t - n). \tag{123}$$

Returning to (21), it will be recognized that (122) and (123) represent the in-phase and quadrature components in a binary amplitude modulation system when pulses of duration T and alternating polarity are transmitted, i.e., $a(n) = (-1)^n$ in (21). The fundamental frequency of such a pulse train is $\bar{\omega} = T/\pi$. Let $A(-\bar{\omega})$ and $\Psi(-\bar{\omega})$ be the amplitude and phase characteristic of the channel at the frequency $-\bar{\omega}$ from ω_0; $A(\bar{\omega})$ and $\Psi(\bar{\omega})$ the corresponding quantities at the frequency $\bar{\omega}$ from ω_0. Solution of (122) and (123) for the above steady state condition of alternate marks and spaces in binary AM gives

$$\alpha_0(x) = A(-\bar{\omega}) \cos[-\bar{\omega}t - \Psi(-\bar{\omega})] + A(\bar{\omega}) \cos[\bar{\omega}t - \Psi(\bar{\omega})], \quad (124)$$

$$\beta_0(x) = -A(\bar{\omega}) \sin[-\bar{\omega}t - \Psi(-\bar{\omega})] + A(\bar{\omega}) \sin[\bar{\omega}t - \Psi(\bar{\omega})]. \quad (125)$$

With (124) and (125) in (50), it turns out by way of check that $U_0(x) = 1$ for a continuing mark for any amplitude and phase characteristic of the channel.

For pulse trains other than continuing marks or spaces, intersymbol interference will be encountered from amplitude and phase distortion. In the following section special cases of phase distortion will be examined further. It will be assumed that the amplitude characteristic has the appropriate shape so that intersymbol interference can be avoided in the absence of phase distortion. To this end it is necessary that $A(-\bar{\omega}) = A(\bar{\omega}) = \frac{1}{2}$ or $\mu = 2$ as shown elsewhere (Ref. 1, Section V). In this case (50) becomes with $U_0 = U$:

$$U(t) = \frac{1}{D}\Bigg[2(\alpha_0^2 + \beta_0^2) - \alpha_0 \cos y - \beta_0 \sin y - \frac{1}{\bar{\omega}}(\alpha_0' \sin y - \beta_0' \cos y) - \frac{2}{\bar{\omega}}(\beta_0'\alpha_0 - \alpha_0'\beta_0)\Bigg], \quad (126)$$

where

$$D = 1 + 4(\alpha_0^2 + \beta_0^2) - 4(\alpha_0 \cos y + \beta_0 \sin y), \quad (127)$$

in which

$$\begin{aligned} \alpha_0' &= d\alpha_0/dt, \\ \beta_0' &= d\beta_0/dt, \\ y &= \pi x + \Psi(-\bar{\omega}), \end{aligned} \quad (128)$$

$$\alpha_0 = \alpha_0(x) = \sum_{n=-\infty}^{\infty} (-1)^n a(n) R_0(x - n), \quad (129)$$

$$\beta_0 = \beta_0(x) = \sum_{n=-\infty}^{\infty} (-1)^n a(n) Q_0(x - n), \quad (130)$$

where

$$x = t/T, \qquad a(n) = \begin{matrix} 0 \\ 1 \end{matrix} \begin{cases} \text{for space} \\ \text{for mark.} \end{cases}$$

5.3 *Even Symmetry Spectrum and Delay Distortion*

When the amplitude characteristic of the channel has even symmetry about ω_0 and the phase characteristic has odd symmetry $Q_0(x + n) = 0$ and (126) simplifies to

$$U(t) = \frac{2\alpha_0^2 - \alpha_0 \cos y - (\alpha_0'/\bar{\omega}) \sin y}{1 + 4\alpha_0^2 - 4\alpha_0 \cos y}. \tag{131}$$

Optimum performance would be expected when a single pulse is sampled at its peak, a condition which is at least closely approximated with $y = 0$. This condition is met when $t = t_0$ is so chosen that

$$t_0/T = x_0 = -\Psi(-\bar{\omega})/\pi. \tag{132}$$

Expression (131) then simplifies to

$$U(t_0) = \frac{\alpha_0(x_0)}{2\alpha_0(x_0) - 1}. \tag{133}$$

For further analysis it is convenient to introduce the quantities

$$\alpha_0^-(x_0) = \sum_{n=1}^{\infty} |(-1)^n R_0(x - n)|^- + |(-1)^n R_0(x + n)|^-, \tag{134}$$

$$\alpha_0^+(x) = \sum_{n=1}^{\infty} |(-1)^n R_0(x - n)|^+ + |(-1)^n R_0(x - n)|^-, \tag{135}$$

where $|\ \ |^-$ designates absolute values when $(-1)^n R_0(x \pm n)$ is negative and $|\ \ |^+$ when it is positive.

It will be recognized that

$$\begin{aligned} R_0(x) + \alpha_0^+(x) - \alpha_0^-(x) &= \sum_{n=\infty}^{\infty} (-1)^n R_0(x - n) \\ &= \alpha_0(x) = \cos y, \end{aligned} \tag{136}$$

where the last relations follow in view of (122), (124) and (128).

During transmission of a space, delay distortion will have an adverse effect only if U as given by (133) is positive, since only in this case is the tolerance to noise reduced. To obtain a positive value of U, it is necessary to have either $\alpha_0 \geqq \frac{1}{2}$ or $\alpha_0 < 0$. For a space, $a(0) = 0$ in (129) and a value of $\alpha_0 \geqq \frac{1}{2}$ can then be excluded for any reasonable delay distortion. It will, therefore, be assumed that $\alpha_0 < 0$. The maximum positive value of U, i.e., the maximum adverse effect of delay distortion, is then obtained with the maximum possible negative value of α_0. This maximum value is obtained by choosing $a(n) = 0$ in (129) whenever $(-1)^n R_0(x - n)$ is positive and choosing $a(n) = 1$ whenever $(-1)^n R_0(x - n)$ is negative. The maximum negative value of $\alpha_0(x)$ thus obtained is given by (134). The corresponding maximum value of $U(t_0)$ in the presence of a space and with sampling at $t = t_0$ as defined by (132) is

$$U_{\max}^{(0)} = \frac{-\alpha_0^-(x_0)}{-2\alpha_0^-(x_0) - 1} = \frac{\alpha_0^-(x_0)}{1 + 2\alpha_0^-(x_0)}. \tag{137}$$

During transmission of a mark delay distortion will have an adverse effect only if $U(t_0) < 1$. This will be the case if $\alpha_0 > 1$ or $\alpha_0 < \frac{1}{2}$ in (133). With $a(0) = 1$ in (129) for a mark, the condition $\alpha_0 < \frac{1}{2}$ will not be encountered with any reasonable delay distortion and only the case $\alpha_0 > 1$ needs to be considered. The minimum positive value of $U(t_0)$ in the presence of a mark is obtained when α_0 is taken as the maximum positive value given by $\alpha_0(x) = R_0(x) + \alpha_0^+(x)$, where α_0^+ is given by (135). In view of (136) it follows that, for $y = 0$,

$$\alpha_0(x_0) = R_0(x_0) + \alpha_0^+(x_0) = 1 + \alpha_0^-(x_0). \tag{138}$$

With (138) in (133) the minimum amplitude of a pulse train in the presence of a mark and at the sampling instant t_0 defined by (132) becomes:

$$U_{\min}^{(1)} = \frac{1 + \alpha_0^-(x_0)}{2[1 + \alpha_0^-(x_0)] - 1} = \frac{1 + \alpha_0^-(x_0)}{1 + 2\alpha_0^-(x_0)}. \tag{139}$$

The optimum slicing level in the presence of delay distortion becomes for conditions as discussed in Section 2.9,

$$L_0 = \tfrac{1}{2}[U_{\max}^{(0)} + U_{\min}^{(1)}] = \tfrac{1}{2}. \tag{140}$$

The minimum amplitude of a pulse train in the presence of a mark relative to the optimum slicing level becomes

$$U_{\min}^{(1)} - L_0 = \frac{1}{2}\,\frac{1}{[1 + 2\alpha_0^-(x_0)]}. \tag{141}$$

The latter expression also applies for the difference between the slicing level and the maximum amplitude of a pulse train at a sampling point in the presence of a space.

Expression (141) shows that the minimum amplitude at a sampling point is smaller than in the absence of delay distortion ($\alpha_0^- = 0$) by the factor

$$\eta_{\min} = \frac{1}{1 + 2\alpha_0^-(x_0)}. \tag{142}$$

TABLE XIV — FACTOR η_{min} FOR RAISED COSINE SPECTRUM AND QUADRATIC DELAY DISTORTION

d/T	0	1	2	3	4
$\Psi(-\bar{\omega})$	0	$-\pi/12$	$-\pi/6$	$-\pi/4$	$-\pi/3$
x_0	0	1/12	1/6	1/4	1/3
$\alpha_0^-(x_0)$	0	0.07	0.125	0.20	0.35
η_{min}	1	0.88	0.80	0.72	0.59

5.4 *Raised Cosine Spectrum and Quadratic Delay Distortion*

In the particular case of a raised cosine spectrum of the pulses at the detector input, as shown in Fig. 1, and quadratic delay distortion, the function $R_0(t/T) = R_0(x + n)$ is given in Table XIX of the Appendix. The phase distortion $\Psi(-\bar{\omega})$ in this case is given by

$$\Psi(-\bar{\omega}) = -\left(\frac{d}{T}\right)\frac{\pi}{12}, \tag{143}$$

where d/T is defined as in Section 4.5.

In Table XIV are given $\Psi(-\bar{\omega})$ together with x_0 as obtained from (132), $\alpha_0^-(x_0)$ as given by (134) and η_{min} as obtained from (142). These factors are shown in Fig. 10, together with the corresponding factor for binary double-sideband AM as obtained from Table IV.

5.5 *Raised Cosine Spectrum and Linear Delay Distortion*

When both the pulse spectrum at the detector input and phase distortion has even symmetry about the frequency ω_0, the following relations apply (see Appendix):

$$R_0(-t) = R_0(t), \qquad Q_0(-t) = Q_0(t); \tag{144}$$

$$R_0'(-t) = -R_0'(t), \qquad Q_0'(-t) = -Q_0'(t). \tag{145}$$

The maximum amplitude of a single pulse in this case is at $t = 0$. Optimum performance is obtained with sampling at $t = 0$, in which case y in (126) and (127) is given by

$$y = y_0 = \Psi(-\bar{\omega}), \tag{146}$$

and:

$$\alpha_0 = \alpha_0(0) = \sum_{n=-\infty}^{\infty} (-1)^n a(n) R_0(-n), \tag{147}$$

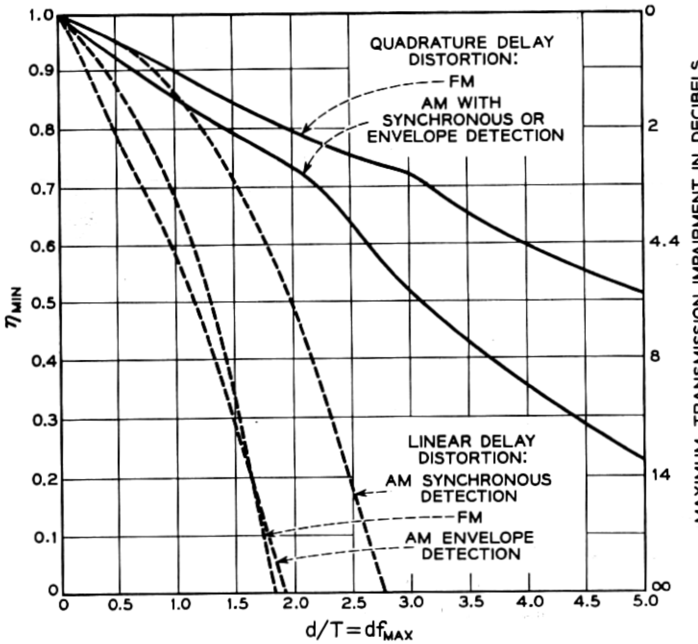

Fig. 10 — Factor $\eta_{\min}$ for binary pulse transmission by FM and double-sideband AM, for quadratic delay distortion as in Fig. 2 and linear delay distortion as in Fig. 4.

$$\beta_0 = \beta_0(0) = \sum_{n=-\infty}^{\infty} (-1)^n a(n) Q_0(-n), \tag{148}$$

$$\alpha_0' = \alpha_0'(0) = \sum_{n=-\infty}^{\infty} (-1)^n a(n) R_0'(n), \tag{149}$$

$$\beta_0' = \beta_0'(0) = \sum_{n=-\infty}^{\infty} (-1)^n a(n) Q_0'(n). \tag{150}$$

For the special case of a raised cosine spectrum and linear delay distortion, the functions R_0 and Q_0 are given in Table XX of the Appendix. The functions $R_0'(n)$ and $Q_0'(n)$ are related to the functions R_1 and Q_1 given in Table XXII by

$$R_0'(t)/\bar{\omega} = \frac{4}{\pi} R_1(t), \qquad Q_0'(t)/\bar{\omega} = \frac{4}{\pi} Q_1(t). \tag{151}$$

The functions R_0 and Q_0 are given in Table XV for integral values of $n = t/T$ and the functions $R_0'/\bar{\omega}$ and $Q_0'/\bar{\omega}$ are given in Table XVI.

In the above case of quadratic phase distortion of the form $\psi(u) = cu^2$

TABLE XV — FUNCTIONS $R_0(nT)$ AND $Q_0(nT)$ FOR LINEAR DELAY DISTORTION

d/T	0.5		1		2	
n	R_0	Q_0	R_0	Q_0	R_0	Q_0
0	0.9516	0.1941	0.8309	0.3306	0.5786	0.3895
±1	0.080	−0.0956	0.0726	−0.1792	0.2596	−0.2439
±2	0.0048	−0.0014	0.0112	0.0107	−0.0383	0.0577
±3	0.0008	0	0.0006	0.0024	−0.0094	−0.0052
±4	0	0	0	0	0	0

and linear delay distortion $\psi'(u) = 2cu$, the phase distortion at $u = -\bar{\omega}$ is given by

$$\Psi(-\bar{\omega}) = \Psi(\bar{\omega}) = \frac{d}{T}\frac{\pi}{4}. \tag{152}$$

Owing to the several quantities α_0, β_0, α_0', β_0', $\beta_0'\alpha_0$, $\alpha_0'\beta_0$ and y_0 involved in (126), it does not appear feasible to derive simple relations for $U_{\max}^{(s)}$ and $U_{\min}^{(m)}$. However, it is possible to determine these by examining several cases, as illustrated below for $d/T = 1$ and $d/T = 2$.

With $d/T = 1$ in (152), relation (146) gives

$$y_0 = \Psi(-\bar{\omega}) = \pi/4$$

and (126) becomes

$$U(0) = N_1/D_1, \tag{153}$$

$$N_1 = 2(\alpha_0^2 + \beta_0^2) - \tfrac{1}{2}\sqrt{2}(\alpha_0 + \beta_0 + \alpha_0'/\bar{\omega} - \beta_0'/\bar{\omega}) - 2[(\alpha_0\beta_0'/\bar{\omega} - \beta_0\alpha_0'/\bar{\omega})], \tag{154}$$

$$D_1 = 1 + 4(\alpha_0^2 + \beta_0^2) - 2\sqrt{2}(\alpha_0 + \beta_0), \tag{155}$$

where α_0, β_0, α_0' and β_0' are given by (147) through (150).

TABLE XVI — FUNCTIONS $R_0'/\bar{\omega} = (4/\pi)R_1$ AND $Q_0'/\bar{\omega} = (4/\pi)Q_1$ FOR LINEAR DELAY DISTORTION

d/T	0.5		1		2	
n	$R_0'/\bar{\omega}$	$Q_0'/\bar{\omega}$	$R_0'/\bar{\omega}$	$Q_0'/\bar{\omega}$	$R_0'/\bar{\omega}$	$Q_0'/\bar{\omega}$
0	0	0	0	0	0	0
±1	±0.1432	±0.0624	±0.2263	±0.0639	±0.3016	±0.1053
±2	±0.0065	±0.0134	±0.0013	±0.0272	±0.0980	±0.0167
±3	±0.0006	±0.0036	±0.0051	±0.0023	±0.0042	±0.0188
±4	0	0	0	0	0	0

TABLE XVII — VALUES OF $U(0)$ FOR RAISED COSINE SPECTRUM AND LINEAR DELAY DISTORTION WITH $d/T = 1$

$a(-2)$	$a(-1)$	$a(0)$	$a(1)$	$a(2)$	α_0	β_0	$\alpha_0'/\bar{\omega}$	$\beta_0'/\bar{\omega}$	N_1	D_1	$U(0)$
0	1	0	1	0	−0.146	0.358	0	0	0.15	1.0	0.15
0	0	0	1	0	−0.073	0.179	−0.227	0.064	0.173	0.87	0.16*
0	1	0	0	0	−0.073	0.179	0.227	−0.064	−0.29	0.87	−0.33
1	1	0	1	1	−0.124	0.38	0	0	0.136	0.90	0.15
0	0	1	0	0	0.831	0.331	0	0	0.78	0.90	0.86
0	1	1	1	0	0.68	0.89	0	0	1.4	1.6	0.87
0	0	1	1	0	0.76	0.51	−0.227	0.064	0.66	0.76	0.87
0	1	1	0	0	0.76	0.51	0.227	−0.064	0.89	0.76	1.16
1	1	1	1	1	0.70	0.91	0	0	1.21	1.43	0.84*

For various combinations of marks and spaces, i.e., $a(n) = 1$ and 0, the results given in Table XVII are obtained.

The maximum value in the presence of a space is $U_{\max}^{(0)} \cong 0.16$ and the minimum value in the presence of a mark is $U_{\min}^{(1)} \cong 0.84$, as indicated by asterisks. The optimum slicing level is $\frac{1}{2}[U_{\max}^{(0)} + U_{\min}^{(1)}] = 0.5$. The factor $\eta_{\min}$ is in this case

$$\eta_{\min} = U_{\min}^{(1)} - U_{\max}^{(0)} \cong 0.68.$$

For $d/T = 2$, (152) gives $\Psi(-\bar{\omega}) = \pi/2 = y_0$. In this case (126) becomes

$$U(0) = N_2/D_2, \tag{156}$$

$$N_2 = 2(\alpha_0^2 + \beta_0^2) - \beta_0 - \alpha_0'/\bar{\omega} - 2(\alpha_0\beta_0'/\bar{\omega} - \beta_0\alpha_0'/\bar{\omega}), \tag{157}$$

$$D_2 = 1 + 4(\alpha_0^2 + \beta_0^2) - 4\beta_0. \tag{158}$$

The results in this case are given in Table XVIII. In this case $U_{\max}^{(0)} = 0.57$, $U_{\min}^{(1)} = 0.42$, $L_0 \cong 0.5$ and $\eta_{\min} = U_{\min}^{(1)} - U_{\max}^{(0)} \cong -0.15$.

TABLE XVIII — VALUES OF $U(0)$ FOR RAISED COSINE SPECTRUM AND LINEAR DELAY DISTORTION WITH $d/T = 2$

$a(-2)$	$a(-1)$	$a(0)$	$a(1)$	$a(2)$	α_0	β_0	$\alpha_0'/\bar{\omega}$	β_0'/ω	N_2	D_2	$U(0)$
0	1	0	1	0	−0.520	0.488	0	0	0.526	1.08	0.49
0	0	0	1	0	−0.260	0.244	−0.302	−0.106	0.09	0.53	0.17
0	1	1	0	0	−0.260	0.244	0.302	0.106	−0.07	0.53	−0.13
1	1	0	1	1	−0.596	0.603	0.07	0.06	0.837	1.96	0.57*
0	0	1	0	0	0.579	0.390	0	0	0.58	1.38	0.42*
0	1	1	1	0	0.069	0.88	0	0	0.66	0.56	1.17
0	0	1	1	0	0.319	0.634	−0.302	−0.106	0.46	0.67	0.69
0	1	1	0	0	0.319	0.634	0.302	−0.106	0.48	0.67	0.71

The factors in Table XVIII are shown in Fig. 10, together with the corresponding factors for binary double-sideband AM with synchronous detection as given in Table V and with envelope detection as given in Table IX.

VI. SUMMARY

6.1 *General*

The shape of pulse trains at the detector input and output in pulse transmission by various methods of carrier modulation and detection has been formulated in terms of a basic function common to all modulation methods: the carrier pulse transmission characteristic. This function is related to the amplitude and phase characteristics of the channel by a Fourier integral, which can be evaluated by numerical integration with the aid of digital computers for any prescribed channel characteristic. In this way can be determined the effect of specified channel gain and phase deviations on the demodulated pulse train for any modulation method, together with the resultant maximum transmission impairment.

The carrier pulse transmission characteristics are given herein for the representative case of pulses with a raised cosine spectrum at the detector input, for two cases of envelope delay distortion over the channel band. In one case delay distortion is assumed to vary linearly with frequency, and in the other case to vary as the second power of frequency from midband, as indicated in Fig. 11. The resultant maximum effect on the demodulated pulse trains at sampling instants has been determined for various carrier modulation and detection methods, together with the corresponding maximum transmission impairment. The maximum transmission impairment is expressed as the maximum increase in signal-to-noise ratio required at the detector input to compensate for the effect of phase distortion, or corresponding envelope delay distortion. The maximum transmission impairments specified here apply as the error probability approaches zero, and actual impairment will be somewhat smaller, depending on error probability.

In evaluating the effect of phase distortion, idealized modulation and demodulation have been assumed, together with ideal implementation in other respects, such as instantaneous sampling of the appropriate instants and optimum slicing levels.

The numerical results are given in various tables and curves, summarized in Fig. 12 and discussed briefly below.

6.2 *Choice of Transmission Delay Parameters*

In the expressions for the carrier pulse transmission characteristic the phase characteristic of the channel is a basic function. Transmission impairments from phase distortion could be expressed in terms of some parameter or set of parameters that would define the type of phase distortion under consideration. Alternatively, any type of phase distortion can be specified in terms of its derivative with respect to frequency, that is, in terms of envelope delay distortion. From the standpoint of engineering applications the latter method is preferable, since variation in transmission delay over the channel band is more readily measured than variation in phase, and it is ordinarily the quantity specified for various existing facilities.

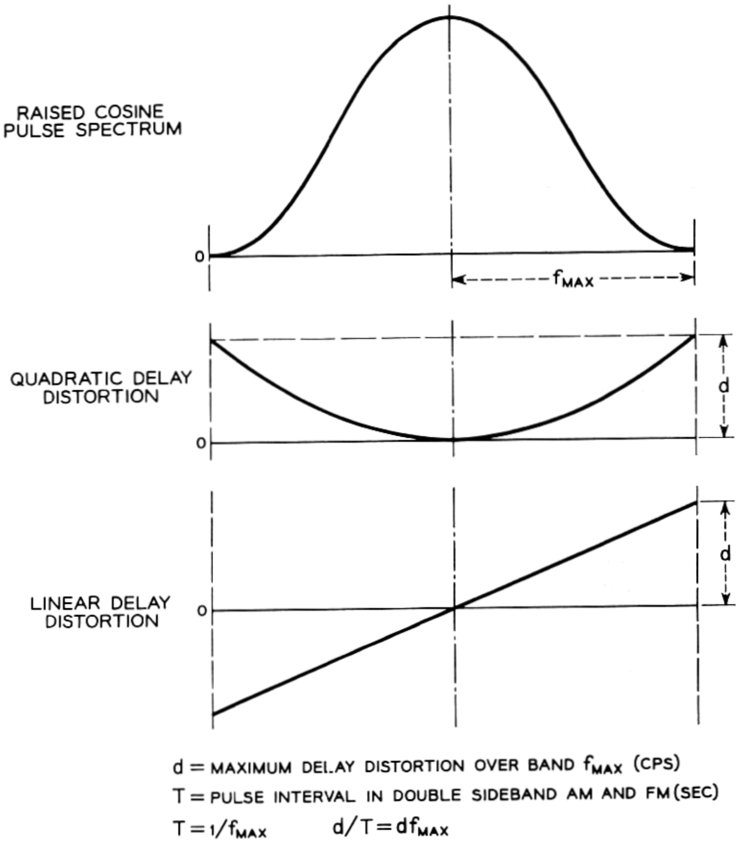

Fig. 11 — Pulse spectrum at detector input and types of delay distortion assumed in comparison of modulation methods.

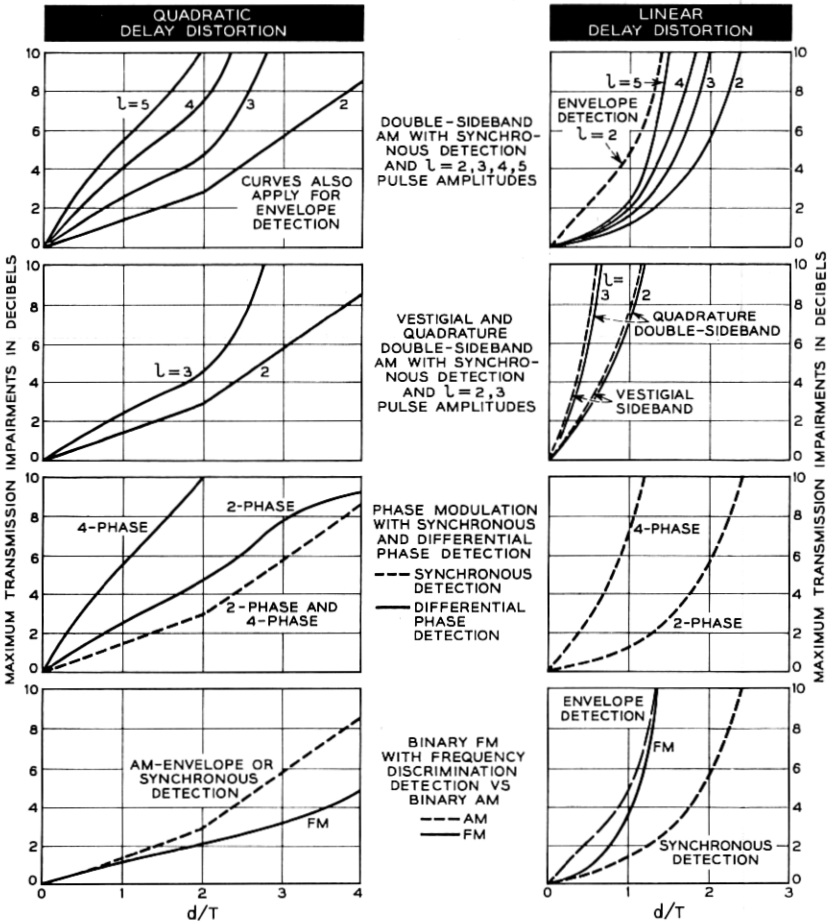

Fig. 12 — Maximum transmission impairments with various modulation methods for raised cosine pulse spectrum with linear and quadratic delay distortion as in Fig. 11.

Linear, quadratic or any other analytically specified delay distortion can be expressed in terms of the difference in transmission delay between any two reference frequencies in the channel band. In the present analysis the difference d in delay between the midband frequency and the maximum frequency $f_{\max}$ from midband, as in Fig. 11, has been taken as a basic parameter. The maximum transmission impairments with various carrier modulation methods have been given in terms of the ratio $d/T = df_{\max}$, where T is the pulse interval in double-sideband AM.

An alternative choice of delay parameter might have been the maximum difference d_{max} in transmission delay between any two frequencies in the channel band. In the case of linear delay distortion $d_{max} = 2d$, while in the case of quadratic delay distortion $d_{max} = d$, where d is defined in Fig. 11. A third choice might have been the difference in delay d between the midband frequency and the mean sideband frequency $\frac{1}{2}f_{max}$, in which case $d = d/2$ for linear and $d = d/4$ for quadratic delay distortion.

It will be recognized that translation from one basic delay parameter to another can readily be made. Also, the question of whether linear or quadratic delay distortion causes greater transmission impairments will depend significantly on the choice of transmission delay parameters.

6.3 *Double-Sideband AM*

Maximum transmission impairments are shown in Fig. 12 for systems employing $l = 2, 3, 4$ and 5 pulse amplitudes and ideal synchronous detection. With envelope detection the transmission impairments are the same as with synchronous detection, for quadratic delay distortion and for any type of delay distortion with even symmetry about the channel midband (carrier) frequency. However, with envelope detection greater transmission impairments are incurred in the case of linear delay distortion, and for any type of delay distortion with odd symmetry about the channel midband frequency. The difference between envelope and synchronous detection in the presence of linear delay distortion is illustrated in Fig. 12 for $l = 2$ pulse amplitudes.

As noted previously, the maximum transmission impairments indicated in Fig. 12 would be encountered for extremely small error probabilities. For error probabilities in the range normally considered, the maximum impairments given in Fig. 12 would be rather closely approached when the impairments are fairly small, say less than 3 db. However, when the maximum impairments are rather high the actual impairments may be significantly smaller. For example, with a maximum impairment of 10 db, the actual impairment would be expected to be about 1.5 db less for an error probability 10^{-5} and about 2 db less for an error probability 10^{-4}.

6.4 *Vestigial-Sideband AM and Quadrature Double-Sideband AM*

Vestigial-sideband AM and quadrature double-sideband AM with synchronous detection are equivalent methods as regards channel bandwidth requirements and signal-to-noise ratios, in the absence of delay distortion. Both methods may be used in preference to double-sideband AM either (a) to realize a two-fold increase in pulse transmission rate for a given bandwidth in exchange for a 3 db penalty in signal-to-noise ratio or (b) to secure a two-fold reduction in bandwidth for a given pulse transmission rate, without a penalty in signal-to-noise ratio.

The maximum transmission impairments shown in Fig. 12 are for the same bandwidths as in double-sideband AM with a two-fold increase in the pulse transmission rate. In this case transmission impairments from quadratic delay distortion are no greater than in double-sideband AM, and this applies for any type of delay distortion with even symmetry about the channel midband frequency.

With linear delay distortion, or any delay distortion with odd symmetry about the channel midband frequency, transmission impairments are not identically the same for vestigial-sideband AM and quadrature double-sideband AM. However, the difference is not significant in the case of linear delay distortion, as indicated in Fig. 12. For practical purposes the two methods can be regarded as equivalent for any type of delay distortion actually expected, as regards channel bandwidth requirements and signal-to-noise ratios for a given error probability, assuming ideal synchronous detection.

With linear delay distortion the transmission impairments with the above two methods are significantly greater than for double-sideband AM as indicated by comparison of the curves in Fig. 12 for the two methods for $l = 2$ and 3 pulse amplitudes. This assumes that the pulse transmission rate is twice as great as in double-sideband AM.

When the pulse transmission rate is the same as in double sideband AM but the bandwidth is halved, delay distortion over the channel band is reduced. In this case vestigial-sideband AM or quadrature double-sideband AM affords an advantage over double-sideband AM in the presence of delay distortion with even symmetry about the channel midband frequency, but not necessarily when delay distortion has odd symmetry. With linear delay distortion the ratio d/T is halved, and in this case there is a slight disadvantage compared to double-sideband AM, for $l = 2$ pulse amplitudes. However, with the type of delay distortion ordinarily encountered vestigial-sideband AM and quadrature double-sideband AM would afford some advantage in signal-to-noise ratio over double-sideband AM for equal pulse transmission rates and with ideal synchronous detection.

6.5 *PM with Synchronous Detection*

Two-phase modulation or phase reversal is equivalent to double-sideband AM with equal amplitudes but opposite polarities of the transmitted pulses. The curves in Fig. 12 for double-sideband AM and $l = 2$

pulse amplitudes apply also for two-phase transmission, for the reason that the transmission impairments for a given peak-to-peak difference between pulse amplitudes is the same regardless of polarities.

Two-phase modulation can also be used in conjunction with vestigial-sideband transmission. The curves in Fig. 12 for vestigial-sideband AM and $l = 2$ pulse amplitudes also apply for two-phase vestigial-sideband modulation.

Four-phase modulation is equivalent to bipolar AM on each of two carriers at quadrature with each other. The curves in Fig. 12 for quadrature double sideband AM and $l = 2$ pulse amplitudes also apply for the special case of four-phase modulation.

The maximum transmission impairments with double-sideband two-phase and four-phase modulation and synchronous detection are shown separately in Fig. 12 for comparison with PM with differential phase detection.

6.6 *PM with Differential Phase Detection*

In phase modulation systems differential phase modulation (described in Section 4.1) may be used in place of synchronous detection. Differential phase detection has been implemented in various ways, which in general involve some transmission impairments from channel bandwidth limitations, even with a linear phase characteristic. Such transmission impairments from channel bandwidth limitation is avoided with the implementation assumed herein (Section 4.1), and only the effect of phase distortion is evaluated. Transmission impairments from delay distortion will be greater with this method than with synchronous detection, as illustrated in Fig. 12 for double-sideband two-phase and four-phase quadrature systems and delay distortion. Transmission impairments from linear delay distortion have not been determined for this case.

6.7 *Binary FM*

With optimum systems design, binary FM, or frequency shift keying, requires the same bandwidth for a given pulse transmission rate as binary double-sideband AM. Maximum transmission impairments with these two methods are compared in Fig. 12. It will be noted that with quadratic delay distortion the impairments are smaller with FM than with AM employing either envelope or synchronous detection. In the case of linear delay distortion, the transmission impairments are greater with FM than with synchronous AM, but are somewhat smaller than with AM employing envelope detection.

The transmission impairments given in Fig. 12 for FM apply without a postdetection low-pass filter for noise reduction, and may involve somewhat greater approximations than for the other modulation methods. Approximately the same impairments from phase distortion would be expected with an appropriate low-pass filter.

6.8 *Comparisons of Carrier Modulation Methods*

Signal-to-noise ratios at the detector input for a given error probability and various methods of carrier modulation are ordinarily compared on the premise of ideal amplitude versus frequency characteristics of the channels, and a linear phase characteristic. The curves in Fig. 12 indicate that transmission impairments resulting from phase distortion depend significantly on the carrier modulation method. The optimum method as regards signal-to-noise ratio will thus depend on the type and degree of phase distortion encountered in a particular application. For example, two-phase modulation with synchronous or with differential phase detection may have a slight advantage in signal-to-noise ratio over binary frequency shift keying in the absence of delay distortion. However, the advantage in signal-to-noise ratio would be expected to be with frequency shift keying in application to channels with pronounced quadratic delay distortion or other types of delay distortion with essentially even symmetry about the carrier frequency.

In comparing the performance of various methods of carrier modulation it is necessary to consider other factors than signal-to-noise ratios and channel bandwidth requirements as discussed here. Among them can be mentioned the adverse effects of sudden or gradual level and phase variations and the complexity of instrumentation.

VII. ACKNOWLEDGMENTS

The writer is indebted to A. P. Stamboulis for pointing out some errors in the original equation (50) and for showing the presence of the third term in (56) and to C. F. Pease for numerical evaluation of integrals in the Appendix with the aid of a 704 digital computer.

APPENDIX

Determination of Carrier Pulse Transmission Characteristics

As mentioned in Section 2.2, the in-phase and quadrature components of the carrier pulse transmission characteristics for any carrier frequency

ω_c can be determined from those for any other carrier frequency ω_0, for example the midband frequency of the channel. Basic Fourier integrals are given here for the carrier pulse transmission characteristics for a reference or carrier frequency ω_0. In addition, special integrals are given, applying for a raised cosine pulse spectrum with linear delay distortion, quadratic delay distortion and the type of delay distortion introduced by flat bandpass filters with sharp cutoffs. For these three cases the carrier pulse transmission characteristics have been determined by numerical integration and are tabulated here.

A.1 *General Formulation*

The shape of $R_0(t)$ and $Q_0(t)$ depends on the shape of the transmitted carrier pulse and on the transmission-frequency characteristic of the channel. If the carrier pulse is assumed of sufficiently short duration, the spectrum will be essentially flat over the channel band, so that the shape of the received spectrum is the same as that of the amplitude characteristic of the channel. The functions R_0 and Q_0 are then obtained from expression given elsewhere (Ref. 2, Section 2) in terms of the amplitude characteristic $A(u)$ of the channel, where u is the frequency measured from the carrier frequency ω_0, as indicated in Fig. 13. In the more general case of carrier pulses of any shape and any channel transmission-frequency characteristic, the functions R_0 and Q_0 are obtained by replacing in the above expressions $A(u)$ with the spectrum $S_0(u)$ of the pulse

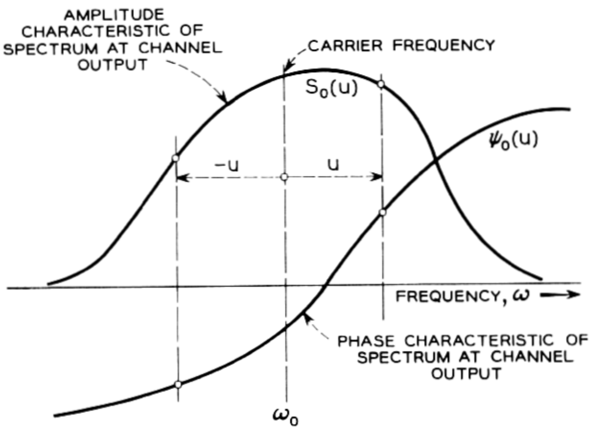

Fig. 13 — Amplitude characteristic $S_0(u)$ and phase characteristic $\Psi_0(u)$ of pulse spectrum at channel output (i.e., detector input) for carrier at frequency ω_0.

envelope at the channel output (detector input). The following expressions are thus obtained in place of (2.10) and (2.11) of Ref. 2:

$$R_0 = R_0^- + R_0^+, \qquad Q_0 = Q_0^- - Q_0^+, \tag{159}$$

$$R_0^- = \frac{1}{\pi} \int_0^{\omega_0} S_0(-u) \cos [ut + \Psi_0(-u)] \, du, \tag{160}$$

$$R_0^+ = \frac{1}{\pi} \int_0^{\infty} S_0(u) \cos [ut - \Psi_0(u)] \, du, \tag{161}$$

$$Q_0^- = \frac{1}{\pi} \int_0^{\omega_0} S_0(-u) \sin [ut + \Psi_0(-u)] \, du, \tag{162}$$

$$Q_0^+ = \frac{1}{\pi} \int_0^{\infty} S_0(u) \sin [ut - \Psi_0(u)] \, du. \tag{163}$$

The various quantities in the above expressions are as shown in Fig. 13. It will be recognized that the upper limit ω_0 in (160) and (162) can for practical purposes be replaced by ∞, since $S_0(-\omega_0) \cong 0$.

A.2 *Even Symmetry Spectrum and Delay Distortion*

Let the spectrum at the detector input have even symmetry about ω_0 and the phase distortion odd symmetry, in which case

$$S_0(-u) = S_0(u), \tag{164}$$

$$\Psi_0(-u) = -\Psi_0(u). \tag{165}$$

Delay distortion will then have even symmetry about ω_0, i.e., $\Psi_0'(-u) = \Psi_0'(u)$.

With (164) and (165) in (159) through (163), the following relations are obtained when the upper limit ω_0 is replaced by ∞:

$$R_0(t) = \frac{2}{\pi} \int_0^{\infty} S_0(u) \cos [ut - \Psi_0(u)] \, du, \tag{166}$$

$$Q_0(t) = 0. \tag{167}$$

A.3 *Even Symmetry Spectrum and Odd Symmetry Delay Distortion*

When the phase characteristic has a component with even symmetry about the frequency ω_0, so that

$$\Psi_0(-u) = \Psi_0(u) \tag{168}$$

the corresponding delay distortion will have odd symmetry.

With (164) and (168) in (159) through (163), the following relations are obtained:

$$R_0(-t) = R_0(t) = \frac{2}{\pi} \int_0^\infty S_0(u) \cos ut \cos \Psi_0(u) \, du, \tag{169}$$

$$Q_0(-t) = Q_0(t) = \frac{2}{\pi} \int_0^\infty S_0(u) \cos ut \sin \Psi_0(u) \, du. \tag{170}$$

A.4 *Raised Cosine Pulse Spectrum*

For reasons discussed elsewhere (Ref. 2, Section 5) it is desirable in pulse systems to employ raised cosine pulse spectra, as shown in Fig. 1 and given by

$$S_0(-u) = S_0(u) = \frac{T}{2} \cos^2 \frac{\pi u}{4\bar{\omega}}, \tag{171}$$

where $\bar{\omega}$ is the mean frequency from midband.

The corresponding carrier pulse transmission characteristic obtained from (159) through (163) with $\Psi_0(u) = 0$ is

$$\bar{P}_0 = R_0(t) = \frac{\sin 2\bar{\omega}t}{2\bar{\omega}t[1 - (2\bar{\omega}t/\pi)^2]}. \tag{172}$$

Pulses can in this case be transmitted without intersymbol interference at intervals T such that

$$\bar{\omega}T = \pi. \tag{173}$$

A.5 *Quadratic Delay Distortion*

It will be assumed that the phase characteristic contains a linear component, which can be disregarded, and a distortion component given by

$$\Psi_0(u) = cu^3, \tag{174}$$

where c is a constant. The corresponding delay distortion is then quadratic or parabolic, as given by

$$\Psi_0'(u) = 3cu^2. \tag{175}$$

In this case $Q_0(t) = 0$ in accordance with (167), while (175) in (166) gives

$$R_0(t) = \frac{4}{\pi} \int_0^{\pi/2} \cos^2 x \cos (ax - bx^3) \, dx, \tag{176}$$

TABLE XIX — FUNCTIONS $R_0(t/T)$ AND $R_1(t/T)$ FOR RAISED COSINE PULSE SPECTRUM AND QUADRATIC DELAY DISTORTION

t/T	$d/T = 0$		$d/T = 1$		$d/T = 2$		$d/T = 3$		$d/T = 4$	
	R_0	$-R_1$	R_0	$-R_1$	R_0	$-R_1$	R_0	$-R_1$	R_0	$-R_1$
−3.00	0	−0.0024	−0.0006	0.0005	0.0025	0.0028	0	−0.0003	0	0
−2.75	0.0020	−0.0006	−0.0003	−0.0011	−0.0023	0.0048	−0.0002	0	0	−0.0003
−2.50	0	0.0042	0.0009	−0.0007	−0.0030	−0.0004	0	0.0002	0.0005	−0.0005
−2.25	−0.0037	0.0013	0.0004	0.0017	0.0005	−0.0009	0.0002	−0.0007	0.0011	−0.0009
−2.00	0	−0.0083	−0.0013	0.0009	0.0011	0.0003	0.0017	−0.0024	0.0028	−0.0029
−1.75	0.0081	−0.0037	0	−0.0038	0.0020	−0.0023	0.0055	−0.0055	0.0079	−0.0079
−1.50	0	0.0208	0.0053	−0.0062	0.0081	−0.0114	0.0145	−0.0139	0.0202	−0.0178
−1.25	−0.0243	0.0164	0.0137	−0.0138	0.0280	−0.0306	0.0374	−0.0344	0.0464	−0.0367
−1.00	0	−0.0833	0.0467	−0.0627	0.0756	−0.0689	0.0891	−0.0721	0.0986	−0.0703
−0.75	0.1698	−0.2603	0.1699	−0.1774	0.1772	−0.1404	0.1879	−0.1282	0.1923	−0.1192
−0.50	0.5000	−0.3750	0.4064	−0.3036	0.3645	−0.2342	0.3492	−0.1943	0.3389	−0.1731
−0.25	0.8488	−0.2829	0.7297	−0.3126	0.6303	−0.2813	0.5692	−0.2373	0.5320	−0.2070
0	1.00	0	0.9633	−0.1228	0.8795	−0.1884	0.7956	−0.1963	0.7336	−0.1830
0.25	0.8488	0.2829	0.9357	0.1809	0.9540	0.0580	0.9182	−0.0276	0.8666	−0.0661
0.50	0.5000	0.3750	0.6338	0.3900	0.7557	0.3255	0.8223	0.2227	0.8364	0.1361
0.75	0.1698	0.2603	0.2366	0.3638	0.3610	0.4237	0.4952	0.4056	0.5936	0.3386
1.00	0	0.0833	−0.0341	0.1634	−0.0098	0.2838	0.0827	0.3792	0.2045	0.4073
1.25	−0.0243	−0.0164	−0.0954	−0.0236	−0.1700	0.0367	−0.1951	0.1547	−0.1501	0.2692
1.50	0	−0.0208	−0.0341	−0.0752	−0.1161	−0.1165	−0.2200	−0.0898	−0.2909	0.0065
1.75	0.0081	0.0037	0.0205	−0.0260	0.0038	−0.0985	−0.0714	−0.1729	−0.1869	−0.1872
2.00	0	0.0083	0.0196	0.0205	0.0543	−0.0016	0.0655	−0.0809	0.0142	−0.1802
2.25	−0.0037	−0.0013	−0.0031	0.0176	0.0242	0.0477	0.0799	0.0436	0.1237	−0.0283
2.50	0	−0.0042	−0.0091	−0.0046	−0.0151	0.0225	0.0120	0.0727	0.0814	0.0945
2.75	0.0020	0.0006	−0.0002	−0.0093	−0.0170	−0.0150	−0.0357	0.0158	−0.0185	0.0834
3.00	0	−0.0024	0.0044	0.0007	0.0020	−0.0167	−0.0231	−0.0327	−0.0584	−0.0583
3.25	−0.0012	−0.0028	0.0061	0.0050	0.0091	0.0028	0.0093	−0.0234	−0.0216	−0.0543
3.50	0	−0.0115	−0.0024	0.0002	0.0011	0.0097	0.0159	0.0090	0.0229	−0.0249
3.75	0.0007	0.0002	−0.0005	−0.0029	−0.0047	0.0007	0.0003	0.0164	0.0229	0.0208
4.00	0	0.0010	0.0014	−0.0004	−0.0014	−0.0053	−0.0087	0	−0.0037	0.0241
4.25	−0.0005	0	0.0004	−0.0017	0.0024	−0.0013	−0.0025	−0.0095	−0.0147	−0.0030
4.50	0	−0.0007	−0.0009	0.0003	0.0011	0.0030	0.0044	−0.0025	−0.0030	−0.0157
4.75	0.0004	0	−0.0003	−0.0011	−0.0014	0.0012	0.0024	0.0051	0.0078	−0.0033
5.00	0	0.0005	0.0006	−0.0003	−0.0008	−0.0008	−0.0022	0.0025	0.0040	0.0086

where

$$a = 4 \frac{t}{T}, \qquad b = \frac{16}{3\pi^2} \frac{d}{T}. \tag{177}$$

The ratio t/T is the time measured in pulse intervals and the ratio d/T the maximum delay distortion measured in pulse intervals, with d defined as in Fig. 2 or Fig. 11.

In certain cases, as in connection with pulse transmission by frequency modulation, the time derivative of $R_0(t)$ is involved. This derivative is given by

$$dR_0/dt = \frac{4}{T} R_1(t), \tag{178}$$

where

$$R_1(t) = dR/da$$

and is given by

$$R_1(t) = -\frac{4}{\pi} \int_0^{\pi/2} x \cos^2 x \sin (ax - bx^3)\, dx. \tag{179}$$

The functions $R_0(t)$ and $R_1(t)$ obtained by numerical integration of (176) and (179) are given in Table XIX. The function $R_0(t/T)$ is shown in Fig. 2.

A.6 *Linear Delay Distortion*

It will be assumed that the phase distortion component is given by

$$\Psi_0(u) = cu^2, \tag{180}$$

which corresponds to a linear delay distortion given by

$$\Psi_0'(u) = 2cu. \tag{181}$$

In this case expressions (169) and (170) give

$$R_0(-t) = R_0(t) = \frac{4}{\pi} \int_0^{\pi/2} \cos^2 x \cos ax \cos bx^2\, dx, \tag{182}$$

$$Q_0(-t) = Q_0(t) = \frac{4}{\pi} \int_0^{\pi/2} \cos^2 x \cos ax \sin bx^2\, dx, \tag{183}$$

where

$$a = 4 \frac{t}{T}, \qquad b = \frac{4}{\pi} \frac{d}{T},$$

in which the delay d is defined as in Fig. 4 or Fig. 11.

TABLE XX — FUNCTIONS $R_0(t/T)$ AND $Q_0(t/T)$ FOR RAISED COSINE PULSE SPECTRUM AND LINEAR DELAY DISTORTION

t/T	$d/T = 0$		$d/T = 0.5$		$d/T = 1$		$d/T = 2$		$d/T = 3$	
	R_0	Q_0	R_0	Q_0	R_0	Q_0	R_0	Q_0	R_0	Q_0
0	1.00	0	0.9516	0.1941	0.8309	0.3306	0.5786	0.3895	0.4537	0.3471
±0.25	0.8488	0	0.8235	0.1277	0.7567	0.2293	0.5871	0.3180	0.4668	0.3138
±0.50	0.5000	0	0.5181	−0.0105	0.5563	0.0096	0.5733	0.1304	0.4921	0.2106
±0.75	0.1698	0	0.2064	−0.1029	0.2968	−0.1592	0.4674	−0.0934	0.4878	0.0423
±1.00	0	0	0.0180	−0.0956	0.0726	−0.1792	0.2596	−0.2439	0.4026	−0.1489
±1.25	−0.0243	0	−0.0331	−0.0336	−0.0447	−0.0917	0.0327	−0.2488	0.2215	−0.2823
±1.50	0	0	−0.0155	0.0101	−0.0538	−0.0030	−0.1035	−0.1334	0.0029	−0.2858
±1.75	0.0081	0	0.0039	0.0122	−0.0147	0.0257	−0.1087	−0.0014	−0.1472	−0.1620
±2.00	0	0	0.0048	−0.0014	0.0112	0.0107	−0.0383	0.0577	−0.1614	−0.0004
±2.25	−0.0037	0	−0.0007	−0.0049	0.0085	−0.0051	0.0216	0.0380	−0.0709	0.0911
±2.50	0	0	−0.0018	0.0003	−0.0023	−0.0048	0.0269	−0.0025	0.0255	0.0767
±2.75	0.0020	0	0.0002	0.0024	−0.0039	0.0021	0.0043	−0.0164	0.0543	0.0109
±3.00	0	0	0.0008	0	0.0006	0.0024	−0.0094	−0.0052	0.0229	−0.0300
±3.25	−0.0012	0							−0.0126	−0.0221
±3.50	0	0							−0.0169	−0.0031
±3.75	0.0007	0							−0.0139	0.0116

The values of R_0 and Q_0 obtained by numerical integration of (182) and (183) are given in Table XX.

It will be noted that $Q_0(0) \neq 0$. From the standpoint of analysis, it may be convenient to modify the phase such that $Q_0(0) = 0$. The modified values are given by

$$\begin{aligned} R_{00}(t) &= [R_0^2(t) + Q_0^2(t)]^{\frac{1}{2}} \cos [\Psi_0(t) - \Psi_0(0)] \\ &= k_1 R_0(t) + k_2 Q_0(t), \end{aligned} \tag{184}$$

$$\begin{aligned} Q_{00}(t) &= [R_0^2(t) + Q_0^2(t)]^{\frac{1}{2}} \sin [\Psi_0(t) - \Psi_0(0)] \\ &= k_1 Q_0(t) - k_2 R_0(t), \end{aligned} \tag{185}$$

where

$$\begin{aligned} k_1 &= \frac{R_0(0)}{[R_0^2(0) + Q_0^2(0)]^{\frac{1}{2}}}, \\ k_2 &= \frac{Q_0(0)}{[R_0^2(0) + Q_0^2(0)]^{\frac{1}{2}}}. \end{aligned} \tag{186}$$

The modified values are given in Table XXI. The functions $R_{00}(t/T)$ and $Q_{00}(t/T)$ are shown in Fig. 4.

The time derivatives of $R_0(t)$ and $Q_0(t)$ are of interest in connection with frequency modulation and given by

$$dR_0/dt = \frac{4}{T}\, dR_0/da = \frac{4}{T}\, R_1(t)\,, \tag{187}$$

$$dQ_0/dt = \frac{4}{T}\, dQ_0/da = \frac{4}{T}\, Q_1(t)\,, \tag{188}$$

where

$$R_1(t) = -\frac{4}{\pi} \int_0^{\pi/2} x \cos^2 x \sin ax \cos bx^2 \, dx\,, \tag{189}$$

$$Q_1(t) = -\frac{4}{\pi} \int_0^{\pi/2} x \cos^2 x \sin ax \sin bx^2 \, dx. \tag{190}$$

The functions R_1 and Q_1 obtained by numerical integration are given in Table XXII.

The following functions occur in connection with binary FM:

$$\frac{1}{\bar{\omega}} \frac{dR_0(t)}{dt} = \frac{4}{\bar{\omega} T} R_1(t) = \frac{4}{\pi} R_1(t)\,, \tag{191}$$

$$\frac{1}{\bar{\omega}} \frac{dQ_0(t)}{dt} = \frac{1}{\bar{\omega} T} Q_1(t) = \frac{4}{\pi} Q_1(t)\,. \tag{192}$$

These functions are given in Table XVI for integral values of $n = t/T$.

TABLE XXI — FUNCTIONS $R_{00}(t/T)$ AND $Q_{00}(t/T)$ FOR RAISED COSINE PULSE SPECTRUM AND LINEAR DELAY DISTORTION

t/T	$d/T = 0$		$d/T = 0.5$		$d/T = 1$		$d/T = 2$		$d/T = 3$	
	R_{00}	Q_{00}	R_{00}	Q_{00}	R_{00}	Q_{00}	R_{00}	Q_{00}	R_{00}	Q_{00}
0	1.00		0.9712	0	0.8943	0	0.6975	0	0.5712	0
±0.25	0.8488	0	0.8324	−0.0395	0.7879	−0.0667	0.6646	−0.0641	0.5614	−0.0344
±0.50	0.5000	0	0.5056	−0.1138	0.5204	−0.1967	0.5484	−0.2120	0.5188	−0.1317
±0.75	0.1698	0	0.1817	−0.1421	0.2169	−0.2577	0.3356	−0.3385	0.4131	−0.2628
±1.00	0	0	−0.0015	−0.0973	0.0012	−0.1933	0.0792	−0.3473	0.2293	−0.3629
±1.25	−0.0243	0	−0.0392	−0.0263	−0.0754	−0.0687	−0.1118	−0.2247	0.0044	−0.3588
±1.50	0	0	−0.0132	0.0130	−0.0511	0.0171	−0.1604	−0.0529	−0.1714	−0.2288
±1.75	0.0081	0	0.0063	0.0111	−0.0042	0.0293	−0.0910	0.0595	−0.2153	−0.0392
±2.00	0	0	0.0044	−0.0023	0.0144	0.0058	0.0005	0.0693	−0.1284	0.0978
±2.25	−0.0037	0	−0.0017	−0.0047	0.0060	−0.0079	0.0391	0.0195	−0.0010	0.1154
±2.50	0	0	−0.0017	0.0007	−0.0039	−0.0036	0.0209	−0.0171	0.0669	0.0454
±2.75	0.0020	0	0.0007	−0.0023	−0.0032	0.0026	0.0056	−0.0160	0.0498	−0.0243
±3.00	0	0	0.0008	−0.0002	0.0014	0.0020	−0.0107	0.0009	−0.0000	−0.0377
±3.25	−0.0012	0							−0.0234	−0.0100
±3.50	0								−0.0115	0.0127
±3.75	0.0007								−0.0040	0.0177

Table XXII — Functions $R_1(t/T)$ and $Q_1(t/T)$ for Raised Cosine Pulse Spectrum and Linear Delay Distortion

$$R_1(-t/T) = -R_1(t/T); \qquad Q_1(-t/T) = -Q_1(t/T)$$

t/T	$d/T = 0$		$d/T = 0.5$		$d/T = 1.0$		$d/T = 2.0$		$d/T = 3.0$	
	R_1	Q_1	R_1	Q_1	R_1	Q_1	R_1	Q_1	R_1	Q_1
0	0	0	0	0	0	0	0	0	0	0
0.25	−0.2829	0	−0.2418	−0.1214	−0.1448	−0.1869	+0.0095	−0.1385	+0.0239	−0.0674
0.50	−0.3750	0	−0.3379	−0.1329	−0.2446	−0.2212	−0.0505	−0.2231	+0.0194	−0.1389
0.75	−0.2603	0	−0.2626	−0.0423	−0.2578	−0.0987	−0.1637	−0.2051	−0.0373	−0.1907
1.00	−0.0833	0	−0.1125	+0.0490	−0.1777	+0.0502	−0.2369	−0.0827	−0.1363	−0.1774
1.25	+0.0164	0	−0.2872	+0.0624	−0.0568	+0.1050	−0.1967	+0.0683	−0.2158	−0.0761
1.50	+0.0208	0	+0.0261	+0.0212	+0.0274	+0.0617	−0.0686	+0.1434	−0.2020	+0.0689
1.75	−0.0037	0	+0.0097	−0.0118	+0.0396	−0.0004	+0.0477	+0.1045	−0.0859	+0.1625
2.00	−0.0083	0	−0.0051	−0.0105	+0.0010	−0.0214	+0.0770	+0.0131	+0.0517	+0.1408
2.25	+0.0013	0	−0.0040	+0.0029	−0.0112	−0.0075	+0.0330	−0.0430	+0.1110	+0.0359
2.50	+0.0042	0	+0.0014	+0.0051	−0.0073	+0.0058	−0.0150	−0.0311	+0.0684	−0.0541
2.75	−0.0006	0	+0.0018	−0.0010	+0.0032	+0.0042	−0.0227	+0.0025	−0.0089	−0.0633
3.00	−0.0024	0	0.0005	−0.0028	+0.0040	−0.0018	−0.0033	+0.0148	−0.0430	−0.0146
3.25									−0.0216	−0.0240
3.50									+0.0103	+0.0201
3.75									+0.0157	−0.0028
4.00									+0.0011	−0.0112

A.7 *Delay Distortion from Flat Bandpass Filters*

Let a bandpass filter have an amplitude characteristic A_0 between $-\omega_c + \omega_0$ and $\omega_0 + \omega_c$ and A_1 outside this band. When the bandwidth $2\omega_c$ is small in relation to the midband frequency ω_0, the phase characteristic is closely approximated by

$$\psi_0(u) = \frac{B}{\pi} \log_\epsilon \frac{1 + u/\omega_c}{1 - u/\omega_c}, \tag{193}$$

where

$$B = \log_\epsilon (A_0/A_1). \tag{194}$$

The corresponding envelope delay distortion is $D(u) = d\psi_0(u)/du$ and delay distortion relative to the midband frequency becomes

$$D_0(u) = D(u) - D(0) = \frac{2B}{\pi\omega_c} \frac{(u/\omega_c)^2}{1 - (u/\omega_c)^2} \tag{195}$$

$$= \frac{2B}{\pi\omega_c}\left[\left(\frac{u}{\omega_c}\right)^2 + \left(\frac{u}{\omega_c}\right)^4 + \left(\frac{u}{\omega_c}\right)^6 + \cdots\right]. \tag{196}$$

It will be noted that the first term in (196) represents quadratic delay distortion, which is approximated for $u/\omega_c \ll 1$.

Let the pulse spectrum at the detector input have a raised cosine shape, as given by (171), in which case the maximum radian frequency to each side of midband is $2\bar{\omega}$. With a phase characteristic as given by (193), the carrier pulse transmission characteristic is in this case obtained with (171) and (193) in (166) and becomes

$$R_0(-t) = R_0(t) = \frac{4}{\pi} \int_0^{\pi/2} \cos^2 x \cos\left[ax - \psi_0(x)\right] dx, \tag{197}$$

where

$$a = 4t/T,$$

$$\psi_0(x) = \frac{B}{\pi} \log_\epsilon \left(\frac{k + \frac{2}{\pi} x}{k - \frac{2}{\pi} x} \right), \tag{198}$$

$$k = \frac{\omega_c}{2\bar{\omega}} = \frac{W_2}{W_1}, \tag{199}$$

in which W_1 is the bandwidth of the raised cosine spectrum and W_2 that of the flat filter, as indicated in Fig. 14.

In Table XXIII are given the values of $R_0(t/T)$ obtained by numeri-

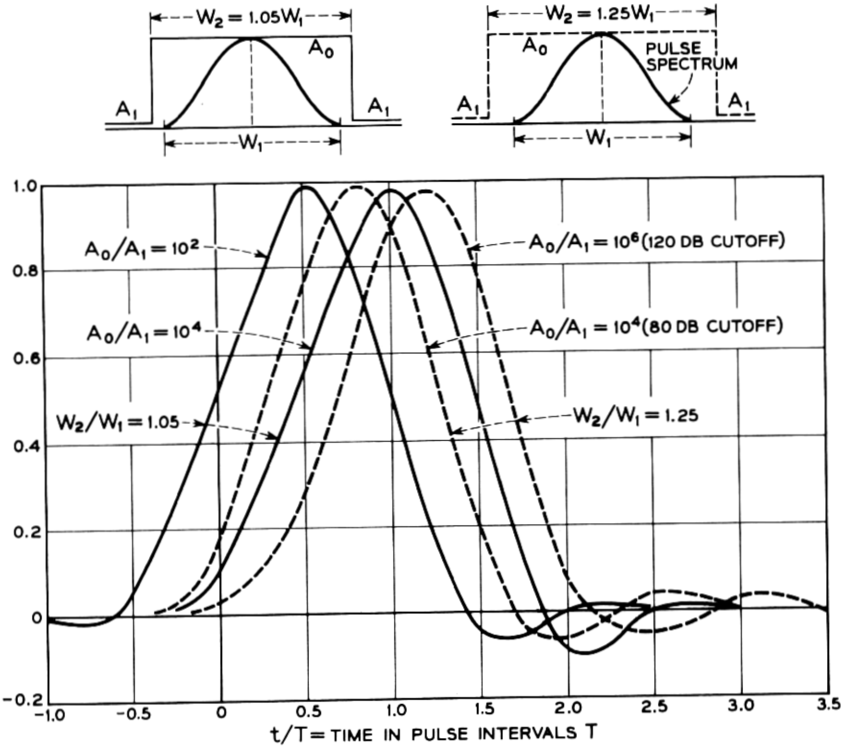

Fig. 14 — Carrier pulse transmission characteristics for raised cosine pulse spectrum and phase distortion resulting from flat filters with sharp cutoffs.

Table XXIII— Function $R_0(t/T)$ for Raised Cosine Spectrum and Phase Distortion Resulting from Flat Filters with Sharp Cutoffs

$k = W_2/W_1$		1.05		1.25	
A_0/A_1		10^2	10^4	10^4	10^6
A_0/A_1, in db		40	80	80	120
t/T	−1.0	−0.002	~0	0.001	~0
	−0.5	0.048	0.001	−0.001	~0
	0	0.525	0.092	0.168	0.018
	0.5	0.994	0.554	0.758	0.285
	1.0	0.481	0.979	0.903	0.888
	1.5	−0.051	0.464	0.192	0.789
	2.0	0.004	−0.110	−0.060	0.059
	2.5	0.003	0.005	0.020	−0.059
	3.0	−0.003	0.001	−0.008	0.028

cal integration of (197) for certain cases as indicated in the table. The functions $R_0(t/T)$ are shown in Fig. 14.

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