# Mode Conversion in Metallic and Helix Waveguide*

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*Helix waveguide, composed of closely wound insulated copper wire covered with an absorptive or reactive jacket, transmits circular electric waves with low loss. Mechanical imperfections, such as curvature and deformation, cause mode conversion and degrade the transmission. Generalized telegraphist's equations describe propagation in an imperfect helix waveguide with coupling between the many propagating waves. The coefficients of coupling depend strongly on the outside jacket. However, the sum of the squares of the coupling coefficients is independent of the jacket for circular electric waves. As a consequence, the average circular electric wave loss in a helix waveguide with random imperfections is also nearly independent of the jacket and the same as in a metallic pipe.*

## I. INTRODUCTION

Helix waveguide consisting of closely wound insulated copper wire covered with an electrically absorptive or reactive jacket (Fig. 1) is a good transmission medium for circular electric waves.[1] In long distance communication with waveguides it is useful as a mode filter, for negotiating bends or particularly as the transmission line proper instead of a metallic waveguide.[2]

The loss of circular electric waves decreases steadily with frequency only in a perfect metallic waveguide.[3,4] A similar situation prevails for helix waveguide. Any deviations from a round and straight guide will add to the transmission loss. At such imperfections power is converted from the circular electric wave into other propagating modes and reconverted. The mode conversion-reconversion effects increase the loss and degrade the transmission characteristics.

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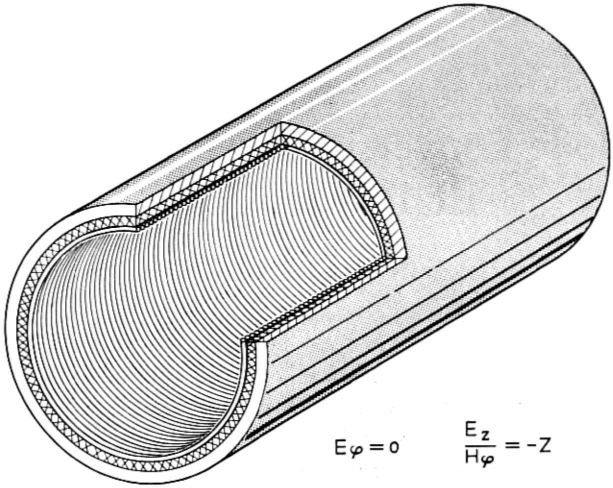


Fig. 1 — Helix waveguide and boundary conditions.

## II. GENERALIZED TELEGRAPHIST'S EQUATIONS AND MATRIX REPRESENTATION

Wave propagation in imperfect helix waveguide is most conveniently described with generalized telegraphist's equations.[5] This is a representation in terms of the normal modes of the perfect waveguide. The solution of the perfect waveguide is perturbed in the imperfect waveguide. Physically the perturbation appears as coupling between the normal modes of the perfect waveguide.

In the perfect metallic waveguide, wave propagation is completely described by a set of independent first-order differential equations

$$\frac{dA_n}{dz} = -\kappa_{nn} A_n , \tag{1}$$

where $A_n$ are the amplitudes of the $TE_{pn}$ or $TM_{pn}$ modes normalized with respect to power and $\kappa_{nn}$ are their propagation constants. $z$ is the axial coordinate. A square matrix with only the diagonal terms $\kappa_{nn} = jh_n$ ,

$$K = \begin{bmatrix} jh_0 & 0 & 0 & \cdots \\ 0 & jh_1 & 0 & \cdots \\ 0 & 0 & jh_2 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}, \tag{2}$$

describes the perfect metallic waveguide completely.

The perfect helix waveguide may be considered a perturbed metallic waveguide. The boundary conditions for the tangential electric field

$$E_\varphi = 0 \qquad \text{and} \qquad E_z = 0 \tag{3}$$

are replaced by

$$E_\varphi = 0 \qquad \text{and} \qquad \frac{E_z}{H_\varphi} = -Z, \tag{4}$$

where $Z$ is the wall impedance that the jacket of the helix waveguide presents at the helix interface.

To describe wave propagation in the helix waveguide with metallic waveguide modes, off-diagonal terms have to be added in the diagonal matrix of the metallic waveguide and the diagonal terms are perturbed:

$$K = \begin{bmatrix} \kappa_{00} & 0 & 0 & 0 \\ 0 & \kappa_{11} & \kappa_{12} & \cdots \\ 0 & \kappa_{12} & \kappa_{22} & \cdots \\ 0 & \vdots & \vdots & \end{bmatrix}. \tag{5}$$

Only the propagation constant $\kappa_{00} = jh_0$ of the circular electric wave stays unperturbed, and its off-diagonal terms remain zero. In other words, to describe wave propagation in helix waveguides with normal modes of metallic waveguide, coupling has to be introduced between these modes, and their propagation constants are modified. Only circular electric waves stay unchanged.

It is now expedient to transform from the normal modes of metallic waveguide $A_n$ to the normal modes of helix waveguide $E_n$ :

$$A = LE. \tag{6}$$

The so-called modal matrix $L$ of $K$ transforms $K$ to its diagonal form $\Gamma$:

$$L^{-1}KL = \Gamma. \tag{7}$$

Since $K$ is symmetrical the modal matrix $L$ is orthonormal. It obeys:

$$L_t = L^{-1}. \tag{8}$$

Its transpose is equal to its inverse. The diagonal terms $\gamma_{nn}$ of $\Gamma$ are the propagation constants of normal modes in helix waveguide. The circular electric wave remains unaffected by this transformation:

$$\gamma_{00} = \kappa_{00} = jh_0 .$$

It is now possible to consider an imperfect helix waveguide, perturbed by curvature or cross-sectional deformations in the same terms. The matrix $K'$ then has off-diagonal elements also in the row and column associated with the circular electric wave:

$$K' = \begin{bmatrix} \kappa_{00} & \kappa_{01} & \kappa_{02} & \cdots \\ \kappa_{01} & \kappa_{11} & \kappa_{12} & \cdots \\ \kappa_{02} & \kappa_{12} & \kappa_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}. \tag{9}$$

These new off-diagonal elements represent coupling between the circular electric wave and other modes in an imperfect helix waveguide. They are the coupling coefficients of generalized telegraphist's equations. When Maxwell's equations for the imperfect helix waveguide are converted into generalized telegraphist's equations, it is found that the coupling coefficients $\kappa_{0n}$ in $K'$ for curvature and cross-sectional deformation are independent of the wall impedance and the same as in metallic waveguide. Generalized telegraphist's equations for deformed helix waveguide are found in the Appendix.

The independence of the $\kappa_{0n}$'s from the wall impedance has an important bearing on the coupling coefficients between the normal modes in helix waveguide. If the new $K'$ for the imperfect helix waveguide is transformed by the previous modal matrix $L$, there results

$$L_t K' L = \Gamma'. \tag{10}$$

The new matrix $\Gamma'$ has off-diagonal elements $c_{0n}$ which are caused by the imperfection. They are the coefficients of coupling between the normal modes of the perfect helix waveguide. In terms of the elements of $K'$ and $L$ they are:

$$c_{0n} = \sum_{m=1}^{\infty} \kappa_{0m} l_{mn} . \tag{11}$$

If this expression is squared and summed over all $n$, then, since the $l$'s are elements of an orthonormal matrix, the resulting expression does not contain any $l$'s:

$$\sum_n c_{0n}^2 = \sum_m \kappa_{0m}^2. \tag{12}$$

It is recalled that the $\kappa_{0m}$ are coupling coefficients between normal modes in an imperfect metallic waveguide. Consequently, the sum of the squares of the coupling coefficients $c_{0n}$ in an imperfect helix waveguide is independent of the wall impedance and the same as in metallic waveguide.

## III. AVERAGE MODE CONVERSION LOSS FROM RANDOM IMPERFECTIONS

The following statement will be proved: The average mode conversion loss of certain kinds of random imperfections in helix waveguide depends only on the sum of the square of all coupling coefficients.

To determine mode conversion, first of all, generalized telegraphist's equations have to be solved. With the elements of the perturbed Γ matrix the coupled line equations are

$$\frac{dE_n}{dz} = -\gamma_n E_n - \sum_m c_{nm} E_m . \tag{13}$$

When only a circular electric wave of unit amplitude, $E_0(0) = 1$, is incident and the imperfections are small, a first-order solution at $z = L$ is:

$$| E_0(L) | = \left| 1 - \sum_n \int_0^L e^{(\gamma_0 - \gamma_n)u} \, du \int_0^{L-u} c_{0n}(s) c_{0n}(s + u) \, ds \right| . \tag{14}$$

The coupling coefficients are proportional to the geometric imperfection $\delta$:

$$c_{0n}(z) = C_{0n} \delta(z). \tag{15}$$

Let the imperfections be a stationary random process with covariance

$$\rho(u) = \langle \delta(z) \delta(z + u) \rangle \tag{16}$$

and spectral distribution

$$S(\xi) = \int_{-\infty}^{+\infty} \rho(u) \, e^{-j2\pi\xi u} \, du, \tag{17}$$

where $\xi$ is the spatial frequency of the geometric imperfection. Then, as Rowe first pointed out,[6] the average output amplitude $\langle | E_0(L) | \rangle$ can be expressed in terms of the covariance $\rho(u)$:

$$\langle | E_0(L) | \rangle = \left| 1 - \sum_n C_{0n}^2 \int_0^L (L - u) \, e^{(\gamma_0 - \gamma_n)u} \rho(u) \, du \right| .$$

If, furthermore, the correlation between imperfections any appreciable distance apart is small the covariance drops off very rapidly with increasing argument. Then in the expression for $\langle | E_0(L) | \rangle$ the exponential and the factor $(L - u)$ are constant for any weight of $\rho(u)$, and one obtains:

$$\langle | E_0(L) | \rangle = \left| 1 - L \int_0^\infty \rho(u) \, du \sum_n C_{0n}^2 \right| . \tag{18}$$

The average mode conversion loss $\langle \alpha \rangle$ can now be written in terms of the spectral density $S$ of the random imperfections $\delta$ and the coupling factors $C_{0n}$ :

$$\langle \alpha \rangle = \tfrac{1}{2} S(0) \operatorname{Re} \left[ \sum_n C_{0n}^2 \right]. \tag{19}$$

With (12) and (15), it may be concluded from (19) that the average loss for circular electric waves in an imperfect helix waveguide is independent of the wall impedance and the same as in metallic waveguide that has the same geometrical imperfections.

The above derivation has assumed the covariance to drop off fast or the correlation distance to be small. This is the case for any imperfection created in the manufacturing process of the waveguide. Any manufacturing imperfections some reasonable distance apart are hardly correlated to each other. The effects of manufacturing imperfections for helix waveguide are therefore the same as in metallic waveguide. It is relatively easy to determine tolerances for metallic waveguide.[4] The above rule lets these tolerances be valid for helix waveguide and the very involved calculations for helix waveguide are not necessary.

Before accepting this rule the range of correlation distance for which it is valid must be examined. As a typical example, the covariance has been assumed exponential:

$$\rho(u) = \langle \delta^2 \rangle e^{-2\pi(|u|/L_0)}. \tag{20}$$

The average $TE_{01}$ loss at 55 kmc has then been calculated for various helix waveguides of 2-inch inside diameter as a function of the correlation distance $L_0$.[7] Fig. 2 shows for deformed helix waveguide the rms of elliptical diameter differences which increase the $TE_{01}$ loss by 10 per cent of the loss in a perfect copper pipe. Up to a correlation distance of one foot the curves almost coincide and indicate independence of the wall impedance.

For a curved helix waveguide the range is even larger. Fig. 3 shows for a curved helix waveguide the rms curvature under the same conditions. Random curvature of up to a 10-foot correlation distance causes nearly the same average $TE_{01}$ loss in helix waveguide and in metallic waveguide. For a correlation distance larger than 10 feet there is an ever growing dependence of curvature loss on wall impedance. But such curvature distributions do not occur in the manufacturing process. They are, however, representative of laying tolerances when the waveguide is installed with long bows to follow right of ways or the contour of the landscape. A properly designed helix waveguide can tolerate much more laying curvature than can metallic waveguide.

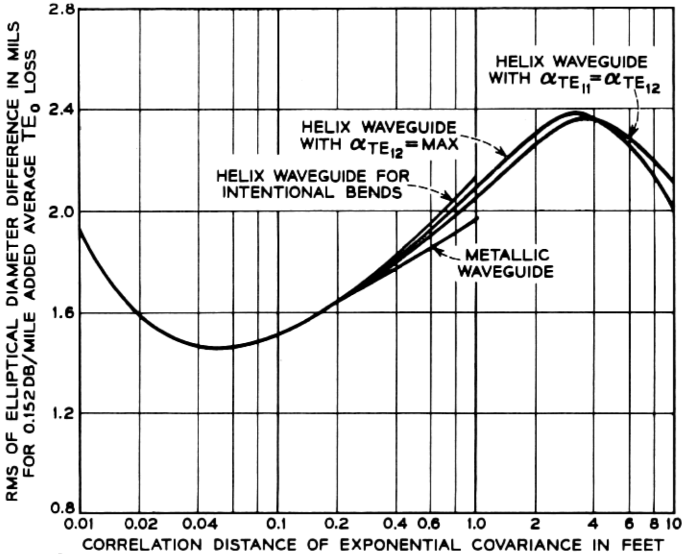


Fig. 2 — $TE_{01}$ loss in round waveguide with random ellipticity, 2-inch inside diameter, at 55.5 kmc.

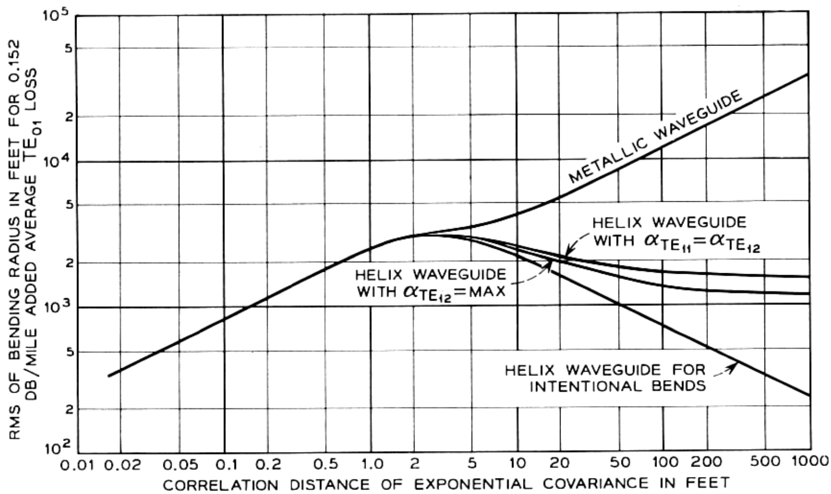


Fig. 3 — $TE_{01}$ loss in round waveguide with random curvature, 2-inch inside diameter, at 55.5 kmc.

## APPENDIX

*Generalized Telegraphist's Equation for Noncylindrical Helix Waveguide*

Maxwell's equations have been converted into generalized telegraphist's equations for curved helix waveguide elsewhere.[2] They have been represented in terms of normal modes of the metallic waveguide as well as in terms of normal modes of helix waveguide. In the former representation the coefficients of curvature coupling between circular electric waves and the modes of metallic waveguide were independent of the wall impedance and the same as in metallic waveguide.

The same is true for any cross-sectional deformation in helix waveguide. To prove this, Maxwell's equations will be converted into generalized telegraphist's equations in terms of modes of metallic waveguide for the deformed helix waveguide. This representation is different from another analysis, where the equations are written in terms of the normal modes of helix waveguide.[8]

If the radius of the deformed guide is

$$a_1 = a(1 + \delta),$$

where $a$ is the nominal radius and the deformation $\delta$ is for the moment assumed to be only a function of $\varphi$ of cylindrical coordinates $(r\varphi z)$, then the boundary conditions at $r = a$, are

$$E_\varphi + E_r \frac{d\delta}{d\varphi} = 0, \tag{21}$$

$$E_z = \frac{-Z}{\sqrt{1 + \left(\frac{d\delta}{d\varphi}\right)^2}} \left(H_\varphi + H_r \frac{d\delta}{d\varphi}\right). \tag{22}$$

The deformation is assumed to be small and smooth:

$$\delta \ll 1 \qquad \text{and} \qquad \frac{d\delta}{d\varphi} \ll 1. \tag{23}$$

The fields at $r = a_1$ can by series expansion be written in terms of the fields at $r = a$. The boundary conditions then take the approximate form:

$$E_\varphi = -\frac{\partial E_\varphi}{\partial r} a\delta - E_r \frac{d\delta}{d\varphi}, \tag{24}$$

$$E_z = -\frac{\partial E_z}{\partial r} a\delta - Z \left(H_\varphi + \frac{\partial H_\varphi}{\partial_r} a\delta + H_r \frac{d\delta}{d\varphi}\right). \tag{25}$$

Maxwell's equations in cylindrical coordinates for exponential time dependence $\epsilon^{j\omega t}$ are

$$\frac{1}{r}\frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z} = -j\omega\mu H_r\,, \tag{26}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -j\omega\mu H_\varphi\,, \tag{27}$$

$$\frac{1}{r}\frac{\partial (rE_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial \varphi} = -j\omega\mu H_z\,, \tag{28}$$

$$\frac{1}{r}\frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} = j\omega\epsilon\, E_r\,, \tag{29}$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = j\omega\epsilon\, E_\varphi\,, \tag{30}$$

$$\frac{1}{r}\frac{\partial (rH_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial H_r}{\partial \varphi} = j\omega\epsilon\, E_z\,, \tag{31}$$

where $\mu$ and $\epsilon$ are permeability and permittivity of the waveguide interior.

The electromagnetic field is derived from two sets of wave functions, $T_{(n)}$ for TM waves and $T_{[n]}$ for TE waves of metallic waveguide:

$$\begin{aligned}
E_r &= \sum_n \left[ V_{(n)} \frac{\partial T_{(n)}}{\partial r} + V_{[n]} \frac{\partial T_{[n]}}{r\partial \varphi} \right], \\
E_\varphi &= \sum_n \left[ V_{(n)} \frac{\partial T_{(n)}}{r\partial \varphi} - V_{[n]} \frac{\partial T_{[n]}}{\partial r} \right], \\
H_r &= \sum_n \left[ -I_{(n)} \frac{\partial T_{(n)}}{r\partial \varphi} + I_{[n]} \frac{\partial T_{[n]}}{\partial r} \right], \\
H_\varphi &= \sum_n \left[ I_{(n)} \frac{\partial T_{(n)}}{\partial r} + I_{[n]} \frac{\partial T_{[n]}}{r\partial \varphi} \right].
\end{aligned} \tag{32}$$

The transverse field distribution is described by $T(r,\varphi)$ while the voltage and current coefficients $V(z)$ and $I(z)$ are functions of the coordinate $z$.

The $T$-functions satisfy the wave equation

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{\partial T}{r\partial \varphi}\right)\right] = -\chi^2 T, \tag{33}$$

where $\chi$ is a separation constant, which takes on discrete values for the various normal modes. The $T$-functions are normalized so that

$$\int_S (\text{grad } T)(\text{grad } T)\, dS = \int_S (\text{flux } T)(\text{flux } T)\, dS = \chi^2 \int_S T^2\, dS = 1, \tag{34}$$

where $S$ is the nominal cross section of the guide and the gradient and flux vectors of $T$ are defined by:

$$\begin{aligned} \text{grad}_r\, T &= \frac{\partial T}{\partial r}, & \text{grad}_\varphi\, T &= \frac{\partial T}{r\partial \varphi}, \\ \text{flux}_r\, T &= \frac{\partial T}{r\partial \varphi}, & \text{flux}_\varphi\, T &= -\frac{\partial T}{\partial r}. \end{aligned} \tag{35}$$

Various orthogonality relations exist among the $T$ functions:

$$\begin{gathered} \int_S T_{(n)} T_{(m)}\, dS = \int_S T_{[n]} T_{[m]}\, dS = 0, \\ \int_S (\text{grad } T_{(n)})(\text{grad } T_{(m)})\, dS = \int_S (\text{flux } T_{(n)})(\text{flux } T_{(m)})\, dS = 0, \\ \int_S (\text{grad } T_{[n]})(\text{grad } T_{[m]})\, dS = \int_S (\text{flux } T_{[n]})(\text{flux } T_{[m]})\, dS = 0 \end{gathered} \tag{36}$$

if $m \neq n$, and

$$\begin{aligned} \int_S (\text{grad } T_{(n)})(\text{flux } T_{[m]})\, dS &= \int_S (\text{grad } T_{[n]})(\text{flux } T_{(m)})\, dS \\ &= \int_S (\text{grad } T_{(n)})(\text{flux } T_{(m)})\, dS = 0 \end{aligned} \tag{37}$$

for all $m$ and $n$.

To transform Maxwell's equations into generalized telegraphist's equations the series expansions (32) are substituted for the field components in (26) through (31). Certain combinations of the equations are integrated over the nominal cross section, and advantage is taken of the orthogonality relations (36) and (37).

For example, adding $-\partial T_{(m)}/r\partial\varphi$ times (26) and $\partial T_{(m)}/\partial r$ times (27) and integrating over the nominal cross section:

$$\frac{dV_{(m)}}{dz} + j\omega\mu\, I_{(m)} = a \int_0^{2\pi} E_z \frac{\partial T_{(m)}}{\partial r}\bigg|_a d\varphi + \chi_{(m)}^2 \int_S E_z\, T_{(m)}\, dS. \tag{38}$$

In the first term on the right-hand side of (38) the boundary condition

(25) is substituted for $E_z$. In the second term (31) is substituted for $E_z$:

$$\frac{dV_{(m)}}{dz} + j\,\frac{{h_{(m)}}^2}{\omega\epsilon}\,I_{(m)} = -Z\sum_n\left[I_{(n)}a\int_0^{2\pi}(1-\delta)\,\frac{\partial T_{(n)}}{\partial r}\,\frac{\partial T_{(m)}}{\partial r}\bigg|_a d\varphi \right. \tag{39}$$
$$\left. + I_{[n]}\int_0^{2\pi}(1-\delta)\,\frac{\partial T_{[n]}}{\partial\varphi}\,\frac{\partial T_{(m)}}{\partial r}\bigg|_a d\varphi\right] - a^2\int_0^{2\pi}\delta\,\frac{\partial E_z}{\partial r}\,\frac{\partial T_{(m)}}{\partial r}\bigg|_a d\varphi,$$

where $h^2 = \omega^2\mu\epsilon - \chi^2$.

Add $\partial T_{[m]}/\partial r$ times (26) and $\partial T_{[m]}/r\partial\varphi$ times (27) and integrate over the nominal cross section:

$$\frac{dV_{[m]}}{dz} + j\omega\mu\,I_{[m]} = \int_0^{2\pi}E_z\,\frac{\partial T_{[m]}}{\partial\varphi}\bigg|_a d\varphi. \tag{40}$$

The boundary condition (25) is substituted for $E_z$:

$$\frac{dV_{[m]}}{dz} + j\omega\mu\,I_{[m]} = -a\int_0^{2\pi}\delta\,\frac{\partial E_z}{\partial r}\,\frac{\partial T_{[m]}}{\partial\varphi}\bigg|_a d\varphi$$
$$-Z\sum_n\left[I_{(n)}\int_0^{2\pi}(1-\delta)\,\frac{\partial T_{(n)}}{\partial r}\,\frac{\partial T_{[m]}}{\partial\varphi}\bigg|_a d\varphi \right. \tag{41}$$
$$\left. + I_{[n]}\int_0^{2\pi}\frac{1-\delta}{a}\,\frac{\partial T_{[n]}}{\partial\varphi}\,\frac{\partial T_{[m]}}{\partial\varphi}\bigg|_a d\varphi\right].$$

Add $-\partial T_{(m)}/\partial r$ times (29) and $-\partial T_{(m)}/r\partial\varphi$ times (30) and integrate over the nominal cross section:

$$\frac{dI_{(m)}}{dz} + j\omega\epsilon\,V_{(m)} = 0. \tag{42}$$

Add $-\partial T_{[m]}/r\partial\varphi$ times (29) and $\partial T_{[m]}/\partial r$ times (30) and integrate over the nominal cross section:

$$\frac{dI_{[m]}}{dz} + j\omega\epsilon\,V_{[m]} = {\chi_{[m]}}^2\int_S H_z\,T_{[m]}\,dS. \tag{43}$$

For the right-hand side of (43) integrate $T_{[m]}$ times (28) over the nominal cross section:

$$-j\omega\mu\int_S H_z\,T_{[m]}\,dS = V_{[m]} + a\int_0^{2\pi}E_\varphi\,T_{[m]}\bigg|_a d\varphi. \tag{44}$$

Substitute the boundary condition (24) for $E_\varphi$ and perform the partial integration:

$$\int_0^{2\pi}E_r\,\frac{d\delta}{d\varphi}\,T_{[m]}\bigg|_a d\varphi = -\int_0^{2\pi}\delta\left(\frac{\partial E_r}{\partial\varphi}\,T_{[m]} + E_r\,\frac{\partial T_{[m]}}{\partial\varphi}\right)\bigg|_a d\varphi. \tag{45}$$

With (44) and (45), (43) reads:

$$\frac{dI_{[m]}}{dz} + j\frac{h_{[m]}^2}{\omega\mu} V_{[m]} = ja\frac{\chi_{[m]}^2}{\omega\mu}\left(\int_0^{2\pi} \delta E_r \frac{\partial T_{[m]}}{\partial \varphi}\bigg|_a d\varphi - a\sum_n V_{[n]}\chi_{[n]}^2 \int_0^{2\pi} \delta T_{[m]} T_{[n]}\bigg|_a d\varphi\right). \tag{46}$$

Equations (39), (41), (42) and (46) describe coupling between all modes. For the present consideration, all terms must be retained which contain the wall impedance $Z$, because in practical helix waveguide the wall impedance may be quite large. The deformation, however, is small, and to analyze circular electric wave transmission only $\delta$ terms need be retained that describe direct coupling between circular electric and other waves. The above equations then reduce to:

$$\frac{dV_{(m)}}{dz} + j\frac{h_{(m)}^2}{\omega\epsilon} I_{(m)} = -Z\sum_n\left(I_{(n)}a\int_0^{2\pi}\frac{\partial T_{(n)}}{\partial r}\frac{\partial T_{(m)}}{\partial r}\,d\varphi + I_{[n]}\int_0^{2\pi}\frac{\partial T_{[n]}}{\partial \varphi}\frac{\partial T_{(m)}}{\partial r}\,d\varphi\right), \tag{47}$$

$$\frac{dV_{[m]}}{dz} + j\omega\mu I_{[m]} = -Z\sum_n\left(I_{(n)}\int_0^{2\pi}\frac{\partial T_{(n)}}{\partial r}\frac{\partial T_{[m]}}{\partial \varphi}\,d\varphi + I_{[n]}\frac{1}{a}\int_0^{2\pi}\frac{\partial T_{[n]}}{\partial \varphi}\frac{\partial T_{[m]}}{\partial \varphi}\,d\varphi\right) \tag{48}$$

$$\frac{dI_{(m)}}{dz} + j\omega\epsilon V_{(m)} = 0 \tag{49}$$

$$\frac{dI_{[m]}}{dz} + j\frac{h_{[m]}^2}{\omega\mu} V_{[m]} = -j\frac{a^2}{\omega\mu}\sum_n V_{[n]}\chi_{[m]}^2\chi_{[n]}^2\int_0^{2\pi}\delta\, T_{[m]}\, T_{[n]}\,d\varphi. \tag{50}$$

All the integrals are taken along the nominal circumference.

Alternatively, in terms of voltages and currents the equations are more conveniently written in terms of amplitudes $A$ of forward- and $B$ of backward-traveling waves. The mode current and voltage are related to the mode amplitudes by

$$V_n = \sqrt{K_n}\,(A_n + B_n),$$
$$I_n = \frac{1}{\sqrt{K_n}}\,(A_n - B_n), \tag{51}$$

where $K_n$ is the wave impedance:

$$K_{(n)} = \frac{h_{(n)}}{\omega\epsilon}, \qquad K_{[n]} = \frac{\omega\mu}{h_{[n]}}. \tag{52}$$

If the currents and voltages in the generalized telegraphist's equations are represented in terms of the traveling wave amplitudes, the system of coupled equations can be written in matrix notation as

$$\frac{dA}{dz} = -K'A. \tag{53}$$

The column matrix

$$A = \begin{bmatrix} A_{[0]} \\ A_{[1]} \\ A_{(1)} \\ A_{[2]} \\ A_{(2)} \\ \vdots \\ B_{[n]} \\ \vdots \end{bmatrix} \tag{54}$$

represents the amplitudes of metallic waveguide modes. The square matrix

$$K' = \begin{bmatrix} \kappa_{00} & \kappa_{01} & \kappa_{02} & \cdots \\ \kappa_{01} & \kappa_{11} & \kappa_{12} & \cdots \\ \kappa_{02} & \kappa_{12} & \kappa_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{55}$$

describes the deformed helix waveguide. To calculate the elements of $K'$ the wave functions of normal modes of metallic waveguide are introduced. The customary double-subscript notation will be used, but TM-waves will still be denoted with parentheses and TE-waves with brackets:

$$\begin{aligned} T_{(pn)} &= \sqrt{\frac{\epsilon_p}{\pi}}\, \frac{J_p(\chi_{(pn)}r)\, \sin p\varphi}{k_{(pn)}J_{p-1}(k_{(pn)})}, \\ T_{[pn]} &= \sqrt{\frac{\epsilon_p}{\pi}}\, \frac{J_p(\chi_{[pn]}r)\, \cos p\varphi}{(k_{[pn]}{}^2 - p^2)^{\frac{1}{2}} J_p(k_{[pn]})}, \end{aligned} \tag{56}$$

where

$$\begin{aligned} k_{(pn)} &= \chi_{(pn)}a, & J_p(k_{(pn)}) &= 0, \\ k_{[pn]} &= \chi_{[pn]}a, & J_p{}'(k_{[pn]}) &= 0 \end{aligned}$$

and

$$\begin{aligned} \epsilon_p &= 1 \quad \text{if} \quad p = 0, \\ \epsilon_p &= 2 \quad \text{if} \quad p \neq 0. \end{aligned}$$

Let the first element of $A$ be the amplitude $A_{[0m]}$ of a circular electric wave. The propagation constant of this wave is

$$\kappa_{00} = jh_{[0m]} = j\sqrt{\omega^2\mu\epsilon - \chi_{[0m]}{}^2}.$$

The elements of the first row and column,

$$\kappa_{0n} = \begin{cases} \kappa_{[0m][pn]} = j\dfrac{\sqrt{\epsilon_p}}{2\pi}\dfrac{k_{[0m]}k_{[pn]}{}^2}{a^2\sqrt{h_{[0m]}h_{[pn]}}\sqrt{k_{[pn]}{}^2 - p^2}}\displaystyle\int_0^{2\pi}\delta\cos p\varphi\, d\varphi, \\ \kappa_{[0m](pn)} = 0, \end{cases} \tag{57}$$

describe the coupling between the circular electric wave and other modes as it is caused by the deformation of the helix waveguide. The $\kappa_{0n}$ are independent of the wall impedance and the same as in metallic waveguide.

The other diagonal elements of $K'$ are propagation constants of other modes:

$$\kappa_{nn} = \begin{cases} \kappa_{(pn)(pn)} = \gamma_{(pn)} = jh_{(pn)} + \dfrac{\epsilon_p}{2a}\dfrac{Z}{K_{(pn)}}, \\ \kappa_{[pn][pn]} = \gamma_{[pn]} = jh_{[pn]} + \dfrac{\epsilon_p}{2a}\dfrac{p^2}{k_{[pn]}{}^2 - p^2}\dfrac{Z}{K_{[pn]}}. \end{cases} \tag{58}$$

The off-diagonal elements describe coupling between the other modes:

$$\kappa_{nm} = \begin{cases} \kappa_{(pn)(pm)} = \dfrac{\epsilon_p}{2a}\dfrac{Z}{\sqrt{K_{(pn)}K_{(pm)}}}, \\ \kappa_{(pn)[pm]} = \kappa_{[pm](pn)} = -\dfrac{\epsilon_p}{2a}\dfrac{p}{\sqrt{k_{[pm]}{}^2 - p^2}}\dfrac{Z}{\sqrt{K_{(pn)}K_{[pm]}}}, \\ \kappa_{[pn][pm]} = \dfrac{\epsilon_p}{2a}\dfrac{p^2}{\sqrt{k_{[pn]}{}^2 - p^2}\sqrt{k_{[pm]}{}^2 - p^2}}\dfrac{Z}{\sqrt{K_{[pn]}K_{[pm]}}}. \end{cases}$$

They all depend on the wall impedance of the helix waveguide.

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