Ultimately Periodic Solutions to a Non-Linear Integrodifferential Equation

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Tychonov's fixed point theorem is used to study the existence of ultimately periodic solutions of an integrodifferential equation that arises in the theory of the phase-controlled oscillator. The principal result describes conditions under which solutions of the equation exist which have a given ultimate period T, not necessarily the minimal period.

I. INTRODUCTION

Let $h(\cdot)$ be an integrable function of integral unity, vanishing for negative argument; let α, ω, and $x(0)$ be constants; and let $f(\cdot)$ be a periodic function of period (say) 2π. It is of interest to know what choices of $h(\cdot)$, α, ω, $x(0)$, and $f(\cdot)$ give rise to asymptotically periodic solutions of the equation

$$\dot{x} = \omega - \alpha \int_0^t h(t-u) f(x(u))\, du \quad (t \geqq 0). \tag{1}$$

This equation arises in the theory of various synchronization phenomena. (Cf. Refs. 1, 2, and 3 and references therein.) For example, the synchronous motor and the phase-controlled oscillator are devices often described by (1). For a specific physical application, we consider the phase-controlled loop depicted in Fig. 1 and described in detail by Goldstein,[3] q.v.

The system is described by the equations, in $t \geqq 0$,

$$\begin{aligned} \dot{\varphi}_o(t) &= \omega_c + \alpha v(t) \\ \dot{\varphi}_i(t) &= \omega_c + \omega(t) \\ x(t) &= \varphi_i(t) - \varphi_o(t) \\ v(t) &= \int_0^t h(t-u) f(x(u))\, du \end{aligned}$$

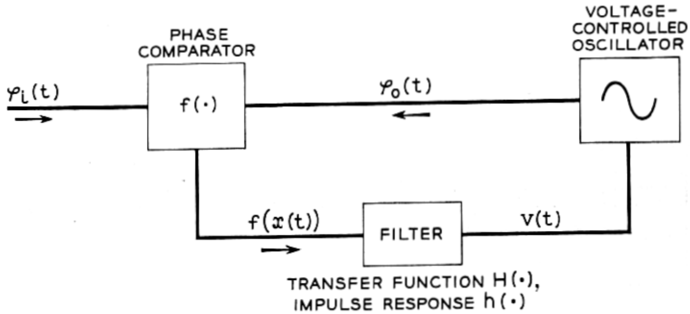

Fig. 1 — Phase-controlled loop.

where ω_c is the center or "free-running" frequency of the oscillator, $\varphi_o(\cdot)$ is the instantaneous phase of the output of a voltage-controlled oscillator, $\varphi_i(\cdot)$ is the instantaneous phase of an input signal (driving function, $h(\cdot)$ is the impulse response of a filter with dc gain unity, $f(\cdot)$ is a periodic phase comparator characteristic, and α is a gain constant. If the initial phase difference $x(0)$ is given, $\omega(t) = 0$ for $t < 0$, and $\omega(t) = \omega$ for $t \geqq 0$, we obtain (1); these conditions describe a sudden step of size ω in the frequency of the input signal.

Since the system described by (1) is autonomous, one may conjecture that a solution $x(\cdot)$ of (1) is always ultimately periodic, if we count identically constant functions as periodic. The present paper attempts to shed light on the question: what choices of ω give rise to an $x(\cdot)$ satisfying (1) with a *given* ultimate period T? We shall investigate solutions of (1) that are *ultimately periodic* in the following sense: a function $y(\cdot)$ is ultimately periodic with period T if there is a periodic function $p_y(\cdot)$ of period T such that

$$\lim_{n\to\infty} y(nT + t) = p_y(t) \quad (t \in [0,T]).$$

In this case we say that $y(\cdot)$ is u.p. [T], and write $p_y(\cdot)$ for the periodic function approached by $y(\cdot)$. The number T is not necessarily the *minimal* period.

If $h(\cdot)$ is the Green's function of a differential operator of low order, the nature of solutions of (1) can be studied by the classical phase-plane method, as by Barnard.[2] To retain maximum generality and to exhibit (to some extent) the core of the problem, however, we shall use Tychonov's fixed point theorem. The possible novelty of our approach lies in using Tychonov's theorem to obtain specific asymptotic information about solutions $x(\cdot)$ of (1) by finding a fixed point (corresponding to a solution) in a relatively small region of a function space. This is achieved by verifying some local properties of the operator whose fixed point is sought, and ensuring that a particular set is mapped into itself. A similar method has been used on (1) by the author in a previous paper[4] discussing the question whether $\dot{x}(t)$ approaches zero for large t.

II. PRELIMINARY CONSIDERATIONS

We rewrite (1) as the functional equation

$$g(t) = f\left(x(0) + \omega t - \alpha \int_0^t \eta(t-u) g(u)\, du\right) \quad (t \geqq 0) \qquad (2)$$

where

$$\begin{aligned} \eta(t) &= \int_0^t h(u)\, du \quad (t \geqq 0) \\ \eta(\infty) &= 1 \\ g(t) &= f(x(t)) \qquad\quad (t \geqq 0) \end{aligned}$$

and we seek an ultimately periodic solution $g(\cdot)$ of (2) in the space $B \frown C$ of bounded continuous functions. However, what periods $T > 0$ should be considered? If we choose a period T arbitrarily and define an operator J by

$$Jg(t) = \begin{cases} f\left(x(0) + \omega t - \alpha \int_0^t \eta(t-u) g(u)\, du\right) & (t \geqq 0) \\ f(x(0)) & (t \leqq 0) \end{cases}$$

then even if $g(\cdot)$ is ultimately periodic with period T, we have no guarantee that the image function $Jg(\cdot)$ is u.p. [T], or that it is ultimately periodic at all. This circumstance is due to the presence of the constant ω in the definition of J, which has no immediate relation to $g(\cdot)$ or to a specific period T of interest.

We next observe heuristically that if $g(\cdot)$ is u.p. [T], then $Jg(\cdot)$ can be u.p. [T] only if ω bears a suitable relation to both the period 2π of $f(\cdot)$ and the desired period T. Roughly speaking, one effect of the integration in (2) is to subtract a linear term $c_0 t$ from the linear term ωt already present; the coefficient c_0 will be proportional to the mean (over a period T) of the periodic function $p_g(\cdot)$ to which $g(\cdot)$ is asymptotic; the remainder of the contribution of the integral will be u.p. [T]. It is intuitively clear that c_0 should have the form

$$c_0 = \frac{\alpha}{T} \int_0^T p_g(u)\, du.$$

Let $a(\cdot)$ be a function of period T. Then certainly the function defined by

$$f(\omega t - c_0 t + a(t))$$

will have period T if

$$\omega = c_0 + \frac{2n\pi}{T} \quad (n \text{ an integer}). \tag{3}$$

For in such a case

$$f[(\omega - c_0)t + (\omega - c_0)T + a(t + T)] = f[(\omega - c_0)t + a(t)].$$

In view of this, we shall allow the "constant" ω in (2) to depend on the function $g(\cdot)$ being mapped according to (3), for a fixed choice of n. That is, we define a transformation $A(\cdot)$ of $g(\cdot)$'s that are u.p. $[T]$ by

$$Ag(t) = \begin{cases} f\left(x(0) + \omega_g t - \alpha \int_0^t \eta(t-u) g(u)\, du\right) & (t \geqq 0) \\ f(x(0)) & (t \leqq 0) \end{cases} \tag{4}$$

$$\omega_g = \frac{2n\pi}{T} + \frac{\alpha}{T} \int_0^T p_g(u)\, du \qquad (n \text{ fixed}).$$

By this device we shall be able to consider an arbitrary period T.

III. SUMMARY OF HYPOTHESES AND RESULTS

If $h(\cdot)$ is the impulse response of a physically realizable network then $\eta(t) = \int_0^t h(u)\, du$ is its response to a unit step-function, and $\eta(\infty)$ is its dc gain, here taken to be unity. The function $\psi(t) = \eta(\infty) - \eta(t)$, $t \geqq 0$, is basic to much of our discussion, and is assumed to be absolutely integrable. The integral

$$\int_0^\infty [\eta(\infty) - \eta(u)]\, du = \int_0^\infty \psi(u)\, du \tag{5}$$

can be invested with physical meaning as follows: a partial integration,

$$\int_0^t uh(u)\, du = t\eta(t) - \int_0^t \eta(u)\, du$$

$$= t[\eta(t) - \eta(\infty)] + \int_0^t \psi(u)\, du$$

and the observation that $\eta(\infty) - \eta(t) = o(t^{-1})$ as $t \to \infty$, show that (5) is the "mean" of $h(\cdot)$, i.e.,

$$\int_0^\infty \psi(u)\, du = \int_0^\infty uh(u)\, du.$$

The integrable function $h(\cdot)$ has a Fourier transform $H(\cdot)$ defined by

$$H(s) = (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{isu} h(u)\, du \quad (s \text{ real})$$

and the absolute convergence of (5) implies that

$$H'(0) = i(2\pi)^{-\frac{1}{2}} \int_0^\infty \psi(u)\, du.$$

The derivative of $H(\cdot)$ at $s = 0$ is closely related to the phase characteristic of the network whose impulse response is $h(\cdot)$. For upon representing $H(\cdot)$ as

$$H(s) = A(s) e^{i\varphi(s)}$$

we find

$$H'(s) = A'(s) e^{i\varphi(s)} + i\varphi'(s) H(s).$$

The amplitude characteristic $A(\cdot)$ is a real, even function, so $A'(0) = 0$. The phase characteristic is representable as an arctangent, so $\varphi'(0) \geqq 0$ in general. Since $A(0) = \eta(\infty)(2\pi)^{-\frac{1}{2}} = (\text{dc-gain})(2\pi)^{-\frac{1}{2}}$, we have

$$H'(0) = \frac{i}{(2\pi)^{\frac{1}{2}}} \times \text{dc-gain} \times \text{coefficient of } s \text{ in Taylor's expansion of } \varphi(\cdot) \text{ around zero}$$

$$= i(2\pi)^{-\frac{1}{2}} \int_0^\infty \psi(u)\, du.$$

We shall call $\varphi'(0)$, the derivative of the phase characteristic of $H(\cdot)$ at the origin, the delay of the network, so that

$$\int_0^\infty \psi(u)\, du = (\text{dc gain})(\text{delay}).$$

A condition on the function $\psi(\cdot)$, stronger than integrability, will be used. This is that

$$h_j = \sup_{0 \leqq t \leqq T} \int_{t+jT}^{t+(j+1)T} |\psi(u)|\, du \quad (j \geqq 0) \tag{6}$$

be an absolutely summable sequence. The magnitude of $\psi(\cdot)$ measures to a certain extent the speed of the response of the network whose impulse response is $h(\cdot)$; our condition requires that this speed be sufficient to make $\sum_{j \geqq 0} h_j$ finite.

We shall assume that the non-linear function $f(\cdot)$ is Lipschitz of order 1 with a constant β,

$$| f(x) - f(y) | \leqq \beta | x - y | .$$

To obtain an estimate of the rate of convergence of a solution $x(\cdot)$ of (1) to a periodic function, and to verify the compactness condition needed for use of Tychonov's theorem, we shall assume that a positive, absolutely summable sequence $\{k_j , j \geqq 0\}$ exists, satisfying the integral inequality

$$\alpha[T \sum_{i \geqq j} k_i + \sup_u | f(u) | \sum_{i>j} h_i + \sum_{i=0}^{j} k_{j-i}h_i] \leqq \beta^{-1}k_j \quad (j \geqq 0). \tag{7}$$

Under these hypotheses we shall prove that for each n, there exist a value of ω, a corresponding solution $x(\cdot)$ of (1), and a function $g(\cdot)$ that is ultimately periodic of period T, such that

$$f(x(t)) = g(t)$$

$$| f(x(jT + t)) - p_g(t) | \leqq k_j \quad (t \in [0,T]) \quad (j \geqq 0)$$

$$\omega = \omega_g = \frac{2n\pi}{T} + \frac{\alpha}{T} \int_0^T p_g(u) \, du .$$

In a corollary we give a condition under which the constants ω for various n are all distinct and lie roughly on a lattice.

IV. TOPOLOGY

In the linear space $B \frown C$ of bounded continuous functions we introduce a topology by means of the metric (distance function):

$$d(g_1 , g_2) = \sum_{n=1}^{\infty} 2^{-n} \max_{-n \leqq x \leqq n} | g_1(x) - g_2(x) |$$
$$+ \sup_{0 \leqq t \leqq T} \limsup_{n \to \infty} | \{g_1(nT + t) - g_2(nT + t)\} | .$$

The sum term defines a metric for the topology of uniform convergence on compact sets, and the other term (so to speak) "strengthens" the topology at infinity. The number $T > 0$ occurring in the metric is a parameter, the period of interest. The d-topology so defined is convenient for studying solutions of (1) that are u.p. $[T]$.

Since the metric $d(\cdot,\cdot)$ depends only on the difference function $g_1 - g_2$, it is invariant under translation to zero

$$d(g_1, g_2) = d(g_1 - g_2, 0).$$

Also it can be verified that for $a > 0$,

$$d(ag_1, ag_2) = ad(g_1, g_2).$$

Let $S_\epsilon(g)$ denote an open sphere of radius ϵ about an element g in $B \frown C$,

$$S_\epsilon(g) = \{g_1 \mid d(g_1, g) < \epsilon\}.$$

Let g_1 and g_2 be elements of $S_\epsilon(g)$ and consider a convex combination

$$a_1 g_1 + a_2 g_2 \quad (a_1, a_2 \geqq 0, \qquad a_1 + a_2 = 1).$$

Then

$$\begin{aligned} d(a_1g_1 + a_2g_2, g) &= d(a_1g_1 - a_1g + a_2g_2 - a_2g, 0) \\ &\leqq d(a_1g_1 - a_1g, 0) + d(a_2g_2 - a_2g, 0) \\ &\leqq a_1d(g_1, g) + a_2d(g_2, g) \\ &< \epsilon. \end{aligned}$$

Hence $S_\epsilon(g)$ is convex. The family of such spheres is a base for the d-topology consisting entirely of convex sets. Hence with the d-topology. $B \frown C$ is a locally convex, linear, topological space (Cf. Ref. 5).

V. PRELIMINARY RESULTS

We define a modulus $m(\cdot)$ of continuity by the equation

$$m(|\epsilon|) = \beta |\epsilon| \left\{\frac{2n\pi}{T} + 2\alpha \sup_u |f(u)| \sup_u |\eta(u)|\right\} + \beta \sup_{|\delta| \leqq |\epsilon|} \int_0^\infty |\psi(t + \delta) - \psi(t)|\, dt.$$

Lemma 1: If $g(\cdot)$ is u.p. $[T]$, and

$$\sup_u |g(u)| \leqq \sup_u |f(u)|$$

then $Ag(\cdot)$ has modulus of continuity $m(\cdot)$.

Proof: The mean value (first Fourier coefficient) of the limit function $p_g(\cdot)$ is at most $\sup_u |f(u)|$ in magnitude. Hence

$$\begin{aligned} |Ag(t+\epsilon) - Ag(t)| &\leqq \beta|\epsilon|\omega_g + \beta\left|\int_t^{t+\epsilon} \eta(t+\epsilon-u)g(u)\,du\right| \\ &\quad + \beta\left|\int_0^t [\eta(t+\epsilon-u) - \eta(t-u)]g(u)\,du\right| \\ &\leqq \beta|\epsilon|\left\{\frac{2n\pi}{T} + 2\alpha \sup_u |f(u)| \sup_u |\eta(u)|\right\} \\ &\quad + \beta\left|\int_0^t [\psi(t+\epsilon-u) - \psi(t-u)]g(u)\,du\right| \\ &\leqq m(|\epsilon|) \end{aligned}$$

by a known result of Lebesgue (Ref. 6, p. 14).

Let S be the subset of $B \frown C$ consisting of the functions $g(\cdot)$ with the properties

(*i*) There is a continuous function $p_g(\cdot)$ of period T such that

$$|g(jT+t) - p_g(t)| \leqq k_j \quad (t \,\varepsilon\, [0,T] \qquad (j \geqq 0).$$

(*ii*) $\sup_u |g(u)| \leqq \sup_u |f(u)|$

(*iii*) $g(\cdot)$ has modulus of continuity $m(\cdot)$.

Lemma 2: S is compact.

Proof: To show S is closed, let $\{x_m\} \subset S$ be a sequence converging to x. The second term of the $d(\cdot,\cdot)$ metric ensures that $p_{x_m}(t)$ converges as $m \to \infty$, uniformly for $t \,\varepsilon\, [0,T]$. Denote the continuous limit function by $p_x(\cdot)$. Then

$$\begin{aligned} |x(jT+t) - p_x(t)| &\leqq |x(jT+t) - x_m(jT+t)| \\ &\quad + |x_m(jT+t) - p_{x_m}(t)| \\ &\quad + |p_{x_m}(t) - p_x(t)|. \end{aligned}$$

With j and $t \,\varepsilon\, [0,T]$ fixed, let $m \to \infty$; the first and third terms on the right go to zero; the second is at most k_j, for all m. Hence

$$|x(jT+t) - p_x(t)| \leqq k_j \quad (t \,\varepsilon\, [0,T]) \quad (j \geqq 0).$$

Also

$$\begin{aligned} |x(t+\epsilon) - x(t)| &\leqq |x(t+\epsilon) - x_m(t+\epsilon)| \\ &\quad + |x(t) - x_m(t)| \\ &\quad + |x_m(t+\epsilon) - x_m(t)|. \end{aligned}$$

Letting $m \to \infty$ for fixed t, the first two terms on the right vanish; the last is at most $m(\epsilon)$ for all $t \geqq 0$. Thus $x(\cdot)$ has modulus of continuity $m(\cdot)$, and so belongs to S. Hence S is closed.

Also, for $y \varepsilon S$ and $t \varepsilon [0,T]$,

$$\begin{aligned} | p_y(t + \epsilon) - p_y(t) | \leqq & | p_y(t + \epsilon) - y(jT + t + \epsilon) | \\ & + | p_y(t) - y(jT + t) | \\ & + | y(jT + t + \epsilon) - y(jT + t) |. \end{aligned}$$

With t and ϵ fixed, let $j \to \infty$; the first two terms on the right vanish, and the last is at most $m(\epsilon)$ for all $j \geqq 0$. Hence, for $y \varepsilon S$, the limiting period function has modulus of continuity $m(\cdot)$.

Now let x_m be an arbitrary sequence of S. From the associated sequence p_{x_m} of periodic functions we can pick a subsequence converging uniformly on $[0,T]$ to a function $p(\cdot)$. From the x_m's associated with this subsequence we can pick, by a standard diagonal argument, a further subsequence $x_{k(i)}$ $i = 1, 2, \cdots$ such that for some $x \varepsilon S$

$$x_{k(i)} \to x \text{ uniformly on any compact set}$$

$$p_{x_{k(i)}} \to p \text{ uniformly on } [0,T].$$

Then $x_{k(i)}$ converges to x in the d-topology, so that S, being closed and sequentially compact, is compact.

Lemma 3: $AS \subset S$

Proof: In view of Lemma 1 and the form of A, it sufficies to show that A preserves the defining property (i) of the set S. Accordingly, let $g(\cdot) \varepsilon S$, and define the periodic function $p_{Ag}(\cdot)$ of period T by

$$\begin{aligned} p_{Ag}(t) = f\bigg(x(0) + \frac{2\pi n}{T} t + \alpha \int_0^\infty [p_g(u) - g(u)]\, du \\ + \alpha \int_0^t [Mp_g - p_g(u)]\, du + \int_0^\infty p_g(t - u)\psi(u)\, du\bigg) \end{aligned}$$

where Mp_g is the mean value of $p_g(\cdot)$. We shall show that

$$| Ag(jT + t) - p_{Ag}(t) | \leqq k_j \quad (t \varepsilon [0,T]) \quad (j \geqq 0).$$

Let us rewrite

$$\omega_g t - \alpha \int_0^t \eta(t - u) g(u)\, du$$

in the form, for $t \geqq 0$,

$$\begin{aligned} \frac{2n\pi}{T} t + \alpha \int_0^t \psi(t-u) p_g(u)\, du + \alpha \int_0^t \psi(t-u)[g(u) - p_g(u)]\, du \\ + \alpha \int_0^t [p_g(u) - g(u)]\, du + \alpha \int_0^t [Mp_g - p_g(u)]\, du. \end{aligned} \tag{8}$$

From the Lipschitz condition satisfied by $f(\cdot)$, we obtain

$$\begin{aligned} | Ag(jT+t) - p_{Ag}(t)| \leqq \beta \Big\{ \int_{jT+t}^{\infty} | p_g(u) - g(u)|\, du + \alpha \\ \int_{jT+t}^{\infty} | p_g(t-u)| \cdot | \psi(u)|\, du \\ + \alpha \int_0^{jT+t} | g(jT+t-u) - p_g(t-u)| \cdot | \psi(u)|\, du \Big\}. \end{aligned}$$

Since $g(\cdot)$ belongs to S, it is true that for $t \,\varepsilon\, [0,T]$

$$\int_{jT+t}^{\infty} | p_g(u) - g(u)|\, du \leqq T \sum_{i \geqq j} k_i ,$$

$$\int_{jT+t}^{\infty} | p_g(t-u)| \cdot | \psi(u)|\, du \leqq \sup_u | f(u)| \cdot \sum_{i \geqq j} h_i ,$$

$$\begin{aligned} \int_0^{jT+t} | g(jT+t-u) - p_g(t-u)| \cdot | \psi(u)|\, du \\ \leqq \int_0^t | g(jT+t-u) - p_g(t-u)| \cdot | \psi(u)|\, du \\ + \sum_{i=1}^{j-1} \int_{iT+t}^{(i+1)T+t} | g(jT+t-u) - p_g(t-u)| \cdot | \psi(u)|\, du \\ \leqq \sum_{i=0}^{j} k_{j-i} h_i . \end{aligned}$$

Lemma 3 now follows from the integral inequality (7).

VI. PRINCIPAL RESULTS

Theorem 1: If $\sum_{i \geqq 0} h_i < \infty$ and $\sum_{i \geqq 0} k_i < \infty$, where $\{h_i\}$ is given by (6) and $\{k_i\}$ satisfies (7), then for each integer n there exist a value of ω, a (corresponding) solution $x(\cdot)$ of (1), and a function $g(\cdot)$ that is u.p. $[T]$, such that

i. $Ag = g$, i.e., $g(\cdot)$ satisfies (4),
ii. $g(t) = f(x(t))$, for all t,
iii. $|f(x(jT + t)) - p_g(t)| \leqq k_j \quad (t \in [0,T]) \quad (j \geqq 0)$,
iv. the periodic function $p_g(\cdot)$ is a solution on $[0,T]$ of the equation

$$p_g(t) = f\left(x(0) + \frac{2\pi n}{T} t + \alpha \int_0^\infty [p_g(u) - g(u)]\, du + \alpha \int_0^t [Mp_g - p_g(u)]\, du + \alpha \int_0^\infty p_g(t-u)\psi(u)\, du\right),$$

v.

$$\omega = \frac{2\pi n}{T} + \frac{\alpha}{T}\int_0^T p_g(u)\, du = \frac{2\pi n}{T} + \alpha M p_g .$$

Proof: We first show that A is a continuous transformation on the set S. Let $g_m \to g$ with $g_m \in S$; then $g \in S$, because S is closed. Let L be a compact set of the line and set $z = \sup_{t \in L} |t|$. For $t \in L$ we have

$$|Ag_m(t) - Ag(t)| \leqq \beta \left| t(\omega_{g_m} - \omega_g) + \alpha \int_0^t \eta(t-u)[g(u) - g_m(u)]\, du \right|$$
$$\leqq \alpha\beta\left(z\,|Mp_{g_m} - Mp_g| + \sup_u |\eta(u)| \cdot \int_0^z |g(u) - g_m(u)|\, du\right).$$

Since $p_{g_m} \to p_g$ uniformly, and $g_m \to g$ uniformly on compact sets, it follows that $Ag_m \to Ag$ uniformly on compact sets. It remains to show that $p_{Ag_m} \to p_{Ag}$ uniformly. Now for $t \in [0,T]$,

$$|p_{Ag_m}(t) - p_{Ag}(t)| \leqq \alpha\beta \left| (Mp_{g_m} - Mp_g)t + \int_0^\infty [p_{g_m}(u) - g_m(u)] - \int_0^\infty [p_g(u) - g(u)]\, du + \int_0^\infty [p_{g_m}(t-u) - p_g(t-u)]\psi(u)\, du \right|. \tag{9}$$

The first and fourth terms on the right of (9) obviously converge to zero uniformly in t, by the uniform convergence of p_{g_m} to p_g. The middle two terms on the right are together at most

$$\int_0^{jT} |p_{g_m}(u) - p_g(u)|\, du + \int_0^{jT} |g(u) - g_m(u)|\, du + 2T \sum_{i \geqq j} k_i$$

in magnitude; this bound can be made arbitrarily small by first choosing j large, and then m large.

The set S is compact and convex, and A is a continuous map of S into itself. Hence by Tychonov's fixed point theorem (Ref. 5, p. 456) there is a fixed point $g \varepsilon S$ with $Ag = g$. Define $x(\cdot)$ by

$$x(t) = \begin{cases} x(0) + \omega_g t - \alpha \int_0^t \eta(t-u)g(u)\,du & (t \geqq 0) \\ x(0) & (t \leqq 0). \end{cases}$$

Then $x(\cdot)$ satisfies (1) almost everywhere in $t \leqq 0$, and $g(t) = f(x(t))$. Since $g \varepsilon S$, there is a function $p_g(\cdot)$ of period T such that

$$| g(jT + t) - p_g(t) | \leqq k_j \quad (t \varepsilon [0,T]) \qquad (j \geqq 0).$$

The limit equation for $p_g(\cdot)$ is obtained by taking the limit as $j \to \infty$ in the equation

$$g(jT + t) = Ag(jT + t) \quad (t \varepsilon [0,T])$$

and using (8) to expand the right-hand side. The second term of (8) approaches

$$\int_0^\infty p_g(t-u)\psi(u)\,du$$

while the third term of (8) goes to zero, by an elementary Abelian result. Finally the value of ω associated with the solution $x(\cdot)$ of (1) is given by (v) because $f[x(\cdot)]$ is a fixed point of A.

Corollary: If $h(\cdot)$ is bounded, and

$$\frac{\pi}{T} > \alpha \sup_u | f(u) | = \gamma$$

then each interval $(2n\pi/T) \pm \gamma$, n an integer, contains a value of ω for which the (unique) solution of (1) is u.p. $[T]$.

Proof: The condition stated guarantees that the values of (v), Theorem 1, for various integers n, are all distinct. Uniqueness of the solution $x(\cdot)$ of (1) can be proved from the boundedness of $h(\cdot)$ and the Lipschitz condition on $f(\cdot)$ by standard methods.

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