Discrimination Against Unwanted Orders in the Fabry-Perot Resonator

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It is proposed here that the usual Fabry-Perot interferometer structure of the optical maser may be modified in a very simple way to provide discrimination against unwanted orders. The modification is an extra reflecting surface suitably positioned outside the maser which can greatly affect the losses of the various orders. A simple one-dimensional analysis is given for the effect, and numerical results are presented for a realistic case, showing that the effect can be large. It is concluded that this technique may be useful in preventing unwanted oscillations in the optical maser.

I. INTRODUCTION

The Fabry-Perot interferometer has recently become important as a resonant cavity for electromagnetic radiation at optical frequencies.[1,2,3,4] The nature of the modes of such a cavity has been discussed by Schawlow and Townes[1] and by Fox and Li.[5] The modes may be specified by three quantum numbers, one of which is the familiar *order number* giving the separation of the plates in units of the half-wavelength. The other two quantum numbers specify the possible field configurations across the plates, which are essentially identical in each order. Fox and Li have investigated these configurations and the corresponding frequencies and losses for interferometers consisting of perfectly reflecting plates in air. In the usual laboratory interferometer the Fox and Li modes cannot be resolved because of insufficient reflectivity of the plates. Therefore the role played by these modes in optical masers is not settled. On the other hand, fine structure which could be due to various *Fabry-Perot orders* has been seen in the output of both the gas[6] and the ruby[7] optical maser. It has been pointed out[1,8] that the optical maser is inherently a multimode device, and that the excitation of many modes can lead to undesirable effects in the noise, stability, and ultimate usefulness of the device. Therefore it is proposed here that it would be useful to discriminate against many of the Fabry-Perot orders which can occur in the output by increasing their losses relative to other "preferred" modes.

The Fabry-Perot orders present a problem only when the fluorescence emission of the maser covers a frequency band wider than $(2\mu d)^{-1}$ wave numbers, where μ is the refractive index and d the separation of the plates. This is the case in the gas maser of Javan, Bennett, and Herriott[3] where $(2\mu d)^{-1} \sim 0.005$ cm^{-1} and the doppler broadened Ne transition would be expected to have a width ~0.05. Also, in the ruby optical maser of Collins et al[2], $(2\mu d)^{-1} \sim 0.1$ while the fluoresence line width at room temperature ~10. The orders cannot be eliminated in these cases by shortening the maser and hence spreading the orders, because the gain would then be insufficient to produce oscillations.[9] In ruby, however, the gain could be increased[9] by more than an order of magnitude by cooling, so that the crystal could be shortened. At the same time, the cooling could decrease the line width by more than an order of magnitude,[10] so that elimination of orders appears possible in ruby optical masers by cooling. These examples show the interrelation of gain, line width, and the occurrence of Fabry-Perot orders in the optical maser output.

The idea of using a Fabry-Perot interferometer to discriminate against unwanted orders in the optical maser has occurred to a number of people.[11] Indeed, if the external beam contains several orders, a Fabry-Perot etalon could be constructed which would transmit only one of them. This, of course, would not necessarily have any effect on the losses of the various modes in the maser. If the etalon were put in the internal beam, elementary considerations do not tell us what to expect for the relative losses of the modes. The structure to be proposed in the next section is equivalent to making the etalon one of the reflecting ends of the maser. It is believed that a detailed discussion of how discrimination comes about in such structures is given here for the first time.

II. A MODIFIED INTERFEROMETER

It is proposed that another reflecting plate parallel to the maser plates be provided outside the maser with a means for adjusting the separation of the new plate from the maser. This would produce a modified interferometer having three essential optical surfaces with the active medium in the space between two of these surfaces. It is expected that the separation of the third surface from the maser will be much less than the length of the maser. The purpose of the extra surface is to provide discrimination between the Fabry-Perot orders of the original maser by making some orders very lossy compared to other orders. The losses may be due to scattering by inhomogeneities in the medium and irregularities on the reflecting surfaces, absorption by processes other than the fluorescence process of the active medium, and transmission through the outer reflecting surfaces. For convenience of discussion, all losses may be ascribed to the last mechanism by assigning suitable effective reflectivities to the outer surfaces. In any case, it is clear that the proposal has meaning only when losses are taken into account, since the only other effect of the extra surface would be to shift the frequencies of the already existing orders by amounts less than $(2\mu d)^{-1}$ and to introduce new frequencies corresponding to the increased over-all length of the modified interferometer. Therefore the performance of the device cannot be deduced in an elementary way by considering the two regions between the surfaces as two interferometers with the shorter preferentially selecting and rejecting certain orders of the longer. The truth is that the modified interferometer has more, not fewer, orders than the original maser, but unlike the latter the orders may have very different losses.

III. ANALYSIS

For analysis it is convenient to consider the one dimensional problem shown schematically in Fig. 1. A medium of real dielectric constant $\epsilon > 1$ and real conductivity σ occupies the region $-a \leqq z \leqq a$. For

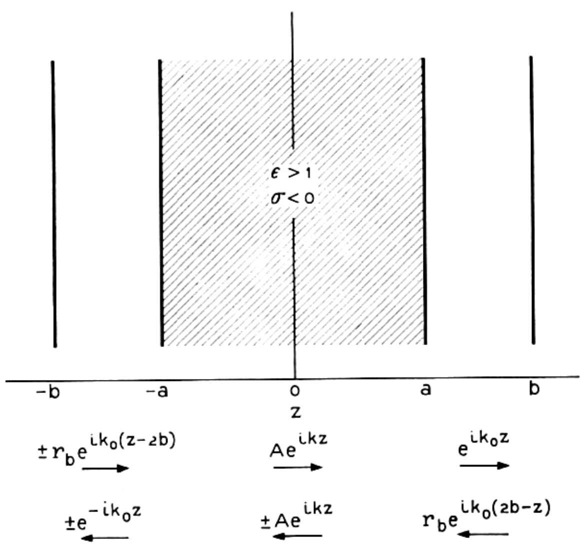

Fig. 1 — Schematic diagram of one-dimensional symmetric structure chosen for analysis of modified interferometer. The value of the constant A is not needed in the analysis.

$|z| > a$ it is assumed that $\epsilon = 1$ and $\sigma = 0$. At $z = \pm b$ are placed reflecting surfaces having the reflectivity for amplitude

$$r_b = e^{-2f}. \tag{1}$$

Since the phase angle of reflection is unimportant here, it has been assumed zero. For later use the quantity

$$T = \tanh f \tag{2}$$

will now be defined. The reflectivity of the surfaces at $z = \pm a$ is

$$r_a = (\sqrt{\epsilon} - 1)/(\sqrt{\epsilon} + 1). \tag{3}$$

From (1), (2) and (3) one can write

$$\begin{aligned} T &= (1 - r_b)/(1 + r_b) \\ 1/\sqrt{\epsilon} &= (1 - r_a)/(1 + r_a). \end{aligned} \tag{4}$$

It is therefore possible in this example to consider arbitrary reflectivities at $z = \pm a, \pm b$ by suitable choices for T and $1\sqrt{\epsilon}$ in the range 0 to 1.

The symmetry of Fig. 1 about a plane at $z = 0$ causes the field to be either even or odd with respect to reflection in this plane. The even solutions are shown by (+) and the odd solutions by (−) signs in Fig. 1. The propagation constants are given by

$$\begin{aligned} k_o &= \omega/c \\ k &= k_o\sqrt{\epsilon}[1 + i(4\pi\sigma/\epsilon\omega)]^{\frac{1}{2}} \\ &= k_o\sqrt{\epsilon} + i(2\pi\sigma/c\sqrt{\epsilon}) + \cdots . \end{aligned} \tag{5}$$

The continuity of the field and its derivative at $z = a$ gives the conditions

$$k \tan(ka) = -k_0 \tan(k_0 b - k_0 a + if) \tag{6}$$

for even modes, and

$$k_0 \tan(ka) = +k \cot(k_0 b - k_0 a + if) \tag{7}$$

for odd modes. These equations give, in general, complex eigenvalues for the angular frequency ω.

It is convenient to require that ω be real and allow σ to assume an appropriate negative value. Both ω and σ are determined by (6) or (7) for even or odd modes respectively. Physically this corresponds to supplying sufficient gain through the negative σ to maintain steady oscillations at frequency ω. The larger the value of $-\sigma$ the greater are the losses of the mode in question. Now let the dimensions be so chosen that

$$n(b - a) = ma\sqrt{\epsilon} \tag{8}$$

where m,n are positive integers. It is then possible to write

$$\begin{aligned} k_0(b - a) &= m\pi + (m/n)\Delta \\ \sqrt{\epsilon} k_0 a &= n\pi + \Delta. \end{aligned} \tag{9}$$

The conductivity to be determined is contained in the parameter

$$\chi = \tanh(2\pi\sigma a/c\sqrt{\epsilon}). \tag{10}$$

In any practical case the ratio k/k_0 occurring in (6) and (7) can be considered real, $k/k_0 \sim \sqrt{\epsilon}$. The equations for the real frequency and conductivity then reduce to

$$\tan\Delta = -\left(\tan\frac{m}{n}\Delta\right)\frac{1 + \sqrt{\epsilon}T\chi}{\sqrt{\epsilon} + T\chi} \tag{11}$$

$$\chi = -T\,\frac{1 - \sqrt{\epsilon}\tan\Delta\tan(m/n)\Delta}{\sqrt{\epsilon} - \tan\Delta\tan(m/n)\Delta} \tag{12}$$

for even modes, and to

$$\tan\Delta = \left(\cot\frac{m}{n}\Delta\right)\frac{\sqrt{\epsilon} + \chi T}{1 + \sqrt{\epsilon}\chi T} \tag{13}$$

$$\chi = -T\,\frac{\sqrt{\epsilon}\tan(m/n)\Delta + \tan\Delta}{\tan(m/n)\Delta + \sqrt{\epsilon}\tan\Delta} \tag{14}$$

for odd modes. When $\tan\Delta$ is eliminated between (11) and (12) or between (13) and (14) the same quadratic equation for χ is obtained, namely

$$\chi^2 + 2p\chi + 1 = 0 \tag{15}$$

where

$$2p = \frac{\epsilon + T^2 + (1 + \epsilon T^2)\tan^2(m/n)\Delta}{T\sqrt{\epsilon}(1 + \tan^2(m/n)\Delta)}. \tag{16}$$

The solution of (15) which reduces properly as $r_b \to 0$ ($T \to 1$) is

$$\chi = -p + |(p^2 - 1)^{\frac{1}{2}}|. \tag{17}$$

When $|\chi| \ll 1$, this reduces to

$$-\chi \sim \frac{T\sqrt{\epsilon}(1 + \tan^2(m/n)\Delta)}{\epsilon + \tan^2(m/n)\Delta}. \tag{18}$$

The most practical method of solution is to find the frequencies by neglecting $T\chi$ in (11) and (13), which gives

$$\sqrt{\epsilon} \tan \Delta = -\tan(m/n)\Delta \qquad \text{(even)} \tag{19}$$

$$\tan \Delta = \sqrt{\epsilon} \cot(m/n)\Delta \quad \text{(odd)} \tag{20}$$

respectively. From these solutions, the values of $\tan(m/n)\Delta$ can be substituted into (17) or (18) to obtain χ.

From (15) and (16) it is seen that χ depends upon Δ through $\tan^2(m/n)\Delta$. As a result of the "tuning" condition (8), $\Delta = 0$ is a solution of (19); this is the "preferred" mode having the lowest loss

$$\chi_{\min} = -T/\sqrt{\epsilon}. \tag{21}$$

The largest losses belong to modes having $\tan^2(n/n)\Delta \gg 1$. The solution (17) gives two results in the limit $\tan^2(m/n)\Delta \to \infty$, depending on whether $\epsilon T^2 < 1$ or > 1

$$\begin{aligned} \chi_{\max} &= -T\sqrt{\epsilon} \qquad (\epsilon T^2 < 1) \\ \chi_{\max} &= -1/(T\sqrt{\epsilon}) \quad (\epsilon T^2 > 1). \end{aligned} \tag{22}$$

Let the quantity

$$R = \chi/\chi_{\min} \tag{23}$$

be called the *discrimination ratio*; then $R_{\max} = \epsilon$ or $1/T^2$, whichever is smaller. Therefore the extra reflecting surface should satisfy

$$r_b \geqq r_a \tag{24}$$

to achieve the maximum discrimination, but there is no advantage in making r_b exceed r_a. It should be noted that the practical approximations (19) and (20) are not valid if $\epsilon T^2 > 1$.

IV. DISCUSSION

The properties of the solutions are best discussed with the aid of an example. For simplicity, a case is chosen in which (19), (20) are valid. Let

$$\begin{aligned} m/n &= 1/5 \\ \sqrt{\epsilon} &= 10 \\ T &= 0.02. \end{aligned} \tag{25}$$

The corresponding reflectivities for amplitude are $r_a = 0.82$, $r_b = 0.96$. According to (21) the loss of the preferred mode $\Delta = 0$ is measured by

$$\chi(0) = \chi_{\min} = -0.002. \tag{26}$$

From (19) it is seen that $\Delta = 5\pi/2$ is a solution with $\tan^2(m/n)\Delta = \infty$ so that according to (22)

$$\chi(5\pi/2) = \chi_{\max} = -0.2. \tag{27}$$

The graphical solution of (19) and (20) is sketched in Fig. 2 with circles representing even solutions and squares odd solutions. The results are summarized in Table I up to $\Delta = 5\pi/2 = 450°$; the remaining roots in the fundamental period of 5π may be obtained from the symmetry about $\Delta = 5\pi/2$. The roots are alternately even and odd as shown in the second column, and $\tan(m/n)\Delta$ in the fourth column rises monotonically from 0 to ∞ corresponding to increasing losses. The discrimination ratio R,

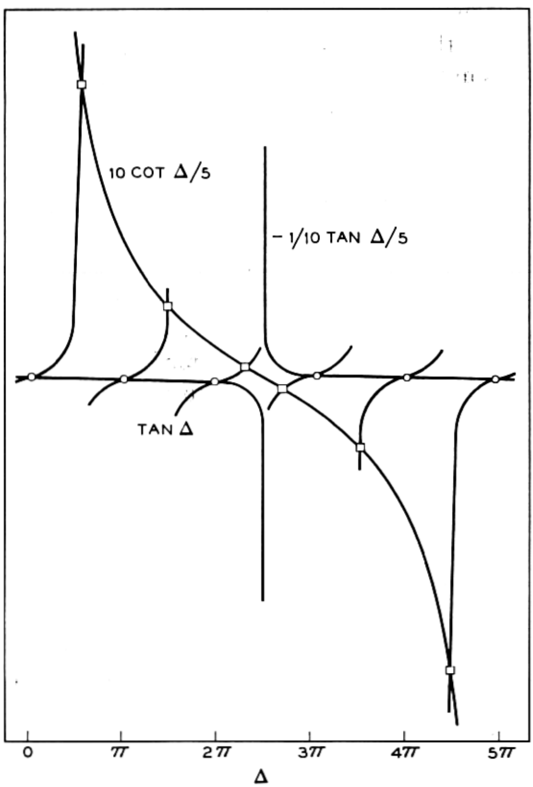

Fig. 2 — Graphical representations of (19) and (20) for $m/n = 1/5$, $\sqrt{\epsilon} = 10$. Odd solutions are indicated by squares and even solutions by circles except at $\Delta = 5\pi/2$, where the intersection is at $\pm\infty$.

Table I—Summary of Results for Numerical Example with $m/n = 1/5$, $\sqrt{\epsilon} = 10$

Δ	type	$\tan \Delta$	$\tan \Delta/5$	R
0	even	0	0	1
88°10′	odd	+31.46	+0.318	1.1
175°58′	even	−0.0705	+0.705	1.49
262°34′	odd	+7.67	+1.304	2.66
345°21′	even	−0.261	+2.61	7.27
412°38′	odd	+1.31	+7.63	37.4
450°	even	∞	∞	100

given in the fifth column, increases from 1 to 100. These results are further summarized in Fig. 3, where the spectrum just calculated is compared with that of the "original" interferometer having no surfaces at $z = \pm b$. The loss in the original interferometer is $\chi = -1/\sqrt{\epsilon} = -0.1$ for all modes. The heights of the spectrum lines in Fig. 3 are proportional to $1/R$ to indicate the relative "Q" of each mode. The total number of frequencies in the fundamental period is twelve compared with ten in the original interferometer for the same period. This is exactly what one would expect, corresponding to the 20 per cent increase in optical length of the modified interferometer. Also as one would expect, the spacing of the preferred modes corresponds to the orders of an ordinary Fabry-Perot interferometer of spacing $d = b - a$.

It will be seen in Fig. 3 that the three modes on either side of a preferred mode have frequencies very close to modes of the original interferometer at $\Delta = \pm\pi/2, \pm\pi, \pm 3\pi/2$. The losses of these modes can be

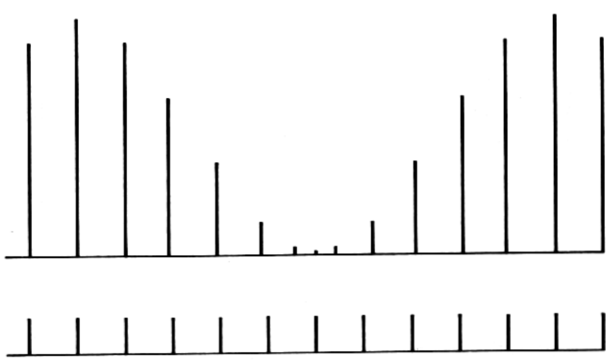

Fig. 3 — The calculated spectrum with the "Q" of each mode indicated by the height of the lines. Shown below for comparison is the spectrum of the "original" interferometer.

calculated to a good approximation from these values of Δ. In general the approximation is

$$\tan(m/n)\Delta \sim \tan[(m/n)N\pi/2] \tag{28}$$

where $N = 0,1,2, \cdots$ but $N \ll n/m$. Using (28) the evaluation of (17) or (18) can then be carried out immediately without solving for all of the frequencies. This is very convenient since only the modes near the preferred mode are expected to be of interest. It will be noted that the extra modes introduced by the extra surface are among the lossy modes. The periodicity in the above example is a result of choosing n/m an integer. If n/m is chosen not an integer, the periodicity is destroyed, but $\Delta = 0$ remains a preferred mode with minimum loss. Except for extra modes in the regions of high loss, the general effect of the extra surface is to impose a modulation of period $(n/m)\pi$ on the original modes. It is of course not essential for the desired effect that this modulation have a period commensurate with the period of the orders of the original interferometer. Greatest advantage in discrimination against unwanted Fabry-Perot orders is obtained by setting

$$b - a \sim (2\Delta\nu)^{-1} \tag{29}$$

where $\Delta\nu$ is the half-width at half-maximum of the fluorescence emission.

V. SUMMARY

The theory of the orders of the modified interferometer has been treated in one dimension by considering the symmetrical structure of Fig. 1. The analysis clearly shows the nature and magnitude of the effects to be expected. These effects do not depend in any essential way upon the symmetry assumed for convenience in the analysis, and similar results would be expected for an unsymmetrical modified interferometer with only one extra reflecting surface. It is clear that details in the analysis could be generalized in various ways without changing the substance of the conclusions. The most important of these would be to allow arbitrary reflection and absorption at the interfaces at $\pm a$ to represent the properties of deposited metal layers. On the basis of what has been presented, however, it can be asserted that a third surface of suitable reflectivity and properly positioned can provide considerable discrimination between the orders of a Fabry-Perot interferometer.

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