# The One-Sided Barrier Problem for Gaussian Noise

By DAVID SLEPIAN

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*This paper is concerned with the probability, $P[T,r(\tau)]$, that a stationary Gaussian process with mean zero and covariance function $r(\tau)$ be nonnegative throughout a given interval of duration $T$. Several strict upper and lower bounds for $P$ are given, along with some comparison theorems that relate $P$'s for different covariance functions. Similar results are given for $F[T,r(\tau)]$, the probability distribution for the interval between two successive zeros of the process.*

## INTRODUCTION

Let $X(t)$ be a real continuous parameter Gaussian process, stationary and continuous in the mean. We shall assume throughout that $EX(t) = 0$ and shall write $r(\tau) = EX(t)X(t + \tau)$. We further assume throughout that we are dealing with a separable, measurable version of the process.

Our main concern in this paper is the probability $P[T,r(\tau)]$ that $X(t)$ be nonnegative for $0 \leqq t \leqq T$. This quantity is of interest as a means of describing the duration of the excursions taken by the process from its mean. From $P[T,r(\tau)]$, the distribution function $F[\lambda,r(\tau)]$ of the interval between successive zeros of the process can be determined by differentiation [see (19)]. This latter quantity is of importance in a variety of engineering applications of noise theory.

Considerable effort has been directed in the past toward the numerical determination of $F[\lambda,r(\tau)]$ both theoretically[1-25] and empirically.[26-32] These researches have resulted in various approximations for $F[\lambda,r(\tau)]$, but many of these are neither upper nor lower bounds for $F$, and exact circumstances under which they are good approximations are not clear. Generally speaking, they are good for small values of $\lambda$ and become nugatory for sufficiently large $\lambda$. There appears to be nothing rigorous in the literature concerning the asymptotic behavior of F for large $\lambda$. (An approximation method is given in Ref. 21.)

In this paper we summarize some known results and present a number of new strict bounds and comparison theorems for $P[T,r(\tau)]$ and $F[\lambda,r(\tau)]$. The most important of these are: Theorem 1, Section 1.3; Theorem 3, Section 1.4; and Theorem 10, Section 1.8. Theorems 12 and 13 (Section 2.7) dealing with class 2 covariances (defined in Section 1.1), though of less importance for our goal, are perhaps of more than passing interest. These and other results presented shed some light on theoretical questions regarding $P$ and $F$. Their utility in numerically determining these quantities will be discussed elsewhere.

The paper is divided into two parts: Part I presents definitions, results, and discussions; Part II contains the more technical aspects of proofs and other supportive material for Part I.

## PART I — DEFINITIONS, RESULTS AND DISCUSSIONS

### 1.1 *Preliminaries*

From its definition, it is clear that $P[T,r(\tau)]$ is a nonincreasing function of T. It assumes the value $\frac{1}{2}$ for $T = 0$. It obeys the scaling laws

$$P[T,\lambda r(\tau)] = P[T,r(\tau)] \tag{1}$$

$$P[T,r(\lambda\tau)] = P[\lambda T,r(\tau)] \tag{2}$$

$$\lambda > 0.$$

In asserting (2) for all $\lambda > 0$ we have assumed $r(\tau)$ given for all $\tau$. This is a convention that will be adhered to throughout this paper. It is to be noted, however, that $P[T,r(\tau)]$ for $0 \leqq T \leqq T_o$ depends only on the "piece" of the covariance function $r(\tau)$, $0 \leqq \tau \leqq T_o$.

The scaling law (1) suggests normalizing the covariances to be considered so that

$$r(0) = 1. \tag{3}$$

We adopt this convention hereafter.

The scaling law (2) suggests that a normalization of the time scale is in order. There does not appear to be a convenient way to do this for the class of all covariances. For processes continuous in the mean, such as are being considered here, all one can say in general about covariances is that they are even continuous nonnegative-definite functions. This is a rather large class of functions containing a great variety of pathologies such as nowhere differentiable continuous functions. In what follows we shall have occasion to consider covariances $r(\tau)$, strictly monotone in some right neighborhood of $\tau = 0$ and such that $r(\tau) - 1$ behaves like a nonnegative power of $|\tau|$ for sufficiently small $|\tau|$. We normalize and define this class as follows: *The continuous covariance $r(\tau)$ is said to be of class $\alpha$ if, as $\tau$ approaches zero,*

$$r(\tau) = 1 - \frac{|\tau|^{\alpha}}{\Gamma(\alpha + 1)} + o(|\tau|^{\alpha}),$$

*and if $r(\tau)$ is strictly monotone in some right neighborhood* $0 < \tau < \tau_0$ *of the origin.* Here necessarily $0 \leqq \alpha \leqq 2$ and $\Gamma(\alpha + 1)$ is the usual gamma function. The normalization is contained in the specific choice of the coefficient of $|\tau|^{\alpha}$.

To the author's knowledge, when the scaling laws (1) and (2) are taken into account, there are only three distinct covariances for which $P[T,r(\tau)]$ is known explicitly. These are:

$$(i) \quad r_1(\tau) = e^{-|\tau|}, \qquad 0 \leqq \tau \leqq \infty,$$

$$P[T,r_1(\tau)] = \frac{2}{\pi} \arcsin e^{-T}, \qquad 0 \leqq T < \infty;$$

$$(ii) \quad r_2(\beta,\tau) = 1 - \beta^2 + \beta^2 \cos\left(\frac{\tau}{\beta}\right), \qquad 0 \leqq \tau < \infty, \qquad 0 \leqq \beta \leqq 1,$$

$$P[T,r_2(\beta,\tau)] = \begin{cases} \dfrac{1}{2} - \dfrac{T}{4\pi} - \dfrac{1}{2\pi} \arcsin\left[\beta \sin\left(\dfrac{T}{2\beta}\right)\right], & 0 \leqq \dfrac{T}{\beta} \leqq 2\pi, \\ \tfrac{1}{2}[1 - \beta], & 2\pi \leqq \dfrac{T}{\beta} < \infty; \end{cases}$$

$$(iii) \quad r_3(\tau) = \begin{cases} 1 - |\tau|, & |\tau| \leqq 1 \\ 0, & |\tau| \geqq 1, \end{cases}$$

$$P[T,r_3(\tau)] = \frac{1}{4} + \frac{1}{2\pi}[\arcsin(1 - T) - \sqrt{T(2 - T)}], \qquad 0 \leqq T \leqq 1.$$

The process with covariance $r_1(\tau)$ is Markovian, and it is this special property that permits determination of $P[T,r_1(\tau)]$ in this case (see Ref. 22 or Ref. 21, Section IX).

Case (*ii*) corresponds to the stochastic process

$$X(t) = A + B \cos\left[\frac{t}{\beta} + \Phi\right],$$

with $A$, $B$ and $\Phi$ independent random variables, the two former being normal with mean zero and variances $1 - \beta^2$ and $\beta^2$ respectively, and the latter being distributed uniformly in $(0,2\pi)$. The determination of $P$ in this case is an exercise in integration and elementary probability theory that will be omitted here. For the obvious generalization of this case, namely,

$$X(t) = A + \sum_1^N B_i \cos[t/\beta_i + \Phi_i],$$

$P[T,r(\tau)]$ can be expressed in principle as a $(2N + 1)$-fold integral. Except in the case $N = 1$ presented, the integrals appear untractable.

The form for $P[T,r_3(\tau)]$ given follows from results found in Ref. 23. Note that it is valid only for $T \leqq 1$. We have been unable to extend $P$ beyond this point.

These examples shed little light on the many questions that naturally arise concerning the behavior of $P[T,r(\tau)]$, both as a function of $T$ and as a functional of $r(\tau)$. What are possible asymptotic behaviors of $P$ for large $T$? What features of $r(\tau)$ determine this behavior? To what extent is $P$ determined by the behavior of $r(\tau)$ in the neighborhood of $\tau = 0$? (For example, if $r(\tau)$ is analytic in the neighborhood of $\tau = 0$, then it can be extended as a covariance in only one way, namely, by its analytic continuation. In this case, then, $P[T,r(\tau)]$ is completely determined by the behavior of $r(\tau)$ near $\tau = 0$.) If $q(\tau)$ is another covariance, in some sense close to $r(\tau)$ for $0 \leqq \tau \leqq T$, is $P[T,r(\tau)]$ close in some sense to $P[T,q(\tau)]$? How can $P[T,r(\tau)]$ be determined numerically for a given covariance $r(\tau)$?

These and many other basic questions await to be answered in full.

### 1.2 *$P[T,r(\tau)]$ as a Limit*

Let $0 = t_1 < t_2 < \cdots < t_n = T$ be a partition of the interval $(0,T)$ into $n - 1$ parts. The $n$ random variables $X(t_1)$, $X(t_2)$, $\cdots$, $X(t_n)$ are jointly Gaussian with covariance matrix $\mathbf{r} = (r_{ij})$, where $r_{ij} = r(t_i - t_j)$. Denote by $P_n(\mathbf{r})$ the probability that these $n$ random variables be nonnegative. Because of the assumed separability of the process,

$$P[T,r(\tau)] = \lim_{n\to\infty} P_n(\mathbf{r}), \tag{4}$$

where it is understood that the limit is taken as the partition is refined with mesh tending to zero. If $r(\tau)$ is positive definite, then $|\mathbf{r}| \neq 0$ for any choice of partition, and one can write explicitly

$$P_n(\mathbf{r}) = (2\pi)^{-n/2} |\mathbf{r}|^{-\frac{1}{2}} \int_0^\infty dx_1 \cdots \int_0^\infty dx_n e^{-\frac{1}{2}\Sigma r_{ij}^{-1} x_i x_j}. \tag{5}$$

It is somewhat surprising that information about $P[T,r(\tau)]$ is so difficult to obtain when it can be expressed as the limit of the apparently not too unwieldy expression on the right of (5). This integral is deceptive. For $n > 3$ it cannot be expressed in terms of elementary functions of the covariance elements $r_{ij}$. Series expansions and upper and lower bounds can be easily written for this integral, but most of the obvious ones yield vacuous results in the limit as the partition is refined.

The integral (5) admits of a simple geometric interpretation obtained by reducing the quadratic form in the exponent to a sum of squares by a linear transformation and performing a radial integration. $P_n(\mathbf{r})$ is the fraction of the unit sphere in Euclidean $n$-space cut out by $n$-hyperplanes through the center of the sphere. The angle $\theta_{ij}$ between the normals to the $i$th and $j$th hyperplanes directed into the cutout region is given by $\cos\theta_{ij} = r_{ij}$, $i,j = 1,2, \cdots, n$. This geometric interpretation of $P_n(\mathbf{r})$ holds even when $|\mathbf{r}| = 0$. For $n = 2$ and 3, this picture gives at once

$$P_2 = \frac{1}{2\pi}[\pi - \theta_{12}] = \frac{1}{4} + \frac{1}{2\pi}\arcsin r_{12} \tag{6}$$

$$\begin{aligned} P_3 &= \frac{1}{4\pi}[2\pi - \theta_{12} - \theta_{13} - \theta_{23}] \\ &= \frac{1}{8} + \frac{1}{4\pi}[\arcsin r_{12} + \arcsin r_{13} + \arcsin r_{23}]. \end{aligned} \tag{7}$$

Seen on the surface of the sphere, the region described above is the generalization of the spherical triangle in three-space and is known as an $n$-dimensional spherical simplex. Geometers have studied the problem of expressing the content of the spherical simplex in terms of the angles between its bounding surfaces.[33-38] Many of their results can be readily derived from known results in probability theory using the connection with $P_n(\mathbf{r})$ just mentioned (see Section 2.1).

It is clear that $P_n(\mathbf{r})$ is an upper bound for $P[T,r(\tau)]$. The result (7) then is a simple upper bound for $P[T,r(\tau)]$, where $r_{12} = r(t_2 - t_1)$, $r_{13} = r(t_3 - t_1)$, $r_{23} = r(t_3 - t_2)$ and $t_1$, $t_2$, $t_3$ are any three points in the interval $(0,T)$. For very small values of $T$, this upper bound can be made close to the true value of $P[T,r(\tau)]$. For large values of $T$, this is generally not the case. If, for example, $r(\tau)$ is never negative, $P_3$ is always greater than $\frac{1}{8}$. If $r(\tau)$ oscillates in sign, there is a minimum value for $P_3$ different from zero (unless $r(\tau)$ achieves the value $-1$) obtainable for any choice of $t_1 \leqq t_2 \leqq t_3$, and hence this bound for $P[T,r(\tau)]$ does not approach zero for large $T$.

### 1.3 *A Comparison Theorem for* $P[T,r(\tau)]$

Recall that in the geometric picture of $P_n(\mathbf{r})$, $r_{ij} = \cos\theta_{ij}$ where $\theta_{ij}$ is the angle between the inward normals to hyperplanes $i$ and $j$. Intuitively, it is clear that if this angle is decreased, i.e., if $r_{ij}$ is increased, $P_n(\mathbf{r})$ should also increase. This is borne out by the following

*Lemma 1*† — *Let* $P_n(\mathbf{r})$ *be the probability that* $n$ *jointly Gaussian variates with mean zero and normalized covariance matrix* $\mathbf{r}(r_{ii} = 1)$ *be nonnegative. Let* $q$ *be another normalized* $n \times n$ *covariance matrix. If* $r_{ij} \geqq q_{ij}$ *for* $i,j = 1, 2, \cdots, n$, *then* $P_n(\mathbf{r}) \geqq P_n(\mathbf{q})$.

Note that the matrices $\mathbf{r}$ and $\mathbf{q}$ need only be nonnegative definite (as distinguished from positive definite).

By regarding $P[T,r(\tau)]$ as a limit of $P_n(\mathbf{r})$, as explained in the preceding section, Lemma 1 can be used to deduce the following comparison theorem.

*Theorem 1* — *If* $r(\tau) \geqq q(\tau)$ *for* $0 \leqq \tau \leqq T_o$, *then* $P[T,r(\tau)] \geqq P[T,q(\tau)]$ *for* $0 \leqq T \leqq T_o$.

The covariance function of a process is generally regarded as a rough measure of how much the process "hangs together." This view is supported by the theorem. A process with a larger covariance function hangs together more and is more likely to maintain the same sign than one with a smaller covariance.

The comparison theorem can be used with the three covariances (Section 1.1) for which $P[T,r(\tau)]$ is known exactly to bound this quantity for other covariances. The theorem is particularly useful for comparing covariances of the same class. Let $r(\tau)$ and $q(\tau)$ both be of class $\alpha$, and suppose that $r(\tau) \geqq q(\tau)$ in some neighborhood of the origin. Then $P[T,r(\tau)] \geqq P[T,q(\tau)]$ in this neighborhood. But, for any $\lambda > 1, q(\tau) \geqq r(\lambda\tau)$ in some sufficiently small neighborhood of the origin, so that also $P[T,q(\tau)] \geqq P[T,r(\lambda\tau)] = P[\lambda T,r(\tau)]$ by the scaling law (2). Choosing $\lambda$ appropriately leads to the following

*Theorem 2* — *Let* $r(\tau)$ *and* $q(\tau)$ *be of class* $\alpha$ *with* $r(\tau) \geqq q(\tau)$ *in some neighborhood of* $\tau = 0$. *Then for some* $T^* > 0$,

$$P[T,r(\tau)] \geqq P[T,q(\tau)] \geqq P[r^{-1}(q(T)),r(\tau)], \quad 0 \leqq T \leqq T^*.$$

The theorem is proved in Section 2.3 where the determination of $T^*$ and the choice of proper branch for $r^{-1}(q)$ are also discussed. Knowledge of $P[T,r(\tau)]$ thus provides both upper and lower bounds for $P[T,q(\tau)]$ near $\tau = 0$.

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† Proved in Section 2.2. A special case of this lemma was proved by J. Chover[4] by a completely different method. He applied his result to obtain a weak version of our Theorem 1. Chover's result inspired much of the present paper.

### 1.4 *Some Related Results Useful for Large T*

From Lemma 1, it is easy to deduce (see Section 2.4)

*Theorem 3 — Let* $T_1 \geqq 0, T_2 \geqq 0, T_3 \geqq 0$ *be such that* $T_1 + T_2 = T_3$. *If* $r(\tau) \geqq 0$ *for* $0 \leqq \tau \leqq T_3$, *then*

$$P[T_3, r(\tau)] \geqq P[T_1, r(\tau)]P[T_2, r(\tau)]. \tag{8}$$

This theorem provides some asymptotic information on $P[T,r(\tau)]$ for covariances that are never negative. It implies for these covariances that $-(1/T) \log P[T,r(\tau)]$ approaches a nonnegative limit as $T$ becomes infinite. In this sense, then, for *nonnegative covariances,* $P[T,r(\tau)]$ *cannot decrease asymptotically more rapidly than exponentially.* An exponential lower bound for these covariances is found by iterating (8). Thus, if $T = NT_o$, $P[T,r(\tau)] = P[NT_o, r(\tau)] \geqq P[T_o, r(\tau)]^N$. One obtains in this manner the exponential bound

$$P[T,r(\tau)] \geqq P_o P_o^{T/T_o} \qquad T \geqq T_o \tag{9}$$

which holds for nonnegative $r(\tau)$ with $P_o = P[T_o, r(\tau)]$, $T_o > 0$.

For covariances for which $P[T,r(\tau)]$ is not known, (9) still gives useful information by replacing $P_o$ by a lower bound. For example, from the lower bounds presented below Theorem 6 in Section 1.6, it follows that for nonnegative $r(\tau)$ of class 2, $P[T,r(\tau)] \geqq f(T)$ where

$$f(T) = \begin{cases} \dfrac{1}{2}\left[1 - \dfrac{T}{\pi}\right], & 0 \leqq T \leqq \dfrac{\pi}{2} \\[2ex] \dfrac{1}{4}\left[\dfrac{3}{2} - \dfrac{T}{\pi}\right], & \dfrac{\pi}{2} \leqq T \leqq \dfrac{3\pi}{2}. \end{cases} \tag{10}$$

By choosing $T_o$ to maximize $f(T_o)^{1/T_o}$ and using this maximum value for $P_o$ in (9), one obtains the following

*Lower Bound — If* $r(\tau)$ *is of class 2 and nonnegative, then*

$$P[T,r(\tau)] \geqq 0.121\, e^{-2.078(T/\pi)}, \qquad T \geqq (1.016)\pi.$$

For a specific nonnegative covariance of class 2, a somewhat smaller exponent can often be obtained by using for $f$ the lower bound of Theorem 6, or a lower bound obtained from the comparison theorem and example (*ii*) of Section 1.1.

For covariances (such as $r_3(\tau)$ of Section 1.1) that are identically zero for $\tau \geqq T_1$ for some $T_1 > 0$, an exponential upper bound can readily be written for $P[T,r(\tau)]$. For example, if $T = (2N - 1)T_1$, then $P[(2N - 1)T_1, r(\tau)]$ is certainly not greater than the probability that the process be nonnegative in the intervals $(0,T_1)$, $(2T_1, 3T_1)$, $\cdots$,

$((2N-2)T_1, (2N-1)T_1)$. But the process in any one of these intervals is independent of the process in the other intervals because of the vanishing of $r(\tau)$ for $\tau \geqq T_1$. Thus, $P[T,r(\tau)] \leqq \{P[T_1, r(\tau)]\}^N$. Arguing in this manner, one arrives at the

*Upper Bound — If $r(\tau)$ vanishes for $\tau \geqq T_1$, then*

$$P[T,r(\tau)] \leqq \frac{1}{\sqrt{P_1}} P_1^{T/2T_1}, \qquad T \geqq T_1,$$

*where $P_1 = P[T_1, r(\tau)]$.*

### 1.5 *Bounds from Rice's Series*

Let $0 = t_1 < t_2 < \cdots < t_n = T$ be a partition of the interval $(0,T)$ into $n-1$ parts. Let $A_i$ denote the event: "$X(t)$ changes sign at least once in the interval $t_i \leqq t < t_{i+1}$," $i = 1,2,\cdots,n-1$. Then, by the method of inclusion and exclusion,

$$\begin{aligned} 2P[T,r(\tau)] = 1 - \sum_i \Pr\{A_i\} + \sum_{i<j} \Pr\{A_i \cap A_j\} \\ - \sum_{i<j<k} \Pr\{A_i \cap A_j \cap A_k\} \\ + \cdots + (-1)^{n-1}\Pr\{A_1 \cap A_2 \cap \cdots \cap A_{n-1}\}, \end{aligned}$$

is the probability that none of the events $A_i$ occur. If $r''(0)$ exists, the above series approaches as a limit as the partition is refined with mesh tending to zero

$$2P[T,r(\tau)] = 1 - \int_0^T q_1(t_1)\, dt_1 + \frac{1}{2!}\int_0^T dt_1 \int_0^T dt_2 q_2(t_1, t_2) - \cdots;$$

(compare Rice,[19] Equation 3.4–11) which we write as

$$\begin{aligned} 2P[T,r(\tau)] &= 1 + \sum_1^\infty \frac{(-1)^n B_n}{n!}, \\ B_n &= \int_0^T dt_1 \cdots \int_0^T dt_n q_n(t_1, \cdots, t_n). \end{aligned} \tag{11}$$

Here $q_n(t_1, \cdots, t_n)dt_1 \cdots dt_n$ is the probability that $X(t)$ has one or more zeros in each of the intervals $(t_1, t_1 + dt_1), \cdots, (t_n + dt_n)$. The existence of $r''(0)$ assures us that $X(t)$ has a derivative almost everywhere in $(0,T)$ for almost all sample functions. One then has

$$\begin{aligned} q_n(t_1, \cdots, t_n) = \int_{-\infty}^{\infty} d\xi_1 \cdots \int_{-\infty}^{\infty} d\xi_n \,|\, \xi_1 \cdots \xi_n \,| \\ \cdot [p(\xi_1, \cdots, \xi_n, x_1, \cdots, x_n)]_{x's=0}. \end{aligned} \tag{12}$$

Here $p(\xi_1, \cdots, \xi_n, x_1, \cdots, x_n)$ is the joint density for the random variables $X'(t_1), \cdots, X'(t_n), X(t_1), \cdots, X(t_n)$ with $\xi_i$ associated with $X'(t_i)$ and $x_i$ associated with $X(t_i), i = 1, 2, \cdots, n$. $X'(t)$ is the derivative of $X(t)$ with respect to $t$.

From the derivation of the method of inclusion and exclusion, successive partial sums of (11) alternately overestimate and underestimate $2P(T)$. We therefore have the sequence of bounds

$$\begin{gathered} 0 \leqq 2P[T, r(\tau)] \leqq 1, \\ 1 - \frac{B_1}{1!} \leqq 2P[T, r(\tau)] \leqq 1 - \frac{B_1}{1!} + \frac{B_2}{2!}, \\ 1 - \frac{B_1}{1!} + \frac{B_2}{2!} - \frac{B_3}{3!} \leqq 2P[T, r(\tau)] \leqq 1 - \frac{B_1}{1!} + \frac{B_2}{2!} - \frac{B_3}{3!} + \frac{B_4}{4!}, \end{gathered} \tag{13}$$

etc. Unfortunately, except for $n = 1, 2, 3$, the integrand $q_n(t_1, \cdots, t_n)$ occurring in the definition of $B_n$ cannot be expressed in terms of elementary functions. For covariances $r(\tau)$ of class 2, one has

$$q_1(t_1) = \frac{1}{\pi},$$

$$q_2(t_1, t_2) = \frac{1}{\pi^2} \frac{\mu^{3/2}[\sqrt{1 - \alpha^2} + \alpha \arcsin \alpha]}{(1 - \alpha^2)^{3/2}},$$

where

$$\mu = (1 - r^2)(1 - r''^2) - r'^2(2 + 2rr'' - r'^2),$$

$$\alpha = [(1 - r^2)r'' + rr'^2]/[1 - r^2 - r'^2],$$

and

$$r = r(t_2 - t_1), \qquad r' = r'(t_2 - t_1), \qquad r'' = r''(t_2 - t_1).$$

The expression for $q_3$ is too complicated to warrant display here.

Bounds given by partial sums such as (13) cannot be expected to yield useful results for large $T$. Typically, for large $T$, $B_n$ behaves like $T^n$: the upper bounds exceed unity for large $T$ and the lower bounds become negative.

For small $T$, however, (13) yields useful information. One has

$$B_1 = \frac{T}{\pi}.$$

If $r(\tau) = 1 - \tau^2/2 + c\tau^4/4! + O(\tau^6)$, a very tedious computation shows that for small $T$,

$$B_2 = \frac{c-1}{24}\frac{T^3}{\pi} + o(T^3),$$

$$B_3 = O(T^6).$$

From this and the inequalities (13) follows

*Theorem 4 — If for small* $\tau$

$$r(\tau) = 1 - \frac{\tau^2}{2} + \frac{c\tau^4}{4!} + O(\tau^6),$$

*then the first three right-hand derivatives of* $P[T,r(\tau)]$ *with respect to* $T$ *exist at* $T = 0$ *and are given by*

$$P[0,r(\tau)] = \tfrac{1}{2},$$

$$P'[0+,r(\tau)] = -\frac{1}{2\pi},$$

$$P''[0+,r(\tau)] = 0,$$

$$P'''[0+,r(\tau)] = \frac{3}{2}\frac{c-1}{48\pi}.$$

The assumed form for $r(\tau)$ in Theorem 4 is important. It has been shown by Longuet-Higgins[14] that if $r(\tau) = 1 - \tau^2/2 + b\,|\,\tau\,|^3 + O(\tau^4)$, $b \neq 0$, then for small $T$, $B_n = O(T^2)$ for $n = 2,3,4,\cdots$. One can only conclude in this case that $P'[0+,r(\tau)] = -1/2\pi$.

The power series $1 + \sum_1^\infty B_n\lambda^n/n!$ can be written formally as

$$\exp \sum_1 c_n\lambda^n/n.$$

Expand the latter in a power series, equate coefficients of like powers of $\lambda$ and set $\lambda = -1$. There results the formal identity using (11)

$$2P[T,r(\tau)] = e^{-c_1+c_2/2!-c_3/3!+\cdots}, \tag{14}$$

where

$$\begin{aligned} c_1 &= B_1 = \frac{T}{\pi} \\ c_2 &= B_2 - B_1^2 \\ c_3 &= B_3 - 3B_1B_2 + 2B_1^3 \\ c_4 &= B_4 - 4B_1B_3 - 3B_2^2 + 12B_1^2B_2 - 6B_1^4, \end{aligned} \tag{15}$$

etc., with the $B$'s given by (11) and (12). Relations (15) are the usual ones connecting semi-invariants with central moments (see Ref. 39, p. 37 or Ref. 40, p. 186). Kuznetsov, Stratonovich and Tikhonov[12] have suggested the use of (14) keeping a finite number of $c$'s as a better approximation to $P$ than series (11). For large $T$, (14) will perhaps yield a better approximation than (11), but it is difficult to see under just what circumstances this will be true. A knowledge of the asymptotic behavior of the $c$'s for large $T$ is needed, but this appears to be a difficult point.

A truncated form of (14) will not in general yield the correct asymptotic behavior of $P[T,r(\tau)]$. For example, retaining only $c_1$, (14) gives $2\,P[T,r(\tau)] \sim e^{-T/\pi}$ for all class 2 covariances. That this is not in general correct can be seen from a family of simple counterexamples. If $q(\tau)$ is of class 2, then so is

$$r^*(\tau) = q(\alpha\tau)\,\frac{\sin\beta\tau}{\beta\tau}, \tag{16}$$

where $\alpha = \sqrt{1 - \beta^2/3}$ and $0 < \beta \leqq \sqrt{3}$. If $X(t)$ has covariance $r^*(\tau)$, then since $r^*(n\pi/\beta) = 0, n = \pm1, \pm2, \cdots$, the random variables

$$X(\pi/\beta),\, X(2\pi/\beta),\, X(3\pi/\beta), \cdots$$

are independent. Set $N = [\beta T/\pi]$. Then

$$\begin{aligned} P[T,r^*(\tau)] &\leqq \Pr\{X(j\pi/\beta) \geqq 0,\, j = 1,\cdots,N\} = (\tfrac{1}{2})^N \\ &\leqq 2(\tfrac{1}{2})^{\beta T/\pi} = 2e^{-(\beta \log 2)T/\pi}. \end{aligned}$$

Thus if

$$\sqrt{3} = 1.732 \geqq \beta > \frac{1}{\log 2} = 1.442, \tag{17}$$

$e^{T/\pi}P[T,r^*(\tau)]$ approaches zero exponentially for large $T$, and the first term in the exponent of (14) yields an incorrect asymptotic behavior.

It is interesting to note that the form $e^{-T/\pi}$ obtained from (14) by retaining only $c_1$ would be correct for a process in which the axis crossings were independent. One would then have $q_n(t_1, \cdots, t_n) = \prod q_1(t_j)$, $B_n = (B_1)^n$ and $c_n = 0, n > 1$. For processes with the covariance (16) with $\beta$ given by (17), $P[T,r^*(\tau)]$ decays even more rapidly. This has nothing to do with the asymptotic behavior of $r^*$: by proper choice of $q(\tau)$, this can be altered at will. One must suppose this rapid decay of $P[T,r^*(\tau)]$ is due to the fact that typically $r^*(\tau)$ takes negative values so that at certain time separations the process is anticorrelated. Indeed, it is tempting to conjecture that for nonnegative class 2 covariances, $e^{T/\pi}P[T,r(\tau)]$ increases without limit for large $T$.

### 1.6 *Some Other Bounds for* $P[T,r(\tau)]$

In this section we list a few miscellaneous bounds on $P[T,r(\tau)]$.

*Theorem 5* —

$$P[T,r(\tau)] \leqq \frac{2}{\pi} \int_0^1 (1 - u) \arcsin r(Tu)\, du.$$

The theorem is proved in Section 2.5. If $\tau \arcsin r(\tau)$ is integrable, the bound in Theorem 5 approaches zero like $1/T$.

Lower bounds for $P[T,r(\tau)]$ are difficult to obtain. One is given by (see Section 2.6)

*Theorem 6 — If* $r(\tau)$ *is of class 2,*

$$P[T,r(\tau)] \geqq \frac{3}{8} - \frac{T}{4\pi} + \frac{1}{4\pi} \arcsin r(T).$$

This bound goes negative for relatively small values of $T$ (at least before $T = 2\pi$). It gives somewhat more information than the bound

$$P[T,r(\tau)] \geqq \frac{1}{2}\left[1 - \frac{T}{\pi}\right], \tag{18}$$

obtained from Rice's series (Section 1.5) by retaining only $B_1$. The bound obtained by retaining $B_1$, $B_2$ and $B_3$ is of course generally much better than that of Theorem 6 but is so complicated that it can be used only with difficulty even with a modern computer. For nonnegative covariances of class 2, Theorem 6 gives $P[T,r(\tau)] \geqq 3/8 - T/4\pi$. This, together with (18), gives (10).

*Theorem 7 — If in the neighborhood of* $\tau = 0$,

$$r(\tau) = 1 - \frac{\tau^2}{2} + \frac{1}{\beta^2}\frac{\tau^4}{4!} + o(\tau^4),$$

*then*

$$P[T,r(\tau)] \leqq \frac{1}{2} - \frac{T}{4\pi} - \frac{1}{2\pi} \arcsin\left[\beta \sin\left(\frac{T}{2\beta}\right)\right], \qquad 0 \leqq T \leqq T_1,$$

*where* $T_1 = \min(\beta\pi,\tau_o)$ *and* $\tau_o$ *is the smallest positive value of* $\tau$ *for which* $r(\tau) = 1 - 2\sqrt{\beta}$. This theorem follows from the comparison Theorem 1, the result (*ii*) of Section 1.1 and the fact (see Theorem 14, p. 494), that for $0 \leqq \tau \leqq T_1$, the covariance of Theorem 7 is dominated by $r_2(\beta,\tau)$.

*Theorem 8 — If $r(\tau)$ is nonnegative and of class 2, then*

$$P[T,r(\tau)] \geqq \frac{1}{2} - \frac{T}{4\pi} - \frac{1}{2\pi}\arcsin\left[\frac{1}{\sqrt{2}}\sin\frac{T}{\sqrt{2}}\right], \qquad 0 \leqq T \leqq \frac{\pi}{\sqrt{2}}.$$

This theorem follows from the comparison Theorem 1, the result (*ii*) of Section 1 and the fact (see Theorem 13 in Section 2.7) that for $0 \leqq \tau \leqq \pi/\sqrt{2}$, every nonnegative covariance of class 2 is greater than $r_2(1/\sqrt{2},\tau)$.

We conclude this section with a rather weak, but sometimes useful, result proved in Section 2.8.

*Theorem 9 — Let $h(\xi)$ be nonnegative for $0 \leqq \xi \leqq \theta$ and let $h(\xi) = 0$ for $\xi < 0$ and $\xi > \theta$. Define*

$$G_\theta(x) = \int_{-\infty}^{\infty} h(x + \xi)h(\xi)\, d\xi$$

*and set*

$$r_\theta(\tau) = \int_{-\infty}^{\infty} r(\tau - x)G_\theta(x)\, dx.$$

*Then*

$$P[T,r_\theta(\tau)] \geqq P[T + \theta,r(\tau)].$$

### 1.7 *Relationship Between $P[T,r(\tau)]$ and $F[\lambda,r(\tau)]$*

If $r''(0)$ exists, then almost all sample functions $X(t)$ possess a derivative almost everywhere. If $r''(0)$ does not exist, then almost all sample functions are nowhere differentiable. In this latter case, if a realization $X(t)$ has a zero at $t = 0$, it almost certainly has infinitely many zeros in every right neighborhood of $t = 0$. In discussing $F[\lambda,r(\tau)]$, the distribution of the interval, $l$, between successive zeros of $X(t)$, *we accordingly restrict our attention to covariances for which $r''(0)$ exists.*

The quantity $P[T,r(\tau)] - P[T + \Delta,r(\tau)]$ is the measure of those sample functions which are nonnegative in $(0,T)$ but are not nonnegative in $(-\Delta,0)$, i.e., the measure of those sample functions that are nonnegative in $(0,T)$ and have at least one axis crossing in $(-\Delta,0)$. Divide this quantity by the probability $\nu\Delta + o(\Delta)$ that $X(t)$ have one or more upward axis crossings in $(-\Delta,0)$ and allow $\Delta$ to approach zero. There results

$$Q[T,r(\tau)] = -\frac{1}{\nu}\frac{d}{dT}P[T,r(\tau)] = 1 - F[T,r(\tau)]. \qquad (19)$$

Here $Q[T,r(\tau)]$ is the conditional probability that $X(t)$ be nonnegative in $(0,T)$ given an upcrossing of the axis at $t = 0$; $F[\lambda,r(\tau)] = \Pr(l \leqq \lambda)$ is the distribution function for the interval $l$ between zeros. One should note carefully that the condition in the definition of $Q$ is in the "horizontal window sense" (see Ref. 10, Section 2 for a more complete discussion of this term). We shall find $Q[T,r(\tau)]$ more convenient to deal with than $F[T,r(\tau)]$.

From its definition, $Q[T,r(\tau)]$ is nonincreasing as a function of $T$. It assumes the value 1 for $T = 0$. Like $P[T,r(\tau)]$, it satisfies the scaling laws

$$\begin{aligned} Q[T,\lambda r(\tau)] &= Q[T,r(\tau)] \\ Q[T,r(\lambda\tau)] &= Q[\lambda T,r(\tau)] \\ \lambda &> 0. \end{aligned} \tag{20}$$

For most purposes, then, it suffices to consider only class 2 covariances. In this case (see Ref. 19, Equation (3.3–10)) $\nu = \dfrac{1}{2\pi}$ and (19) becomes

$$Q[T,r(\tau)] = -2\pi \frac{d}{dT} P[T,r(\tau)]. \tag{21}$$

Clearly upper and lower bounds on $Q[T,r(\tau)]$, say

$$\begin{aligned} Q_U[T,r(\tau)] &\geqq Q[T,r(\tau)], \qquad 0 \leqq T \leqq T_o \\ Q_L[T,r(\tau)] &\leqq Q[T,r(\tau)], \qquad 0 \leqq T \leqq T_o, \end{aligned}$$

furnish bounds on $P[T,r(\tau)]$ by integration:

$$\frac{1}{2} - \frac{1}{2\pi}\int_0^T Q_U[x,r(\tau)]\,dx \leqq P[T,r(\tau)] \leqq \frac{1}{2} - \frac{1}{2\pi}\int_0^T Q_L[x,r(\tau)]\,dx, \qquad 0 \leqq T \leqq T_0.$$

However, since $Q$ is nonincreasing, it is also possible to obtain weak bounds on $Q$ from known bounds on $P$. For example, since $Q$ is nonincreasing, if $b > a \geqq 0$,

$$(b-a)Q[a,r(\tau)] \geqq \int_a^b Q[\gamma,r(\tau)]\,d\gamma \geqq (b-a)Q[b,r(\tau)],$$

or from (21)

$$Q[a,r(\tau)] \geqq 2\pi \frac{P[a,r(\tau)] - P[b,r(\tau)]}{b-a} \geqq Q[b,r(\tau)]. \tag{22}$$

Thus if $P_U(T)$ and $P_L(T)$ are respectively upper and lower bounds for $P[T,r(\tau)]$ valid for all $T$,

$$\max_{x \geqq T} 2\pi \frac{P_L(T) - P_U(x)}{x - T} \leqq Q[T,r(\tau)] \leqq \min_{0 \leqq x \leqq T} 2\pi \frac{P_U(x) - P_L(T)}{T - x}. \tag{23}$$

Note that the left inequality of (22) for $a = 0$, $b = T$ again gives (18). Also from (21) and the fact that $Q$ is nonincreasing, it follows that $P[T,r(\tau)]$ for class 2 covariances must be convex downward.

To the author's knowledge, when the scaling laws (20) are taken into account, the only covariance for which $Q[T,r(\tau)]$ is known explicitly is $r_2(\beta,\tau)$ of *(ii)*, Section 1.1. One has

$$r_2(\beta,\tau) = 1 - \beta^2 + \beta^2 \cos\left(\frac{\tau}{\beta}\right), \qquad 0 < \beta \leqq 1,$$

$$Q[T,r_2(\tau)] = \begin{cases} \dfrac{1}{2}\left[1 + \dfrac{\cos\left(\dfrac{T}{2\beta}\right)}{\sqrt{1 - \beta^2 \sin^2\left(\dfrac{T}{2\beta}\right)}}\right], & 0 \leqq \dfrac{T}{\beta} \leqq 2\pi, \\ 0, & 2\pi \leqq \dfrac{T}{\beta} \leqq \infty. \end{cases}$$

## 1.8 *A Comparison Theorem for* $Q[T,r(\tau)]$

Imposing the condition that $X(t)$ have an upcrossing at $t = 0$ in the horizontal window sense greatly complicates computation of probabilities associated with the process. For instance, when $X(t)$ is conditioned in this manner, the random variables $X(t_1),X(t_2),\cdots,X(t_n)$ are no longer jointly Gaussian. If $r(\tau)$ is of class 2, their joint density is

$$2\pi \int_0^\infty d\xi \, \xi p(\xi, x_0, x_1, \cdots, x_n)_{x_0=0},$$

where $p(\xi, x_o, x_1, \cdots, x_n)$ is the Gaussian density of the unconditioned variables $X'(0)$, $X(0)$, $X(t_1),\cdots,X(t_n)$.

It is possible, nevertheless, to derive a comparison theorem for $Q[T,r(\tau)]$ and $Q[T,q(\tau)]$ for class 2 covariances somewhat in the spirit of Theorem 1. (See Section 2.9 for proof.) The function $g(t) = q^{-1}[r(t)]$ plays a role here. Writing $\tau = g(t)$, then $q(\tau) = r(t)$. For a given value of $t$, we choose $g(t)$ as the smallest positive value of $\tau$ for which $q(\tau) = r(t)$. At $t = 0$, we have $\tau = 0$. As $t$ increases from 0, so does $\tau$. One of two difficulties can occur as $t$ increases: $r(t)$ may reach a local minimum $r(t_o)$ at $t = t_o$ before $q(\tau)$ has reached its first local minimum, say $q(\tau_1)$; $\tau$ may assume the value $\tau_1$ when $t$ assumes the value $t_1 \leqq t_o$. In the former case we define $g(t)$ only for $0 \leqq t < t_o$; in the latter case, we define $g(t)$ only for $0 \leqq t \leqq t_1$. The comparison theorem can now be stated as follows:

*Theorem 10 — Let $r(\tau)$ and $q(\tau)$ be of class 2 and let $g(t) = q^{-1}[r(t)]$ be defined as above. If for all nonnegative $x$ and $y$ with $x + y \leqq T_o$,*

$$g(x) + g(y) \geqq g(x + y), \tag{24}$$

*then for $0 \leqq T \leqq T_o$*

$$Q[T,r(\tau)] \leqq Q[g(T),q(\tau)]. \tag{25}$$

It is easy to show that if $r(\tau) \geqq q(\tau)$ in some neighborhood of the origin, then $g(t)$ has the subadditive property (24) in some sufficiently small neighborhood of the origin so that the theorem is not vacuous.

The steps which led from Theorem 1 to Theorems 2 and 3 are no longer valid when $X(t)$ is conditioned to have an upcrossing at $t = 0$. We have found no analogue of these theorems for $Q[T,r(\tau)]$.

By using (21), one can integrate the inequality (25) to obtain a more complicated comparison theorem for $P[T,r(\tau)]$, namely

$$P[T,r(\tau)] \geq \tfrac{1}{2} + \int_0^{g(T)} h'(\xi)\frac{d}{d\xi}P[\xi,q(\tau)]\,d\xi = P[g(T),q(\tau)]/g'(T) - \int_0^{g(T)} P[\xi,q(\tau)]h''(\xi)\,d\xi,$$

valid for $0 \leqq T \leqq T_o$. Here $h(\xi) = g^{-1}(\xi) = r^{-1}[q(\xi)]$.

## PART II — PROOFS AND SUPPORTIVE MATERIAL

### 2.1† *The Geometric Approach to $P_n$*

We wish to consider the probability $P_n(\mathbf{r})$ that $n$ jointly normal variates, each with mean zero and normalized covariance matrix $\mathbf{r}$, be nonnegative. Throughout this section we assume that $\mathbf{r}$ is nonsingular. Then $P_n(\mathbf{r})$ can be written as in (5). Denote the eigenvalues and nor-

† The material in this section was developed in 1952. Many of the results have been obtained independently by other workers and have been reported in the literature. Cf. Plackett[41] in particular.

malized eigenvectors of $\mathbf{r}$ by $\lambda_i$ and $\psi^i = (\psi_1{}^i,\psi_2{}^i,\cdots,\psi_n{}^i), i = 1,2,\cdots,n$. One has

$$\begin{aligned} \sum_k r_{ik}\psi_k{}^j &= \lambda_j\psi_i{}^j, \\ \sum_k \psi_k{}^i\psi_k{}^j &= \sum_k \psi_i{}^k\psi_j{}^k = \delta_{ij}, \\ r_{ij} &= \sum_k \lambda_k\psi_i{}^k\psi_j{}^k, \\ i,j &= 1,2,\cdots,n. \end{aligned} \tag{26}$$

In (5) make the substitution $x_i = \sum_k \psi_i{}^k\sqrt{\lambda_k}y_k$. There results

$$P_n(\mathbf{r}) = (2\pi)^{-n/2}\int_R \cdots \int dy_1 \cdots dy_n e^{-\frac{1}{2}\Sigma y_k^2},$$

where the region $R$ is defined by

$$H_i \equiv \sum_k \psi_i{}^k\sqrt{\lambda_k}y_k \geqq 0, \qquad i = 1,2,\cdots,n.$$

Denote by $A_n$ the $(n-1)$-dimensional content of the intersection of this region with the surface of the unit sphere having center at the origin. Then, by changing to a spherical coordinate system,

$$P_n = (2\pi)^{-n/2}A_n\int_0^\infty dr\, r^{n-1}e^{-r^2/2} = \frac{A_n}{S_n},$$

where $S_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere. Thus, $P_n$ is the fraction of the unit sphere on the positive side of the $n$ hyperplanes $H_i = 0$. The unit normal $\mathbf{a}^i$ to $H_i$ directed into $R$ has components $a_k{}^i = \psi_i{}^k\sqrt{\lambda_k}$. From the last of (26), we find for the angle $\theta_{ij}$ between $\mathbf{a}^i$ and $\mathbf{a}^j$, $\cos\theta_{ij} = \mathbf{a}^i\cdot\mathbf{a}^j = r_{ij}$.

As mentioned in Section 1.2, expressions for the content $A_n$ of the spherical simplex in terms of the angles between its bounding surfaces are not known for $n > 3$. However, for the determination of $P[T,r(\tau)]$ one is concerned with the limit as $n \to \infty$ of $P_n$ where the angles $\theta_{ij}{}^{(n)}$ are given, for example, by $\cos\theta_{ij}{}^{(n)} = r[(i-j)T/n]$ with $r(\tau)$ a given positive definite function. Thus, sufficiently tight bounds for $P_n$ might in the limit yield useful results concerning $P[T,r(\tau)]$. The geometric picture suggests a large number of such bounds. Unfortunately, none has been found which yields useful limits. Since, however, approximations for the $n$-variable normal integral $P_n$ are of interest in their own right, we digress to mention several such bounds which may be useful. (See Ref. 42 for a bibliography on the multivariate normal integral.)

Circular cones with vertices at the origin can be inscribed and circumscribed about the region $R$. The half-angle of the inscribed cone is found to be given by

$$\sin \theta_i = \frac{1}{\sqrt{\sum_{ij} r_{ij}^{-1}}}, \tag{27}$$

and the half-angle of the circumscribed cone is given by

$$\cos \theta_c = \frac{1}{\sqrt{\sum_{ij} r_{ij}\sqrt{r_{ii}^{-1}}\sqrt{r_{jj}^{-1}}}}. \tag{28}$$

The fraction of the unit sphere cut out by a circular cone of half-angle $\theta$ is

$$F_n(\theta) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\theta d\varphi \sin^{n-2} \varphi = \tfrac{1}{2} I_{\sin^2\theta}\left(\frac{n-1}{2}, \frac{1}{2}\right) \tag{29}$$

where $I$ is Pearson's incomplete beta function.[43] One has

$$F_n(\theta_i) \leqq P_n \leqq F_n(\theta_c). \tag{30}$$

Bounds for $P_n$ can also be written in terms of inscribed and circumscribed Euclidean simplexes. The planes $H_i = 0$ intersect $n - 1$ at a time in lines which pass through the origin and a vertex of the spherical simplex. Let $\mathbf{b}^i$ denote the unit vector from the origin to the vertex not contained in $H_i = 0$. One finds for the components $b_k^i = \psi_i^k(\lambda_k r_{ii}^{-1})^{-1/2}$ and for the content of the Euclidean simplex determined by the origin and the end points of the $\mathbf{b}^i$,

$$G_n = \frac{1}{n! \sqrt{|\mathbf{r}|} \sqrt{\Pi r_{ii}^{-1}}}. \tag{31}$$

This simplex lies within the region of interest. The hyperplane through the end points of the vectors $\mathbf{b}^i \sec \theta_c$ is tangent to the unit sphere. The Euclidean simplex determined by the origin and the ends of these vectors therefore contains the region of interest. Thus,

$$\frac{G_n}{V_n} \leqq P_n \leqq \frac{\sec^n \theta_c G_n}{V_n}, \tag{32}$$

where $V_n = \pi^{n/2}/\Gamma(n/2 + 1)$ is the content of the unit sphere, $\theta_c$ is given by (28) and $G_n$ by (31). Incidentally, for the cosines of the angles between the $\mathbf{b}$'s one finds the interesting reciprocal relations

$$s_{ij} \equiv \mathbf{b}^i \cdot \mathbf{b}^j = \frac{r_{ij}^{-1}}{\sqrt{r_{ii}^{-1} r_{jj}^{-1}}}, \qquad r_{ij} = \frac{s_{ij}^{-1}}{\sqrt{s_{ii}^{-1} s_{jj}^{-1}}},$$

which is the natural generalization of the usual relationship between the sides and angles of a spherical triangle in three-space.

One can expect the bounds in (30) to be close to each other when the $\mathbf{b}^i$ are nearly coplanar, e.g., when all the entries of $\mathbf{r}$ are near unity. One can expect the bounds in (32) to be close to each other when the $\mathbf{b}^i$ are nearly codirectional, e.g., when all the entries of $\mathbf{r}^{-1}$ are nearly equal.

An important differential recursion relation first derived by Schläfli[33] for the content of the spherical simplex can be obtained in an analytic manner from the expression (5) for $P_n$. We write

$$P_n(\mathbf{r}) = \int_0^\infty dx_1 \cdots \int_0^\infty dx_n g_n(x_1, \cdots, x_n; \mathbf{r}) \tag{33}$$

where the $n$-variate Gaussian density is given in terms of its characteristic function by

$$g_n(x_1, \cdots, x_n; \mathbf{r}) = \int_{-\infty}^\infty d\xi_1 \cdots \int_{-\infty}^\infty d\xi_n \, e^{i\Sigma x_j \xi_j} e^{-\frac{1}{2}\Sigma r_{jk}\xi_j\xi_k}.$$

From this latter expression it follows that

$$\frac{\partial g_n}{\partial r_{jk}} = \frac{\partial^2 g_n}{\partial x_j \partial x_k}, \qquad k > j. \tag{34}$$

Here we regard $g_n$ as a function of the $n(n-1)/2$ variables $r_{jk}$, $k > j$, and recall that $r_{ii} = 1$, $r_{jk} = r_{kj}$. Regarding $P_n$ as a function of this same set of variables, we find from (33) and (34)

$$\frac{\partial P_n(\mathbf{r})}{\partial r_{12}} = \int_0^\infty dx_1 \cdots \int_0^\infty dx_n \frac{\partial^2}{\partial x_1 \partial x_2} g_n(x_1, \cdots, x_n; \mathbf{r}).$$

Perform the integrations indicated on $x_1$ and $x_2$. There results

$$\frac{\partial P_n(\mathbf{r})}{\partial r_{12}} = \int_0^\infty dx_3 \cdots \int_0^\infty dx_n g_n(0,0,x_3, \cdots, x_n; \mathbf{r}) \geqq 0. \tag{35}$$

Now if $g_n$ is the density for the random variables $X_1, \cdots, X_n$,

$$g_n(x_1, \cdots, x_n; \mathbf{r}) = p(x_1, x_2) p(x_3, \cdots, x_n \mid x_1, x_2),$$

where $p(x_1, x_2)$ is the joint density for $X_1$ and $X_2$ and

$$p(x_3, \cdots, x_n \mid x_1, x_2)$$

is the conditional density of $X_3, \cdots, X_n$ given that $X_1 = x_1$ and $X_2 = x_2$. In the case of Gaussian variates, these densities are well known Evaluating this expression at $x_1 = x_2 = 0$, one finds

$$g_n(0,0,x_3, \cdots, x_n; \mathbf{r}) = \frac{1}{2\pi\sqrt{1 - r_{12}^2}} g_{n-2}(x_3, \cdots, x_n; \mathbf{r}_{\cdot 12}).$$

When combined with (35) and generalized for arbitrary indices, this yields

$$\frac{\partial P_n(\mathbf{r})}{\partial r_{jk}} = \frac{1}{2\pi\sqrt{1 - r_{jk}^2}} P_{n-2}(\mathbf{r}_{\cdot jk}) \geqq 0. \tag{36}$$

Here $\mathbf{r}_{\cdot jk}$ is the customary notation of the statistician for partial correlation coefficients (see Ref. 40, Section 23.4 and pp. 318–319), so that, for example with $\mu \neq j,k$, $\nu \neq j,k$

$$r_{\mu\nu\cdot jk} = \frac{\begin{vmatrix} r_{\mu\nu} & r_{\mu j} & r_{\mu k} \\ r_{j\nu} & 1 & r_{jk} \\ r_{k\nu} & r_{kj} & 1 \end{vmatrix}}{\begin{vmatrix} 1 & r_{\mu j} & r_{\mu k} \\ r_{j\mu} & 1 & r_{jk} \\ r_{k\mu} & r_{kj} & 1 \end{vmatrix}^{\frac{1}{2}} \begin{vmatrix} 1 & r_{\nu j} & r_{\nu k} \\ r_{j\nu} & 1 & r_{jk} \\ r_{k\nu} & r_{kj} & 1 \end{vmatrix}^{\frac{1}{2}}}.$$

Equation (36) is Schläfli's celebrated differential recursion formula. His many relations connecting the angles of the boundary simplexes are familiar to the statistician as identities among partial correlation coefficients.

We close this section with a simple demonstration that for odd $n$, $P_n$ can be expressed in terms of the content of lower dimensional simplexes. Let $p_i$ denote the probability that $X_i$ be nonnegative, $p_{ij}$ denote the probability that $X_i$ and $X_j$ be nonnegative, etc. Then $P_n = p_{12\cdots n}$. Set $M_1 = \Sigma p_i$, $M_2 = \sum_{i<j} p_{ij}$, etc. Then from the well-known inclusion and exclusion formula, the probability $Q_n$ that none of the variates be nonnegative is

$$Q_n = 1 - M_1 + M_2 - \cdots + (-1)^n M_n .$$

But from symmetry, $P_n = Q_n = M_n$ so that

$$[1 - (-1)^n]P_n = 1 - M_1 + M_2 - \cdots + (-1)^{n-1} M_{n-1} .$$

(Cf. Sommerville,[35] Chapter IX, Section 1.9.) No recursion is known for even $n$.

### 2.2 *Proof of Lemma 1*

Lemma 1 follows directly from (35). Note that in the derivation of this result, it was not necessary to normalize the covariance matrix. This result thus states that if $\boldsymbol{\varrho}$ is a position definite symmetric matrix, then

$$\frac{\partial P_n(\boldsymbol{\varrho})}{\partial \rho_{ij}} \geqq 0, \qquad j > i, \tag{37}$$

with $P_n(\boldsymbol{\varrho})$ defined by (5).

Now let $\mathbf{r}$ and $\mathbf{q}$ be nonnegative definite $n \times n$ symmetric matrices with $r_{ii} = q_{ii} = 1$. Then $\boldsymbol{\varrho} = \lambda\mathbf{r} + (1 - \lambda)\mathbf{q} + \epsilon\mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ unit matrix, is positive definite for each $\epsilon > 0$ and each $\lambda$ satisfying $0 \leqq \lambda \leqq 1$. Consider $P_n(\boldsymbol{\varrho})$ as a function of $\lambda$. It is readily established that $P_n(\boldsymbol{\varrho})$ possesses a continuous derivative and indeed that

$$\begin{aligned}\frac{dP_n(\boldsymbol{\varrho})}{d\lambda} &= \sum_{j>i} \frac{\partial P_n(\boldsymbol{\varrho})}{\partial \rho_{ij}} \frac{d\rho_{ij}}{d\lambda} \\ &= \sum_{j>i} \frac{\partial P_n(\boldsymbol{\varrho})}{\partial \rho_{ij}} (r_{ij} - q_{ij}).\end{aligned}$$

If now $r_{ij} \geqq q_{ij}$, $j > i$, (37) then gives

$$\frac{dP_n(\boldsymbol{\varrho})}{d\lambda} \geqq 0.$$

Integration on $\lambda$ from 0 to 1 yields $P_n(\mathbf{r} + \epsilon\mathbf{I}) \geqq P_n(\mathbf{q} + \epsilon\,\mathbf{I})$. From well-known continuity theorems (see Cramer,[40] Section 24.3 and 10.7), Lemma 1 follows by letting $\epsilon$ tend to zero.

### 2.3 *Proof of Theorem 2*

Let $r(\tau)$ and $q(\tau)$ both be of class $\alpha > 0$ and suppose that $r(\tau) \geqq q(\tau)$ for $0 \leqq \tau \leqq T_o$. Then for any $\lambda > 1$,

$$\begin{gathered} r(\tau) \geqq q(\tau) \geqq r(\lambda\tau) \\ 0 < \tau \leqq \tau_1(\lambda), \end{gathered} \tag{38}$$

for some suitable $\tau_1(\lambda)$. By Theorem 1, then, and the scaling law (2)

$$\begin{gathered} P[T,r(\tau)] \geqq P[T,q(\tau)] \geqq P[\lambda T,r(\tau)] \\ 0 \leqq T \leqq \tau_1(\lambda). \end{gathered} \tag{39}$$

To see how best to choose $\lambda$ to obtain a good lower bound for $P[T,q(\tau)]$, it is convenient to define a version of $h(\tau) = r^{-1}[q(\tau)]$. Let $\tau_q$ be the smallest value of $\tau > 0$ for which $q(\tau)$ is not decreasing. (Strictly speaking, $\tau_q$ = inf. of those $T$ for which $q(\tau)$ is not strictly monotone for $0 < \tau \leqq T$. If this $T$ set is empty, define $\tau_q = \infty$.) Define $\tau_r$ in an analogous manner. The function $r^{-1}(q)$ is defined for $1 \geqq q \geqq r(\tau_r)$ by the branch having values between 0 and $\tau_r$. Similarly we define $q^{-1}(r)$ for $1 \geqq r \geqq q(\tau_q)$ by the branch having values between 0 and $\tau_q$. If $q(\tau_q) \leqq r(\tau_r)$, we define $h(\tau) = r^{-1}[q(\tau)]$ only for $0 \leqq \tau < q^{-1}[r(\tau_r)]$. If $q(\tau_q) \geqq r(\tau_r)$, we define $h(\tau)$ for $0 \leqq \tau < \tau_q$. Clearly $h(0) = 0$. As $\tau$ increases from zero, $h(\tau)$ is at first at least as large as $\tau$, since $r(\tau) \geqq q(\tau)$ near $\tau = 0$. For small $\tau$, $r(h) = q(\tau)$, so that $h'(\tau)r'(h) = q'(\tau)$ or

$$h'(0+) = \lim_{t \to 0+} \frac{q'(\tau)}{r'(h)} = \lim_{t \to 0+} \frac{\tau^{\alpha-1}}{h^{\alpha-1}} = \lim_{t \to 0+} \left(\frac{h(\tau)}{\tau}\right)^{1-\alpha} = h'(0+)^{1-\alpha},$$

so that $h'(0+) = 1$. Three typical curves for $y = h(\tau)$ are shown in Fig. 1. Note that $h(\tau)$ is strictly monotone in its domain of definition.

Consider now the plots of $y = h(\tau)$ and $y = \lambda\tau$ as shown on Fig. 1. For all values of $\lambda$, these curves have the origin as a point in common. When $\lambda = 1$, the straight line $y = \lambda\tau$ is tangent to $y = h(\tau)$ at the origin. As $\lambda$ is increased from 1, a second point of intersection moves out from the origin. It may happen, as in Fig. 1(a), that the line $y = \lambda\tau$ becomes tangent to $y = h(\tau)$. If so, we denote by $T^*$ the abscissa of the first such point of tangency as $\lambda$ increases from unity and we denote the corresponding value of $\lambda$ by $\lambda^*$. If no such tangency occurs, we denote by $T^*$ the largest value of $\tau$ in the domain $h(\tau)$. In this case we set $\lambda^* = h(T^*)/T^*$. (Note that $\lambda^*$ may be infinite.) Observe that for a given $\lambda < \lambda^*$, the abscissa of the first point of intersection of $y = \lambda\tau$ with $y = h(\tau)$ to the right of the origin, say $\tau_1$, satisfies $h(\tau_1) = \lambda\tau_1$ or $q(\tau_1) = r(\lambda\tau_1)$. For $\tau \leqq \tau_1$, the right inequality of (38) maintains; for $\tau = \tau_1 + \epsilon, r(\lambda\tau) > q(\tau)$ for small positive $\epsilon$.

The lower bound $P[\lambda T, r(\tau)]$ on the right of (38) is a nonincreasing function of $\lambda$ for a fixed $T$. For a given $T \leqq T^*$, then, this bound is made as large as possible by choosing $\lambda$ as the smallest value greater than unity for which $q(T) = r(\lambda T)$. With this choice, $\lambda T$ has the value $h(T)$ and Theorem 2 is proved. The largest $T^*$ for which the theorem as stated in Section 1.3 is true is the value $T^*$ defined in the previous paragraph.

Note that if $r(\tau)$ and $q(\tau)$ cross at $\tau_o > 0$, i.e., $r(\tau_o) = q(\tau_o)$, $T^*$ is necessarily less than $\tau_o$, for in this case, $y = h(\tau)$ crosses $y = \tau$ at $\tau_o$ as in Fig. 1(a) and a tangency occurs as indicated.

### 2.4 *Proof of Theorem 3*

Let $T_1 > 0$ and $T_2 > 0$ be given and set $T_3 = T_1 + T_2$. Consider the approximation to $P[T_3, r(\tau)]$ given by the probability $P_n(\mathbf{r})$ that

$$X(t_1),\cdots,X(t_{n_1}),\ X(\tau_1),\cdots,X(\tau_{n_2})$$

all be nonnegative. Here $0 = t_1 < t_2 < \cdots < t_{n_1} = T_1$ is a partition of $(0,T_1)$ and $T_1 < \tau_1 < \tau_2 < \cdots < \tau_{n_2} = T_3$ is a partition of $(T_1, T_1 + T_2)$ and $n_1 + n_2 = n$. The covariance matrix $\mathbf{r}$ can be written in block form

$$\mathbf{r} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{C} \end{pmatrix},$$

where $\mathbf{A}$ is an $n_1 \times n_1$ normalized covariance matrix with elements $r(t_i - t_j)$ $\mathbf{C}$ is an $n_2 \times n_2$ normalized covariance matrix with elements

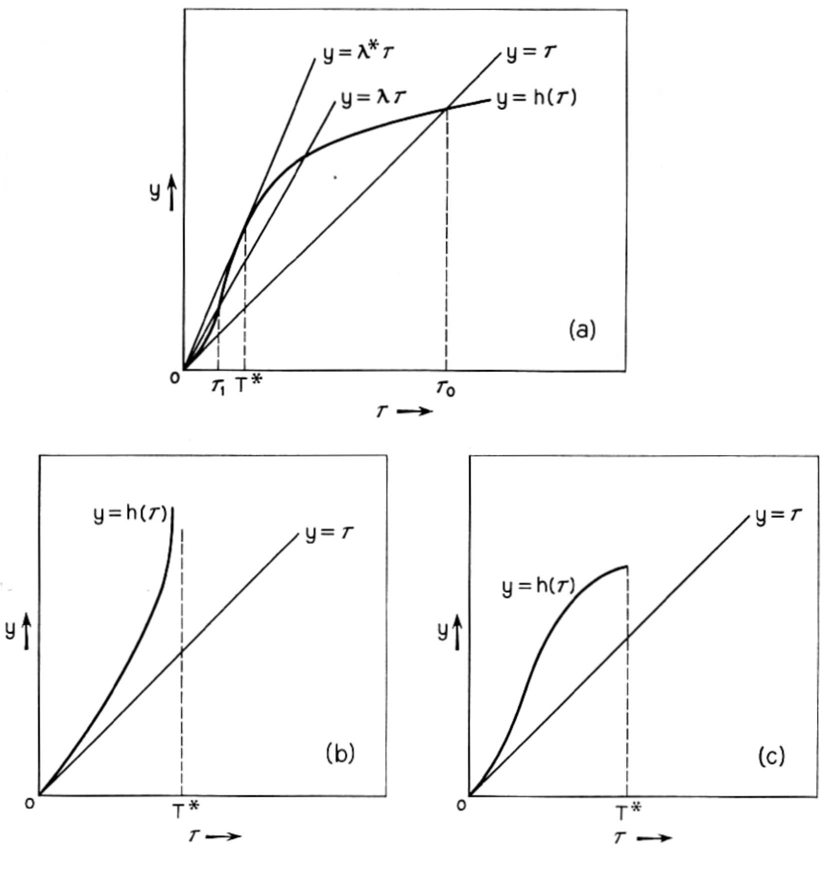


Fig. 1 — The curve $y = h(\tau)$.

$r(\tau_i - \tau_j)$, and $\mathbf{B}$ has $n_1$ rows and $n_2$ columns and elements $r(t_i - \tau_j)$. Now

$$\hat{\mathbf{r}} = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{C} \end{pmatrix},$$

is also a covariance matrix, and if $r(\tau) \geqq 0$ for $0 \leqq \tau \leqq T_3$, the elements of $\mathbf{r}$ are not less than the corresponding elements of $\hat{\mathbf{r}}$. From Lemma 1, it follows that $P_n(\mathbf{r}) \geqq P_n(\hat{\mathbf{r}})$. But $\hat{\mathbf{r}}$ is the covariance matrix for two independent sets of random variables so that

$$P_n(\mathbf{r}) \geqq P_n(\hat{\mathbf{r}}) = P_{n_1}(\mathbf{A})P_{n_2}(\mathbf{C}).$$

By refining the partition with mesh tending to zero, one has $P[T_3, r(\tau)] \geqq P[T_1, r(\tau)]P[T_2, r(\tau)]$ and the theorem is established. (It is trivially true if $T_1$ or $T_2$ or both are zero.)

### 2.5 *Proof of Theorem 5*

Theorem 5 is a consequence of the following more general

*Theorem 11 — Let the random variables $X_1, X_2, \cdots, X_n$, $n > 2$ have a joint density $p(x_1, \cdots, x_n)$ with the property $p(-x_1, \cdots, -x_n) = p(x_1, \cdots, x_n)$. Then*

$$\Pr\{X_i \geqq 0, i = 1,2,\cdots,n\} \leqq -\frac{1}{2} + \frac{4}{n(n-2)} \sum_{i<j} \Pr\{X_i \geqq 0, X_j \geqq 0\}.$$

The proof of this theorem follows that of a theorem by Gaddum[44] concerning spherical simplexes and their angle sums. We introduce the following notations: $P_{ij} = \Pr(X_i \geqq 0, X_j \geqq 0)$, $P = \Pr\{X_i \geqq 0, i = 1,2,\cdots,n\}$, $R(a_1, a_2, \cdots, a_n) = \Pr\{a_1X_1 \geqq 0, a_2X_2 \geqq 0, \cdots, a_nX_n \geqq 0\}$, $a_i = \pm 1$, $i = 1,\cdots,n$. Thus $P = R(1,1,\cdots,1)$ and

$$\sum_{a_1,\cdots,a_n} R(a_1, a_2, \cdots, a_n) = 1,$$

where in the sum each $a$ takes values $+1$ and $-1$. The $2^n$ symbols $R$ are equal in pairs;

$$R(a_1, a_2, \cdots, a_n) = R(-a_1, -a_2, \cdots, -a_n).$$

We call $R(-a_1, -a_2, \cdots, -a_n)$ the complement of $R(a_1, a_n, \cdots, a_n)$.

One has

$$\begin{aligned} P_{12} &= P + \Sigma' R(1,1,a_3, a_4, \cdots, a_n) \\ P_{13} &= P + \Sigma' R(1,a_2, 1,a_4, \cdots, a_n) \\ &\vdots \\ P_{n(n-1)} &= P + \Sigma' R(a_1, a_2, \cdots, a_{n-2}, 1,1). \end{aligned} \tag{40}$$

Here the $R$ symbol on the right of the equation having $P_{ij}$ as left member has a 1 in the $i^{\text{th}}$ and $j^{\text{th}}$ places and $a$'s elsewhere. In each equation, the sum is over all combinations of plus and minus 1 for the $a$'s except for the combination all $a$'s plus 1.

Now consider adding the $n(n-1)/2$ equations (40). One has

$$\sum_{i<j} P_{ij} = [n(n-1)/2]P + S,$$

where $S$ is the sum of all the sums of $R$ symbols on the right of (40). A given $R$ symbol with precisely $j$ of its arguments $+1$ will occur $j(j-1)/2$ times in $S$, $j = 2,3,\cdots,n-1$. Denote by $T_j$ the sum of all $R$ symbols that have precisely $j$ of their arguments $+1$. Then

$$\sum_{i<j} P_{ij} = \frac{n(n-1)}{2} P + \frac{(n-1)(n-2)}{2} T_{n-1} + \sum_{j=2}^{n-2} \frac{j(j-1)}{2} T_j. \tag{41}$$

Now

$$\sum_{j=2}^{n-2} \frac{j(j-1)}{2} T_j = \sum_{j=2}^{n-2} \frac{(n-j)(n-j-1)}{2} T_{n-j},$$

so that

$$\sum_{j=2}^{n-2} \frac{j(j-1)}{2} T_j = \frac{1}{2} \sum_{j=2}^{n-2} \left[\frac{j(j-1)}{2} T_j + \frac{(n-j)(n-j-1)}{2} T_{n-j}\right].$$

But since an $R$ symbol and its complement are numerically equal, $T_j = T_{n-j}$, so that (41) becomes

$$\sum_{i<j} P_{ij} = \frac{n(n-1)}{2} P + \frac{(n-1)(n-1)}{2} T_{n-1} + \frac{1}{2} \sum_{j=2}^{n-2} \left[\frac{j(j-1)}{2} + \frac{(n-j)(n-j-1)}{2}\right] T_j.$$

Now, for $j = 2,3,\cdots,n-2$,

$$\frac{j(j-1)}{2} + \frac{(n-j)(n-j-1)}{2} \geqq \frac{n(n-2)}{4},$$

so that

$$\sum_{i<j} P_{ij} \geqq \frac{n(n-1)}{2} P + \frac{(n-1)(n-2)}{2} T_{n-1} + \frac{n(n-2)}{8} \sum_{j=2}^{n-2} T_j \geqq \frac{n(n-2)}{2} P + \frac{n(n-2)}{8} \sum_{j=1}^{n-1} T_j.$$

However, the last appearing sum is $1 - 2P$ and Theorem 11 follows directly.

In the case of a Gaussian process $X(t)$ with normalized covariance function $r(\tau)$, we consider the application of Theorem 11 to the random variables $X_i = X(iT/n), i = 1,2,\cdots,n$. Then from (6), $P_{ij} = \frac{1}{4} + 1/2\pi \arcsin r[(i - j)T/n]$. By taking limits as $n$ becomes infinite, Theorem 11 then yields

$$P[T,r(\tau)] \leqq \frac{2}{\pi}\frac{1}{T^2}\int_0^T dy \int_0^y dx \arcsin r(y - x).$$

Elementary manipulations then lead to the result stated as Theorem 5.

## 2.6 *Proof of Theorem 6*

Consider $n$ random variables, $X_1, X_2, \cdots, X_n$, and the following mutually exclusive events: ($A$) the variables are all nonnegative; ($B_j$) the first $j$ variables are nonnegative and the $(j + 1)^{\text{st}}$ is negative, $j = 1,2,3,\cdots,n - 1$. The union $C$ of these events is the event $X_1 \geqq 0$. We suppose $\Pr\{C\} = \frac{1}{2}$ and write $P_n = \Pr\{A\}, V_j = \Pr\{B_j\}, j = 1,2,\cdots,n - 1$ so that

$$P_n = \tfrac{1}{2} - \sum_{j=1}^{n-1} V_j .$$

But $V_j \leqq \Pr\{X_1 \geqq 0, X_j \geqq 0, X_{j+1} < 0\}, j = 2,\cdots,n - 1$ so that

$$P_n \geqq \tfrac{1}{2} - \Pr\{X_1 \geqq 0, X_2 \leqq 0\} - \sum_{j=2}^{n-1} \Pr\{X_1 \geqq 0, X_j \geqq 0, X_{j+1} < 0\}. \quad (42)$$

Consider a stationary Gaussian process $X(t)$ with a class 2 covariance $r(\tau)$. In (42) set $X_j = X(jT/n)$. From (7), one obtains

$$\Pr\{X_1 \geqq 0, X_j \geqq 0, X_{j+1} < 0\}$$

$$= \frac{1}{8} + \frac{1}{4\pi}\left[\arcsin r\left[(j - 1)\frac{T}{n}\right] - \arcsin r\left[j\frac{T}{n}\right] - \arcsin r\left[\frac{T}{n}\right]\right],$$

and from (6)

$$\Pr\{X_1 \geqq 0, X_2 \leqq 0\} = \frac{1}{4} - \frac{1}{2\pi}\arcsin r\left(\frac{T}{n}\right).$$

Insert these values in (42) and pass to the limit as $n$ becomes infinite. Theorem 6 results.

### 2.7 *On Class 2 Covariances*

Let $r(\tau)$ be a class 2 covariance. From the Bochner representation

$$r(\tau) = \int_0^\infty \cos \lambda\tau \, dF(\lambda),$$

where we now have

$$1 = \int_0^\infty dF(\lambda) = \int_0^\infty \lambda^2 dF(\lambda),$$

it is not hard to show that $r$ is continuous, that $r'(\tau)$ exists everywhere and is continuous, and that $r''(\tau)$ exists and is continuous everywhere except perhaps at $\tau = 0$.

If the process $X(t)$ with mean zero has $r(\tau)$ as its covariance function, then the four random variables $X(0), X'(0), X(t), X'(t)$ have covariance matrix

$$\begin{pmatrix} 1 & 0 & r & r' \\ 0 & 1 & -r' & -r'' \\ r & -r' & 1 & 0 \\ r' & -r'' & 0 & 1 \end{pmatrix}$$

where we write $r = r(t), r' = d/dt\, r(t), r'' = d^2/dt^2\, r(t)$. For this to be a nonnegative definite matrix it is necessary that the determinant of all major diagonal submatrices be nonnegative. Evaluating these determinants, one finds the system of differential inequalities

$$(1 - r^2 - r'^2)(1 - r'^2 - r''^2) - (rr' + r'r'')^2 \geqq 0, \tag{43}$$

$$1 - r^2 - r'^2 \geqq 0, \tag{44}$$

$$1 - r^2 - r''^2 \geqq 0, \qquad 1 - r'^2 - r''^2 \geqq 0,$$

$$1 - r^2 \geqq 0, \qquad 1 - r'^2 \geqq 0, \qquad 1 - r''^2 \geqq 0.$$

These inequalities can also be derived without raising the question of existence of the derivative process by demanding that the covariance matrix of the four random variables $X(0), X(\epsilon) - X(0), X(t), X(t+\epsilon) - X(t)$ be nonnegative definite for arbitrarily small values of $\epsilon$.

Consider now the family of covariances

$$r_2(\beta,\tau) = 1 - \beta^2 + \beta^2 \cos\left(\frac{\tau}{\beta}\right), \qquad 0 \leqq \beta \leqq 1, \tag{45}$$

introduced in Section 1.1. In what follows, we shall be concerned with the family, $F$, of curves $r = r_2(\beta,\tau)$, where for each $\beta$ with $0 < \beta \leqq 1$ we restrict our attention to the interval $0 \leqq \tau \leqq \pi\beta$. Several members of the family are shown in Fig. 2. The following statements, evident from the figure, are easy to prove analytically. (1) The curves of the family do not intersect each other except at $\tau = 0$. (2) A horizontal line $r = r_o$ with $| r_o | < 1$ intersects exactly once each member of $F$ with parameter value in the range $1 \geqq \beta \geqq \sqrt{(1 - r_o)/2}$. For each value of $\alpha$ satisfying $-\sqrt{1 - r_o^2} \leqq \alpha \leqq 0$, there is a unique member of the famly that intersects the line $r = r_o$ with slope $\alpha$. If $\beta(\alpha)$ denotes the parameter value of this member of $F$, $\beta(\alpha)$ is a continuous strictly monotone decreasing function of $\alpha$, $-\sqrt{1 - r_o^2} \leqq \alpha \leqq 0$.

We shall say that the curve $r = r(\tau)$ intersects the curve $r = g(\tau)$ from below if at the point of intersection $r' > g'$.

*Lemma 2 — Let* $r(\tau)$ *be of class* 2.

a. *If the first local minimum of* $r(\tau)$ *is at* $\tau_1$, *then* $r = r(\tau)$ *cannot intersect from below any member of the family* $F$,

$$r = r_2(\beta,\tau) = 1 - \beta^2 + \beta^2 \cos\left(\frac{\tau}{\beta}\right), \qquad 0 \leqq \tau \leqq \pi\beta, \qquad 0 \leqq \beta \leqq 1,$$

*in the interval* $0 \leqq \tau \leqq \tau_1$.

b. *If* $r = r(\tau)$ *passes down through the point* $(\tau_o, r_o)$ *with slope* $r_o'$ *satisfying* $-\sqrt{1 - r_o^2} \leqq r_o' \leqq 0$, *then there is a unique translated member of* $F$, *say* $r = r_2(\beta_o, \tau - \mu)$ *which passes through* $(\tau_o, r_o)$ *with slope* $r_o'$. *If* $r_2(\beta_o, \tau - \mu)$ *and* $r(\tau)$ *are nonincreasing for* $\bar{\tau} \leqq \tau \leqq \tau_o$, *then* $r(\tau) \leqq r_2(\beta_o, \tau - \mu)$ *for* $\bar{\tau} \leqq \tau \leqq \tau_o$.

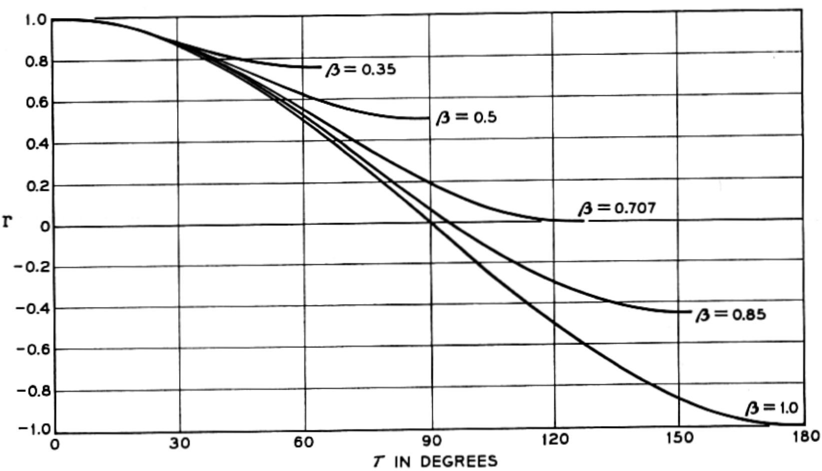


Fig. 2 — The family $F$.

Proof — Part $a$ of the lemma will be deduced from part $b$. The first conclusion of part $b$ is the remark (2) above. The second conclusion of part b follows from the inequality (43). If $|r| \neq 1$, this latter can be written by elementary algebraic manipulations as

$$-\frac{1 - r^2 - r'^2}{1 - r^2} \leqq r'' + \frac{rr'^2}{1 - r^2} \leqq \frac{1 - r^2 - r'^2}{1 - r^2}.$$

The right-hand inequality can be rewritten as

$$\frac{r''}{(1 - r)^2} + \frac{r'^2}{(1 - r)^3} \leqq \frac{1}{(1 - r)^2},$$

or, if $r' \leqq 0$, as

$$\frac{2r'r''}{(1 - r)^2} + \frac{2r'^3}{(1 - r)^3} \geqq \frac{2r'}{(1 - r)^2},$$

or

$$\frac{d}{d\tau}\frac{r'^2}{(1 - r)^2} \geqq 2\frac{d}{d\tau}\frac{1}{1 - r}.$$

Integrate this expression from $\tau$ to $\tau_o$ with $\tau < \tau_o$ to obtain

$$\frac{r'^2}{(1 - r)^2} - \frac{2}{1 - r} \leqq \frac{r_o'^2}{(1 - r_o)^2} - \frac{2}{1 - r_o}, \tag{46}$$

where the subscript $o$ refers to quantities evaluated at $\tau_o$. Denote the right member of this inequality by $-1/h^2$, and note that, as is indicated by the notation,

$$\frac{1}{h^2} = \frac{2(1 - r_o) - r_o'^2}{(1 - r_o)^2} \geqq \frac{(1 + r_o)(1 - r_o) - r_o'^2}{(1 - r_o)^2} = \frac{1 - r_o^2 - r_o'^2}{(1 - r_o)^2} \geqq 0,$$

by (44). Inequality (46) now becomes

$$r'^2 - 2(1 - r) \leqq -\frac{1}{h^2}(1 - r)^2,$$

or what is the same

$$r'^2 \leqq \frac{1}{h^2}(1 - r)(r - \lambda),$$

where

$$\lambda = 1 - 2h^2. \tag{47}$$

It follows then that

$$\frac{r'}{\sqrt{(1 - r)(r - \lambda)}} \geqq \frac{1}{h},$$

with $h$ a nonnegative quantity. Integrate this again from $\tau$ to $\tau_o$ to obtain

$$\arcsin \frac{r_o - (1 + \lambda)/2}{(1 - \lambda)/2} - \arcsin \frac{r - (1 + \lambda)/2}{(1 - \lambda)/2} \geqq -\frac{\tau_o - \tau}{h}.$$

Thus one finds

$$r(\tau) \leqq \frac{1 + \lambda}{2} + \frac{1 - \lambda}{2} \sin \left[\frac{\tau_o - \tau}{h} + \arcsin \frac{r_o - (1 + \lambda)/2}{(1 - \lambda)/2}\right] \equiv q(\tau). \tag{48}$$

This inequality is valid in a $\tau$-range to the left of $\tau_o$ until either $q(\tau)$ or $r(\tau)$ has a local maximum.

Now by (47), $q(\tau)$ can be written

$$q(\tau) = 1 - h^2 + h^2 \cos\left(\frac{\tau - \mu}{h}\right),$$

for suitably defined $\mu$, and one finds by using the various definitions

$$q(\tau_o) = r_o$$

$$q'(\tau_o) = r_o'.$$

Thus $q(\tau)$ is the member of the family $F$ which, when translated in the $\tau$-direction, passes through the point $(\tau_o, r_o)$ with slope $r_o'$. To the left of $\tau_o$, the curve $r = r(\tau)$ remains below this translated member of $F$. Part $b$ is thus proved.

Now suppose that $r = r(\tau)$ intersects a member of the family $F$ from below, say at $(\tau_o, r_o)$ with $\tau_o \leqq \tau_1$. Let the parameter value of this member of $F$ be $\beta_o$. Since $0 \geqq r'(\tau_o) > r_2'(\beta_o, \tau_o)$, the translated member of $F$ passing through $(\tau_o, r_o)$ with slope $r'(\tau_o)$ has a parameter value $\beta = \beta_1 < \beta_o$. This translated version of $r = r_2(\beta_1, \tau)$ has no local maximum in the interval $(0, \tau_o)$, and its value at $\tau = 0$ is less than unity. One thus has the contradiction $r(0) < 1$ and the lemma is proved.

*Theorem 12 — Let $r(\tau)$ be a class 2 covariance. Then*

$$r(\tau) \geqq \cos \tau, \qquad 0 \leqq \tau \leqq \pi.$$

Proof: In a region where $r'(\tau) \leqq 0$, inequality (44) implies

$$-1 \leqq -\frac{r'}{\sqrt{1 - r^2}} \leqq 1.$$

Integrating from $\tau_o$ to $\tau > \tau_o$ assuming that $r'(\tau) \leqq 0$ throughout $(\tau_o, \tau)$, one finds

$$-(\tau - \tau_o) + \arccos r_o \leqq \arccos r \leqq (\tau - \tau_o) + \arccos r_o,$$

where $r_o = r(\tau_o)$. This in turn implies $\cos[\tau - \tau_o - \arccos r_o] \geqq r(\tau)$ and $r(\tau) \geqq \cos[\tau - \tau_o + \arccos r_o]$, where the former inequality holds from $\tau = \tau_o$ until the cosine assumes the value unity, and the latter inequality holds from $\tau = \tau_o$ until the cosine assumes the value minus unity. The result may be stated as follows: Let the class 2 covariance $r(\tau)$ pass downward (= not upward) through the point $(\tau_o, r_o)$ in the $\tau$-$r$ plane. The curve $r = \cos \tau$ can be translated in the $\tau$-direction to pass downward through $(\tau_o, r_o)$. Then to the right of $\tau_o$, $r = r(\tau)$ lies above this translated cosine curve until either the cosine curve or $r(\tau)$ has its next local minimum. Similarly, a cosine curve can be translated to pass up through $(\tau_o, r_o)$. To the right of $\tau_o$, $r = r(\tau)$ lies below this translated cosine curve until either $r(\tau)$ has its next local minimum or the cosine curve has its next maximum.

A similar result holds if $r(\tau)$ increases through $(\tau_o, r_o)$.

Now let $\tau_o = 0$, $r_o = 1$. Then $r = r(\tau)$ lies above $r = \cos \tau$ until the first minimum of either. If the first minimum of $r(\tau)$ occurs at $\tau_1 \geqq \pi$, the theorem is proved. Suppose now $\tau_1 < \pi$ and that $r = r(\tau)$ crosses $r = \cos \tau$ in $(0,\pi)$. The first such crossing must be downward, since $r(\tau) \geqq \cos \tau$ from 0 to $\tau_1$. If the crossing is at $\bar{\tau}$, then $r(\bar{\tau}) = \cos \bar{\tau}$, and $r'(\bar{\tau}) \leqq -\sin \bar{\tau}$. If indeed $r'(\bar{\tau}) < -\sin \bar{\tau}$, one obtains from (43) the contradiction $1 \geqq r^2(\bar{\tau}) + r'^2(\bar{\tau}) > \cos^2\bar{\tau} + \sin^2\bar{\tau} = 1$. On the other hand, if the crossing takes place with $r'(\bar{\tau}) = -\sin \bar{\tau}$, then $b$ of Lemma 2 shows that $r(\tau) \leqq \cos \tau$ for $\tau < \bar{\tau}$ which contradicts the assumption that the crossing was downward. Thus, the theorem is proved.

*Theorem 13 — If $r(\tau)$ is of class 2 and*

$$r(\tau) \geqq 0, \qquad 0 \leqq \tau \leqq \frac{\pi}{\sqrt{2}},$$

*then*

$$r(\tau) \geqq r_2\left(\frac{1}{\sqrt{2}}, \tau\right) = \tfrac{1}{2} + \tfrac{1}{2} \cos \sqrt{2}\,\tau = \cos^2\left(\frac{\tau}{\sqrt{2}}\right)$$

*for*

$$0 \leqq \tau \leqq \frac{\pi}{\sqrt{2}}.$$

The theorem is a consequence of repeated applications of Lemma 2. We prove the theorem by supposing it false and then arrive at a contradiction. We refer to the curve $r = r_2(1/\sqrt{2},\tau)$, $0 \leqq \tau \leqq \pi/\sqrt{2}$ as $C$.

Suppose now that $r(\tau) \geqq 0$ for $0 \leqq \tau \leqq \pi/\sqrt{2}$ and that some point $P_o$ on $r = r(\tau)$, say $(\tau_o, r_o)$, lies below $C$. Denote $r'(\tau_o)$ by $r_o'$. We can suppose $P_o$ chosen so that $r_o' < 0$, since $r = r(\tau)$ cannot be nondecreasing at all points where it lies below $C$. Let the horizontal line $r = r_o$ through $P_o$ intersect $C$ at $P_1$ and denote the slope of $C$ at $P_1$ by $C'(r_o)$. The point $P_1$ has larger abscissa than the point $P_o$. The curve $r = r(\tau)$ possesses a continuous derivative. As the height $r_o$ of the horizontal line $r = r_o$ is continuously decreased to zero from its initial value, a value must be found with $P_o$ to the left of $P_1$ and $r_o' \geqq C'(r_o)$. By $b$ of Lemma 2, a curve of the family $F$ with parameter value $\beta \leqq 1/\sqrt{2}$ can be translated to the left to pass through $P_o$ with slope $r_o'$. In the interval $0 \leqq \tau \leqq \tau_o$, this translated member of $F$ lies strictly below $C$ and is monotone. The first local maximum of $r = r(\tau)$ to the left of $P_o$ therefore lies below $C$ as must also the local minimum just preceding this maximum. A curve of $F$ can then be translated to pass through this local minimum with slope zero, and repetition of the argument shows that all local maxima of $r = r(\tau)$ for $0 \leqq \tau \leqq \tau_o$ lie below $C$. In particular $r(0) < 1$, which contradicts the initial assumption concerning $r(\tau)$. Q.E.D.

*Theorem 14 — Let the covariance $r(\tau)$ have the behavior*

$$r(\tau) = 1 - \frac{\tau^2}{2} + m\frac{\tau^4}{4!} + o(\tau^4),$$

*near $\tau = 0$. Then*

$$r(\tau) \leqq r_2\left(\frac{1}{\sqrt{m}}, \tau\right), \qquad 0 \leqq \tau \leqq T_1,$$

*with $r_2(\beta,\tau)$ given by (45). Here $T_1 = \min(\beta\pi,\tau_o)$ and $\tau_o$ is the smallest positive value of $\tau$ for which $r(\tau) = 1 - 2/m$.*

Proof — The first four derivatives of $r(\tau)$ exist at $\tau = 0$. From the Bochner representation for $r(\tau)$, it is easy to show using Schwarz's inequality that

$$v^2 \equiv m - 1 \geqq 0. \tag{49}$$

It also follows that $r''(\tau)$ exists everywhere and is continuous.

The Gaussian process $X(t)$ having covariance $r(\tau)$ has first and second derivates $X'(t)$ and $X''(t)$ almost everywhere with probability 1. The covariance matrix of the random variables $X(0),X(t),X'(t),X''(t)$ is

$$\begin{vmatrix} 1 & r & r & r'' \\ r & 1 & 0 & -1 \\ r' & 0 & 1 & 0 \\ r'' & -1 & 0 & m \end{vmatrix}.$$

The determinant of this matrix cannot be negative. This is equivalent to the inequalities

$$-v \leqq \frac{r + r''}{\sqrt{1 - r^2 - r'^2}} \leqq v.$$

In any region where $r' \leqq 0$, the right-hand inequality gives

$$\frac{r'(r + r'')}{\sqrt{1 - r^2 - r'^2}} = -\frac{d}{d\tau}\sqrt{1 - r^2 - r'^2} \geqq vr'.$$

Integrate this from 0 to $\tau$ to obtain

$$\sqrt{1 - r^2 - r'^2} \leqq v(1 - r). \tag{50}$$

Note that if $\tau_1$ is the first positive value of $\tau$ for which $r'(\tau) = 0$, (50) gives

$$r(\tau_1) \leqq \frac{v^2 - 1}{v^2 + 1}.$$

Thus we have the interesting side result that if $r(\tau)$ is everywhere non-negative $v^2 \geqq 1$ or $m \geqq 2$.

Squaring the inequality (50) and rearranging the terms, one finds

$$r'^2 \geqq (1 + v^2)(1 - r)(r - \alpha),$$

where

$$\alpha = \frac{v^2 - 1}{v^2 + 1} < 1. \tag{51}$$

Since $r' \leqq 0$, this implies

$$\frac{r'}{\sqrt{(1 - r)(r - \alpha)}} \leqq -\sqrt{1 + v^2},$$

if $r > \alpha$. Integration from 0 to $\tau$ yields

$$\arcsin\frac{r - (1 - \alpha)/2}{(1 - \alpha)/2} - \frac{\pi}{2} \leqq -\sqrt{1 + v^2}\,\tau,$$

where it is assumed that $\tau \leqq \tau_o$. If then

$$\left| \frac{\pi}{2} - \sqrt{1 + v^2}\,\tau \right| \leqq \frac{\pi}{2},$$

$$\frac{r - (1 + \alpha)/2}{(1 - \alpha)/2} \leqq \sin\left(\frac{\pi}{2} - \sqrt{1 + v^2}\,\tau\right),$$

or, what is the same thing in virtue of the definitions (49) and (51),

$$r(\tau) \leqq 1 - \frac{1}{m^2} + \frac{1}{m^2}\cos(m\tau).$$

The theorem is thus proved.

### 2.8 *Proof of Theorem 9*

Let $h(\xi)$ be nonnegative for $0 \leqq \xi \leqq \theta$ and zero elsewhere. Then

$$Y(t) = \int_{t-\theta}^{t} h(t - t')X(t')\,dt' = \int_{-\infty}^{\infty} du\, h(u)X(t - u)\,du,$$

will certainly be nonnegative for $0 \leqq t \leqq T$ whenever $X(t)$ is nonnegative for $-\theta \leqq t \leqq T$. The probability that the $Y$ process be nonnegative in $(0,T)$ is therefore not less than the probability that the $X$ process be nonnegative in $(-\theta,T)$. If $X$ is Gaussian with mean zero and covariance $r(\tau)$, then $Y$ is Gaussian with mean zero and covariance

$$\begin{aligned} r_\theta(\tau) &= EY(t)Y(t + \tau) = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\; h(u)h(v)EX(t - u)X(t + \tau - v) \\ &= \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\; h(u)h(v)r(\tau - u + v) \\ &= \int_{-\infty}^{\infty} dx\; r(\tau - x) \int_{-\infty}^{\infty} d\xi\; h(x + \xi)h(\xi). \end{aligned}$$

One has then $P[T,r_\theta(\tau)] \geqq P[T + \theta, r(\tau)]$, which is Theorem 9.

### 2.9 *Proof of Theorem 10*

Let $0 = t_1 < t_2 < \cdots < t_n = T$ be a partition of $(0,T)$. Define $Q_n(\mathbf{r})$ by

$$Q_n(\mathbf{r}) = \frac{\Pr\,(X(t_1) < 0, X(t_i) \geqq 0, i = 2, 3, \cdots, n)}{\Pr\,(X(t_1) < 0, X(t_2) \geqq 0)}, \tag{52}$$

where $X(t)$ is a Gaussian process with zero mean and class 2 covariance

$r(\tau)$. As the partition is refined with mesh tending to zero, $Q_n(\mathbf{r})$ approaches $Q[T,r(\tau)]$ as a limit. The numerator on the right of (52) is $P_n(\hat{\mathbf{r}})$ where

$$\hat{\mathbf{r}} = \begin{vmatrix} 1 & -r(t_2) & -r(t_3) & \cdots & -r(t_n) \\ -r(t_2) & 1 & r(t_3 - t_2) & \cdots & r(t_n - t_2) \\ -r(t_3) & r(t_3 - t_2) & 1 & \cdots & r(t_n - t_3) \\ \vdots & & & & \vdots \\ -r(t_n) & r(t_n - t_2) & r(t_n - t_3) & \cdots & 1 \end{vmatrix}, \quad (53)$$

and as usual $P_n(\mathbf{r})$ denotes the probability that $n$ normal variates of mean zero and covariance matrix $\mathbf{r}$ be nonnegative. Note that the denominator of the right of (52) depends only on $r(t_2)$.

Let another Gaussian process, $Y(t)$, have class 2 covariance $q(\tau)$. We define $r^{-1}(\tau), q^{-1}(\tau), h(\tau) = r^{-1}[q(\tau)]$ as in Section 2.3 and set $g(t) = q^{-1}[r(t)] = h^{-1}(t)$. Note that $g(t)$ is strictly monotone within its domain of definition. Assume that $T$ is within the domain of definition of $g$. With the points $t_i$ given as in (52), set $\tau_i = g(t_i), i = 1,2,\cdots,n$. The points $0 = \tau_1 < \tau_2 < \cdots < \tau_n = g(T)$ form a partition of the interval $(0, g(T))$. The mesh of this partition tends to zero with the mesh of the $t_i$ partition.

Consider now the approximation to $Q[g(T),q(\tau)]$ given by

$$Q_n(\mathbf{q}) = \frac{\Pr\{Y(\tau_1) < 0, Y(\tau_i) \geqq 0, i = 1, 2, 3, \cdots, n\}}{\Pr\{Y(\tau_1) < 0, Y(\tau_2) \geqq 0\}}. \quad (54)$$

The numerator here is $P_n(\hat{\mathbf{q}})$ where $\hat{\mathbf{q}}$ is given by (53) with $r$ replaced by $q$ and $t$ replaced by $\tau$. Since $\tau_i = g(t_i), q(\tau_i) = r(t_i), i = 1,2,\cdots,n$, so that the first row and column of $\hat{\mathbf{r}}$ are the same as the first row and column of $\hat{\mathbf{q}}$. For any other entry of $\hat{\mathbf{r}}$ with $t_i \geqq t_j$, one has

$$\begin{aligned} r(t_i - t_j) &= q[g(t_i - t_j)] \\ &= q[\tau_i - \tau_j + \{g(t_i - t_j) - g(t_i) + g(t_j)\}]. \end{aligned}$$

Since $q(\tau)$ is nonincreasing

$$r(t_i - t_j) \leqq q(\tau_i - \tau_j)$$

and hence by Lemma 1

$$P_n(\hat{\mathbf{r}}) \leqq P_n(\hat{\mathbf{q}}),$$

provided

$$g(t_i - t_j) - g(t_i) + g(t_j) \geqq 0.$$

or what is the same thing, provided

$$g(x) + g(y) \geqq g(x + y), \tag{55}$$

where $0 \leqq x = t_j < t_i = x + y$.

When (55) is satisfied, the numerator of (54) is not less than the numerator of (52). The denominators of these expressions are equal since they are the same function of $r(t_2) = q(\tau_2)$. Therefore, $Q_n(\mathbf{q}) \geqq Q_n(\mathbf{r})$. The conclusion of Theorem 10 results by passing to the limit as the $t$ partition is refined.

### 2.10 *Generalizations*

A number of the results presented in this paper can be generalized in a direct manner. We only mention here an obvious extension of Theorem 1.

In the derivation of Lemma 1, the lower limit of integration for $x_i$ in (33) can be replaced by $a_i$. Now choose $a_i = a(t_i)$ with $a(t)$ a given function defined for $0 \leqq t \leqq T$, and where the points $t_i$ form a partition of $(0,T)$. Proceeding as in the derivation of Theorem 1, one arrives at the following more general result. Let $X(t)$ be a Gaussian process with $EX(t) = 0$, $EX(t)X(s) = r(s,t)$. Let $Y(t)$ be a Gaussian process with $EY(t) = 0$, $EY(t)Y(s) = q(s,t)$. Then if

$$r(s,s) = q(s,s), \qquad 0 \leqq s \leqq T$$

and

$$r(s,t) \geqq q(s,t), \qquad 0 \leqq s,t \leqq T$$

$$\Pr\{X(t) \geqq a(t), 0 \leqq t \leqq T\} \geqq \Pr\{Y(t) \geqq a(t), 0 \leqq t \leqq T\}.$$

### 2.11 *Asymptotics*

As already remarked in the introduction of this paper, there appears to be little in the literature concerning the asymptotic behavior of $P[T,r(\tau)]$ for large $T$. Intuition would indicate exponential falloff for a wide class of covariances. Example (*ii*) of Section 1.1, though special in nature since $r_2(\beta,\tau)$ is periodic, provides a counterexample to exponential behavior, and so the class must be carefully defined. Here, by the two bounds presented in Section 1.4, we have shown exponential behavior for nonnegative covariances that vanish identically for $\tau$ greater than some $\tau_o > 0$. Recently, by using Theorem 1, M. Rosenblatt has established an asymptotic exponential upper bound for $P[T,r(\tau)]$ for all covariances which are ultimately majorized by a decaying exponential. This, together with the lower bound of Section 1.4, establishes the asymptotic exponential behavior of $P[T,r(\tau)]$ for all nonnegative covariances that themselves decay exponentially. Professor Rosenblatt has also established that if $r(\tau) \rightarrow 0$ with increasing $\tau$, then $T^n P[T,r(\tau)] \rightarrow 0$ with increasing $T$ for every $n > 0$.

We conclude with the remark that from (23) of Section 1.7, one can show that asymptotic exponential behavior of $P[T,r(\tau)]$ implies asymptotic exponential behavior for $Q[T,r(\tau)]$.

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