Properties and Design of the Phase-Controlled Oscillator with a Sawtooth Comparator

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A sawtooth phase comparator has advantages over the more common sinusoidal comparator in a phase-controlled oscillator because its output is linear for larger values of phase error. For some applications, it is no more complex or expensive than the sinusoidal comparator.

This paper analyzes properties of the phase-controlled oscillator with a sawtooth comparator that have been mentioned in the literature for sinusoidal comparators. In addition, there is new theoretical material on the effect of fast jitter and noise.

The properties of the circuit are presented in a manner which is convenient for design.

Since it is easier to analyze the circuit with a sawtooth comparator, many applications of the device have been considered. Because of this wide viewpoint, the paper may be helpful in understanding the phase-controlled oscillator in general.

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I. INTRODUCTION

The phase-controlled oscillator (see Fig. 1), otherwise known as the phase-locked oscillator, is often used to produce a signal whose frequency and phase are controlled by an input signal. The literature[1,2,3] on the subject assumes that the phase comparator, which is the error detector of the loop, produces an output which is proportional to the sine of the phase difference.

This paper considers the case of the sawtooth comparator, whose output is a linear function of the phase difference over a periodic range (see Fig. 2a). Because of this linearity, the sawtooth comparator is superior in operation to the sinusoidal comparator for some applications. In general, the sinusoidal comparator is simpler and cheaper, but in applications involving digital signals, the two are comparable in cost and complexity.

The purpose of this paper is to present a comprehensive survey of many properties of the phase-controlled oscillator, relating to many different applications. We have drawn heavily on the literature, modifying the analysis to make it apply to the sawtooth comparator. In addition, there is new theoretical material on the effect of fast jitter and noise. New results derived by A. J. Goldstein in a companion paper[4] are presented in an abbreviated form, more suitable for design.

Most of the properties are presented in a graphical form which facilitates design.

II. DESCRIPTION

2.1 *General*

The block diagram of a phase-controlled oscillator is shown in Fig. 1. Notice the resemblance to a negative feedback amplifier or a servo loop.[2] There is a forward gain path, a feedback path, and a subtracting or error detecting device.

The input and output signals are not the voltages themselves, but are the phases of the nearly periodic voltages. If the input and output voltages are at different frequencies, dividers or multipliers must be used to bring them to a common frequency at the phase comparator. In this paper, we will assume that the output and the input are at the same frequency. We will however, consider the use of dividers to allow the comparator to operate on the Nth submultiple of the input and output frequency. We will measure phase of the submultiple signals in radians of the original frequency.

2.2 *Phase Comparator*

The phase comparator is the error detector of the servo loop. It produces a voltage which depends on the phase difference between the input submultiple and the output submultiple.

Of course, the comparator cannot distinguish between different cycles

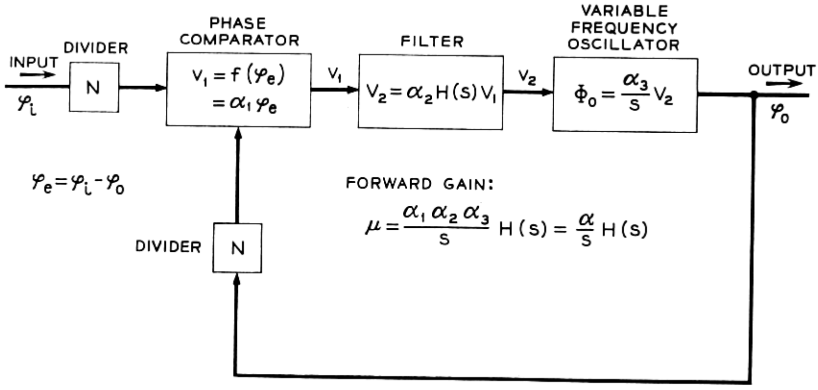


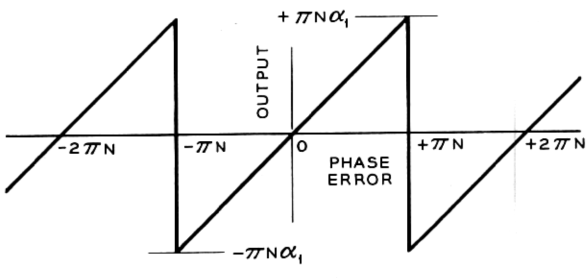
Fig. 1 — Block diagram of the phase-controlled oscillator.

of the input and output submultiples. Therefore, its output must be a periodic function of the phase difference between input and output, with a period equal to one cycle of the submultiple frequency or N cycles of the input and output frequency. We see that the greater the divider ratio, the greater the range of the phase comparator, in cycles of the input and output frequency.

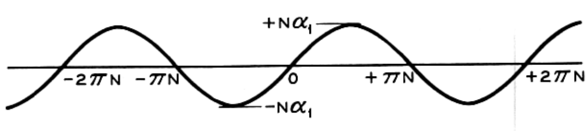
The sawtooth and sinusoidal comparator functions are shown in Fig. 2. The phase error is measured in radians of the input and output frequency. The gains have been adjusted so that the slopes at zero are identical. This means that the functions have the same small-signal performance at zero quiescent phase error. Note that the peak output of the sawtooth comparator is π times the peak of the sinusoidal comparator.

The sampler and mixer types of sinusoidal comparator are described in the literature.[5]

Since the sawtooth characteristic is not common, we will describe one method of building such a comparator. We assume that the input and output signals are available as short pulses. If the signals are originally sinusoids, the pulses can be obtained from zero crossings. As shown in Fig. 3, these pulses control a flip-flop. The input is sent into the set terminal of the flip-flop and the output is sent into a comple-



(a) SAWTOOTH CHARACTERISTIC



(b) SINUSOIDAL CHARACTERISTIC

Fig. 2 — Characteristics of the sawtooth and sinusoidal phase comparators.

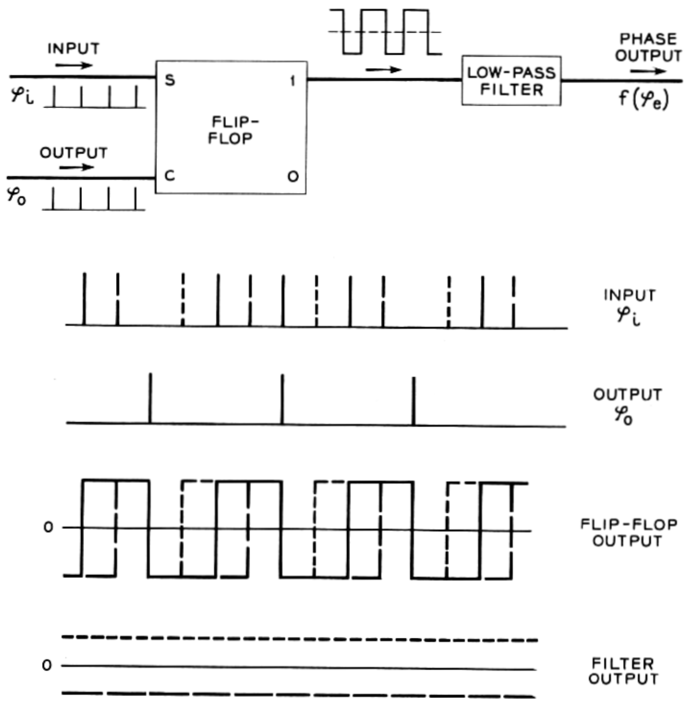


Fig. 3 — Flip-flop sawtooth phase comparator.

ment (or count) terminal. Therefore the time spent in the set state will be the time between the input pulse and the output pulse.

If the flip-flop puts out a positive voltage in the set state and an equal negative voltage in the reset state, the average output voltage will be a linear function of the phase error. The average output will be zero when the pulses are 180° out of phase. Therefore one of the signals should be inverted before pulse forming if the output is desired to be in phase with the input.

If the phase error exceeds $\pm\pi$, the pulses will pass each other. There will be a sudden discontinuity, and the voltage will change quickly from one extreme to the other.

If the input signal is turned off, the flip-flop acts like a binary counter, driven by the output signal. The average output voltage will be zero.

The average voltage will be extracted from the flip-flop output by the low-pass filter. It should have a cutoff frequency low enough to remove signal components near the submultiple frequency.

Since this type of comparator works on zero crossings, its conversion gain is independent of signal amplitude.

A sampler comparator can also have a sawtooth characteristic, if the input has a sawtooth waveform.

Because of the operation of the phase comparator, the phase-controlled oscillator is really a sampled system. E. G. Kimme has shown[6] that the phase-controlled oscillator can be treated as a continuous system if the sampling frequency is so high that its effects are strongly attenuated by the closed loop. We will assume this to be the case throughout this paper.

2.3 *Filter*

The filter has a low pass characteristic to attenuate fast changes in the phase error due to noise in the input signal. It also helps to smooth out the high frequency component of the phase comparator output. Usually a simple RC filter or a phase lag filter is used, as shown in Fig. 4.

2.4 *Oscillator*

The variable oscillator produces the output signal. When its input voltage v_2 is zero, the output frequency is the design center frequency ω_c. If v_2 is not zero, the output frequency varies in proportion to v_2. Since the important property of the output is the phase, which is the integral of the frequency, the variable oscillator acts like a perfect integrator.

III. OPERATION

Readers who have a background in servo systems may find it helpful to think of the phase-controlled oscillator as a type 1 servo system, such

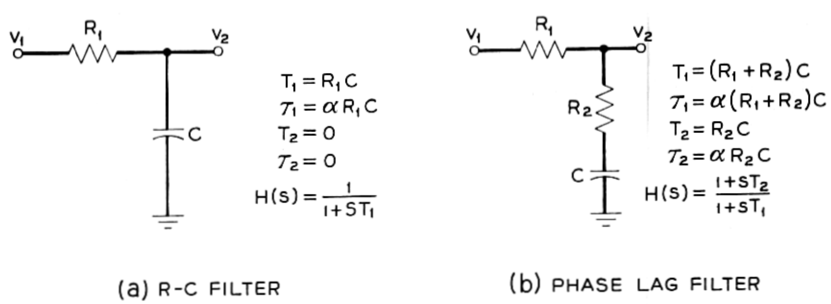


Fig. 4 — Filters.

as a velocity motor with position feedback.[2] The analogy is clear from Fig. 1.

3.1 *Aligned Operation*

Let the frequency of the input signal be identical to the center frequency of the oscillator and let the phase error be zero. Then the input to the oscillator is zero and its frequency will be identical to that of the input.

Now let us quickly advance the input phase by a small amount and continue at the center frequency. There will be a positive error voltage which will increase the output frequency. The output phase will advance until it catches up to the input. The circuit cannot settle down until the output phase is identical to the input phase, because of the integrating action of the oscillator.

3.2 *Mistuning*

Assume that the input frequency increases a little, causing the input phase to continually advance. As before, a positive error signal will result, increasing the output frequency. Therefore the output phase will continually advance. When the circuit settles down to a steady state, the phase error will be constant, and just sufficient to detune the oscillator so that its frequency will be *identical* to the input frequency. The greater the phase-to-frequency gain of the forward path, the less phase error will result from a given input frequency deviation.

3.3 *Jitter*

Now let the average input frequency be constant, but assume that the phase is jittering back and forth. Suppose the jitter is very rapid. Even if there were no filter, the integrating action of the oscillator would smooth out the jitter so that the output would be more stable than the input. The low-pass filter, of course, smooths the error signal before it gets to the oscillator and attenuates the jitter even more.

If the amplitude of the jitter is too great, the phase comparator will go through a discontinuity, and when the circuit settles down again, it will have slipped N cycles of the input, either ahead or behind.

As the rate of jitter decreases, the operation of the loop becomes more complex. Because of the integration, jitter in the oscillator phase lags the fluctuations in its input voltage by 90°. If the low pass filter also has about 90° phase lag at some frequency of jitter, we see that we have positive feedback instead of negative feedback. The open loop phase gain is the ratio of a change in output phase to a change in phase error. If this is large enough at a frequency where we have positive feedback, we can actually have an increase in jitter, or even a jitter oscillation[1] which would destroy the usefulness of the device for most purposes.

If the jitter is very slow compared to the loop time constants, the servo loop will track it, and the jitter will be passed on to the output.

If the jitter is distributed in a wide band, such as that caused by the addition of white noise to a coherent signal, the circuit will respond only to that jitter resulting from noise components near the frequency of the coherent signal. Therefore the circuit can be used to enhance the signal-to-noise ratio of a phase-modulated carrier. This property also allows the circuit to lock on a coherent signal of approximately known frequency although it is surrounded by strong wide-band noise.

3.4 *Phase Modulation*

The error signal v_1 (see Fig. 1) is essentially proportional to the phase modulation of the input at frequencies higher than the circuit can track, and to the frequency modulation at frequencies that can be tracked. The signal at v_2 is filtered to reduce noise. Therefore the circuit can be used as a demodulator of phase or frequency modulated signals in noise.

The circuit can also be used as a phase modulator. The carrier is connected to the input. The modulating voltage is added to the output of the comparator. The feedback tends to keep the oscillator input voltage small. Therefore the comparator output must be nearly equal to the negative of the modulating voltage. This means that the output phase is nearly proportional to the modulating voltage. At high frequencies, the loop gain drops, and these relations are no longer valid.

3.5 *Quieting*

If the input signal is smooth, but the oscillator itself is jittery because of internal noise, the oscillator will be quieted by the feedback, especially at low frequencies where the problem is likely to be most serious.

3.6 *Discontinuities*

We have looked at the small-signal linear performance of the phase-controlled oscillator; now let us examine its operation when it is passing through discontinuities. Suppose we increase the input frequency until the phase error is nearly equal to $+N\pi$, where N is the divider ratio. A small further increase will cause the phase comparator to go through a discontinuity, making the error $-N\pi$. This will start to decrease the oscillator frequency and the error will rapidly return to $+N\pi$, and then jump to $-N\pi$ again. After a short time, the error will settle down to a periodic behavior, with discontinuities at regular intervals. Since the average error must be somewhat less than $+N\pi$, the average output frequency will be somewhat less than the input frequency, and the frequency of the phase error will be the beat frequency between input and output, divided by N.

3.7 *Pull-in*

As the input frequency is reduced in this "flickering" state, the beat frequency decreases. Finally, the phase error does not quite hit a discontinuity at its highest excursion, and the error settles down to a static value. We say the loop has pulled into lock with the input.

Depending on the nature of the filter, there may or may not be hysteresis in the pull-in action. If there is hysteresis the pull-in frequency deviation will be less than the deviation which can be held in lock, once lock has been established.

IV. APPLICATIONS

The phase-locked oscillator has many interesting capabilities, and consequently has found many diverse applications.[1] Some of the functions and examples of use are:

a. Locking a high frequency signal to a submultiple; television sync signals are locked to the power frequency.

b. Locking a strong steady signal to a weak, intermittent signal; television color carrier recovery.

c. Locating and locking on a weak coherent signal in wide-band noise; space communication.

d. Detecting phase or frequency shifts in a signal; space communication.

e. Smoothing a jittery signal; smoothing jitter in a digital signal.

f. Locking a high-power oscillator to a more stable low-power oscillator; microwave generation.

g. Phase modulation of a reference carrier.

h. Frequency synthesis.

Each of these applications requires a different viewpoint in analyzing the circuit. An optimization process for one application may be useless in another. Even an expression such as noise bandwidth may not have the same meaning with a jitter reducing circuit as with a microwave source.

The application we have foremost in mind is that of capturing and smoothing a jittering timing signal for a digital channel. Most of the properties we analyze are chosen for this application. However, we present additional material which is needed for other applications. We have attempted to be explicit in revealing our viewpoint when we define noise bandwidth, figure of merit, etc.

V. QUIESCENT OPERATION

5.1 *Steady-State Error*

If a phase-locked oscillator is synchronized with a signal whose frequency is not identical with the oscillator's center frequency, there must be a steady phase error. The comparator converts this phase error into the voltage required to tune the oscillator so that its output frequency will be identical to the input frequency.

The gain α is the low frequency conversion gain from phase error to frequency (see Fig. 1). It is the change in output frequency (in radians per second) that results from a change in phase error of one radian. The mistuning frequency ω_m is the difference between the input frequency and the oscillator center frequency. Then the steady phase error is

$$\varphi_e = \frac{\omega_m}{\alpha}. \tag{1}$$

The phase error is directly proportional to the mistuning. With a given mistuning, the error may be made as small as desired by increasing the gain, α. However, we shall see that high gain has undesirable effects also.

5.2 *Lock Frequency*

The greatest frequency mistuning that can be locked in synchronism is determined by the maximum output of the phase comparator. At the limit,

$$|\,\omega_m\,| = \omega_L = N\pi\alpha. \tag{2}$$

We call ω_L the *lock frequency.*

5.3 *Phase Error Margin*

One of the advantages of the sawtooth comparator over a sinusoidal comparator is that the small-signal performance is independent of the steady mistuning, since the gain does not depend on the phase error. However, mistuning reduces the margin between the steady phase error and the error which will cause a discontinuity. This limits the permissible peak jitter amplitude, if no discontinuities are allowed.

The phase error margin is

$$\varphi_{er} = N\pi - \frac{|\omega_m|}{\alpha}. \tag{3}$$

VI. RESPONSE IN THE LINEAR REGION

As long as the circuit is in synchronism and the phase error does not exceed the bounds of $\pm N\pi$, the phase controlled oscillator acts like a linear feedback system.

6.1 *Phase Response*

From Fig. 1, we see that the forward gain of the loop is the product of the gains of the comparator, filter, and oscillator:

$$\begin{aligned} \mu &= [\alpha_1][\alpha_2 H(s)]\left[\frac{\alpha_3}{s}\right] \\ &= \alpha \frac{H(s)}{s}, \end{aligned} \tag{4}$$

where $\alpha = \alpha_1\alpha_2\alpha_3$.

The feedback is

$$\beta = 1. \tag{5}$$

The response of the output phase to changes in the input phase is given by the familiar negative feedback equation:

$$Y = \frac{\Phi_o}{\Phi_i} = \frac{\mu}{1 + \mu\beta} = \frac{\alpha H(s)}{s + \alpha H(s)}. \tag{6}$$

The signals Φ_i and Φ_o are phases of the input and output voltages.

The phase error, as a function of the input phase, is

$$\Phi_e = \Phi_i - \Phi_o = \frac{s}{s + \alpha H(s)} \Phi_i . \tag{7}$$

Notice that we measure phase of the submultiple signals in radians of the original signals.

The filter is usually either an *RC* filter or a phase lag filter, as shown in Fig. 4. For the phase lag, the more general case,

$$H(s) = \frac{1 + sT_2}{1 + sT_1}, \tag{8}$$

where

$$T_1 \geqq T_2 .$$

In the RC case, $T_2 = 0$.

When we substitute (8) into (6), the transfer ratio becomes

$$Y = \frac{1 + s\dfrac{\tau_2}{\alpha}}{1 + s\dfrac{1+\tau_2}{\alpha} + s^2\dfrac{\tau_1}{\alpha^2}}, \tag{9}$$

where

$$\tau_1 = \alpha T_1 , \qquad \tau_2 = \alpha T_2 .$$

The phase error response is found from (7):

$$\Phi_e = \frac{\dfrac{s}{\alpha}\left(1 + s\dfrac{\tau_1}{\alpha}\right)}{1 + s\dfrac{1+\tau_2}{\alpha} + s^2\dfrac{\tau_1}{\alpha^2}}\Phi_i . \tag{10}$$

Note that the denominator of transfer functions (9) and (10) is a second order polynomial, of the form

$$1 + s\frac{2\xi}{\omega_n} + s^2\left(\frac{1}{\omega_n}\right)^2,$$

where:

$$\omega_n = \frac{\alpha}{\sqrt{\tau_1}}, \qquad \xi = \frac{1}{2}\frac{\tau_2 + 1}{\sqrt{\tau_1}}. \tag{11}$$

Equations (9) and (10) appear frequently in the literature, but have been included here for completeness. Some of the literature[1,2] uses the natural frequency ω_n and the damping ratio ξ as defining parameters of the system. We shall use α, τ_1 and τ_2 more often, because they are more closely related to physical quantities.

Most of the important properties of the phase-controlled oscillator can be expressed as normalized ratios which are independent of α. Therefore we shall present these properties as functions of the two remaining design parameters, τ_1 and τ_2. As an example of our method of presentation, contours of constant damping ratio ξ are shown on a plot of τ_2 vs τ_1 in Fig. 5. We will call this the filter plot. Properties of the filter plot are discussed in Section IX.

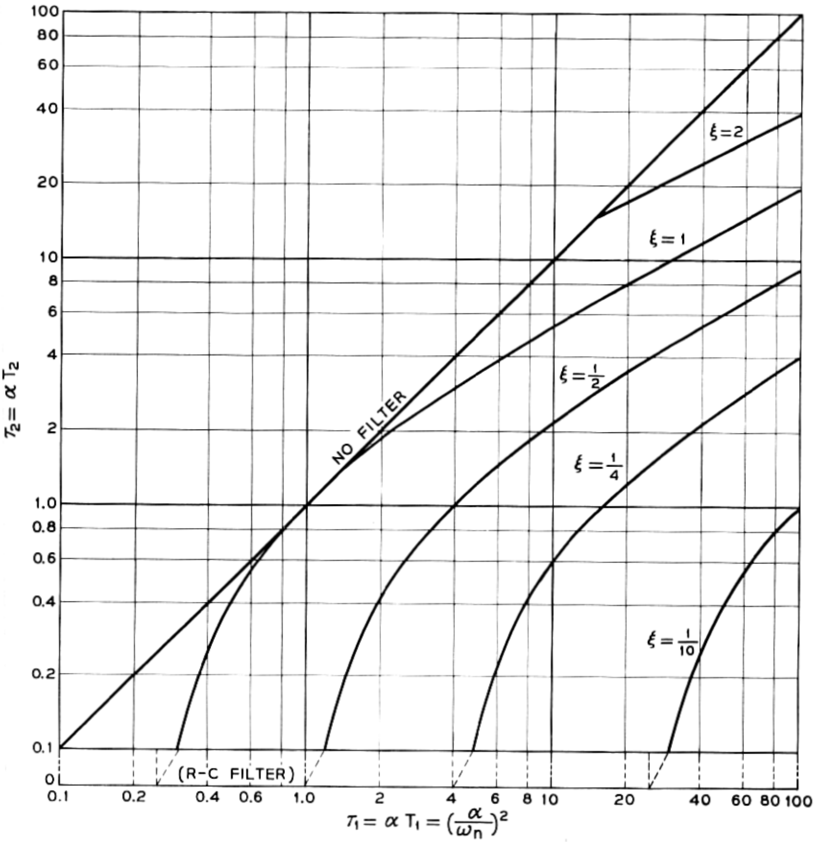


Fig. 5 — Contours of constant damping ratio on the filter plot.

6.2 *Voltage Response*

As we have mentioned, the phase-locked oscillator can be used as a phase modulator by adding a modulating voltage v_M to the error voltage v_1. The response of the output phase is:

$$\Phi_o = \frac{1}{\alpha_1} Y V_M . \tag{12}$$

where Y is given by (6) and (9). Note that we have used V_M for the transform of v_M. Examination of (9) shows that the output phase will follow the input voltage as long as the modulating frequency is low enough, since Y approaches unity as s approaches zero.

If the phase-locked oscillator is used as a demodulator the output can be taken before or after the filter. Therefore we present the response equations for the voltages at each point (see Fig. 1).

$$V_1 = \alpha_1(1 - Y)\Phi_i \tag{13}$$

$$V_2 = \frac{1}{\alpha_3} sY\Phi_i. \tag{14}$$

VII. SMALL-SIGNAL PROPERTIES

The small-signal properties we shall analyze are the response of the output phase to sinusoidal jitter of the input phase, the noise bandwidth, the peak jitter gain, the response to a step change in phase, and the response to a step change in frequency. All of these effects are not pertinent to every system, but each is useful in some of the applications.

7.1 *Sine Wave Jitter Response*

The small signal transfer ratio Y between input phase jitter and output phase jitter was given in (9). For sinusoidal jitter, the squared magnitude (power gain) of $Y(\omega)$ is

$$|Y(\omega)|^2 = \frac{1 + \left(\frac{\omega}{\alpha}\right)^2 \tau_2^2}{1 + \left(\frac{\omega}{\alpha}\right)^2 [1 - 2(\tau_1 - \tau_2) + \tau_2^2] + \left(\frac{\omega}{\alpha}\right)^4 \tau_1^2}. \tag{15}$$

The phase of $Y(\omega)$ is

$$\theta(\omega) = \tan^{-1}\left(\frac{\omega}{\alpha}\right)\tau_2 - \tan^{-1}\frac{\left(\frac{\omega}{\alpha}\right)(1 + \tau_2)}{1 - \left(\frac{\omega}{\alpha}\right)^2 \tau_1}. \tag{16}$$

The jitter attenuation curves for several sets of filter parameters are plotted in Fig. 6.

In Case I, no filter, we have simply an integrator with unity feedback around it. At low frequencies the jitter is not attenuated; at high frequencies there is a 6 db per octave roll-off. When an RC filter is added, the additional high frequency attenuation produces a 12 db per octave roll-off. When the filter time constant is very large, the phase shift in

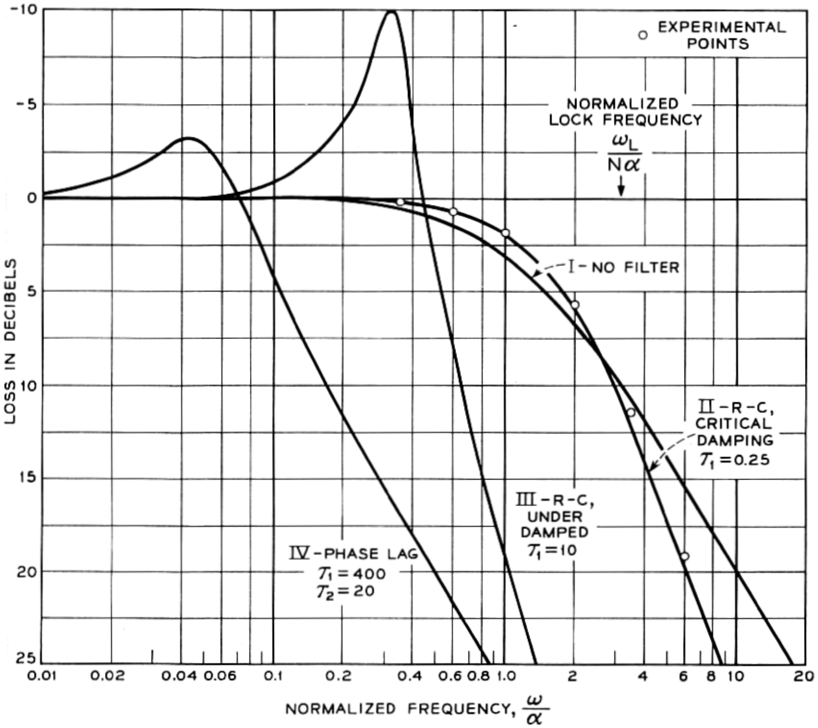


Fig. 6 — Jitter attenuation with various filter parameters.

the forward loop results in positive feedback, and causes a region where jitter is amplified.

When a phase lag filter is used, the second break point caused by the resistor in series with the capacitor can be used to stabilize the feedback loop and reduce the peak jitter gain. Since the attenuation of the phase lag filter is constant at high frequencies, the final slope is 6 db per octave.

7.2 *Noise Bandwidth*

One of the functions of a phase-controlled oscillator is to reduce noise. In the absence of better information, it is usual to assume that somewhere in the system the noise is white and Gaussian. Since most of the noise at the output is usually restricted to a narrow band by the filtering action of the circuit, it is convenient to express the amount of noise that remains as the bandwidth of an ideal filter (i.e., rectangular filter) that would pass the same mean square noise. The familiar formula for computing noise bandwidth is

$$B = \int_0^\infty |\, G(\omega) \,|^2 \, d\omega, \tag{17}$$

where $G(\omega)$ is the normalized transfer function between noise input and noise output, and B is in radians per second. The transfer function which is used for $G(\omega)$ will depend on where the noise input and output are, and this in turn will depend on the application.

When the phase-controlled oscillator is used to clear up jitter in digital signals, the appropriate transfer function is Y, the ratio of output phase shift to input phase shift, as given in (9). We will call the noise bandwidth of Y the jitter bandwidth B_j. When we substitute (9) into (17), we have

$$\frac{B_j}{\pi\alpha} = \frac{1}{2} \frac{1 + \dfrac{\tau_2^{\,2}}{\tau_1}}{1 + \tau_2}. \tag{18}$$

We recall that $N\pi\alpha$ is the lock frequency. An increase in N increases the lock frequency without changing B_j.

For no filter, or for any RC filter, the normalized jitter bandwidth is $\frac{1}{2}$. For $\tau_2^{\,2}/\tau_1$ much greater than 1, the normalized jitter bandwidth approaches $\frac{1}{2}(\tau_2/\tau_1)$. The jitter bandwidth is shown on the filter plot in Fig. 7.

With a sawtooth phase comparator, the jitter bandwidth is independent of the mistuning. This is not true of the sinusoidal comparator. The jitter bandwidth for the sinusoidal case is

$$\frac{B_j}{\pi\alpha} = \frac{1}{2} \cos \varphi_e \frac{1 + \dfrac{\tau_2^{\,2}}{\tau_1} \cos \varphi_e}{1 + \tau_2 \cos \varphi_e} \tag{19}$$

where α is the gain at zero error.

Equation (19) can be obtained from (18) by replacing α by ($\alpha \cos \varphi_e$), the small signal gain at a quiescent phase error φ_e. Note that α is a factor in τ_1 and τ_2. Notice that the jitter bandwidth for the sinusoidal comparator decreases as the mistuning (and therefore φ_e) increases.

Now let us consider the effect of interference due to broad-band noise added to the input signal. To justify a small signal analysis, we must assume that filtering limits the total energy of the interference, to keep it well below the signal level. However, we assume that the filtered noise is essentially flat in a band around the signal which is much wider than the interference noise bandwidth which we shall derive.

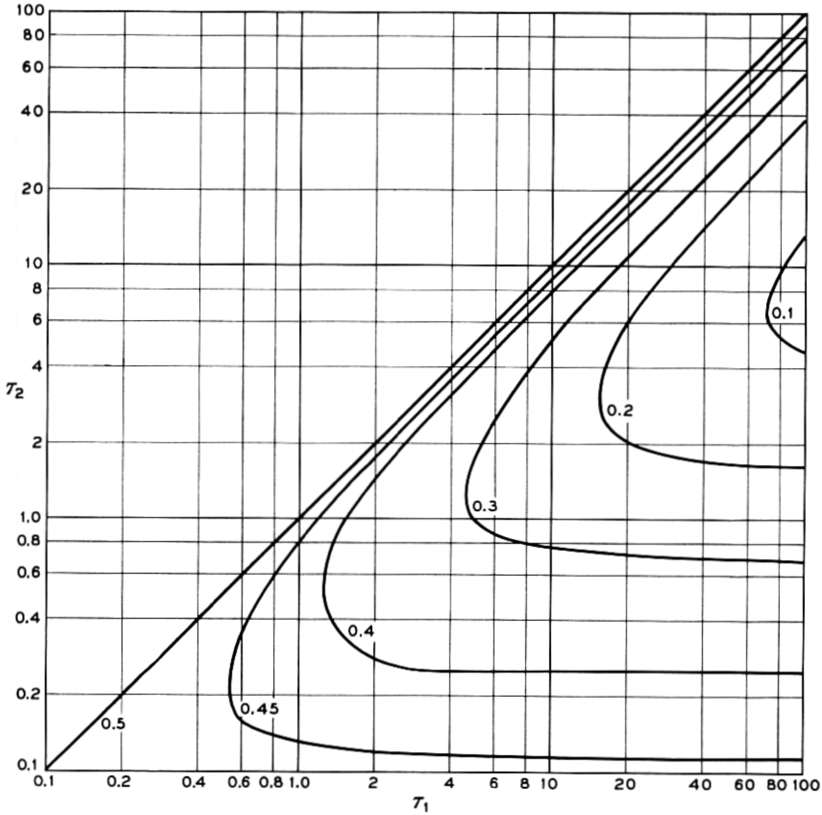


Fig. 7 — Contours of normalized jitter noise bandwidth on the filter plot.

The effect of interference depends strongly on the type of phase comparator in the system. We shall analyze the linear zero-crossing case and the sinusoidal mixer or sampler case.

Interference noise disturbs both the phase and the amplitude of the input signal. When a zero-crossing comparator is used, only the phase disturbance is detected. If the noise power density is $\overline{v_n^2}$ (volts^2 per radians per second) and the input sinusoid has a peak v_i, the jitter "power" density for phase in radians is $\overline{(v_n/v_i)^2}$ (radians^2 per radians per second).

The output jitter will be

$$\overline{\varphi_o^2} = B_j \frac{\overline{v_n^2}}{v_i^2} \tag{20}$$

The effect of broad band input noise is quite different when a sinusoidal sampler or mixer phase comparator is used. The following discussion assumes that the reader is familiar with the literature of the sinusoidal comparator. With this type of comparator, noise at the input produces a voltage at the output of the comparator which is independent of the amplitude and phase of the input signal. The noise density of the comparator output voltage is $(\alpha_1/v_i)^2\overline{v_n^2}$ where v_i is the *expected* peak signal amplitude used in computing the expected α_1 at zero error (if no limiting is used with this type of comparator, the gain depends on the signal amplitude). When the comparator is connected in a feedback loop, the appropriate method of analysis is to consider the interference noise injected at the *output* of the phase comparator. The appropriate transfer ratio is that previously used for modulation in (12).

The interference bandwidth B_i can be found by substituting (12) into (17);

$$\frac{B_i}{\pi \propto} = \frac{1}{2}(\cos\varphi_e)^{-1}\frac{1 + \left(\frac{\tau_2^2}{\tau_1}\cos\varphi_e\right)}{1 + (\tau_2\cos\varphi_e)} \tag{21}$$

This is the noise bandwidth given by Rey.[1]

Notice that the interference bandwidth B_i increases as the phase error increases, while the jitter bandwidth B_j decreases. The reason for the difference is that the sampler and mixer comparators are sensitive to the amplitude of the input signal as well as the phase.

Now we can compare the output phase noise performance of the linear zero-crossing comparator with the sinusoidal sampler or multiplier type. If they have the same gain at zero error, they will have the same response to jitter and interference at zero error. In the presence of mistuning, however, the sinusoidal comparator will be more sensitive to interference and less sensitive to jitter while the linear comparator will not change.

When the phase-controlled oscillator is used as a demodulator, still another definition of noise bandwidth is required. If we take the output signal after the filter, which cuts off some of the noise, and assume that interference noise is added to the input signal, we have for the zero-crossing detector,

$$v_2 = \left[\frac{s}{\alpha}Y\right]\frac{\alpha_1\alpha_2}{v_1}V_n. \tag{22}$$

By substituting the expression in brackets into (17), we can find the demodulator noise bandwidth, B_D. This bandwidth is not finite for the phase lag filter, because the transfer ratio does not approach zero at high frequencies. Therefore higher order filters are desirable for this application.

7.3 *Peak Jitter Gain*

We have shown in Fig. 6 that it is possible for the jitter transfer ratio to be greater than unity. In most systems, this is not very harmful. However, where phase-controlled oscillators are connected in cascade, gain can be very troublesome.

We can find the peak gain $|\hat{Y}|$ by examining (15) for its maximum. The frequency at the peak is

$$\left(\frac{\hat{\omega}}{\alpha}\right)^2 = \frac{1}{\tau_2^2}\left[\left(1 + \left(\frac{\tau_2}{\tau_1}\right)^2 [2(\tau_1 - \tau_2) - 1]\right)^{\frac{1}{2}} - 1\right]. \tag{23}$$

A. J. Goldstein[4] has shown that the square of the peak magnitude can be written

$$|\hat{Y}|^2 = \frac{1}{1 - \tau_1^2\left(\frac{\hat{\omega}}{\alpha}\right)^4}. \tag{24}$$

An examination of (23) shows that there is no peak, and the gain is never greater than unity if

$$\tau_1 - \tau_2 < \tfrac{1}{2}. \tag{25}$$

The peak gain is shown on the filter plot in Fig. 8.

7.4 *Response to a Step Change in Phase*

Fast phase changes can occur because of quick changes in the transmission path or because the signal has been deliberately modulated. When a step in phase occurs there is a sudden change in the phase error, since the phase of the oscillator cannot change instantaneously. The error signal controls the oscillator so that the error returns eventually to its quiescent value.

To act like a step change, the phase shift does not have to be instantaneous, as long as the rise time is much less than the shortest time constant of the phase-controlled loop. Therefore if the phase comparator works from a subharmonic of the input frequency the amplitude of the phase change can be several input periods, as long as the change is slow enough for the subharmonic generator (counter, etc.) to follow, but faster than the loop time constants.

If a counter is used as a subharmonic generator, an error in the counter,

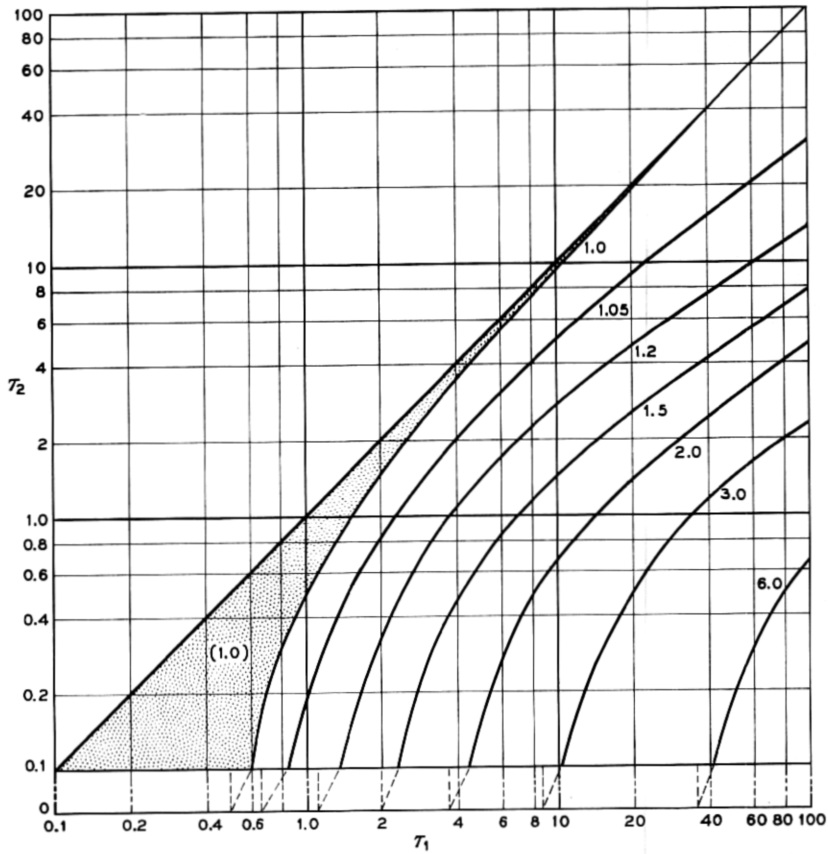


Fig. 8 — Contours of peak jitter gain on the filter plot.

or an extraneous pulse introduced into the counter, will act like a step change in input phase.

The response of the phase error to the phase input is given in (10). When the input phase is a step of amplitude $\Delta\varphi_i$, the time response of the phase error can be shown to be

$$\varphi_e = \Delta\varphi_i e^{-\xi\omega_n t}\left[\cosh\left(\sqrt{\xi^2 - 1}\,\omega_n t\right) - \frac{\xi - \dfrac{\omega_n}{\alpha}}{\sqrt{\xi^2 - 1}} \cdot \sinh\left(\sqrt{\xi^2 - 1}\,\omega_n t\right)\right]. \tag{26}$$

For the underdamped case ($\xi < 1$) the hyperbolic functions in (26) become trigonometric functions. The damping ratio ξ and the natural frequency ω_n have been defined in (11).

At $t = 0$, just after the step, we see that the phase error equals the change in input phase. If we examine the initial derivative of (26), we find that it is never positive. This means that the phase error will never exceed its initial value.

Some examples of the phase error response to a step change in phase are shown in Fig. 9.

7.5 *Response to a Step Change in Frequency*

A sudden change in frequency can occur because of a change from one source to another, because of malfunction, or because the signal has been modulated. When a frequency step occurs, the error signal builds up until the oscillator frequency catches up to the input frequency,

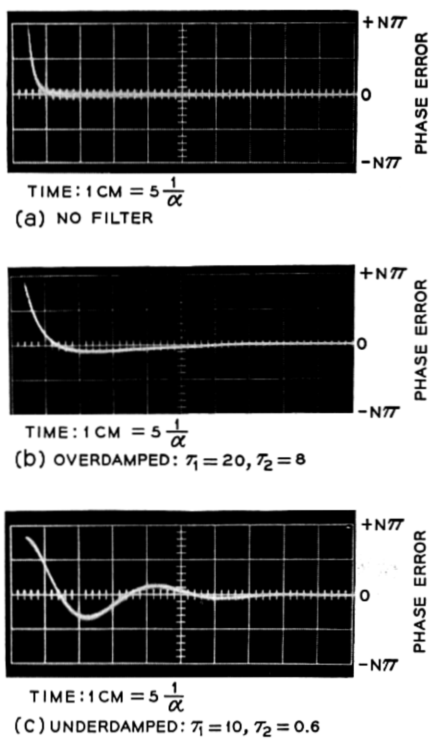


Fig. 9 — Scope traces of the response of the phase error to a step change in phase of the input.

leaving a static change in phase error. If a low-pass filter is used between the phase comparator and the oscillator, the transient phase error can have a peak value much greater than the quiescent phase change.

Let us assume a frequency change $\Delta\omega_i$. This is equivalent to a ramp phase input, $\Delta\omega_i t$. We can use (10) to find the response of the phase error:

$$\varphi_e = \frac{\Delta\omega_i}{\alpha}\left\{1 - e^{-\xi\omega_n t}\left[\cosh\left(\sqrt{\xi^2 - 1}\,\omega_n t\right) - \frac{\dfrac{\alpha}{\omega_n} - \xi}{\sqrt{\xi^2 - 1}}\sinh\left(\sqrt{\xi^2 - 1}\,\omega_n t\right)\right]\right\}. \tag{27}$$

Some examples of the phase error response to a step change in input frequency are shown in Fig. 10.

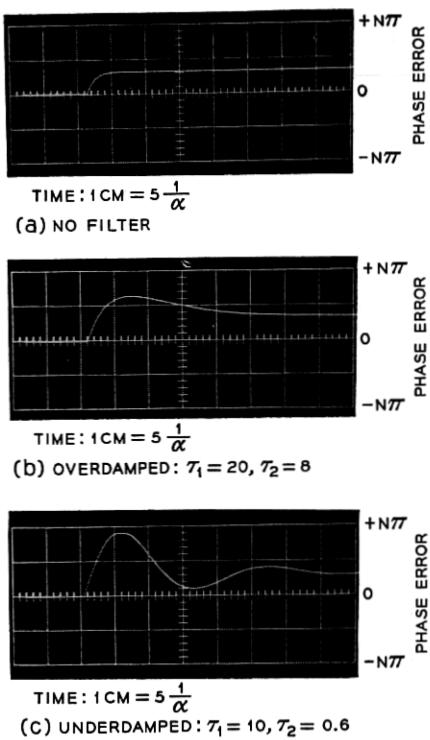


Fig. 10 — Scope traces of the response of the phase error to a step change in frequency of the input.

The peak phase error is of particular interest. For the overdamped case it is

$$\hat{\varphi}_e = \frac{\Delta\omega_i}{\alpha}\left[1 + (\tau_1 - \tau_2)^{\frac{1}{2}} \exp\left(-\frac{\xi}{\sqrt{\xi^2 - 1}} \tanh^{-1} \frac{\sqrt{\xi^2 - 1}}{\xi - \dfrac{\omega_n}{\alpha}}\right)\right]. \quad (28)$$

For the underdamped case, the inverse hyperbolic tangent is replaced by the inverse trigonometric tangent. The value of this angle is between zero and π. An expression closely related to (28) has been derived by R. D. Barnard,[7] as a capture condition.

A large value of τ_1 can result in overshoot which is many times the quiescent phase error. This means that the response of such a system to a sudden frequency shift looks like a pulse. This characteristic is useful in demodulation of a frequency shift signal.

The large overshoot can throw the loop out of synchronism if it exceeds the capacity of the phase comparator. This effect will be discussed more fully in Section 8.5.

The normalized peak phase error is shown on the filter plot in Fig. 11.

VIII. LARGE-SIGNAL PROPERTIES

We have examined the operation of the synchronized phase-controlled oscillator when the error is within the range of the phase comparator. For this "small-signal" case, the problem was completely linear. When the circuit is not in synchronism, or when disturbances of the input signal are large enough to produce a phase error which exceeds the range of the comparator, discontinuities are present in the output and the problem becomes nonlinear. Despite this difficulty, we have been able to analyze certain large-signal properties of the phase-controlled loop with a sawtooth comparator. These are the pull-in frequency, the seize frequency, the settling time, the maximum allowed frequency shift, and the effect of certain types of jitter on large-signal performance.

8.1 *Pull-in Frequency*

A very important property of the phase-locked loop is the range of frequencies that can pull the oscillator into synchronism. In general, this range is smaller than the range of frequencies which can be held in lock. When the system is not synchronized, the phase comparator goes through periodic discontinuities, which prevent the loop from synchronizing. Whether or not a loop will pull a given frequency into lock depends on the past history of the loop and the jitter of the signal.

We define the *pull-in frequency* as the maximum steady mistuning

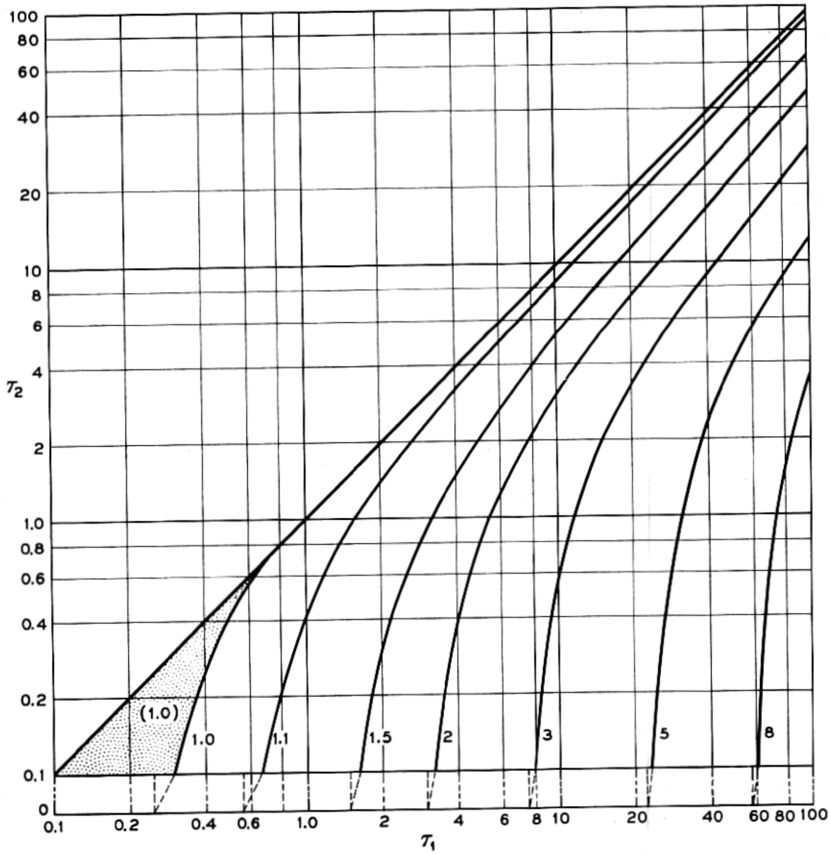


Fig. 11 — Contours of normalized peak phase error caused by a step change in frequency.

of the input frequency that will always pull the circuit into synchronism. Frequencies outside of the pull-in range but inside the lock range may or may not be pulled in, depending on the initial conditions.

We can determine the pull-in frequency experimentally by mistuning the input beyond the lock frequency and then slowly reducing the mistuning until the circuit locks. When the mistuning exceeds the lock range, there are frequent discontinuities in the phase error; it appears to "flicker." As the mistuning is slowly decreased, the flicker rate decreases.

When the mistuning is brought down to the pull-in frequency, the flicker mode becomes unstable. With the mistuning then held constant, just under the pull-in frequency, the phase error trajectory from discontinuity to discontinuity slowly changes as shown in Fig. 12. Finally, the error misses a discontinuity and synchronism is achieved.

The pull-in frequency, then, is the mistuning for which the stable asynchronous mode disappears. For lower values of mistuning, the circuit must eventually reach a synchronous condition since there is no asynchronous solution.

A. J. Goldstein[4] has found an exact answer for the pull-in frequency ω_p ;

$$\frac{\omega_p}{N\pi\alpha} = \frac{1 - D}{\tanh \frac{1}{2}\xi\omega_n T_0} + (D) \tanh \frac{1}{2}\xi\omega_n T_0 . \qquad (29)$$

where T_0 , the critical flicker period, is the smallest positive solution of

$$\sqrt{\tau_1}\sqrt{\xi^2 - 1}\,\frac{\tanh \frac{1}{2}\xi\omega_n T_0}{\tanh \frac{1}{2}\sqrt{\xi^2 - 1}\,\omega_n T_0} = \sqrt{\tau_1}\,\xi - \frac{\tau_1}{\tau_2}\left(1 - \sqrt{1 - \frac{\tau_2}{\tau_1}}\right) = c_1 ,$$

and D is given by

$$D = \frac{c_1(\sqrt{\tau_1}\,\xi - 1) - \tau_1(\xi^2 - 1)}{c_1^2 - \tau_1(\xi^2 - 1)} .$$

For the underdamped case (damping ratio $\xi < 1$) the hyperbolic tangent is replaced by the trigonometric tangent.

A. J. Goldstein[4] has used a digital computer to evaluate (29). The data is presented on the filter plot in Fig. 13.

We can see from (29) that the pull-in frequency is directly proportional to the lock frequency $N\pi\alpha$, for a given set of parameters τ_1 and τ_2 . We will call $\omega_p/N\pi\alpha$ the pull-in to lock ratio, or the relative pull-in.

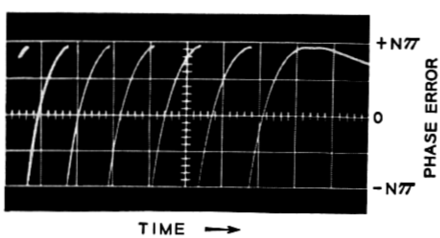


Fig. 12 — Scope trace of the phase error after the mistuning is brought just below the pull-in frequency. The flicker mode becomes unstable.

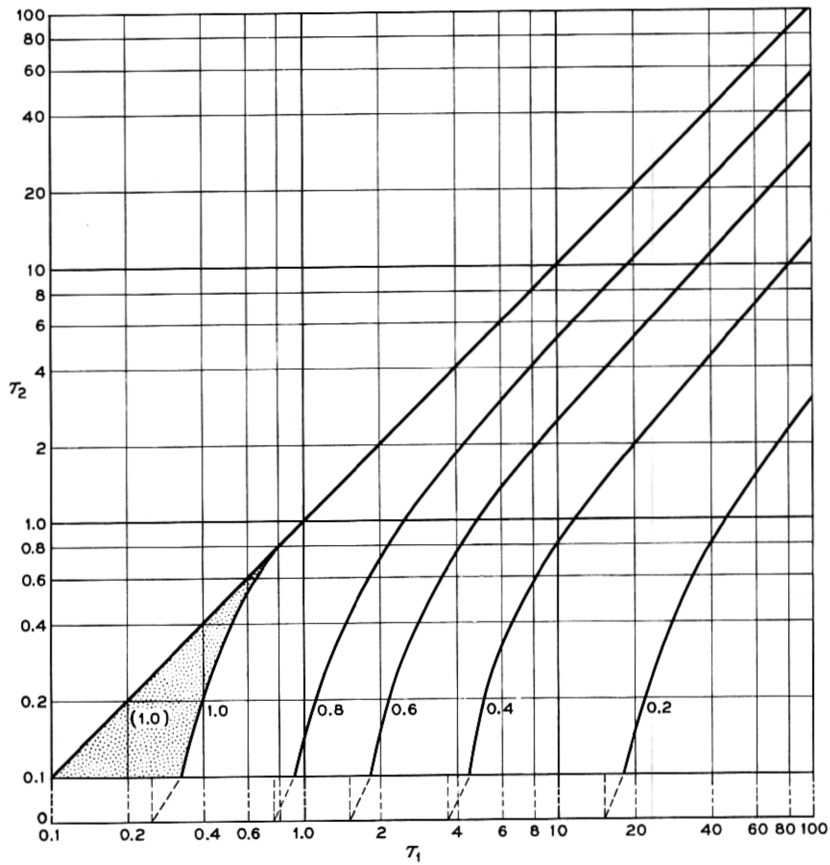


Fig. 13 — Contours of the pull-in to lock ratio on the filter plot.

We have shown that the small-signal properties of the phase-controlled oscillator are completely specified by the parameters τ_1, τ_2 and α. Therefore, for constant small-signal performance (such as noise bandwidth), the pull-in range is proportional to the count ratio N. We can get any pull-in frequency we wish by using a large enough count ratio.

There are two limitations on increasing the count ratio. The first is economy; high counts require more equipment. The second is theoretical. The comparator supplies data only once every period of the submultiple frequency. For our analysis to be valid, the submultiple frequency should be much higher than the cutoff frequency of the forward path, which is of the order of ω_n. This limits the maximum count.

For $\tau_2 \gg 1$, and $\tau_2/\tau_1 < 0.5$, the pull-in frequency approaches

$$\frac{\omega_p}{N\pi\alpha} \cong \frac{2}{\sqrt{3}} \sqrt{\frac{\tau_2}{\tau_1}}. \tag{30}$$

It is interesting to compare the pull-in frequency of a sawtooth comparator to that of a sinusoidal comparator[1] with the same gain at zero error. The normalized pull-in frequencies for both types of comparator are shown in Fig. 14, for a damping ratio ξ of $\frac{1}{2}$.

Fig. 14 shows that the sawtooth phase detector has a pull-in frequency at least twice that of a sinusoidal detector which has the same small-signal performance.

8.2 *Figure of Merit*

In most applications, a large pull-in frequency and a small noise bandwidth are desired. Unfortunately, these requirements are antagonistic, since a small noise bandwidth means that the loop cannot react to a rapidly flickering phase error. Examination of the formulas for pull-in (29) and jitter noise bandwidth (18) shows that both are proportional to the gain, α. Therefore a natural figure of merit is the ratio of pull-in frequency to the jitter noise bandwidth:

$$M = \frac{\omega_p}{B_j}. \tag{31}$$

Since the pull-in frequency is proportional to the count ratio N while

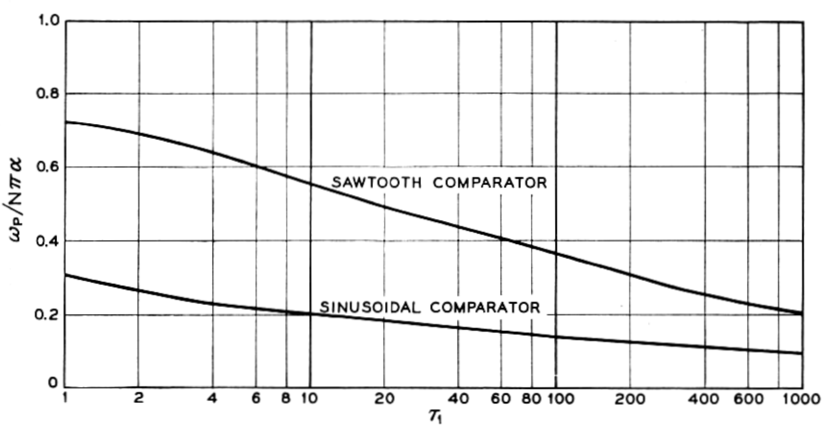


Fig. 14 — Normalized pull-in of the sinusoidal and sawtooth comparators for a damping ratio of 1/2.

the noise bandwidth is independent of N, the figure of merit is proportional to N. This means that we can get as large a value of pull-in as we wish for a given noise bandwidth, if we are willing to use a large count ratio.

The normalized figure of merit M/N is shown on the filter plot in Fig. 15.

D. Richman[8] has defined a different figure of merit, since he wished to compromise between noise bandwidth and gain. His figure of merit is equivalent to our normalized noise bandwidth (18), plotted in Fig. 7.

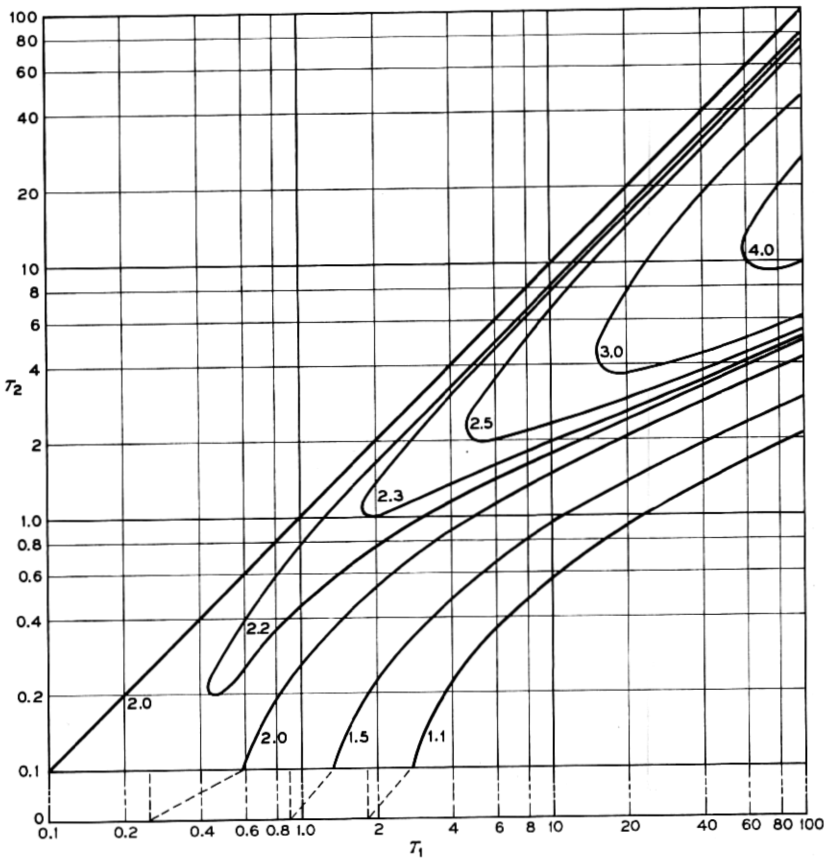


Fig. 15 — Contours of normalized figure of merit on the contour plot. The normalized figure of merit is the ratio of the pull-in to the noise bandwidth, divided by the count ratio N.

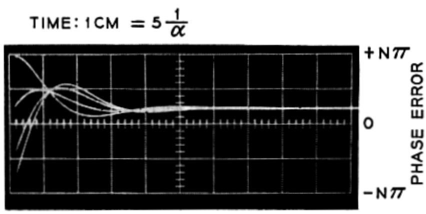
8.3 *Seize Frequency*

As long as the mistuning of a signal is less than the pull-in frequency, we can be sure the circuit will lock; but it may flicker for a long while before it does.

For some applications, it is important that the circuit synchronize immediately on a signal that has just started, without flickering through discontinuities. We define the *seize frequency* ω_s as the maximum mistuning of a suddenly connected signal that cannot cause a discontinuity after the initial phase jump (see Fig. 16).

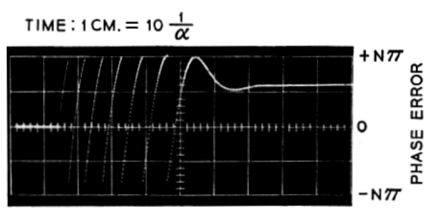
We have described a phase comparator which produces a zero error signal when there is no input signal. With such a device, the effect of suddenly connecting a signal is equivalent to a step phase shift of an arbitrary value between $-N\pi$ and $+N\pi$ and a step change in frequency equal to the mistuning of the signal ω_m.

In the marginal case, the phase error between the oscillator and the signal at the instant of connection is nearly $N\pi$. The seize frequency is the value of mistuning for which the initial derivative of the phase error



(a) SEIZE

MISTUNING SLIGHTLY LESS THAN SEIZE FREQUENCY. SEVERAL VALUES OF INITIAL PHASE ARE SHOWN. $\tau_1 = 10$, $\tau_2 = 2$



(b) PULL-IN

MISTUNING GREATER THAN SEIZE BUT LESS THAN PULL-IN FREQUENCY. NOTE THE DISCONTINUITIES. $\tau_1 = 10$, $\tau_2 = 2$

Fig. 16 — Scope traces of the phase error during capture.

is zero, so that no discontinuity results. It is easily shown that

$$\frac{\omega_s}{N\pi\alpha} = \frac{\tau_2}{\tau_1}. \tag{32}$$

Note that a circuit with an RC filter ($\tau_2 = 0$) may go through a discontinuity for any nonzero mistuning, if the initial phase shift is large enough.

According to Richman[8] the seize frequency for the sinusoidal comparator is $\alpha(\tau_2/\tau_1)$. As indicated by a comparison with (32), the seize frequency in general is simply τ_2/τ_1 times the lock frequency.

8.4 *Settling Time*

The settling time is the time required for the phase error to settle nearly to its steady state value after a change in input conditions. If no discontinuity occurs, the settling time t_s may be estimated to be the time at which the damping term $e^{-\xi\omega_n t_s}$ [in (26) and (27)] decays to 0.1. Then, substituting for ξ according to (11),

$$t_s = \frac{4.6}{\tau_2 + 1} T_1 . \tag{33}$$

If a discontinuity is crossed, an additional time will be required to allow the flickering to die out. During each flicker period a small charge is added to the filter capacitor, bringing the average output frequency of the oscillator closer to the input frequency. Finally, the circuit locks.

The flicker time for a given mistuning depends on the initial conditions, especially on the initial capacitor voltage. For the special case of a suddenly connected signal (initial capacitor voltage zero), D. Richman has derived[8] an approximation for t_F, the time in the flicker state, for the sinusoidal comparator.

He assumes that the capacitor voltage does not change appreciably during a single flicker period; in effect, he replaces the capacitor with a variable battery. Further, Richman neglects the effect of the initial phase. By applying his methods to the sawtooth comparator, we obtain:

$$\frac{t_F}{T_2} = \int_{\omega_m/\omega_L}^{\tau_2/\tau_1} \frac{\dfrac{\tau_1}{\tau_2}\, d\left(\dfrac{\omega_I}{\omega_L}\right)}{\dfrac{\omega_m}{\omega_L} - \left(\dfrac{\tau_1}{\tau_2}\dfrac{\omega_I}{\omega_L}\right) + \left(1 - \dfrac{\tau_2}{\tau_1}\right)\left[\coth^{-1}\left(\dfrac{\tau_1}{\tau_2}\dfrac{\omega_I}{\omega_L}\right)\right]^{-1}}, \tag{34}$$

where ω_I is an "instantaneous mistuning" parameter introduced by Richman.

Equation (34) is a good approximation for $\tau_2 \gg 1$ and $t_F \gg t_s$.

To carry out the integration, we must use numerical methods. We have plotted t_F/T_2 against ω_m/ω_L for various values of τ_2/τ_1 in Fig. 17. Experimental results are also shown in Fig. 17.

Note that t_F goes to zero as ω_m approaches the seize frequency and to infinity as ω_m approaches the pull-in frequency. If a short pull-in time is important, the mistuning frequency should not be allowed to approach the pull-in frequency.

8.5 *Maximum Frequency Shift*

Consider a phase-controlled oscillator which is locked on an input which is frequency modulated by a digital signal (frequency-shift key-

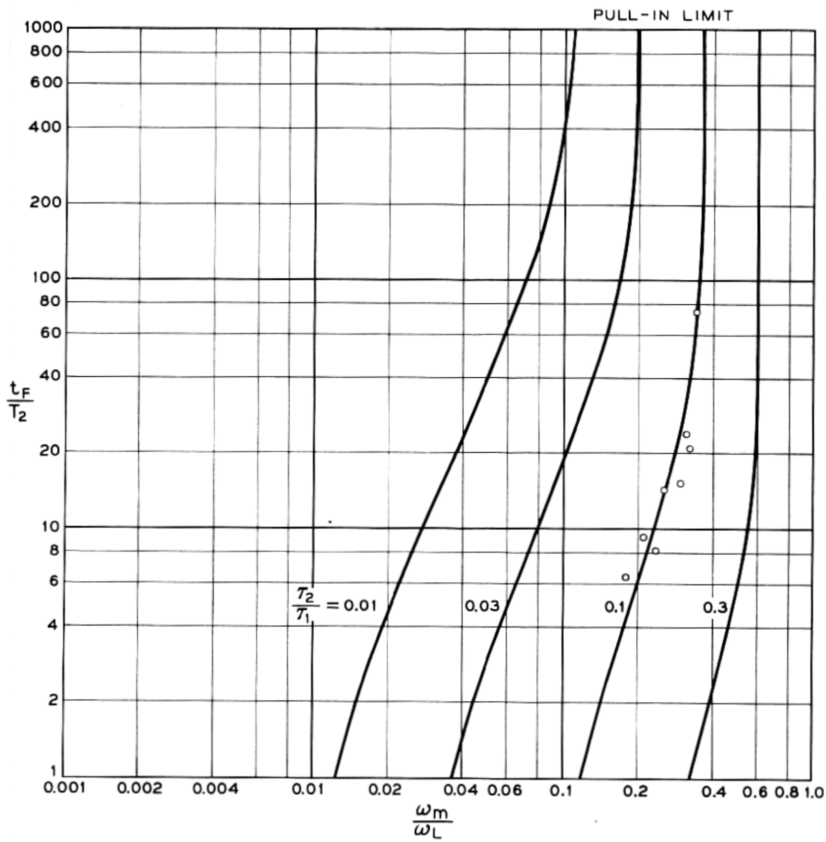


Fig. 17 — Flicker time during pull-in. The time is zero for mistuning less than the seize frequency and infinity for mistuning greater than the pull-in frequency.

ing). Let us find the maximum frequency shift that will not cause the phase error to cross a comparator discontinuity. We assume that the center frequency of the oscillator is set midway between the two signal frequencies. We further assume that the time constants of the phase-controlled oscillator are much smaller than the maximum time between shifts, so that the circuit may be in steady state before the next shift occurs.

We will consider the case of a sudden increase of frequency $\Delta\omega_i$. The initial phase error is $-(\Delta\omega_i/2\alpha)$. The maximum allowed phase error is $+N\pi$. Thus the peak change in phase error $\hat{\varphi}_e$, caused by the maximum allowable change of input frequency $\Delta\hat{\omega}_i$, is

$$\hat{\varphi}_e = N\pi + \frac{\Delta\hat{\omega}_i}{2\alpha}. \tag{35}$$

The error $\hat{\varphi}_e$ has been given in (28). Solving (35) for $\Delta\hat{\omega}_i$ in terms of $\hat{\varphi}_e/\Delta\hat{\omega}_i$, we have:

$$\frac{\Delta\hat{\omega}_i}{N\pi\alpha} = \frac{1}{\dfrac{\alpha\hat{\varphi}_e}{\Delta\hat{\omega}_i} - \dfrac{1}{2}}. \tag{36}$$

In the presence of mistuning, $N\pi$ in (35) and (36) is replaced by the margin φ_{er}, given in (3). Values of $\alpha\hat{\varphi}_e/\Delta\omega_i$ have been plotted in Fig. 11.

8.6 *Effective Comparator Characteristic in the Presence of Fast Jitter*

One of the functions of a phase-locked oscillator is to produce a steady output despite jitter and noise in the input signal. Therefore, we can expect that a major part of the phase comparator output will have frequencies much higher than the oscillator can follow. In such a situation only the low frequency component of the comparator output is significant in controlling the circuit.

The low-frequency component of the comparator output is the time average taken over a time interval which is longer than the period of the predominant jitter, but shorter than the response time of the circuit. The following analysis assumes that such an intermediate time range exists; i.e., that there is very little jitter whose frequency is low enough to cause the circuit to respond.

Let us write the phase error as the sum of a low-frequency component φ_{e0} and a fast jitter component φ_{ej}. Then the instantaneous output of the phase comparator is $f(\varphi_{e0} + \varphi_{ej})$. The average output of the comparator is

$$\overline{v_1} = \frac{1}{T_a} \int_0^{T_a} f(\varphi_{e0} + \varphi_{ej})\, dt, \tag{37}$$

where T_a is the averaging time.

Let us define an effective comparator characteristic in the presence of jitter:

$$\overline{v_1} = f_j(\varphi_{e0}). \tag{38}$$

This new characteristic governs the response of the circuit to slow phase changes in the presence of fast jitter.

Now we assume that the time of integration is such that the time spent at each value of φ_{ej} is proportional to the probability density of φ_{ej} at the value. For random processes, this requires that T_a be much greater than the correlation time of the process. If φ_{ej} is periodic, it is sufficient that T_a be equal to one period.

If this assumption is valid, we can replace the time integral (37) by an ensemble integral:

$$f_j(\varphi_{e0}) = \int_{-\infty}^{+\infty} f(\varphi_{e0} + \varphi_{ej}) p(\varphi_{ej})\, d\varphi_{ej}\,, \tag{39}$$

where $p(\varphi_{ej})$ is the probability density of the jitter.

Equation (39) represents a smoothing operation by the jitter probability function upon the comparator characteristic. If the density function has even symmetry, (39) is analogous to the general filter equation

$$v_{\text{out}}(t) = \int_{-\infty}^{+\infty} v_{\text{in}}(t - \tau) i(\tau)\, d\tau \tag{40}$$

where $i(\tau)$ is the impulse response of a hypothetical filter.

The effective sawtooth comparator characteristic for Gaussian, sinusoidal, and square wave jitter is shown in Fig. 18. These photographs were obtained by opening the phase-controlled oscillator loop and allowing the oscillator to free run. This means that the average phase error φ_{e0} increases linearly with time. The phase comparator output was passed through a low-pass filter to obtain $f_j(\varphi_{e0})$.

Note that jitter always decreases the peak comparator output voltage.

For Gaussian noise, we can evaluate (39) by neglecting the possibility of jitter crossing two or more discontinuities. Then the effective comparator characteristic for $(-N\pi < \varphi_{e0} < +N\pi)$ is

$$f_j(\varphi_{e0}) = \varphi_{e0} + N2\pi \left[\int_{-\infty}^{-x_1} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)}\, dx - \int_{x_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx \right] \tag{41}$$

where

$$x_1 = \frac{N\pi + \varphi_{e0}}{(\varphi_e)\mathrm{rms}},$$

$$x_2 = \frac{N\pi - \varphi_{e0}}{(\varphi_e)\mathrm{rms}}, \quad \text{and}$$

$(\varphi_e)rms$ is the root mean square phase error due to fast jitter.

The peak effective comparator output for Gaussian jitter is plotted in Fig. 19.

The effective comparator function for the sinusoidal comparator is very easy to find, using the filter analogy:

$$\begin{aligned} f(\varphi_e) &= \sin \varphi_e , \\ f_j(\varphi_{e0}) &= e^{-\frac{1}{2}(\varphi_e)\mathrm{rms}^2} \sin \varphi_{e0} . \end{aligned} \tag{42}$$

Therefore the effect of high-frequency Gaussian jitter for the sinusoidal comparator is simply to reduce the loop gain.

In general, the presence of fast jitter causes a deterioration of large

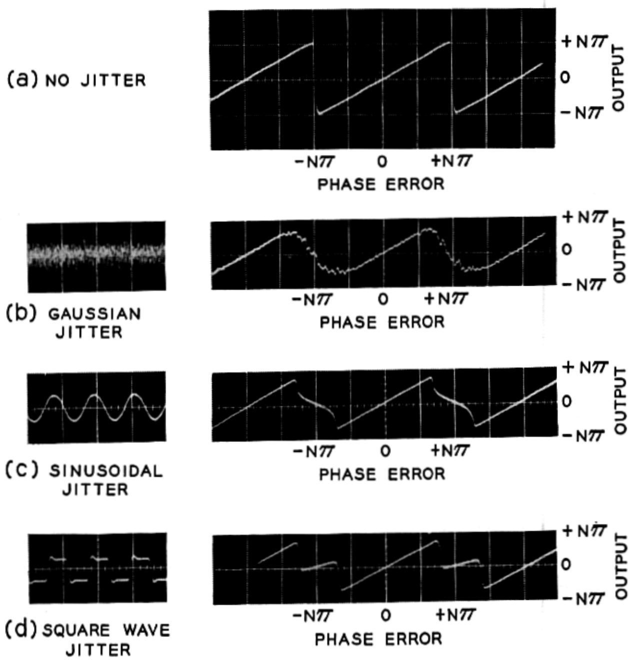


Fig. 18 — The phase comparator characteristic in the presence of fast jitter.

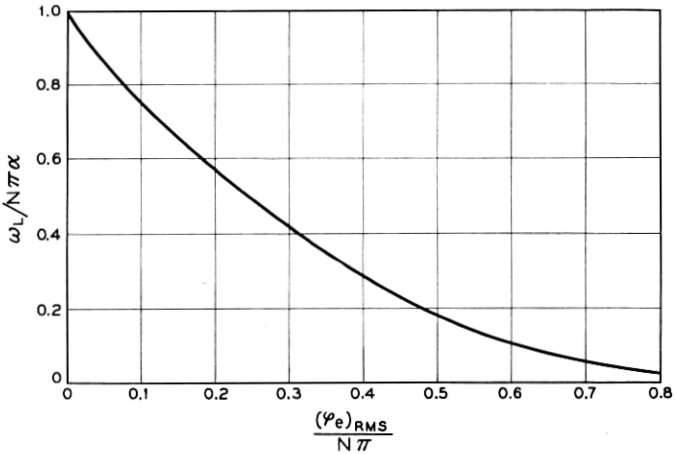


Fig. 19 — Normalized lock frequency (peak comparator output) in the presence of fast Gaussian jitter.

signal performance. For example the lock frequency depends directly upon the peak comparator output, which decreases as jitter increases.

8.7 *False Synchronization Mode*

As shown in Fig. 18(d), the sawtooth comparator characteristic can have a region with positive slope centered on an average phase error $N\pi$. This means that the circuit can synchronize in this region instead of the region near zero error. In this false mode the jitter continually crosses and recrosses the discontinuity.

Fortunately, this undesirable mode is possible only for certain types and amplitudes of jitter. We can test for the possibility of the false mode by examining the slope of $f_j(\varphi_{e0})$ at $N\pi$. We can write $f(\varphi_e)$ in the vicinity of $N\pi$ as $\varphi_e - 2N\pi U(\varphi_e - N\pi)$, where U is the unit step function. Substituting this in (39) and taking the derivative, we have

$$\left.\frac{df_j(\varphi_{e0})}{d(\varphi_{e0})}\right|_{N\pi} = 1 - 2N\pi p(0), \tag{43}$$

where $p(0)$ is the probability density of φ_{ej} at 0. Therefore the false mode is possible when $p(0) < 1/2N\pi$.

For square wave jitter, $p(0) = 0$. Therefore the false mode is always possible.

For sine wave jitter with an amplitude $A, p(0) = 1/A\pi$. Therefore the false mode is possible only when $A > 2N$. Since the comparator can accommodate only jitter error amplitudes less than πN in the normal mode, we are not likely to encounter sinusoidal jitter large enough to support the false mode.

It can be shown by using the filter analogy that Gaussian jitter cannot produce the false mode; the slope of $f_j(N\pi)$ is always negative. The $p(0)$ criterion is not applicable in the case of Gaussian jitter because more than one discontinuity is involved.

We see that the false mode need be considered only for signals with jitter such that $p(0)$ is very small.

Even if the false mode has been established, a lull in the jitter will cause the circuit to jump to the normal mode. It will stay in the normal mode even if jitter returns, as long as no discontinuities occur.

IX. DESIGN METHODS

We have analyzed many properties of the phase-controlled oscillator with a sawtooth comparator. Some of these properties, notably the lock range, pull-in range, and noise bandwidth are significant in nearly all applications of the device. Others, such as peak jitter gain, seize frequency, and settling time are important only for certain specific applications.

Usually, in a particular design problem, two or three of the properties will be of prime importance and the rest can be neglected. Then the problem is to find the values of the design parameters (α, τ_1, τ_2) which yield the best combination of the important properties. If the properties are simple, like the lock frequency ($N\pi\alpha$), it is easy to find the best design.

9.1 *Filter Plot*

Unfortunately, most of the properties of the phase-controlled oscillator turn out to be complicated transcendental functions of the design parameters τ_1 and τ_2. Therefore we have presented many of the properties as contour curves on a plot of τ_1 vs. τ_2, which we call the filter plot (Figs. 5, 7, 8, 11, 13, and 15). Most of the properties are normalized through division by the gain constant α. In some cases, the count ratio N is also used as a normalizing factor.

τ_1 and τ_2 are the time constants of the phase lag filter (Fig. 4b), multiplied by the gain α. We have plotted τ_1 and τ_2 on logarithmic scales, to allow the presentation of large ranges. A useful property of these scales is that a given percentage change in τ_1 or τ_2 appears as a constant displacement on the plot. This facilitates estimating the effect of variations of the parameters.

τ_1 is always larger than τ_2 ; therefore the possible designs are restricted to the region below the 45° line on the plot. Points along this 45° line are identical, and correspond to the case of no filter. When τ_2 is zero, the phase lag filter degenerates to the RC filter. Since this case is of some interest, we have provided a zero τ_2 axis below the plot and indicated the intersection of the various contours with this line.

9.2 *Approximate Relations*

An examination of the filter plots shows that there are large regions where the contours approach straight lines. It is possible to derive simplified formulas for these regions. A summary of these approximations and the conditions for their validity is given below.

Pull-in frequency: $$\frac{\omega_p}{N\pi\alpha} \cong \frac{2}{\sqrt{3}}\sqrt{\frac{\tau_2}{\tau_1}} \quad (\tau_2 \gg 1) \tag{44}$$

Noise bandwidth: $$\frac{B_j}{\pi\alpha} \cong \frac{1}{2}\frac{\tau_2}{\tau_1} \quad (\tau_2^2/\tau_1 \gg 1) \tag{45}$$

Figure of merit: $$\frac{M}{N} \cong \frac{4}{\sqrt{3}}\sqrt{\frac{\tau_1}{\tau_2}} \quad (\tau_2^2/\tau_1 \gg 1) \tag{46}$$

Peak error, frequency step: $$\frac{\alpha\hat{\varphi}_e}{\Delta\omega_i} = \frac{\tau_1}{\tau_2} \quad (\tau_2^2/\tau_1 \gg 1) \tag{47}$$

Equation (44) has been derived from (29) by A. J. Goldstein.[4] Equation (45) can easily be found from (18). Equation (46) is found by dividing (44) by (45), according to the definition (31). Equation (47) can be derived from (28).

These approximations sometimes allow analytic methods to be used to find an approximate optimum solution. This requires justification of operating in the region where the approximations are valid.

9.3 *Optimization Techniques*

There are several types of optimization methods, which we shall discuss in order of increasing difficulty.

The simplest method optimizes one property by varying one parameter, all other parameters being fixed. This yields a class of designs which has one less parameter than the general case. The remaining parameters can be assigned to satisfy requirements on other properties, in confidence that the final design will have high performance for the optimized property.

An example of this approach has appeared in the literature.[1,9] The gain α and the time constant τ_1 (which together specify the resonant frequency) are held constant and the time constant τ_2 is varied to minimize the noise bandwidth B_j. This process restricts the design to

$$\tau_2 + 1 = \sqrt{\tau_1 + 1}. \tag{48}$$

For large values of τ_1, the damping ratio ξ approaches 0.5. Equation (48) is plotted in Fig. 20, against the figure of merit contours.

Let us describe one procedure for designing a circuit using (48). The gain α can be set to give the proper lock frequency. Then τ_1 and τ_2 can be set to give the required pull-in frequency, while satisfying (48).

This approach is good, and yields rather useful designs. However, it does not necessarily produce the best possible design for a given set of requirements.

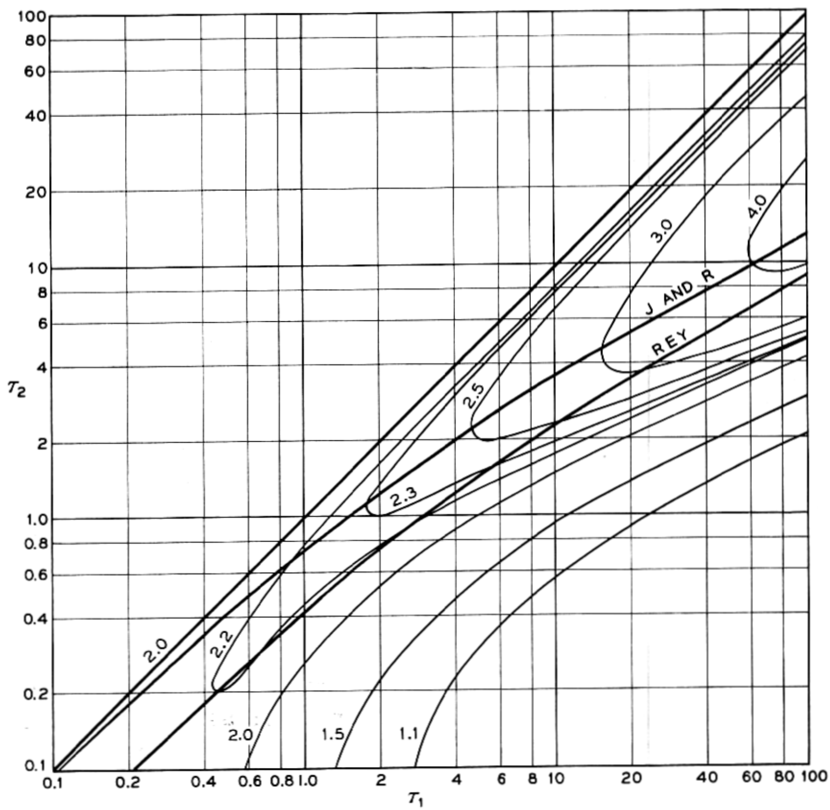


Fig. 20 — "Optimized" designs of Jaffe and Rechtin[10] and T. Rey,[1] with the figure of merit contours on the filter plot.

For example suppose the lock and pull-in frequencies have been specified, with a pull-in to lock ratio of 0.5. Following the above procedure, we compare Figs. 13 and 20 to find that τ_1 and τ_2 should be 18 and 3.2 to satisfy (48) and have $\omega_p/\omega_L = 0.5$. From Fig. 7 we find that the normalized noise bandwidth is 0.19.

To see that a better design than this is possible, suppose that τ_1 and τ_2 were 100 and 20. Then the noise bandwidth would be 0.12, a large improvement.

A more powerful technique is possible when some properties are specified by system requirements and another property should be optimized. The specified properties are used to restrict the range of the design parameters. Then the remaining range is examined to seek the optimum design.

For example, suppose that the lock range has been specified, and the normalized noise bandwidth is required to be less than 0.2. It is desired to maximize the pull-in frequency. A comparison of Fig. 7 and Fig. 13 shows that the design should lie on the upper part of the 0.2 noise bandwidth contour, and τ_1 and τ_2 should be as large as possible.

The most common problems require a compromise design which yields good results for two or more properties. Sometimes it is possible to express the relative importance of the properties mathematically. Then the optimum design can be derived analytically. A good example of this is given by Jaffe and Rechtin,[10] where the desirable properties are low-noise bandwidth and a high peak phase error due to a frequency step. Their design curve is shown in Fig. 20.

More often the relative importance of the properties is indistinctly known, and the engineer must use his judgment in striking a compromise. The filter plots are intended to aid this process by giving the engineer a "feel" for the circuit properties over the entire range of the parameters.

9.4 *Numerical Example*

To show how the design aids we have presented can be used in practice, we will do a realistic problem.

A phase-controlled oscillator is to be designed to smooth jitter in a 1.5 megacycle signal. In the worst case of mistuning, the circuit must pull itself into synchronism. We wish to design a circuit with low jitter noise bandwidth.

The uncertainty of the input signal is $\pm 10^{-5}$ or ± 15 cycles per second.

The oscillator center frequency is controlled by a frequency determining element, which we shall assume to be a crystal, and by the surrounding circuit. We take the range of the crystal as $\pm 10^{-5}$ or ± 15 cps. The effect of variations in the circuit will depend on the control the circuit has on the crystal, which is in turn related to the gain α. We assume that the range of center frequency due to circuit variations is $\pm 0.2 N\pi\alpha$.

The count ratio N is 4.

Let us make the following definitions:

δ_s — maximum deviation of the signal frequency (rad per sec)

δ_0 — maximum deviation of the crystal tuning (rad per sec)

ϵ — maximum deviation of the oscillator center frequency due to circuit variations, divided by $N\pi\alpha$.

Then the maximum mistuning (which determines the pull-in frequency) is

$$\hat{\omega}_m = \omega_p = \delta_s + \delta_0 + \epsilon N\pi\alpha. \tag{49}$$

If we assume that the final design will be in a region where the approximate relations hold, we can use (44) and (45) for the pull-in frequency ω_p and the jitter noise bandwidth B_j.

When (44) and (49) are combined, we find

$$\frac{\tau_2}{\tau_1} = \frac{3}{4}\left(\frac{\delta_s + \delta_0}{N\pi\alpha} + \epsilon\right)^2. \tag{50}$$

For this value of τ_2/τ_1, the jitter bandwidth is

$$B_j = \frac{3}{8}\pi\alpha\left(\frac{\delta_s + \delta_0}{N\pi\alpha} + \epsilon\right)^2. \tag{51}$$

Note that the only variable is α. When we minimize B_j by varying α, we obtain

$$\begin{aligned}
\alpha &= \frac{\delta_s + \delta_0}{N\pi\epsilon}, \\
\frac{\tau_2}{\tau_1} &= 3\epsilon^2, \\
\omega_p &= 2(\delta_s + \delta_0), \quad \text{and} \\
B_j &= \frac{3}{2}\frac{(\delta_s + \delta_0)\epsilon}{N}.
\end{aligned} \tag{52}$$

When the numerical substitutions are made, we have

$$\begin{aligned} \alpha &= 75 \text{ rad/sec per radian}, \\ \frac{\tau_2}{\tau_1} &= 0.12, \\ \omega_p &= 377 \text{ rad/sec, or 60 cps,} \quad \text{and} \\ B_j &= 14 \text{ rad/sec, or 2.25 cps.} \end{aligned} \tag{53}$$

Now we have not yet completely specified the design, because we only have the ratio of τ_2 and τ_1. We can be confident of the numbers above for any value of τ_1, as long as we have the proper value of τ_2/τ_1 and as long as we stay in the region where the approximate relations are valid.

If we make τ_1 very large, we will require a very long time constant in the filter. Therefore we will make τ_1 just large enough to satisfy the condition for the approximate noise bandwidth relation, $\tau_2^2/\tau_1 \gg 1$. Let us set $\tau_2^2/\tau_1 = 4$. Then, from (53)

$$\begin{aligned} \tau_2 &= 33, \\ \tau_1 &= 275, \\ T_2 &= \frac{\tau_2}{\alpha} = 0.44 \text{ sec,} \quad \text{and} \\ T_1 &= \frac{\tau_1}{\alpha} = 3.67 \text{ sec.} \end{aligned} \tag{54}$$

If high accuracy is required, the values of τ_2 and τ_1 given in (54) can be used to find the exact values of ω_p and B_j, instead of using the approximate values given in (53).

X. CIRCUIT MODIFICATIONS

A two mode system has often been used[11] to increase the pull-in frequency. In this system, a frequency detector as well as a phase detector is used; the output of the frequency detector adjusts the oscillator tuning until the phase-controlled loop can synchronize. This scheme greatly extends the pull-in range, but requires additional hardware.

Another means of extending the pull-in frequency has been published by R. Ley.[9] Back-to-back diodes are placed across the series filter resistor R_1. When the circuit is in synchronism and the jitter is small, the diodes do not conduct. The small signal properties are just as we have analyzed them. However, if the circuit is not synchronized, the flickering error voltage will cause the diodes to conduct, shorting out R_1. This will bring the pull-in frequency up near the lock frequency.

The major drawback of this method is that large jitter error voltages will make the diodes conduct, and be passed on to the oscillator.

Either or both of these methods may be used to greatly extend the pull-in range if the other system requirements permit their use.

XI. SUMMARY

Nearly all the properties of the phase-controlled oscillator which have appeared in the literature have been analyzed for the case of the sawtooth comparator and the phase lag filter.

New theoretical material has been introduced on the effects of fast noise and jitter.

The sawtooth comparator has advantages over the sinusoidal comparator for many applications. The reason for this is that the gain of the sawtooth comparator remains constant over a broader range of operation.

The properties of the phase-controlled oscillator are presented in a manner which facilitates design without unnecessary restrictions. Various methods of design are discussed, and numerical examples are provided to illustrate the methods.

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GLOSSARY OF IMPORTANT SYMBOLS

A Laplace transform is denoted by capitalizing the symbol.

B_j	jitter noise bandwidth
B_i	interference noise bandwidth
B_D	demodulator noise bandwidth
$f(\varphi_e)$	comparator function
$f_j(\varphi_{e0})$	effective comparator function
$G(\omega)$	any normalized noise transfer function

$H(s)$	filter transfer function
α_1	dc gain, comparator
α_2	dc gain, filter
α_3	frequency to voltage ratio, oscillator
$\alpha = \alpha_1\alpha_2\alpha_3$	open loop dc gain
$M = \dfrac{\omega_p}{B_j}$	figure of merit
N	count ratio
T_1	large filter time constant
T_2	small filter time constant
t_s	settling time
t_F	flicker time
v_1	comparator output voltage
v_2	oscillator input voltage
v_n	interference noise density
v_i	signal voltage amplitude
v_M	modulating voltage
Y	jitter transfer function
$\hat{Y}$	peak jitter gain
$\xi = \dfrac{1}{2}\dfrac{\tau_2 + 1}{\sqrt{\tau_1}}$	damping ratio
φ_i	input phase
$\Delta\varphi_i$	change in input phase
φ_o	output phase
$\varphi_e = \varphi_i - \varphi_o$	phase error
$\hat{\varphi}_e$	peak phase error
φ_{er}	phase error margin
φ_{e0}	short-time average phase error
φ_{ej}	phase error due to fast jitter
$(\varphi_e)_{rms}$	root mean square of φ_{ej}
$\tau_1 = \alpha T_1$	large filter time constant (normalized)
$\tau_2 = \alpha T_2$	small filter time constant (normalized)
ω_i	input frequency
$\Delta\omega_i$	change in input frequency
$\hat{\Delta}\omega_i$	maximum frequency shift
ω_m	mistuning frequency
$\omega_n = \dfrac{\alpha}{\sqrt{\tau_1}}$	natural frequency
ω_L	lock frequency
ω_p	pull-in frequency
ω_s	seize frequency

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