Analysis of the Phase-Controlled Loop with a Sawtooth Comparator

By A. JAY GOLDSTEIN

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Because of the recent interest in phase-controlled oscillators, a discussion of the phase-controlled loop with a sawtooth comparator is presented. The main emphasis is on finding the pull-in range of the loop. A companion paper in this issue (Ref. 4) deals with applications and shows how design parameters can be obtained from results developed here.

I. INTRODUCTION

The phase-controlled oscillator has evoked much interest in recent years. Some of its applications are to synchronism in television,[1,2] synchronization to a harmonic of a crystal oscillator,[3] elimination of jitter in pulse code modulation,[4] tracking filters, etc.

The general phase-controlled oscillator loop is given in Fig. 1. The incoming signal and the variable oscillator have the same free-running frequency ω_c. The phase comparator has as its output some function f of the phase difference $\varphi_e = \varphi_i - \varphi_0$. As examples of $f(\varphi_e)$ we have

the linear case: $f(\varphi_e) = \varphi_e$

the sinusoidal case: $f(\varphi_e) = \sin \varphi_e$

the sawtoothed case $f(\varphi_e) = \varphi_e \quad \text{for} \quad -\frac{d}{2} < \varphi_e < \frac{d}{2}$

(see Fig. 2): $f(\varphi_e + nd) = f(\varphi_e) \quad \text{for} \quad n = \cdots -1,0,1, \cdots .$

The output of the phase comparator passes through a filter whose impulse response is $h(t)$. The output of the filter $v(t)$ controls the variable oscillator according to the equation

$$\frac{d\varphi_0}{dt} = \alpha v(t). \tag{1}$$

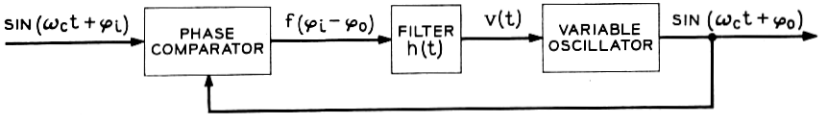

Fig. 1 — The general phase-controlled loop.

Thus, the frequency of the controlled oscillator is

$$\omega_c + \frac{d\varphi_0}{dt} = \omega_c + \alpha v(t).$$

In a companion paper in this issue, C. J. Byrne[4] discusses the engineering origins and applications of the sawtoothed comparator and shows how design parameters can be obtained from the results of this article.

This article is primarily concerned with finding the pull-in range of the loop. This is defined precisely in Section III. Briefly it is *the maximum asymptotic (in time) value of the mistuning $d\varphi_i/dt$ for which the slave oscillator eventually synchronizes or locks to the input frequency*. All of the literature cited in the references deals with this problem for the case of a sinusoidal or linear phase comparator. The linear case is easily solved since the resulting differential equation is linear. (See in particular Labin[5] for a detailed discussion.) In the sinusoidal case the differential equation of the system is nonlinear. Only in the cases of no filter and an ideal integrator has the equation, up to the present, been solved in closed form. See Labin[5] for an excellent discussion of the no-filter case. In order to handle the nontrivial filter, many authors have used methods of phase plane analysis.[6,7,8] Phase plane analysis is restricted to the problem of *capture range* in which the mistuning and phase error are zero for negative time, and the mistuning is constant for positive time. This

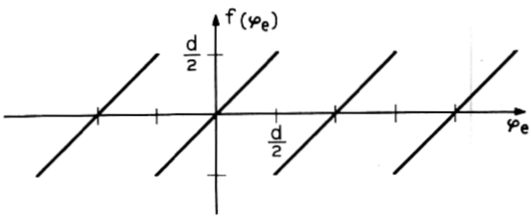

Fig. 2 — The sawtoothed phase comparator characteristic. The phase error φ_e is difference between the input and output phases of the loop.

kind of analysis gives only upper and lower bounds for the capture range and is restricted to a lag filter (Fig. 3). For an RC filter ($R_2 = 0$), Barnard[8] shows how phase plane analysis can give exact results.

To obviate the mathematical complexities, people have resorted to making various hypotheses about the nature of the solution of the nonlinear differential equation. These assumptions are based upon physical intuition and gross behavior observed in the laboratory. Different assumptions have led to different approximate solutions for the capture range. Moreover, they deal primarily with the lag filter, since it leads to a second-order differential equation while a more general filter gives a higher-order differential equation.

The loop equation when expressed as an integral equation is

$$\frac{d\varphi_e}{dt} = -\alpha \int_0^t f[\varphi_e(t')]h(t - t')\, dt' + \frac{d\varphi_i}{dt} - \alpha v_0(t).$$

It is surprisingly tractable for the sawtooth comparator, and the pull-in range can be computed for any filter. Fig. 4 shows the excellent agreement between theory and experiment for the lag filter. These experimental results were obtained by C. J. Byrne.

To obtain our results, we too must make an assumption. While the assumptions other authors have made deal with the behavior (in steady state) when far outside the pull-in range, ours deals with the behavior just outside of the pull-in range (see Section 4.4). This hypothesis is easily verified experimentally and has been so verified by C. J. Byrne for a representative selection of RC filters.

A brief description of each section follows.

Section II gives the basic integro-differential equation of the loop.

Section III defines the lock and pull-in range. The former is called by some the pull-out range. The lock range is the maximum frequency difference that the loop can lock to. It is given by

$$\omega_L = \alpha f_{\max} H(0)$$

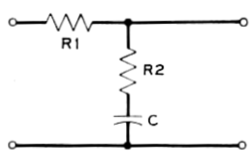

Fig. 3 — The integral compensating or lag filter. The normalized time constants are $\tau_1 = \alpha(R_1 + R_2)C$ and $\tau_2 = \alpha R_2 C$. For an RC filter $\tau_2 = 0$. α = (V. F. O. output frequency shift)/(V. F. O. input voltage).

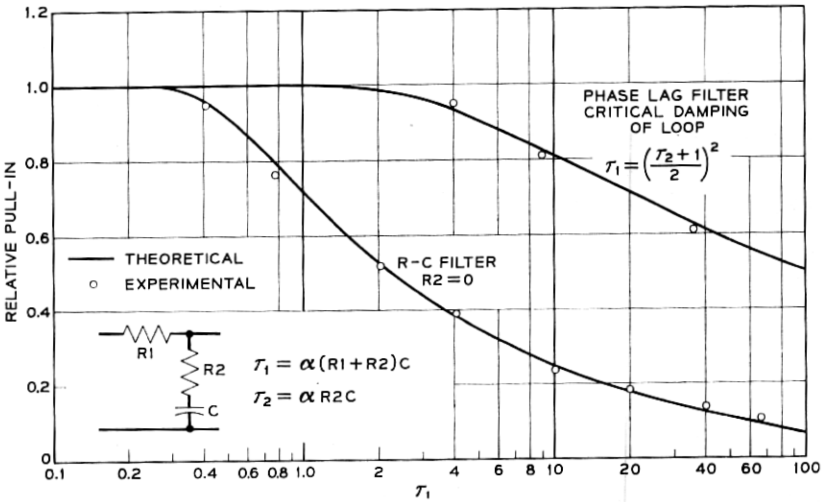

Fig. 4 — The relative pull-in range. For critical damping $(\tau_2 + 1)^2/4 = \tau_1$ and for the RC filter $R_2 = 0$.

where $H(0)$ is the dc gain of the filter and $f_{\max}$ is $d/2$ for the sawtooth comparator.

Section 4.1 gives the solution of the basic loop equation. This solution is the sum of (1) the solution of the linear phase comparator problem, (2) a series of step functions, and (3) a series of damped exponentials. The solution is obtained by representing the phase comparator function as the sum of the phase difference [giving (1)] and a series of translated unit step functions [giving (2) and (3)].

Section 4.2 gives the steady-state solution when *not captured*. In this case the output of the phase comparator is a periodic function whose period for a fixed filter depends on the asymptotic relative mistuning (Fig. 5). By examining this non-capture situation we obtain the pull-in range. We observe that in non-capture state the period and relative mistuning *must* correspond to a point on a curve typified in Fig. 5. Hence a relative mistuning lying below the minimum point of the curve corresponds to a capture or synchronized situation, and the height of the minimum gives the ratio of pull-in to lock range (the relative pull-in γ_p).

Section V gives all the explicit design formulae for the lag filter. For the special case of the RC filter ($R_2 = 0$ in Fig. 3) an explicit formula for relative pull-in can be given, namely

$$\gamma_p = \begin{cases} \tanh \dfrac{\pi}{4} (\alpha R_1 C - \frac{1}{4})^{-\frac{1}{2}} & (\alpha R_1 C \geqq \frac{1}{4}) \\ 1 & (\alpha R_1 C \leqq \frac{1}{4}). \end{cases}$$

In all other cases we must find the roots of a transcendental equation by numerical approximation methods.

Byrne[4] gives graphs of the results of Section V for the lag filter. These are graphs of relative pull-in (Fig. 13), noise bandwidth (small signal) (Fig. 7), figure of merit (relative pull-in/noise bandwidth) (Fig. 15), and maximum loop gain (small signal) (Fig. 8).

The noise bandwidth is a measure of the ability of the loop to reject small phase noise. More explicitly, the noise bandwidth N of a network is defined to be the bandwidth of that ideal low-pass filter which passes the same white noise power as the given network.

There are many possible ways of defining a single measure of the performance of the system, depending on the particular application in mind. We have chosen the figure of merit γ_p/N, i.e., a large figure of merit implies high noise rejection and large relative pull-in.

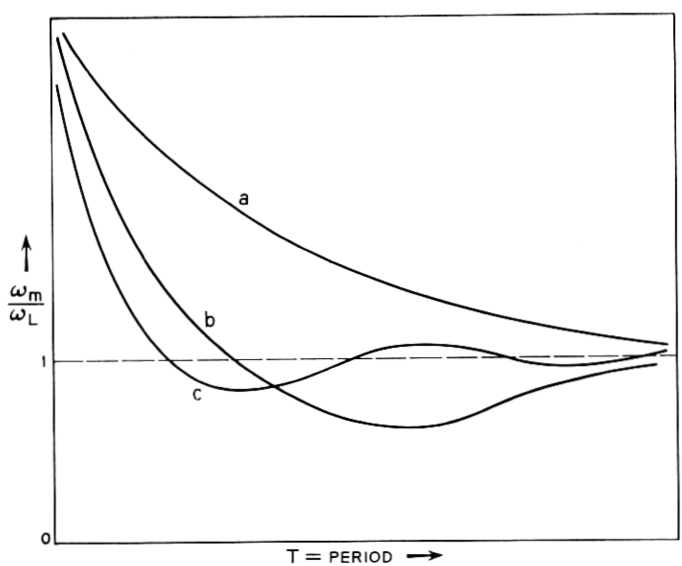

Fig. 5 — Relative mistuning ω_m/ω_L in a non-synchronized steady state vs the period T of the comparator output. (a) no filter, (a) and (b) overdamped loop and (c) underdamped loop.

For small phase deviations of the input, the comparator can be considered linear. We can then discuss the gain of the loop as a function of the frequency of the phase deviation. The maximum of the loop gain is denoted by $\hat{Y}$. In some applications $\hat{Y}$ is restricted by stability considerations to be less than unity.

Section VI is devoted to the derivation of several interesting asymptotic results for the lag filter. A simple formula is obtained for the relative pull-in for large values of the filter time constants. It is also shown that if the maximum loop gain is allowed to have a fixed value greater than unity, then, by appropriate choice of the time constants, arbitrarily large values of the figure of merit can be obtained.

This work could not have been completed without the aid of M. Karnaugh who suggested the problem, E. G. Kimme who proved that the sawtooth comparator is a continuous approximation to the original discrete sample data system, C. J. Byrne whose experimental work confirmed the formulae derived here, D. E. Rowlinson who constructed the contour curves from the computer data, and R. D. Barnard with whom many fruitful discussions were held.

II. THE BASIC LOOP EQUATION

We obtain an integro-differential equation for the loop by noting that the output of the filter can be written as a convolution plus initial conditions

$$v(t) = \int_0^t f[\varphi_e(t')]h(t - t')\,dt' + v_0(t)$$

where $v_0(t)$ is the filter output due to residual charges and fluxes in the filter at time zero. $v_0(t)$ damps out exponentially in all filters of interest. Substituting this into (1) and replacing φ_0 by $\varphi_i - \varphi_e$ we obtain

$$\frac{d\varphi_e}{dt} = -\alpha \int_0^t f[\varphi_e(t')]h(t - t')\,dt' + \frac{d\varphi_i}{dt} - \alpha v_0(t). \qquad (2)$$

In order that the derivations which follow not be unduly complicated by inessential parameters, we make the following normalizations

$$\begin{aligned} x(t) &= \varphi_e(t)/f_{\max} \quad (f_{\max} = d/2) \\ C(x(t)) &= f(\varphi_e(t))/f_{\max} \\ \varphi(t) &= \varphi_i(t)/f_{\max}\,. \end{aligned}$$

The normalized form of (2) becomes

$$\frac{dx}{dt} = -\alpha \int_0^t C[x(t')]h(t - t')\,dt' + \frac{d\varphi}{dt} - \alpha v_0(t)/f_{\max}\,. \qquad (3)$$

III. DEFINITIONS OF LOCK RANGE AND RELATIVE PULL-IN

If the input frequency $\omega_c + d\varphi_i/dt$ is increased "very slowly" to a value which is not too large, the output frequency $\omega_c + d\varphi_0/dt$ will follow it (i.e., be always equal to, or locked to, the input frequency). The maximum value of $d\varphi_i/dt$ for which lock-in will occur is called the *lock range* and is denoted by ω_L. More precisely, ω_L will be determined from (1) when the maximum dc voltage v is obtained. This maximum value is clearly the product of α, $f_{\max}$ the maximum value of the comparator function f and $H(0)$ the dc gain of the filter.*

$$\omega_L = \alpha f_{\max} H(0). \tag{4}$$

Suppose that the input frequency is not increased slowly, but in some sudden or erratic manner. Suppose moreover that the input frequency approaches a limiting value, ω_m, the *mistuning;* i.e.

$$\lim_{t\to\infty} \frac{d\varphi_i}{dt} = \omega_m .$$

In general, even if $0 < \omega_m < \omega_L$ (that is, we are in the lock range), the output frequency will not asymptotically lock to the input frequency (that is, be captured), but will be a modulated frequency. We define the relative *pull-in range* γ_p to be that normalized maximum frequency difference such that

$$-\gamma_p\omega_L < \lim_{t\to\infty} \frac{d\varphi_i}{dt} = \omega_m < \gamma_p\omega_L \tag{5}$$

implies

$$\lim_{t\to\infty} \frac{d\varphi_0}{dt} = \omega_m . \tag{6}$$

Notice that we make no restriction on how $d\varphi/dt$ approaches ω_m, as long as $|\omega_m| < \gamma_p\omega_L$.

IV. DERIVATION OF RESULTS†

4.1 *Basic Equation*

Let

$$0 < t_0 < t_1 \cdots < t_n < \cdots$$

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* We shall use capital letters to denote the Laplace transform of the function denoted by corresponding lower-case letters.

† From here on we are dealing with the sawtooth comparator.

be all the instants (called discontinuity points) at which the phase difference $x(t)$ crosses the discontinuity of C, i.e.,

$$\lim_{\substack{\Delta \to 0 \\ \Delta > 0}} x(t_n - \Delta) = x(t_n -) = 1 + 2n'$$

where the first equality is a definition of $x(t_n -)$ and where n' is an integer dependent on n. Let

$$a_n = 1 \text{ if } x \text{ is increasing at } t_n$$

$$a_n = -1 \text{ if } x \text{ is decreasing at } t_n$$

$$a_n = 0 \text{ if } x \text{ is stationary at } t_n .$$

Using the unit step function

$$u(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t \geqq 0 \end{cases}$$

we can express $C(x(t))$ in the analytically useful form

$$\frac{C(x(t))}{2} = \frac{x(t)}{2} - n_0 - \sum_{j=0}^{\infty} a_j u(t - t_j) \tag{7}$$

where n_0 is an integer so chosen that this equation holds at $t = 0$.

We note here for future reference that

$$x(t_n -) = n_0 \pm \tfrac{1}{2} + \sum_{j=0}^{n} a_j . \tag{8}$$

Substituting (7) into the loop equation (3), we obtain

$$\begin{aligned} \frac{1}{2}\frac{dx}{dt} = -\frac{\alpha}{2}\int_0^t x(t')h(t - t')\,dt' + \alpha n_0 \int_0^t h(t')\,dt' \\ + \alpha \sum_{j=0}^{\infty} a_j \int_0^t u(t' - t_j)h(t - t')\,dt' \\ + \frac{1}{2}\frac{d\varphi}{dt} - \alpha v_0(t)/2f_{\max} . \end{aligned} \tag{9}$$

Solving this by Laplace transform methods we obtain

$$\begin{aligned} \tfrac{1}{2}X(s) = \frac{s\Phi(s) - [\varphi_0(0) + \alpha V_0(s)]/f_{\max}}{2(s + \alpha H(s))} + \frac{n_0}{s}\frac{\alpha H(s)}{s + \alpha H(s)} \\ + \sum_{j=0}^{\infty} \frac{a_j e^{-st_j}}{s}\frac{\alpha H(s)}{s + \alpha H(s)} . \end{aligned}$$

Letting

$$R(s) = \frac{1}{s + \alpha H(s)} \tag{10}$$

we have

$$1 - sR(s) = \frac{\alpha H(s)}{s + \alpha H(s)}. \tag{11}$$

Note that $sR(s)$ is the transfer function between input phase φ_i and comparator output phase φ_e for the linear comparator case. $r(t)$ is then the phase response at the linear comparator output due to a step in input phase. Since applications will require the system to synchronize to a step in phase, we will assume that $r(t) \to 0$ as $t \to \infty$.

Using this equation and taking inverse transforms in the equation for $X(s)$ we obtain

$$\frac{x(t)}{2} = \frac{x_L(t)}{2} + n_0(1 + r(t)) + \sum_{j=0}^{\infty} a_j u(t - t_j) - \sum_{j=0}^{\infty} a_j r(t - t_j) \tag{12}$$

where

$$X_L(s) = \frac{s\Phi(s) - [\varphi_0(0) + \alpha V_0(s)]/f_{\max}}{s + \alpha H(s)}.$$

$x_L(t)$ is the solution of the loop equation in the case of a linear comparator function $f(x) = x$.

Using the final value theorem[10] we have

$$\begin{aligned} x_L(\infty) &= \lim_{t\to\infty} x_L(t) = \lim_{s\to 0} sX_L(s) = \lim_{t\to\infty} \frac{\varphi'(t)}{\alpha H(0)} \\ x_L(\infty) &= \frac{2\omega_m/d}{\alpha H(0)} = \frac{\omega_m}{\omega_L} \end{aligned} \tag{13}$$

From (12) and (7) we have for the comparator output

$$\frac{C(x(t))}{2} = \frac{x_L(t)}{2} - \sum_{j=0}^{\infty} a_j r(t - t_j) + n_0 r(t). \tag{14}$$

In a steady-state condition this reduces to

$$C(x(t)) = \frac{\omega_m}{\omega_L} - 2\sum_{-\infty}^{+\infty} a_j r(t - t_j) \tag{15}$$

where the $n_0 r(t)$ term vanishes because of the remarks following the definition of $R(s)$.

4.2 *Steady-State Solution When Not Captured*

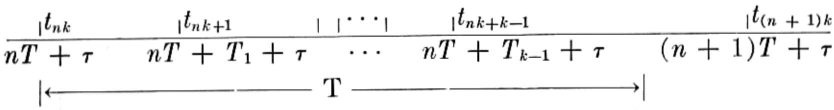
When we are not locked and in steady state, the output of the phase comparator will be a periodic function.* We give here a simplified heuristic derivation of the steady-state periodic solution. A rigorous derivation is easily obtained using the heuristics as a guide. In steady state, the normalized comparator output $y(t) = C(x(t))$ will be periodic with a period which we will call T. In a given period there may be many discontinuity points t_j ; let us suppose there are k. Then assuming we are in steady state, we can write

$$t_{nk+i} = nT + T_i + \tau, \quad \begin{cases} n = \cdots -1,0,1, \cdots \\ i = 0,1, \cdots ,k-1 \end{cases} \tag{16}$$

where

$$0 = T_0 < T_1 < \cdots < T_{k-1} < T.$$

These relations are illustrated below.



The a_n's will be periodic in steady state and we let

$$a_{nk+i} = A_i \quad \begin{cases} n = \cdots -1,0,1, \cdots \\ i = 0,1, \cdots ,k-1 \end{cases}. \tag{17}$$

It is no restriction to assume a time shift so that $\tau = 0$. Then, let

$$t = mT + u \ (0 < u \leqq T) \tag{18}$$

and combine the above three equations with (14). We obtain

$$\begin{aligned} y(t) &= C[x(t)] = C[x(mT + u)] \\ &= \omega_m/\omega_L - 2 \sum a_{nk+i} r(mT + u - t_{nk+i}) \\ &= \omega_m/\omega_L - 2 \sum_{i=0}^{k-1} A_i \sum_{n=-\infty}^{m} r[(m - n)T + u - T_i]. \end{aligned}$$

(The second summation has the upper limit m because $r(t) = 0$ for $t \leqq 0$.) Letting $j = m - n$, we obtain

$$y(t) = \frac{\omega_m}{\omega_L} - 2 \sum_{i=0}^{k-1} A_i \sum_{j=0}^{\infty} r(jT + u - T_i). \tag{19}$$

* A mathematical proof is not at hand. Indications of its truth are given in Benes[9] and experimental observations confirm this.

Let us define a periodic function

$$p(t,T) = \begin{cases} \sum_{j=0}^{\infty} r(t + jT) & (0 < t \leqq T) \\ p(t - nT,T) & (nT < t \leqq (n + 1)T) \end{cases} \tag{20}$$

or

$$p(t,T) = \sum_{-\infty}^{+\infty} r(t + jT).$$

$(r(t)) = 0$ for $t \leqq 0$ makes $p(t,T)$ a well defined function.) With this definition, the normalized steady-state comparator output, when not locked, can be written

$$y(t) = \frac{\omega_m}{\omega_L} - 2 \sum_{i=0}^{k-1} A_i p(t - T_i ,T). \tag{21}$$

The expression for $p(t,T)$ is familiar to those in the field of sample data systems.* Though superficially formidable, it can be expressed in closed form quite easily for the only important class of the filter transfer functions $H(s)$, namely rational functions. In that case $R(s)$ is a rational function too. Hence $r(t)$ is a linear combination of exponentials of the form $t^m e^{\beta t}$ (real part of β negative). Then $p(t,T)$ for $0 < t \leqq T$ is a linear combination of geometric series, each of the form

$$\begin{aligned} z(t) &= \sum_{j=0}^{\infty} (t + jT)^m e^{\beta(t+jT)} \\ &= \frac{d^m}{d\beta^m} \sum_{j=0}^{\infty} e^{\beta(t+jT)} \\ &= \frac{d^m}{d\beta^m} \frac{e^{\beta t}}{1 - e^{\beta T}} . \end{aligned} \tag{22}$$

This steady-state solution consists of a constant term $2\omega_m/d\alpha H(0)$, which is the normalized steady-state output for a linear phase comparator plus a linear combination (with coefficients ± 1) of time translates of the function $p(t,T)$, which is periodic of period T. The derivation shows that every steady-state periodic solution of the loop equation has the form of (21).

Equation (21) hides several pitfalls. These are:

1. We must have $| y(t) | \leqq 1$. Hence only certain T and T_i are admissible.

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* It is the response of a filter $R(s)$ to an input $\sum_{j=-\infty}^{+\infty} \delta(t + jT)$.

2. Are the solutions represented by (21) physically realizable?
3. Are the solutions represented by (21) stable with respect to small noise perturbations?

These three topics are grouped under the title Boundary Conditions and will be discussed following a discussion of the pull-in range.

4.3 *Relative Pull-in*

From the definition of T, $y(T-) = \pm 1$ and by an appropriate choice of τ in (16) (if $\omega_m > 0$) we may assume $y(T) = 1$. Then from (21)

$$\frac{\omega_m}{\omega_L} = 1 + 2 \sum_{i=0}^{k-1} A_i p(T - T_i, T). \tag{23}$$

Now the minimum value of $\omega_m > 0$ for which we have a non-constant periodic steady-state stable solution is by definition $\gamma_p \omega_L$, hence

$$\gamma_p = 1 + 2 \min \sum_{i=0}^{k-1} A_i p(T - T_i, T). \tag{24}$$

where the minimum is taken over all T and over all steady-state solutions satisfying conditions 1, 2 and 3 above.

4.4 *Boundary Conditions*

4.4.1 *Discontinuity Point Condition*

$y(t)$, being the normalized phase comparator output, satisfies $-1 \leqq y(t) \leqq 1$. Also $y(t'-) = \pm 1$ if and only if for some n and i, $t' = T_i + nT$, or $y(t)$ is stationary at t' (i.e., $y'(t') = 0$ and y at t' is increasing if $y(t') = -1$ or decreasing if $y(t') = 1$). These are equivalent to

$$\sum A_i(p(t' - T_i, T) - p(T - T_i, T)) = 0$$

if and only if $t' = nT + T_i$ or $y(t') - y(T)$ is stationary at t'. This restriction will be called the *discontinuity point condition.*

To analytically determine whether this condition is satisfied, in a general case, is clearly very difficult. For the case of the lag filter we can solve the problem analytically but must rely on an experimental fact. C. J. Byrne has found experimentally, in a large class of RC filters, that there is just one discontinuity per period T, i.e., the k in (21) is one. We will call this the *Experimental Hypothesis.* Thus

$$y(t) = \frac{\omega_m}{\omega_L} - 2p(t,T) \tag{25}$$

and

$$\gamma_p = 1 + 2 \min_T p(T,T). \tag{26}$$

In the section on the lag filter we show that if

$$p(T',T') = \min_T p(T,T) \tag{27}$$

then $p(t,T')$ satisfies the discontinuity point condition. Thus if $p(t,T')$ is realizable (it is — see below) and is stable under noise (we do not know, but have some evidence — see below) then

$$\gamma_p = 1 + 2p(T',T') \tag{28}$$

for the lag filter.

4.4.2 *Realizability Condition*

Does there exist, for each of the steady-state functions represented in (21) satisfying the discontinuity point condition, a corresponding input function $\varphi(t)$? That is, are the $y(t)$ in (21) physically realizable?

In Appendix A we prove realizability for any filter but not in quite the form stated above. We do the following:

(a) A particular input $\varphi(t) = 2\omega_m/d$ is injected.
(b) The loop is broken at the output of the phase comparator.
(c) Into the filter, at this point, is injected a voltage which asymptotically has the form (21).
(d) One shows that the output of the phase comparator has asymptotically the same form.
(e) In steady state the loop is closed.

4.4.3 *Non-Synchronous Stability*

Are the solutions stable? By this we mean: Will a steady-state solution be thrown into synchronism by a "small" noise? In formal terms, we suppose that a solution $y(t)$ has a discontinuity point, say t_0 shifted by noise to $t_0 + \Delta_0$. Each of the following discontinuity points $t_1, t_2, \cdots, t_n, \cdots$ is shifted to $t_1 + \Delta_1, t_2 + \Delta_2, \cdots, t_n + \Delta_n, \cdots$. It suffices for our purposes that the $(t_n + \Delta_n)$'s be asymptotically periodic (i.e., the noise sends us into another periodic solution and not into synchronism). The best we have been able to prove is that

$$\lim_{n\to\infty} \left[\frac{d\Delta_n}{d\Delta_0} \bigg|_{\Delta_0 = 0} \right] = c < \infty.$$

This has been done for the lag filter using the experimental assumption that $k = 1$ and that $T' - \epsilon < T < T'$, for ϵ sufficiently small, where

T' is given in (28). Now it would suffice for stability to show that Δ_n is bounded for Δ_0 sufficiently small, but the above does not imply this, for all it says is that

$$\Delta_n = c\Delta_0 + \epsilon_n \Delta_0^2$$

and we do not know that ϵ_n is bounded.

V. LAG (INTEGRAL COMPENSATING) FILTER

5.1 *General Results*

This section gives all the explicit formulae for design procedures in the case of the lag filter (Fig. 3). We assume the experimental hypothesis (see Section 4.4) throughout this section.

The transfer function of the filter is

$$H(s) = \frac{t_2 s + 1}{t_1 s + 1}$$

where

$$t_1 = (R_1 + R_2)C.$$

$$t_2 = R_2 C$$

Hence

$$R(s) = \frac{1}{s + \alpha H(s)} = \frac{t_1 s + 1}{t_1 s^2 + (\alpha t_2 + 1)s + \alpha}$$

$$= \frac{p_1 + \frac{1}{t_1}}{p_1 - p_2} \frac{1}{s - p_1} - \frac{p_2 + \frac{1}{t_2}}{p_1 - p_2} \frac{1}{s - p_2}$$

where p_1 and p_2 are the roots of denominator of $R(s)$. In particular, introducing the normalized dimensionless time constants

$$\tau_i = \alpha t_i, \qquad i = 1,2$$

we have for the roots

$$p_i = \frac{1}{t_1}(a + (-1)^i b)$$

where

$$a = (\tau_2 + 1)/2 \geqq \tfrac{1}{2}$$

$$b^2 = a^2 - \tau_1 .^*$$

* The real or imaginary part of b is non-negative.

The denominator of $R(s)$ can be written in the form

$$s^2 + 2\omega_n \xi s + \omega_n^2$$

where

$$\omega_n^2 = (\alpha/t_1)$$

and ξ, the damping factor, is

$$\xi = (\tau_2 + 1)/2\sqrt{\tau_1} = a/(a^2 - b^2)^{\frac{1}{2}}.$$

In this notation we obtain

$$r(t) = \frac{1}{2b}[-(a - b - 1)\exp(-(a - b)t/t_1) + (a + b - 1)\exp(-(a + b)t/t_1)].$$

Because $r(t)$ is a linear combination of exponentials, we can easily sum the infinite series for $p(t,T)$, obtaining

$$p(t,T) = \tilde{p}(\eta',\eta) = \frac{1}{2b}\left[-(a - b - 1)\frac{\exp[-(a - b)\eta']}{1 - \exp[-(a - b)\eta]} + (a + b - 1)\frac{\exp[-(a + b)\eta']}{1 - \exp[-(a + b)\eta]}\right] \tag{29}$$

where $\eta' = t/t_1$ and $\eta = T/t_1$ are dimensionless time variables.

To obtain γ_p using the results of (27) we must find

$$\min_T p(T,T)$$

or the roots of

$$0 = \frac{d\tilde{p}(\eta,\eta)}{d\eta}.$$

Differentiating the expression for $\tilde{p}(\eta,\eta)$ we obtain $\eta \neq 0$ and

$$\frac{\sinh^2 (a - b)\eta/2}{\sinh^2 (a + b)\eta/2} = \frac{(a - b)(a - b - 1)}{(a + b)(a + b - 1)} \tag{30}$$

or $\eta = \infty$. And upon using the addition formula for the hyperbolic sine, we have

$$\frac{\tanh a\eta/2}{\frac{1}{b}\tanh b\eta/2} + \frac{b\tanh b\eta/2}{\tanh a\eta/2} = 2\frac{a^2 + b^2 - a}{2a - 1} = 2c \tag{31}$$

which defines c, or $\eta = \infty$.

Use of the quadratic formula gives

$$\frac{\tanh a\eta/2}{\frac{1}{b}\tanh b\eta/2} = c + \sqrt{c^2 - b^2} = c_1(a,b) \tag{32}$$

or $\eta = \infty$.* In special cases considered it was found that the minimum of $\tilde{p}(\eta,\eta)$ occurs at the first positive zero of its derivative (or at $\eta = \infty$).

5.2 *Critical Damping*

From (31) we see that as b approaches zero (damping factor equals one),

$$\begin{aligned} \frac{\tanh a\eta/2}{\eta/2} &= 2\,\frac{a(a-1)}{2a-1} = c_1(a,0), \quad \text{if} \quad a > 1 \\ \eta &= \infty, \quad \text{if} \quad \tfrac{1}{2} < a < 1. \end{aligned} \tag{33}$$

Thus $\gamma_p = 1$ for $b = 0$ and $\frac{1}{2} < a < 1$.

5.3 *No Filter and RC Filter*

The filter parameters satisfy

$$0 \leqq \tau_2 \leqq \tau_1$$

which upon conversion to the a and b parameters become

$$(a - 1)^2 \geqq b^2$$

and

$$a \geqq \tfrac{1}{2}.$$

Equality holds in the first case, when $R_1 = 0$ or $C = 0$ (i.e., there is no filter) and in the second case, when $R_2 = 0$, (i.e., a simple RC filter.)

For no filter, $a + b - 1 = 0$ or $a - b - 1 = 0$, and referring to (30) we have only $\eta = \infty$. Thus min $p(T,T) = 0$ and $\gamma_p = 1$.

For the RC filter $R_2 = 0$, $a = \frac{1}{2}$, we obtain from (30) $\eta \neq 0$ and $\sinh b\eta = 0$ or $\eta = \infty$. If b is real, $\eta = \infty$ and $\gamma_p = 1$. If b is imaginary

$$\eta = m\pi/b \qquad m = 1,2, \cdots$$

* If the negative sign were used in the quadratic formula then η would be negative (complex) when b was imaginary (real).

and we easily find that $\tilde{p}(m\pi/b, m\pi/b)$ is minimum at $m = 1$, giving finally

$$\gamma_p = \begin{cases} \tanh \dfrac{\pi}{4}\left(\tau_1 - \dfrac{1}{4}\right)^{-\frac{1}{2}} & \text{if} \quad \tau_1 > \dfrac{1}{4} \\ 1 & \text{if} \quad \tau_1 \leqq \dfrac{1}{4} \end{cases}. \tag{34}$$

The results of these special cases are graphically summarized in Fig. 6. (Also see Fig. 13 of Byrne, Ref. 4.) In the shaded area of Fig. 6 the

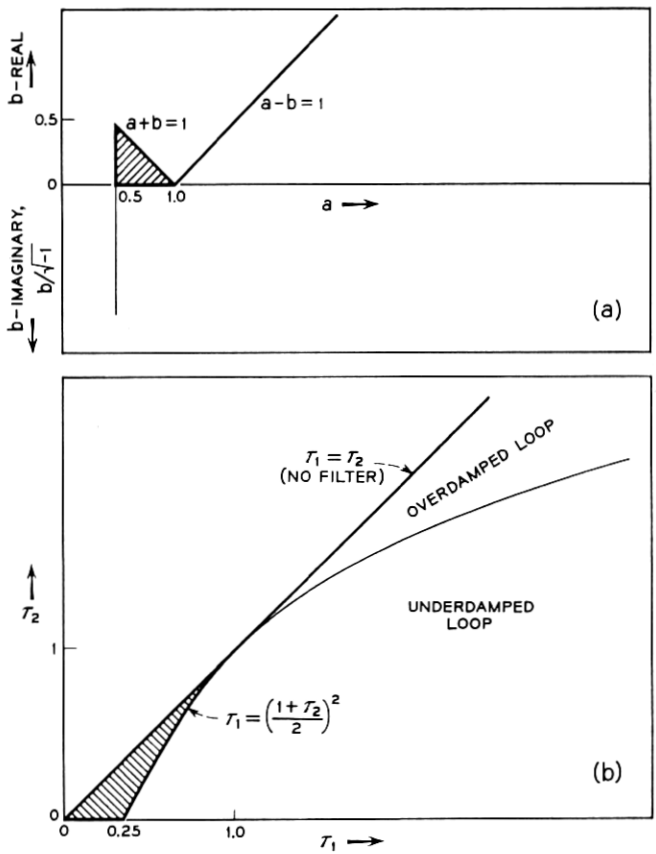

Fig. 6 — In part (a) the parameters a and b are restricted to lie below and/or to the right of the polygonal curve. The heavy lines and the shaded area give values of a and b for which the relative pull-in is unity. In part (b) the same information is given for the normalized time constants τ_1 and τ_2.

relative pull-in is unity. This follows from the fact that the left-hand side of (31) is bounded below by $2b$* while

$$2c - 2b = 2(a - b - 1)(a - b)/(2a - 1)$$

is negative in that region. Hence in (31) we must have $\eta = \infty$.

5.4 *Computational Procedures*

Except in the special cases of no filter ($R_1 = 0$) and the RC filter ($R_2 = 0$), there is no simple way of computing the relative pull-in. We must solve (32) by an iterative procedure and substitute the result into the equation for $\tilde{p}(\eta,\eta)$. If η is the solution of (32) or (33) we have a simpler equation for γ_p, namely

$$\gamma_p = [1 - D \operatorname{sech}^2 a\eta/2]/\tanh a\eta/2$$

where

$$D = \frac{(a - 1)c_1 - b^2}{c_1^2 - b^2} \quad (b \neq 0)$$

and

$$D = (a - \tfrac{1}{2})/a \quad (b = 0).$$

An upper bound for η is obtained from (32) and (33). Using the fact that $\tanh x < 1$, we obtain

$$\eta < \begin{cases} 2(\tanh^{-1} b/c_1)/b & (b \neq 0) \\ 2/c_1 & (b = 0) \end{cases}. \tag{35}$$

A lower bound for η in the case b is real is obtained by using the inequalities

$$z - z^3/3 \leqq \tanh z \leqq z.$$

Using this in the equation for η we have

$$a\eta/2 - (a\eta/2)^3/3 \leqq \tanh a\eta/2 = \begin{Bmatrix} c_1(a,b)\,\dfrac{\tanh b\eta/2}{b} \\ c_1(a,0)\eta/2 \end{Bmatrix} \leqq c_1\eta/2$$

giving the lower bound

$$2\left(3\,\frac{a - c_1}{a^3}\right)^{\frac{1}{2}} \leqq \eta.$$

* The left-hand side of (31) is of the form $b(x + 1/x)$. For x positive this is bounded below by $2b$.

We note here for future reference that if b is imaginary, $b = ib'$ then (35) implies the inequality

$$0 \leqslant \eta b' < \pi. \tag{36}$$

5.5 *Discontinuity Point Condition for the Lag Filter*

To prove that this condition is satisfied, it suffices to show that

$$\frac{dy}{dt} \neq 0 \quad \text{for} \quad 0 < t < T. \tag{37}$$

For, since we may suppose

$$y(T) = 1,$$

it follows that if

$$y(t') = 1 \quad (0 < t' < t)$$

then Rolle's theorem tells us that there exists a t'' with $t' < t'' < T$ such that $y'(t'') = 0$. This contradicts (37). It suffices also to prove (37) for that T which minimizes $p(T,T)$.

Recall that we are assuming we have a lag filter and that $k = 1$ in (21) (experimental hypothesis). Assuming (37) false, we obtain from (29) after some calculation

$$\frac{e^{-(a+b)\eta'/2}}{e^{-(a-b)\eta'/2}} = \frac{1 - e^{-(a+b)\eta/2}}{1 - e^{-(a-b)\eta/2}} \tag{38}$$

where η minimizes $\tilde{p}(u,u)$. Note that $0 < \eta' < \eta$.

Case 1, b real. Then $a > b$ and

$$\begin{aligned} \frac{e^{-(a+b)\eta'/2}}{e^{-(a-b)\eta'/2}} &= e^{-b\eta'} \\ &> e^{-b\eta} \\ &= \frac{e^{-(a+b)\eta/2}}{e^{-(a-b)\eta/2}} \\ &> \frac{1 - e^{-(a+b)\eta/2}}{1 - e^{-(a-b)\eta/2}}.* \end{aligned}$$

Hence (38) is false.

---

* If $0 < x < y < 1$, then $x/y > x - 1/y - 1$, for $-x > -y$ implies $xy - x > xy - y$; hence in factoring and dividing we obtain the desired inequality.

Case 2, b imaginary. Let $b = ib'$, then (38) becomes

$$-b'\eta'/2 + m\pi = \arg(e^{-(a+ib')\eta/2} - 1). \tag{39}$$

Now the real part of $e^{-(a+ib')\eta/2} - 1$ is negative and the imaginary part is negative (since by (36), $0 < b'\eta/2 < \pi/2$). Hence the right-hand side is an angle in the third quadrant. But the left-hand side is an angle which can only be in the second or fourth quadrant, since

$$0 < b'\eta' < b'\eta < \pi.$$

Hence (39) is false, proving the discontinuity point condition.

5.6 *Small-Signal Properties of the Loop*

In this section we give formulae for design parameters of the loop when we are operating on the linear portion of the phase comparator. Then the closed loop transfer function Y is

$$Y(s) = \frac{\alpha(t_2 s + 1)}{s^2 t_1 + (\alpha t_2 + 1)s + \alpha}.$$

Restricting our attention to real frequencies and normalizing the frequency ω by

$$\Omega = \omega/\alpha$$

and recalling that

$$\tau_1 = \alpha t_1 , \qquad \tau_2 = \alpha t_2$$

we obtain

$$|Y(\Omega)|^2 = \frac{\tau_2^2\Omega^2 + 1}{(\tau_2 + 1)^2\Omega^2 + (1 - \Omega^2\tau_1)^2}.$$

With the phase shift

$$\theta = -\arctan \frac{(1 + \tau_1\tau_2\Omega^2)\Omega}{(1 - \tau_1\Omega^2) + \tau_2(1 + \tau_2)^2\Omega^2} + \begin{cases} 0 \text{ if denominator positive} \\ \pi \text{ if denominator negative} \end{cases} *.$$

Important parameters for design are the maximum gain and the frequency and phase shift at which it occurs and the range of frequencies for which the gain exceeds one. Differentiating $|Y(\Omega)|^2$ and solving for its zero gives

* The arctan is an angle in the first or fourth quadrant.

$$\Omega_{\max}^2 = \begin{cases} (2\tau_1 - 1)/2\tau_1^2 & \text{if} \quad \tau_2 = 0, \quad \tau_1 \geqq \tfrac{1}{2} \\ \{[1 + (2(\tau_1 - \tau_2) - 1)\tau_2^2/\tau_1^2]^{\frac{1}{2}} - 1\}/\tau_2^2 & \text{if} \quad \tau_1 - \tau_2 \geqq \tfrac{1}{2} \\ 0 \quad \text{if} \quad \tau_1 - \tau_2 \leqq \tfrac{1}{2}. \end{cases}$$

Solving $| Y(\Omega) |^2 \geqq 1$ gives

$$\Omega^2 \leqq \Omega_1^2$$

where

$$\Omega_1^2 = \begin{cases} \dfrac{2(\tau_1 - \tau_2) - 1}{\tau_1^2} & \text{if} \quad \tau_1 - \tau_2 \geqq \tfrac{1}{2} \\ 0 & \text{if} \quad \tau_1 - \tau_2 \leqq \tfrac{1}{2} \end{cases}.$$

We also have the interesting inequality

$$\sqrt{2}\, \Omega_{\max} \leqq \Omega_1$$

with equality when $\tau_2 = 0$. The cases $\tau_2 = 0$ and $\tau_1 - \tau_2 \leqq \frac{1}{2}$ are immediate. The case $\tau_1 - \tau_2 \geqq \frac{1}{2}$ gives

$$\begin{aligned} \Omega_{\max}^2 &= \{[1 + \tau_2^2\Omega_1^2]^{\frac{1}{2}} - 1\}/\tau_2^2 \\ &= \frac{\Omega_1^2}{[1 + \tau_2^2\Omega_1^2]^{\frac{1}{2}} + 1} \\ &\leqq \frac{\Omega_1^2}{2} \end{aligned}$$

proving the result in this case.

We wish to emphasize that the maximum gain is unity if and only if $\tau_1 - \tau_2 \leqq \frac{1}{2}$. Peak gain = constant contours are given in Fig. 8 of Ref. 4.

The 3 *db* point occurs at $\Omega = \Omega_{\frac{1}{2}}$ where

$$| Y(\Omega_{\frac{1}{2}}) |^2 = \tfrac{1}{2}$$

from which we obtain

$$\Omega_{\frac{1}{2}}^2 = B + (B^2 + \tau_1^{-2})^{\frac{1}{2}}$$

where

$$B = (\tau_2^2 + 2(\tau_1 - \tau_2) - 1)/2\tau_2^2.$$

The noise bandwidth N is defined by[4]

$$N = \int_0^\infty | Y(\omega) |^2 \, d\omega.$$

It can be evaluated in various ways, for example see Ref. 10. One obtains

$$N = \pi\alpha(1 + \tau_2^2/\tau_1)/2(\tau_2 + 1).$$

In the no-filter case ($\tau_2 = \tau_1 = 0$) and RC case ($\tau_2 = R_2 = 0$) we have $N = \pi\alpha/2$. $N =$ constant contours are given in Ref. 4, Fig. 7.

As discussed in the introduction, the figure of merit was chosen to be the ratio N/γ_p. $N/\gamma_p =$ constant contours are given in Ref. 4, Fig. 15.

VI. ASYMPTOTIC RESULTS

In this section we obtain the asymptotic results stated in the introduction. Since the derivations are tedious, the results are first summarized.

From computer data, the contour curves of relative pull-in $\gamma_p =$ constant with ordinate and abscissa the normalized time constants

$$\tau_1 = \alpha(R_1 + R_2)C$$

$$\tau_2 = \alpha R_2 C$$

seem to be asymptotic to straight lines for large values of the normalized parameters. (See Fig. 13 in Ref. 4.) This observation led to the conjecture that for fixed γ_p and large τ_2

$$\tau_1 = K(\tau_2 + 1).$$

In Appendix C we prove this and show that

$$1/K = 1 - (1/\gamma_p - \gamma_p)^2(\tanh^{-1}\gamma_p)^2.$$

With respect to the figure of merit (see Fig. 15 in Ref. 4), the following very important results are derived in Appendix B for the lag filter. Suppose the peak small-signal phase gain $\hat{Y}$ of the loop is restricted to be unity (it is always unity at dc). Then the maximum merit obtainable for filters giving the unity peak loop gain is 2.27. If, however, we permit a fixed peak gain greater than unity, we can have an arbitrarily large merit figure. This usually results in very poor transient response. More precisely, the following results are derived in Appendix B. Let us consider those lag filters for which the peak small-signal (phase) gain is fixed at $\hat{Y}$. Define M by

$$M^2 = 1 - \hat{Y}^{-2}$$

Then for a filter with normalized time constants τ_1 and τ_2 and normalized frequency $\Omega = \omega/\alpha$, for which the loop has peak gain $\hat{Y}$ occurring at frequency $\Omega_{\max}$, we have

$$\Omega_{\max}^2 = M/\tau_1$$

and

$$\tau_1 = (M^2\tau_2^2 + 2\tau_2 + 1)/2(1 + M).$$

Asymptotically for τ_2 large we obtain for the noise bandwidth (with $a = (\tau_2 + 1)/2$)

$$N/\pi\alpha = \left(1 + \frac{2(1 - M)}{M^2}\right) \Big/ 4a + 0(a^{-2})*$$

and for the relative pull-in range

$$\gamma_p = a^{-\frac{1}{2}} \frac{2}{\sqrt{3}} \frac{\sqrt{M - 1}}{M} + 0(a^{-\frac{3}{2}})$$

$$= \frac{2}{\sqrt{3}} \left(\frac{\tau_2 + 1}{\tau_1}\right)^{\frac{1}{2}} + 0((\tau_2/\tau_1)^{\frac{3}{2}}).$$

Thus the noise bandwidth decreases as a^{-1} while the relative pull-in decreases as $a^{-\frac{1}{2}}$. Hence the figure of merit increase as $a^{\frac{1}{2}}$.

The derivations of the preceding results are given in Appendices B and C.

APPENDIX A

Realizability of Steady-State Solutions

Recall that (assuming $d = 2$)

$$y(t) = \frac{\omega_m}{\alpha H(0)} - 2 \sum_{i=1}^{k-1} A_i \sum_{n=0}^{\infty} r(t - T_i - nT) \tag{40}$$

where [see (13)]

$$x_L = x_L(\infty) = \frac{\omega_m}{\alpha H(0)}.$$

Since we assume $y(t)$ satisfies the discontinuity point condition

$$\frac{\omega_m}{\alpha H(0)} = 1 + 2 \sum_{i=0}^{k-1} A_i p(T - T_i, T).$$

Break the loop at the output of the phase comparator, inject $y(t)$

* Two functions $f(x)$ and $g(x)$ satisfy $f(x) = 0(g(x))$ if and only if $| f(x)/g(x) | \leq$ constant $< \infty$ for x sufficiently large.

into the filter, and let the input phase be $\omega_m t + x_L - c$ (where c is defined below). The phase output of the oscillator is given by

$$\frac{d\varphi_0(t)}{dt} = \alpha \int_0^\infty y(t')h(t-t')\,dt'$$

and upon integrating once and substituting (40),

$$\varphi_0(t) = \frac{\omega_m}{H(0)} \int_0^t \int_0^{t'} h(t'')\,dt''\,dt' - 2\sum_{i=0}^{k-1} A_i \sum_{n=0}^{\infty} \int_0^t \int_0^{t'} \alpha r(t'' - T_i - nT)h(t' - t'')\,dt''\,dt'.$$

By taking the Laplace transform of the double integral in the summation and by using the relations in (10) and (11), we find

$$\varphi_0(t) = \frac{\omega_m}{H(0)} \int_0^t \int_0^{t'} h(t'')\,dt''\,dt' - 2\sum_{i=0}^{k-1} A_i \left\{ \sum_{n=0}^{\infty} u(t - T_i - nT) - \sum_{n=0}^{\infty} r(t - T_i - nT) \right\}.$$

Now the remaining double integral is the integral of the step response of the filter and for large t is of the form $H(0)t + c$. Using this and the definition of $y(t)$, we obtain for large t

$$\varphi_0(t) \sim \omega_m t + c - 2\sum_{i=0}^{k-1} A_i \sum_{n=0}^{\infty} u(t - T_i - nT) - y(t) + x_L\,.$$

Now using the discontinuity point condition and the representation of the comparator in (7) we find the comparator output is asymptotically $y(t)$. Hence in steady state we may close the circuit without any disturbance.

APPENDIX B

Figure of Merit for Constant Peak Gain and Large Time Constants (Lag Filter)

From Section 5.6 we have for the closed loop small-signal (phase) gain

$$|\,Y(\Omega)\,|^2 = \frac{\tau_2^2\Omega^2 + 1}{(\tau_1 + 1)^2\Omega^2 + (1 - \Omega^2\tau_1)^2}. \tag{41}$$

Differentiating with respect to Ω and equating the result to zero gives

$$\tau_1^2\tau_2^2\Omega_{\max}^4 + 2\tau_1^2\Omega_{\max}^2 - [2(\tau_1 - \tau_2) - 1] = 0. \tag{42}$$

We can also represent the square of peak gain $\hat{Y}^2$ as the ratio of the derivatives of the numerator and denominator of (41) evaluated at $\Omega_{\max}$.*

$$\hat{Y}^2 = \frac{\tau_2^2}{(\tau_2 + 1)^2 - 2\tau_1(1 - \tau_1\Omega_{\max}^2)}$$

$$= \frac{\tau_2^2}{\tau_2^2 - [2(\tau_1 - \tau_2) - 1] + 2\tau_1^2\Omega_{\max}^2}.$$

This, after using (42), gives

$$\hat{Y}^2 = \frac{1}{1 - \tau_1^2\Omega_{\max}^4}. \tag{43}$$

Defining $M \geqq 0$ by

$$M^2 = 1 - \hat{Y}^{-2},$$

we have

$$0 \leqq M < 1, \quad \text{since} \quad 1 \leqq \hat{Y} < \infty.$$

Also (43) gives

$$\tau_1\Omega_{\max}^2 = M. \tag{44}$$

Substitute (44) into (42) and solve for τ_1. Then

$$\tau_1 = (M^2\tau_2^2 + 2\tau_2 + 1)/2(1 - M).$$

Using this result in the formula for the noise bandwidth (Section 5.6), we have for $\hat{Y}$ constant and τ_2 large (and hence $a = (\tau_2 + 1)/2$ is large)

$$N = \frac{\pi\alpha}{4a}\left(1 + \frac{2(1 - M)}{M^2}\right) + 0(a^{-2}). \tag{45}$$

We now turn to the problem of obtaining asymptotic expressions for the relative pull-in range for $\hat{Y}$ fixed and greater than unity.

We can rewrite the expression for τ_1 as

$$\tau_1 = \frac{2M^2}{1 - M}a^2 + 2(1 + M)a - \frac{M + 1}{2}. \tag{46}$$

* If $f(x) = p(x)/q(x)$, then $f'(x_0) = 0$ implies $f(x_0) = p'(x_0)/q'(x_0)$. One obtains this result by logarithmic differentiation of $f(x)$.

Using the definition $b^2 = a^2 - \tau_1$, (46) and the binomial expansion, we have for large a

$$\frac{b}{a} = \left(1 - \frac{2M^2}{1 - M}\right)^{\frac{1}{2}} \left(1 - \frac{1 - M}{1 - 2M} \frac{1}{a} + 0(a^{-2})\right) \tag{47}$$

if $M \neq \frac{1}{2}$ and

$$b^2 = -3(a - \tfrac{1}{4}) \tag{48}$$

if $M = \frac{1}{2}$. *In the following we suppose* $M \neq \frac{1}{2}$. Recall (31) that to find the relative pull-in we need the root of

$$\tanh a\eta/2 = c_1 \frac{\tanh b\eta/2}{b}$$

where

$$c_1 = c + (c^2 - b^2)^{\frac{1}{2}}$$

and

$$c = (a^2 + b^2 - a)/(2a - 1).$$

Hence

$$\begin{aligned} c &= (2a^2 - a - \tau_1)/(2a - 1) \\ &= a[1 - \tau_1/a(2a - 1)] = a - \frac{\tau_1}{\tau_2} \end{aligned} \tag{49}$$

$$c = a\left[1 - \frac{M^2}{1 - M} + 0(a^{-1})\right].$$

Also

$$\begin{aligned} c^2 - b^2 &= \left(a - \frac{\tau_1}{\tau_2}\right)^2 - (a^2 - \tau_1) \\ &= \left(\frac{\tau_1}{\tau_2}\right)^2 \left(1 - \frac{\tau_2}{\tau_1}\right) \end{aligned} \tag{50}$$

giving

$$(c^2 - b^2)^{\frac{1}{2}} = \frac{\tau_1}{\tau_2}\left[1 - \frac{1}{2}\left(\frac{\tau_2}{\tau_1}\right) - \frac{1}{8}\left(\frac{\tau_2}{\tau_1}\right)^2 + 0\left(\frac{\tau_2^3}{\tau_1^3}\right)\right].$$

Finally

$$c_1 = \left(a - \frac{\tau_1}{\tau_2}\right) + (c^2 - b^2)^{\frac{1}{2}}$$
$$c_1 = a\left[1 - \frac{1}{2a} + 0(a^{-2})\right]. \tag{51}$$

Setting $z = a\eta/2$, we have

$$\tanh z = \frac{c_1}{b} \tanh \frac{b}{a} z. \tag{52}$$

We will show that for large a, z is small and then obtain an approximation to z by using a power series expansion for $\tanh z$. First note that the derivative at zero of the right-hand side of (52) is

$$\frac{c_1}{a} = 1 - \frac{1}{2a} + 0(a^{-2})$$

which approaches 1 from below for large a. Also

$$\frac{c_1}{b} = \frac{a}{b} + 0(b^{-1})$$
$$= \left(1 - \frac{2M^2}{1 - M}\right)^{-\frac{1}{2}} + 0(b^{-1}).$$

Hence $| c_1/b |$ is bounded away from 1 (and greater than 1).

A sketch of the curves of the two sides of (52) with the above two facts shows that

$$\lim_{a\to\infty} z = 0.$$

Using power series expansions in (52) we obtain for large a

$$z - \frac{z^3}{3} = \frac{c_1}{b}\left(z\left(\frac{b}{a}\right) - \frac{1}{3} z^3 \left(\frac{b}{a}\right)^3\right)$$

or

$$z^2 = 3\,\frac{1 - c_1/a}{1 - c_1 b^2/a_3}$$
$$= 3\,\frac{\frac{1}{2a} + 0(a^{-2})}{\frac{2M^2}{1 - M} + 0(a^{-1})} \tag{53}$$
$$z = \frac{(3(1 - M))^{\frac{1}{2}}}{2M} a^{-\frac{1}{2}} + 0(a^{-1}).$$

From Section 5.4 the relative pull-in is

$$\gamma_p = \frac{1 - D}{\tanh\left(\frac{a\eta}{2}\right)} + D \tanh\left(\frac{a\eta}{2}\right)$$

where

$$D = ((a - 1)c_1 - b^2)/(c_1^2 - b^2).$$

Then

$$1 - D = \frac{(c_1 - a + 1)}{c_1 - b^2/c_1}$$

and

$$1 - D = \frac{c_1 - a + 1}{2c} \tag{54}$$

since

$$c_1 + b^2/c_1 = 2c.$$

Using (49) and (51) we have

$$1 - D = \frac{1 - M}{4aM^2} + 0(a^{-2}). \tag{55}$$

We now obtain the asymptotic formula for γ_p by substitution into the formula for γ_p the approximations for D, $1 - D$ and the approximation $\tanh(a\eta/2) \sim a\eta/2 = z$ with z approximated as in (53).

$$\begin{aligned} \gamma_p &= \frac{2(1 - M)^{\frac{1}{2}}}{3^{1/2}M} a^{-\frac{1}{2}} + 0(a^{-\frac{3}{2}}) \\ &= \frac{2}{3^{\frac{1}{2}}} ((\tau_2 + 1)/\tau_1)^{\frac{1}{2}} + 0(\tau_2^{-\frac{3}{2}}). \end{aligned} \tag{56}$$

APPENDIX C

Relative Pull-in (Lag Filter, Large τ_1 and τ_2)

Assuming that for a large a

$$\tau_1 = 2Ka + L + 0(a^{-1}) \tag{57}$$

we obtain from the definition

$$b^2 = a^2 - \tau_1$$

that

$$b = (a - K)\left[1 - \frac{L + K^2}{(a - K)^2} + 0(a^{-3})\right]^{\frac{1}{2}}.$$

Expanding the square root we obtain

$$b = (a - K)\left[1 - \frac{1}{2}\frac{L + K^2}{(a - K)^2} + 0(a^{-3})\right] \tag{58}$$

and

$$\frac{b}{a} = 1 - \frac{K}{a} + 0(a^{-2}). \tag{59}$$

From (57) since

$$a = (\tau_2 + 1)/2$$

we have

$$\frac{\tau_1}{\tau_2} = K + \frac{L + K}{\tau_2} + 0(\tau_2^{-2}) \tag{60}$$

From (50) we have

$$c^2 - b^2 = \left(\frac{\tau_1}{\tau_2}\right)^2 - \left(\frac{\tau_1}{\tau_2}\right)$$

and by using (60) we have

$$c^2 - b^2 = (K^2 - K)\left[1 + \frac{(L + K)(2K - 1)}{K(K - 1)}\frac{1}{\tau_2} + 0(\tau_2^{-2})\right].$$

Using the binomial expansion

$$(c^2 - b^2)^{\frac{1}{2}} = (K^2 - K)^{\frac{1}{2}}\left[1 + \frac{1}{2}\frac{(L + K)(2K - 1)}{K^2 - K}\frac{1}{\tau_2} + 0(\tau_2^{-2})\right]. \tag{61}$$

From (49)

$$c = a - \frac{\tau_1}{\tau_2}$$

and using (60), we obtain

$$c = a - K - \frac{L + K}{\tau_2} + 0(\tau_2^{-2}). \tag{62}$$

Then

$$\begin{aligned} c_1 &= c + (c^2 - b^2)^{\frac{1}{2}} \\ &= (a - K) + (K^2 - K)^{\frac{1}{2}} + 0(\tau_2^{-1}). \end{aligned} \tag{63}$$

Finally from (58) and (63)

$$\begin{aligned} \frac{c_1}{b} &= \frac{c_1/(a - K)}{b/(a - K)} \\ &= 1 + \frac{(K^2 - K)^{\frac{1}{2}}}{a} + 0(a^{-2}). \end{aligned} \tag{64}$$

Letting $z = a\eta/2$, (31) becomes

$$\tanh z = [1 + (K^2 - K)^{\frac{1}{2}}/a + 0(a^{-2})] \tanh (1 - K/a + 0(a^{-2}))z. \tag{65}$$

Using the addition formula for $\tanh (A + B)z$ and simplifying, we have

$$\begin{aligned} &\tanh^2 z \tanh (K/a + 0(a^{-2}))z - [(K^2 - K)^{\frac{1}{2}}/a + 0(a^{-2})] \tanh z \\ &\quad + [1 + (K^2 - K)^{\frac{1}{2}}/a + 0(a^{-2})] \tanh (K/a + 0(a^{-2}))z = 0. \end{aligned} \tag{66}$$

We show that z/a approaches zero with a and use this to simplify (66). From (35)

$$\begin{aligned} z &< \frac{2 \tanh^{-1} [1 + (K^2 - K)^{\frac{1}{2}}/a + 0(a^{-2})]}{1 - K/a + 0(a^{-2})} \\ &= \frac{\ln \left(\dfrac{2a}{(K^2 - K)^{\frac{1}{2}}} + 1 + 0(a^{-1}) \right)}{1 - K/a + 0(a^{-2})}. \end{aligned}$$

Since

$$\lim_{u \to \infty} \ln u/u = 0$$

we have

$$\lim_{a \to \infty} z/a = 0.$$

Returning to (66), we now have asymptotically

$$\tanh^2 z + \left(\frac{K - 1}{K} \right)^{\frac{1}{2}} \frac{\tanh z}{z} - 1 = 0.$$

Solving for K we obtain

$$1/K = 1 - \left[\frac{1}{\tanh z} - \tanh z \right]^2 z^2. \tag{67}$$

Now the relative pull-in given in Section 5.4 is

$$\gamma_p = \frac{1 - D}{\tanh z} + D \tanh z$$

and we easily show that [using (54)]

$$\begin{aligned} 1 - D &= \frac{c_1 - a + 1}{2c} \\ &= \frac{(K^2 - K)^{\frac{1}{2}} - K + 1 + 0(a^{-1})}{a - K + 0(a^{-1})} \\ &= 0(a^{-1}). \end{aligned}$$

Hence asymptotically for fixed z,

$$\gamma_p = \tanh z + 0(a^{-1}). \tag{68}$$

Thus for given relative pull-in, the above gives us z and $\tanh z$, and then (67) gives K from which (57) gives for large τ_1

$$\tau_1 = K(\tau_2 + 1). \tag{69}$$

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